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► **To cite this version:**

Christophe Lacave, Évelyne Miot. The vortex-wave system with gyroscopic effects. 2019. hal-02056936v1

**HAL Id: hal-02056936**

**<https://hal.science/hal-02056936v1>**

Preprint submitted on 4 Mar 2019 (v1), last revised 28 Aug 2019 (v2)

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# THE VORTEX-WAVE SYSTEM WITH GYROSCOPIC EFFECTS

*by*

Christophe Lacave & Evelyne Miot

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**Abstract.** — In this paper, we study the well-posedness for a coupled PDE/ODE system describing the interaction of several massive point vortices moving within a vorticity background in a 2D ideal incompressible fluid. The points are driven by the velocity induced by the background vorticity, by the other vortices, and by a Kutta-Joukowski-type lift force creating an additional gyroscopic effect. This system reduces to the so-called vortex-wave system, introduced by Marchioro and Pulvirenti [13, 14], when the point vortices are massless.

On the one hand, we establish existence of a weak solution before the first collision. We show moreover that the background vorticity is transported by the flow associated to the total velocity field. On the other hand, we establish uniqueness in the case where the vorticity is initially constant in a neighborhood of the points vortices. When all the densities of the point vortices have the same sign, no collision occurs in finite time and our results are then global in time. Our proofs strongly rely on the definition of a suitable energy functional.

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**2000 Mathematics Subject Classification.** — 35Q35, 76B03.

**Key words and phrases.** — Euler equations, small body, uniqueness.

## 1. Introduction

The purpose of this article is to investigate the well-posedness of the following PDE/ODE system:

$$(1.1) \quad \begin{cases} \partial_t \omega + \operatorname{div} \left[ \left( u + \sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - h_k)^\perp}{|x - h_k|^2} \right) \omega \right] = 0, \\ u = K * \omega, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \\ m_k \ddot{h}_k = \gamma_k \left( \dot{h}_k - u(t, h_k) - \sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{(h_k - h_j)^\perp}{|h_k - h_j|^2} \right)^\perp \text{ for } k = 1, \dots, N, \end{cases}$$

where

$$\omega : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad h_k : [0, T] \rightarrow \mathbb{R}^2 \text{ for } k = 1, \dots, N,$$

and where

$$(m_k, \gamma_k) \in \mathbb{R}_*^+ \times \mathbb{R} \text{ for } k = 1, \dots, N.$$

We supplement (1.1) with the initial conditions

$$(1.2) \quad \begin{aligned} \omega(0, \cdot) &= \omega_0 \in L^\infty(\mathbb{R}^2) \text{ compactly supported in some } B(0, R_0), \\ (h_k, h'_k)(0) &= (h_{k,0}, \ell_{k,0}) \text{ for } k = 1, \dots, N, \text{ with } h_{k,0} \text{ distinct.} \end{aligned}$$

System (1.1) for  $N = 1$  was derived by Glass, Lacave and Sueur [7] as an asymptotical system for the dynamics of a body immersed in a 2D perfect incompressible fluid, when the size of the body vanishes whereas the mass is assumed to be constant. The position of the body at time  $t$  is represented by the position  $h(t)$ , the fluid is described by its divergence-free velocity  $u(t, x)$  and vorticity  $\omega(t, x) = \operatorname{curl} u(t, x)$ . Under suitable decay assumptions, the divergence free condition enables to recover the velocity explicitly in terms of the vorticity by the Biot-Savart law [14]:  $u = x^\perp / (2\pi|x|^2) * \omega$ . The quantities  $m$  and  $\gamma$  are reminiscent of the mass of the body and of the circulation of the velocity around the body, respectively. The second order differential equation verified by  $h$  means that the body is accelerated by a force that is orthogonal to the difference between the body speed and the fluid velocity at this point. This gyroscopic force is similar to the well-known Kutta-Joukowski-type lift force revealed in the case of a single body in an irrotational unbounded flow, see for instance [10, 14, 20]. Therefore, a byproduct of [7] is the existence of a global weak solution of (1.1) when  $N = 1$ .

In the case  $N > 1$ , it is not known whether the previous convergence result holds. The main goal of this paper is to establish the existence and the uniqueness of solutions for any  $N \geq 1$ . In particular, we will prove that the trajectories of the points  $h_k$  never collide if all the circulations  $\gamma_k$  have the same sign. Such a result is important for example to justify the 2D spray inviscid

model established by Moussa and Sueur [16], which was derived as a mean-field limit  $N \rightarrow \infty$  of (1.1). We refer to that article for physical motivations.

Before giving the precise statements of our theorems, we mention that (1.1) reduces to the so-called vortex-wave system when setting  $m_k = 0$ :

$$(1.3) \quad \begin{cases} \partial_t \omega + \operatorname{div} \left[ \left( u + \sum_{k=1}^N \frac{\gamma_k (x - h_k)^\perp}{2\pi |x - h_k|^2} \right) \omega \right] = 0, \\ u = K * \omega, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \\ \dot{h}_k = u(t, h_k) + \sum_{j \neq k} \frac{\gamma_j (h_k - h_j)^\perp}{2\pi |h_k - h_j|^2} \text{ for } k = 1, \dots, N. \end{cases}$$

And indeed, for  $N = 1$ , Glass, Lacave and Sueur showed in [8] that the asymptotical dynamics of a small solid with vanishing mass evolving in a 2D incompressible fluid is governed by the vortex-wave system. The vortex-wave system was previously derived by Marchioro and Pulvirenti [13, 14] to describe the interaction of a background vorticity  $\omega$  interacting with one or several point vortices  $h_k$  with circulations  $\gamma_k$ . Very recently, Nguyen and Nguyen have also justified the vortex-wave system as the inviscid limit of the Navier-Stokes equations [17]. For System (1.3), existence of a weak solution (according to Definition 1.1 above) is proved up to the first collision time between the vortex trajectories. Concerning uniqueness, it is open in general, and it holds in the particular case when the vorticity  $\omega$  is initially constant near the point vortices (namely the condition appearing in Theorem 1.7 below), as suggested in [14, 13] and proved in [9, 15]. It is also proved in [13] that if all the  $\gamma_k$  have the same sign then no collision occurs in finite time therefore global existence holds.

As for the spray model, these results are the first key to get a time of existence that is independent of  $N$ , in order to consider the homogenized limit (or mean-field limit)  $N \rightarrow \infty$ , for instance, used by Schochet [18] to justify the vortex method in  $\mathbb{R}^2$ . The main goal of this paper is to establish the corresponding existence and uniqueness results for the vortex-wave system with gyroscopic effects (1.1). From now on we will refer to the points  $h_k$  in (1.1) as “massive” point vortices.

**Main results.** — The first part of our analysis focuses on the existence issue for (1.1).

**Definition 1.1.** — Let  $T > 0$ . We say that  $(\omega, \{h_k\}_{1 \leq k \leq N})$  is a weak solution of (1.1) on  $[0, T]$ , with initial data given by (1.2), if:

- $\omega \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^2)) \cap C([0, T], L^\infty - w^*)$ ,  $h_k \in C^2([0, T])$  for  $k = 1, \dots, N$ ,

- the PDE in (1.1) is satisfied in the sense of distributions, and the ODEs in (1.1) are satisfied in the classical sense.

**Theorem 1.2.** — *Let  $\omega_0$  and  $(\{h_{k,0}\}, \{\ell_{k,0}\})$  be as in (1.2). There exists  $T_* > 0$  such that for any  $T \in (0, T_*)$ , there exists a weak solution  $(\omega, \{h_k\})$  to (1.1) on  $[0, T]$ . Moreover, if we assume that  $\gamma_k$  have the same sign for all  $k = 1, \dots, N$ , then  $T_* = +\infty$ .*

**Remark 1.3.** — The maximal time  $T_*$  corresponds to the first collision between some of the massive points, and we will prove that no collision occurs in finite time if all the  $\gamma_k$  have the same sign.

**Remark 1.4.** — If the initial vorticity  $\omega_0$  was only assumed to be in  $L_c^p(\mathbb{R}^2)$  for some  $p > 2$ , then one could still prove (global if all  $\gamma_k$  have the same sign) existence of a weak solution to (1.1) such that  $\omega \in L^\infty(L^p)$ . However in this case no uniqueness result is known.

We will furthermore show that any weak solution as in Theorem 1.2 is transported by the regular Lagrangian flow relative to the total velocity field. We refer to the recent papers [1, 2, 3, 4] for the subsequent definition of regular Lagrangian flow:

**Definition 1.5.** — Let  $T > 0$  and let  $b \in L_{\text{loc}}^1([0, T] \times \mathbb{R}^2)$ . We say that  $X : [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$  is a regular Lagrangian flow relative to  $b$  if

- For a.e.  $x \in \mathbb{R}^2$ , the map  $t \mapsto X(t, x)$  is an absolutely continuous solution to the ODE  $\frac{d}{dt}X(t, x) = b(t, X(t, x))$  with  $X(0, x) = x$ , i.e. a continuous function verifying  $X(t, x) = x + \int_0^t b(s, X(s, x)) ds$  for all  $t \in [0, T]$ ;
- There exists a constant  $L > 0$  independent of  $t$  such that

$$\mathcal{L}^2(X(t, \cdot)^{-1}(A)) \leq L\mathcal{L}^2(A), \quad \forall t \in [0, T], \forall A \text{ Borel set of } \mathbb{R}^2,$$

where  $\mathcal{L}^2$  is the Lebesgue measure on  $\mathbb{R}^2$ .

Such a definition is intended to generalize the classical notion of flow associated to smooth vector fields. It was proved by Ambrosio [1] that such flow exists and is unique under BV-type regularity for the vector field. In [9, 4], a similar result was established for vector fields composed of a smooth part and of a part with a specific localized singularity. In the present setting, where the total velocity field in (1.1) contains singularities created by the point vortices, we will rely on those last results to establish the following general result.

**Theorem 1.6.** — *Let  $\{h_k\}$  be any given maps belonging to  $W^{2,\infty}([0, T]; \mathbb{R}^2)$  without collision:*

$$\min_{k \neq p} \min_{t \in [0, T]} |h_k(t) - h_p(t)| \geq \rho > 0.$$

For  $\omega_0 \in L_c^\infty(\mathbb{R}^2)$ , let  $\omega$  be a weak solution on  $[0, T]$  (in the class of Definition 1.1) to

$$(1.4) \quad \begin{cases} \partial_t \omega + \operatorname{div} \left[ \left( u + \sum_{k=1}^N \frac{\gamma_k (x - h_k)^\perp}{2\pi |x - h_k|^2} \right) \omega \right] = 0, \\ u = K * \omega, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \end{cases}$$

such that  $\omega(0, \cdot) = \omega_0$ . Then, there exists a unique regular Lagrangian flow  $X$  relative to the total velocity field

$$u(t, x) + \sum_{k=1}^N \gamma_k K(x - h_k(t)),$$

and  $\omega$  is transported by this flow:  $\omega(t, \cdot) = X(t, \cdot)_\# \omega_0$ .

Moreover, the vorticity  $\omega(t, \cdot)$  is compactly supported in  $B(0, R_T)$  for all  $t \in [0, T]$ , where  $R_T$  depends on  $T$  and the initial data.

Furthermore, we have the additional non collision information:

$$\text{for a.e. } x \in \mathbb{R}^2, \quad X(t, x) \neq h_k(t), \quad \forall t \in [0, T], \quad \forall k = 1, \dots, N.$$

Finally, if we assume

$$(1.5) \quad \omega_0 = \alpha_k \quad \text{on } B(h_k(0), \delta_0), \quad \forall k = 1, \dots, N$$

for some  $\alpha_k \in \mathbb{R}$  and  $\delta_0 > 0$ , there exists a positive  $\delta$  depending only on  $T$ ,  $\delta_0$ ,  $\|\omega_0\|_{L^\infty}$ ,  $\|h_k\|_{W^{2,\infty}([0,T])}$  and  $R_0$ , such that

$$\omega(t, \cdot) = \alpha_k \quad \text{on } B(h_k(t), \delta), \quad \forall t \in [0, T].$$

We emphasize that the previous theorem do not use the equation verified by the point vortices and can be applied for more general system than (1.1).

We now turn to the uniqueness issue, which is settled under Assumption (1.5), as is the case for the vortex-wave system.

**Theorem 1.7.** — Let  $\omega_0$  and  $(\{h_{k,0}\}, \{\ell_{k,0}\})$  be as in (1.2). Assume moreover that

$$\omega_0 = \alpha_k \quad \text{on } B(h_{k,0}, \delta_0), \quad \forall k = 1, \dots, N$$

for some  $\alpha_k \in \mathbb{R}$  and  $\delta_0 > 0$ . Then for any  $T > 0$ , there exists at most one weak solution  $(\omega, \{h_k\})$  to (1.1) on  $[0, T]$  with this initial condition.

This theorem together with Theorem 1.2 thus implies global existence and uniqueness if all the  $\gamma_k$  have the same sign, and existence and uniqueness up to the first collision otherwise.

The plan of this paper is as follows. In the next section, we prove Theorem 1.2 after collecting a few well-known properties. Then in Section 3 we

establish Theorem 1.6. Finally, in Section 4 we show how it implies Theorem 1.7 through a straightforward adaptation of the arguments of [9, 15]. For simplicity we focus for this on the case of one point, but the case of  $N \geq 1$  points is similar. The last section is devoted to some perspectives.

With respect to the above-mentioned previous works, the main novelty for the proofs here is the use of a new local energy functional

$$F_k(t) = \sum_{j=1}^N \frac{\gamma_j}{2\pi} \ln |X(t, x) - h_j(t)| + \varphi(t, X(t, x)) + \langle X(t, x), \dot{h}_k^\perp(t) \rangle,$$

defined as long as  $X(t, x) \neq h_j(t)$ , where  $\varphi$  is the stream function associated to  $u$  (namely  $u = \nabla^\perp \varphi$ ). Proving that no collision between the fluid particles and the point vortices occurs, and controlling the behavior of  $\omega(t, \cdot)$  near the massive point vortices actually amount to controlling  $F_k(t)$ . This will be done in Proposition 3.3 by means of a Gronwall estimate for  $F_k(t)$ .

**Notations.** From now on  $C$  will refer to a constant depending only on  $T$  and on the initial data  $(R_0, m_k, \gamma_k, h_{k,0}, \ell_{k,0}$  and  $\|\omega_0\|_{L^\infty}$ ), but not on  $\delta_0$ . It will possibly changing value from one line to another.

**Acknowledgements.** The authors are partially supported by the Agence Nationale de la Recherche, Project SINGFLOWS, grant ANR-18-CE40-0027-01. C.L. was also partially supported by the CNRS, program Tellus, and the ANR project IFSMACS, grant ANR-15-CE40-0010. E.M. acknowledges French ANR project INFAMIE ANR-15-CE40-01. Both authors thank warmly Olivier Glass and Franck Sueur for interesting discussions.

## 2. Proof of Theorem 1.2

**2.1. Some general regularity properties.** — We start with the following well-known property, see [14, Appendix 2.3] for instance.

**Proposition 2.1.** — *Let  $\Omega \in L^\infty([0, T], L^1 \cap L^\infty(\mathbb{R}^2))$ . Let  $U = K * \Omega$ . Then we have*

$$\|U\|_{L^\infty} \leq C \|\Omega\|_{L^\infty(L^1)}^{1/2} \|\Omega\|_{L^\infty}^{1/2}.$$

Moreover,  $U$  is almost-Lipschitz uniformly in time:

$$\|U(\cdot, x) - U(\cdot, y)\|_{L^\infty(0, T)} \leq C(\|\Omega\|_{L^\infty(L^1 \cap L^\infty)}) |x - y| (1 + |\ln |x - y||), \quad \forall (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

We also have the Calderón-Zygmund inequality [19, Chapter II, Theorem 3]

**Proposition 2.2.** — *There exists  $C$  such that for all  $p \geq 2$*

$$\|\nabla U(t, \cdot)\|_{L^p} \leq Cp \|\Omega(t, \cdot)\|_{L^p} \quad \text{for all } t \in [0, T].$$

In particular, it follows that any such velocity field satisfies

$$(2.1) \quad U \in L^\infty([0, T], L^\infty(\mathbb{R}^2)) \cap L^\infty([0, T], W_{\text{loc}}^{1,1}(\mathbb{R}^2)), \quad \operatorname{div}(U) = 0.$$

**2.2. Some basic properties for weak solutions of (1.1)-(1.2).** — In all this paragraph,  $(\omega, \{h_k\})$  denotes a weak solution of (1.1)-(1.2) on  $[0, T]$ , so that in particular  $u$  satisfies Proposition 2.1 and the regularity property (2.1). We assume *moreover* that  $\omega(t, \cdot)$  is compactly supported in some  $B(0, R)$  for all  $t \in [0, T]$ .

We introduce the stream function

$$(2.2) \quad \varphi(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \omega(t, y) dy,$$

so that

$$u(t, x) = \nabla^\perp \varphi(t, x).$$

For the subsequent computations we introduce a regularized version of the stream function: for  $\varepsilon > 0$  and  $\ln_\varepsilon$  a smooth function coinciding with  $\ln$  on  $[\varepsilon, +\infty)$  and satisfying  $|\ln'_\varepsilon(r)| \leq C/r$  for all  $r > 0$ , we set

$$(2.3) \quad \varphi_\varepsilon(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln_\varepsilon|x-y| \omega(t, y) dy.$$

Note that by assumption on the support of  $\omega(t, \cdot)$  the following estimate holds for  $\varphi_\varepsilon$ :

$$(2.4) \quad |\varphi_\varepsilon(t, x)| \leq C \ln(2 + |x|),$$

with  $C$  also independent of  $\varepsilon$ .

The following bound will be useful in order to establish a bound on the local energies in Proposition 3.3:

**Proposition 2.3.** — *There exists  $C_R$  depending only on  $R$ ,  $\|\omega\|_{L^\infty}$ ,  $\|h_k\|_{L^\infty}$ , and the initial data, such that*

$$\|\partial_t \varphi_\varepsilon\|_{L^\infty} \leq C_R.$$

*Proof.* — Using the weak formulation for  $\omega$  in (1.1), we have

$$\begin{aligned} & \partial_t \varphi_\varepsilon(t, x) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln'_\varepsilon(|x-y|) \frac{y-x}{|y-x|} \cdot \left( u(t, y) + \sum_{k=1}^N \gamma_k K(y - h_k(t)) \right) \omega(t, y) dy, \end{aligned}$$

therefore

$$|\partial_t \varphi_\varepsilon(t, x)| \leq C \|u\|_{L^\infty} \int_{\mathbb{R}^2} \frac{|\omega(t, y)|}{|x-y|} dy$$



$$+ C \sum_{k=1}^N \int_{\mathbb{R}^2} \left| \frac{(x-y) \cdot (y-h_k(t))^\perp}{|x-y|^2 |y-h_k(t)|^2} \omega(t, y) \right| dy.$$

By the estimates (1.39) to (1.43) in [12], there exists a constant  $C$  depending only on  $R$ , on  $\|\omega\|_{L^\infty}$ , and on  $\|h_k(t)\|_{L^\infty}$ , such that

$$|\partial_t \varphi_\varepsilon(t, x)| \leq C_R.$$

The conclusion follows. □

**Proposition 2.4.** — *We have*

$$\nabla^\perp \varphi_\varepsilon = u + R_\varepsilon,$$

where  $R_\varepsilon$  satisfies

$$\|R_\varepsilon\|_{L^\infty} \leq \varepsilon C \|\omega\|_{L^\infty}.$$

*Proof.* — We have

$$\begin{aligned} \nabla^\perp \varphi_\varepsilon(t, x) &= u(t, x) + \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} (\ln'_\varepsilon |x-y| |x-y| - 1) \omega(t, y) dy \\ &= u(t, x) + R_\varepsilon(t, x), \end{aligned}$$

where

$$|R_\varepsilon(t, x)| \leq C \int_{|y-x| \leq \varepsilon} \frac{1}{|x-y|} |\omega(t, y)| dy \leq C \|\omega(t, \cdot)\|_{L^\infty} \varepsilon.$$

□

**2.3. Proof of Theorem 1.2.** — The proof of Theorem 1.2 is divided into two steps.

**Step 1: iterative scheme.** Let  $\rho \in (0, \min_{k \neq p} |h_{k,0} - h_{p,0}|)$  which will be fixed later. We consider the following iterative scheme: for  $n \in \mathbb{N}^*$ , given

$$\omega_{n-1} \in L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2)),$$

and given  $N$  trajectories  $h_{k,n-1}$  in  $C^2([0, T_{n-1}])$  such that

$$\min_{t \in [0, T_{n-1}]} \min_{k \neq p} |h_{k,n-1} - h_{p,n-1}| > 0,$$

for some  $T_{n-1} > 0$ , we set

$$u_{n-1} = K * \omega_{n-1},$$

having in mind to solve the linear PDE

$$(2.5) \quad \begin{cases} \partial_t \omega_n + \operatorname{div} \left[ \left( u_{n-1} + \sum_{k=1}^N \frac{\gamma_k (x - h_{k,n-1}(t))^\perp}{2\pi |x - h_{k,n-1}(t)|^2} \right) \omega_n \right] = 0 \\ \omega_n(0) = \omega_0, \end{cases}$$

and the linear system of ODEs: for  $k = 1, \dots, N$ ,

$$(2.6) \quad \begin{cases} \ddot{h}_{k,n}(t) = \frac{\gamma_k}{m_k} \left( \dot{h}_{k,n}(t) - u_{n-1}(t, h_{k,n}(t)) \right. \\ \left. - \sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{(h_{k,n}(t) - h_{j,n}(t))^\perp}{|h_{k,n}(t) - h_{j,n}(t)|^2} \right)^\perp \\ (h_{k,n}(0), \dot{h}_{k,n}(0)) = (h_{k,0}, \ell_{k,0}), \end{cases}$$

on  $[0, T_n]$ , where  $T_n \in (0, T_{n-1}]$  will be chosen such that

$$(2.7) \quad \min_{t \in [0, T_n]} \min_{k \neq p} |h_{k,n}(t) - h_{p,n}(t)| \geq \rho.$$

For  $n = 0$  we take  $\omega_0$  and  $(h_{k,0}, \ell_{k,0})$  as data (with  $T_0 = +\infty$ ).

**Proposition 2.5.** — *For all  $n \in \mathbb{N}$ , there exist  $T_n \in (0, T_{n-1}]$  and a unique weak solution  $\omega_n$  to (2.5) and  $\{h_{k,n}\}$  to (2.6) on  $[0, T_n]$  such that (2.7) is satisfied.*

Moreover,

$$\|\omega_n(t, \cdot)\|_{L^1 \cap L^\infty} \leq \|\omega_0\|_{L^1 \cap L^\infty} \quad \forall t \in [0, T_n]$$

and there exists  $\tilde{T}$  depending only on  $\rho, h_{k,0}, \ell_{k,0}, R_0$  and  $\|\omega_0\|_{L^\infty}$  such that  $T_n \geq \tilde{T}$  for all  $n$ .

Finally, if all the  $\gamma_k$  have the same sign, then for any  $T > 0$ , one can choose  $\rho$  depending on  $T$  (and on  $h_{k,0}, \ell_{k,0}$  and  $\|\omega_0\|_{L^\infty}, R_0$ ) such that  $T_n = T$  for all  $n \in \mathbb{N}$ .

*Proof.* — Given  $(\omega_{n-1}, \{h_{k,n-1}\})$  satisfying the bound of Proposition 2.5, we solve the linear transport equation (2.5) with initial data  $\omega_0$  and velocity field given by

$$b_{n-1}(t, x) = u_{n-1}(t, x) + \sum_{k=1}^N \frac{\gamma_k}{2\pi} \frac{(x - h_{k,n-1}(t))^\perp}{|x - h_{k,n-1}(t)|^2}.$$

The existence of such a weak solution  $\omega_n \in L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2))$  follows from classical arguments on linear transport equation. For the uniqueness issue, we refer to [15, Chapter 1, Lemme 1.5], which proves that any field  $b_{n-1}$  given as above, with  $u_{n-1}$  satisfying the regularity property (2.1) and with the maps  $h_{k,n-1}$  Lipschitz continuous and not intersecting on  $[0, T_{n-1}]$ , has the renormalization property (see [5, Definition 1.5] of the notion of renormalization). By the usual arguments for linear transport equations, see [6], uniqueness therefore holds in  $L^\infty([0, T_{n-1}], L^1 \cap L^\infty(\mathbb{R}^2))$  for the linear transport equation associated to  $b_{n-1}$ . We remark that although [15, Lemme 1.5] is established in order to be applied to the vortex-wave system, the result holds for the linear transport equation with any vector field  $b_{n-1}$  having the form

above, without using the dynamics of the  $h_{k,n-1}$ . So it applies to the present case.

Moreover, it is also proved in [15] that the norms  $\|\omega_n(t, \cdot)\|_{L^p}$  are constant in time for all  $p$ , therefore we get the desired bound for  $\|\omega_n\|_{L^1 \cap L^\infty}$ . Recalling Proposition 2.1, it follows that  $\|u_n\|_{L^\infty} \leq C$ . Furthermore, the weak-time continuity for  $\omega_n$  established in [9, Proposition 4.1] or in [15] implies that  $u_n$  is uniformly continuous in space-time.

Next, in view of the almost-Lipschitz property and the time regularity for  $u_{n-1}$ , Osgood's lemma ensures that there exists a unique solution  $\{h_{k,n}\}$  to (2.6) on some maximal open interval  $I_n \subset [0, T_{n-1}]$  such that

$$\min_{k \neq p} |h_{k,n}(t) - h_{p,n}(t)| > 0 \quad \text{on } I_n.$$

We consider then  $T_n \leq T_{n-1}$  such that  $[0, T_n) \subset I_n$  and  $T_n$  is the largest time for which

$$\min_{k \neq p} |h_{k,n}(t) - h_{p,n}(t)| > \rho \quad \text{on } [0, T_n).$$

Taking the scalar product of (2.6) with  $\dot{h}_{k,n}(t)$  and using Proposition 2.1 and (2.7), we get on  $[0, T_n]$ :

$$m_k \frac{d|\dot{h}_{k,n}(t)|^2}{dt} \leq C |\dot{h}_{k,n}(t)| \leq C |\dot{h}_{k,n}(t)|^2 + 1$$

hence we deduce by Gronwall that

$$(2.8) \quad |\dot{h}_{k,n}(t)| \leq C \quad \text{on } [0, \min(T_n, 1)],$$

so

$$T_n \geq \tilde{T} := \min \left( 1, \frac{\min_{k \neq p} |h_{k,0} - h_{p,0}| - \rho}{2C} \right).$$

It remains to study the case where all  $\gamma_k$  have the same sign (say positive), where we have to derive an inequality like (2.8) which is independent of  $\rho$ . We fix  $T > 0$  and we assume that  $T_{n-1} = T$ . We want to show that  $T_n = T$ . In the sequel of this proof,  $C$  depends only on the initial data and  $T$ . We introduce the quantity

$$\mathcal{H}_n(t) = \sum_{j \neq k} \frac{\gamma_j \gamma_k}{2\pi} \ln |h_{j,n}(t) - h_{k,n}(t)| - \sum_{k=1}^N m_k |\dot{h}_{k,n}(t)|^2,$$

defined on  $[0, T_n]$ , and we compute

$$\dot{\mathcal{H}}_n(t) = 2 \sum_{k=1}^N \gamma_k \dot{h}_{k,n}(t) \cdot u_{n-1}(t, h_{k,n}(t))^\perp,$$

hence, using the bound  $\|u_{n-1}\|_\infty \leq C$  we get

$$(2.9) \quad |\dot{\mathcal{H}}_n(t)| \leq C \sum_{k=1}^N \gamma_k |\dot{h}_{k,n}(t)|.$$

On the other hand, we notice that for all  $k$ , for all  $t \in [0, T_n]$ , using that  $\ln|x-y| \leq |x| + |y|$  we have

$$\begin{aligned} |\dot{h}_{k,n}(t)| &\leq C |\mathcal{H}_n(t)|^{1/2} + C \left( \sum_{j=1}^N \gamma_j |h_{j,n}(t)| \right)^{1/2} \\ &\leq C |\mathcal{H}_n(t)|^{1/2} + C \left( \sum_{j=1}^N \gamma_j |h_{j,n}(0)| + \sum_{j=1}^N \gamma_j \int_0^t |\dot{h}_{j,n}(\tau)| d\tau \right)^{1/2} \\ &\leq C |\mathcal{H}_n(t)|^{1/2} + C \left( \sum_{j=1}^N \gamma_j |h_{j,0}| + \max_k \max_{[0, T_n]} |\dot{h}_{k,n}| \right)^{1/2}, \end{aligned}$$

hence

$$(2.10) \quad \begin{aligned} \max_k \max_{[0, T_n]} |\dot{h}_{k,n}| &\leq C \max_{[0, T_n]} |\mathcal{H}_n|^{1/2} + C \sqrt{1 + \max_k \max_{[0, T_n]} |\dot{h}_{k,n}|} \\ &\leq C \max_{[0, T_n]} |\mathcal{H}_n|^{1/2} + C. \end{aligned}$$

Inserting this inequality in (2.9) we also obtain

$$\max_{[0, T_n]} |\dot{\mathcal{H}}_n| \leq C \left( 1 + \max_{[0, T_n]} |\mathcal{H}_n|^{1/2} \right),$$

therefore we get

$$\max_{[0, T_n]} |\mathcal{H}_n| \leq C \quad \text{and} \quad \max_{[0, T_n]} |\dot{\mathcal{H}}_n| \leq C.$$

Coming back to (2.10), it follows that

$$(2.11) \quad \max_k \max_{[0, T_n]} |\dot{h}_{k,n}| \leq C,$$

so that

$$\max_k \max_{[0, T_n]} |h_{k,n}| \leq C.$$

Finally, by the definition of  $\mathcal{H}_n(t)$  and by the previous bounds, using again that  $\ln|x-y| \leq |x| + |y|$  we have for all  $j \neq k$ :

$$\gamma_j \gamma_k \ln |h_{j,n}(t) - h_{k,n}(t)| \geq -2\pi |\mathcal{H}_n(t)| - \sum_{p,\ell=1}^N \gamma_p \gamma_\ell (|h_{p,n}(t)| + |h_{\ell,n}(t)|)$$

$$\geq -C,$$

which means that there exists  $\rho > 0$  such that

$$\min_{j \neq k} |h_{j,n}(t) - h_{k,n}(t)| > \rho > 0, \quad \forall t \in [0, \min(T_n, T)].$$

Choosing this  $\rho$ , we conclude that  $T_n = T$ , and that the proposition is proved.  $\square$

**Step 2: Passing to the limit** We only sketch the subsequent arguments. By the previous estimates, extracting if necessary, we find that  $\{\omega_n\}_{n \in \mathbb{N}}$  converges to some  $\omega$  in  $L^\infty$  weak-\* on  $[0, \tilde{T}] \times \mathbb{R}^2$ . Moreover, setting  $u = K * \omega$ , we infer that  $\{u_n\}_{n \in \mathbb{N}}$  converges to  $u$  locally uniformly on  $[0, \tilde{T}] \times \mathbb{R}^2$  (see for instance [7, Sect. 6.1]). On the other hand, the bounds (2.7)-(2.8) (or (2.11)) imply that each sequence  $\{\ddot{h}_{k,n}\}_{n \in \mathbb{N}}$  is uniformly bounded on  $[0, \tilde{T}]$ . By Ascoli's theorem, extracting again if necessary, we obtain that each  $\{(h_{i,n}, \dot{h}_{i,n})\}_{n \in \mathbb{N}}$  converges uniformly to some  $(h_i, \dot{h}_i)$  on  $[0, \tilde{T}]$ , and passing to the limit in (2.6), we see that the points  $\{h_k\}$  satisfy the desired system of ODE in (1.1). Note in particular that they satisfy

$$(2.12) \quad \max_k \max_{[0, \tilde{T}]} |h_k| \leq C, \quad \min_{j \neq k} \min_{[0, \tilde{T}]} |h_j - h_k| \geq \rho > 0$$

and

$$(2.13) \quad \max_k \max_{[0, \tilde{T}]} |\dot{h}_k| \leq C.$$

Finally, coming back to (2.5), we can pass to the limit exploiting the previous types of convergence to show that  $\omega$  is a weak solution of the first PDE in (1.1) on  $[0, \tilde{T}]$ .

Iterating this construction we reach existence up to the first time of collisions.

If all the circulations have the same sign, we take  $T > 0$ , and we can replace  $\tilde{T}$  by  $T$  in all the arguments above since for all  $n$  we have a solution  $\omega_n$  and  $\{h_{k,n}\}$  on  $[0, T_n = T]$ . This shows that no collision occurs in finite time.

### 3. Proof of Theorem 1.6

In all this section,  $\omega$  denotes any weak solution of (1.4) on  $[0, T]$ , where  $\{h_k\}$  are given trajectories belonging in  $W^{2,\infty}([0, T])$  with initial data satisfying (1.2). We assume that no collision occurs, i.e.

$$(3.1) \quad \min_{t \in [0, T]} \min_{k \neq p} |h_k(t) - h_p(t)| \geq \rho$$

for some  $\rho > 0$ . The purpose of this section is to show Theorem 1.6. We emphasize that the proof does not use the dynamics of  $\{h_k\}$ .

**3.1. Regular Lagrangian flow.** — We show here that there exists a unique regular Lagrangian flow as defined in Definition 1.1.

Recall the general following abstract result by Ambrosio [1, Theorems 3.3 and 3.5]. Given a vector field  $b$  in  $L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$ , if existence and uniqueness for the continuity equation

$$\partial_t \omega + \operatorname{div}(b\omega) = 0, \quad \omega(0, \cdot) = \omega_0 \in L^1 \cap L^\infty$$

hold in  $L^\infty([0, T], L^1 \cap L^\infty)$  then the regular Lagrangian flow  $X$  for  $b$  exists and is unique, and the unique solution is then given by  $\omega(t, \cdot) = X(t, \cdot)_\# \omega_0$ .

In order to apply this result to the present setting, we introduce the divergence-free field

$$(3.2) \quad b(t, x) = u(t, x) + \sum_{j=1}^N \gamma_j K(x - h_j(t)).$$

It was proved in [15, Chapter 1, Lemme 1.5]<sup>(1)</sup> (see also [9] for the case of one point) that the transport equation associated to  $b$  admits a unique solution (which is renormalized: for any continuous function  $\beta$  growing not too fast at infinity, the function  $\beta(\omega)$  is also a solution). Therefore Ambrosio's result yields the existence and uniqueness of the regular Lagrangian flow  $X$  associated to  $b$ , and we have  $\omega(t, \cdot) = X(t, \cdot)_\# \omega_0$ . This proves the first claim of Theorem 1.6.

We mention that, as noted in [15, Chapter 1, Remark 1.3] or in [9, Lemma 3.2] for the case of one point, the renormalization property ensures that

$$(3.3) \quad \|\omega(t, \cdot)\|_{L^p} = \|\omega_0\|_{L^p}, \quad 1 \leq p \leq +\infty.$$

We derive first the following property:

**Proposition 3.1.** — *There exists  $C$  depending only on  $T$  and the initial data such that*

$$\sup_{t \in [0, T]} |X(t, x)| \leq |x| + C.$$

*Proof.* — Let  $C_1$  be such that  $\max_k \max_{t \in [0, T]} |h_k(t)| \leq C_1$ . Let  $x \in \mathbb{R}^2$  such that  $X(\cdot, x)$  is absolutely continuous on  $[0, T]$ . Take  $C_2 > C_1$  sufficiently large, to be determined later. If  $|X(t, x)| \leq |x| + 4C_2$  for all  $t \in [0, T]$ , the claim is proved. Otherwise, assume that  $|X(t_0, x)| > |x| + 4C_2$  for some  $t_0$ . By continuity, we have  $|X(t, x)| \geq |x| + 2C_2$  on  $(t_1, t_2) \subset (0, T)$  maximal, with

<sup>(1)</sup>We emphasize that, although [15, Lemme 1.5] is established in the setting of the vortex-wave system, the result holds for the linear transport equation with any vector field  $b$  given by (3.2), where  $u$  satisfies the regularity properties (2.1) and where the  $h'_j$ 's are Lipschitz continuous on  $[0, T]$  and do not intersect. Since their precise dynamics is not used to show the renormalization property, the result of [15] holds in the present case.

$|X(t_1, x)| = |x| + 2C_2$  and  $t_0 \in (t_1, t_2)$ . Since  $|X(s, x) - h_j(s)| \geq C_1$  for all  $j = 1, \dots, N$  on  $(t_1, t_2)$ ,

$$\begin{aligned} |x| + 4C_2 &< |X(t_0, x)| \leq |X(t_1, x)| + \int_{t_1}^{t_0} |b(s, X(s, x))| ds \\ &\leq |x| + 2C_2 + T(\|u\|_{L^\infty} + \sum_{j=1}^N \gamma_j (2\pi C_1)^{-1}). \end{aligned}$$

Setting  $2C_2 > \max(2C_1, T(\|u\|_{L^\infty} + \sum_j \gamma_j (2\pi C_1)^{-1}))$  yields a contradiction, which proves the claim.  $\square$

The following corollary gives the second point in Theorem 1.6.

**Corollary 3.2.** — *The vorticity  $\omega(t, \cdot)$  is compactly supported for all  $t \in [0, T]$ , with  $\text{supp}(\omega(t, \cdot)) \subset B(0, R_T)$  for some  $R_T$  depending only on the initial data and on  $T$ .*

*Proof.* — We have  $\omega(t, \cdot) = X(t, \cdot) \# \omega_0$  and  $\omega_0$  is compactly supported in  $B(0, R_0)$ , so it follows from Proposition 3.1 that  $\omega(t, \cdot)$  is compactly supported for all  $t \in [0, T]$ , with  $\text{supp}(\omega(t, \cdot)) \subset B(0, R_T)$  for  $R_T = R_0 + C$  (with  $C$  given in Proposition 3.1).  $\square$

**3.2. Vorticity trajectories.** — For the third point in Theorem 1.6, we have to show that for almost every  $x \in \mathbb{R}^2$ , we have  $X(t, x) \neq h_k(t)$  for all  $t \in [0, T]$  and for all  $k = 1, \dots, N$ .

For almost every  $x \in \mathbb{R}^2 \setminus \cup_j \{h_{j,0}\}$  such that  $X(\cdot, x)$  is an absolutely continuous solution on  $[0, T]$  to the ODE with field  $b$  defined in (3.2), by time continuity, there exists  $T^*(x)$  such that

$$\min_j \min_{t \in [0, T^*(x)]} |X(t, x) - h_j(t)| > 0.$$

We may then consider the local microscopic energies near the points  $h_k(t)$  on  $[0, T^*(x)]$ :

$$F_k(t) = \sum_{j=1}^N \frac{\gamma_j}{2\pi} \ln |X(t, x) - h_j(t)| + \varphi_\varepsilon(t, X(t, x)) + \langle X(t, x), \dot{h}_k^\perp(t) \rangle,$$

where we recall that  $\varphi_\varepsilon$  denotes the regularization of the stream function, see (2.3).

On the other hand, the result in [9, Proposition 4.1] states the continuity of  $u$  on  $[0, T] \times \mathbb{R}^2$ . Therefore, the field  $b(\cdot, X(\cdot, x))$  is continuous on  $[0, T^*(x))$ . So we infer that  $X(\cdot, x)$  is differentiable on  $[0, T^*(x))$  with  $\frac{d}{dt}X(t, x) = b(t, X(t, x))$ . This enables to perform the following estimate on the local energies.

**Proposition 3.3.** — We have for  $t \in [0, T^*(x))$  and for all  $k = 1, \dots, N$ ,

$$|F'_k(t)| \leq C \left( 1 + |x| + \varepsilon |X(t, x) - h_k(t)|^{-1} + \sum_{j \neq k} |X(t, x) - h_j(t)|^{-1} \right).$$

In the previous statement,  $C$  is independent of  $\varepsilon$  whereas  $F_k$  depends on  $\varepsilon$ .

*Proof.* — In the subsequent proof we set for clarity:

$$X = X(t, x), \quad u = u(t, X(t, x)), \quad \varphi_\varepsilon = \varphi_\varepsilon(t, X(t, x)), \quad h_k = h_k(t),$$

and we compute on  $[0, T^*(x))$

$$\begin{aligned} F'_k &= \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \sum_{m=1}^N \gamma_m K(X - h_m) + u - \dot{h}_j \right\rangle \\ &\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle \\ &= \left\langle \sum_{j=1}^N \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \sum_{m=1}^N \frac{\gamma_m}{2\pi} \frac{(X - h_m)^\perp}{|X - h_m|^2} \right\rangle + \sum_{j=1}^N \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_j \right\rangle \\ &\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \left\langle \dot{X}, \dot{h}_k^\perp \right\rangle + \langle X, \ddot{h}_k^\perp \rangle \\ &= \left\langle \frac{\gamma_k}{2\pi} \frac{X - h_k}{|X - h_k|^2}, u - \dot{h}_k \right\rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_j \right\rangle \\ &\quad + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \left\langle \dot{X}, \dot{h}_k^\perp \right\rangle + \langle X, \ddot{h}_k^\perp \rangle. \end{aligned}$$

Next, using again that  $X$  satisfies the ODE with field  $b$  defined in (3.2), we have

$$(3.4) \quad \frac{\gamma_k}{2\pi} \frac{X - h_k}{|X - h_k|^2} = -\dot{X}^\perp + u^\perp - \sum_{j \neq k} \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2},$$

hence we get

$$\begin{aligned} F'_k &= \langle -\dot{X}^\perp + u^\perp, u - \dot{h}_k \rangle - \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_k \right\rangle \\ &\quad + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, u - \dot{h}_j \right\rangle + \partial_t \varphi_\varepsilon + \langle \dot{X}, \nabla \varphi_\varepsilon \rangle + \langle \dot{X}, \dot{h}_k^\perp \rangle + \langle X, \ddot{h}_k^\perp \rangle \\ &= \langle \dot{X}^\perp, -u + \nabla^\perp \varphi_\varepsilon \rangle - \langle u^\perp, \dot{h}_k \rangle \\ &\quad + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \dot{h}_k - \dot{h}_j \right\rangle + \partial_t \varphi_\varepsilon + \langle X, \ddot{h}_k^\perp \rangle. \end{aligned}$$



Hence, plugging the equality  $\nabla^\perp \varphi_\varepsilon = u + R_\varepsilon$ , with  $R_\varepsilon$  defined in Proposition 2.4, we have

$$F'_k = \langle \dot{X}^\perp, R_\varepsilon \rangle - \langle u^\perp, \dot{h}_k \rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \dot{h}_k - \dot{h}_j \right\rangle + \partial_t \varphi_\varepsilon + \langle X, \ddot{h}_k^\perp \rangle.$$

By this proposition together with (3.3) and (3.4) that

$$\left| \langle \dot{X}^\perp, R_\varepsilon \rangle \right| \leq C\varepsilon \|\omega_0\|_{L^\infty} \left( \sum_{j=1}^N |X - h_j|^{-1} + \|u\|_{L^\infty} \right).$$

On the other hand, as  $h_k \in W^{2,\infty}$ , we obtain by Proposition 3.1

$$\begin{aligned} \left| -\langle u^\perp, \dot{h}_k \rangle + \sum_{j \neq k} \left\langle \frac{\gamma_j}{2\pi} \frac{X - h_j}{|X - h_j|^2}, \dot{h}_k - \dot{h}_j \right\rangle + \langle X, \ddot{h}_k^\perp \rangle \right| \\ \leq C \left( \|u\|_{L^\infty} + \sum_{j \neq k} |X - h_j|^{-1} + |x| + 1 \right). \end{aligned}$$

Finally, recalling that  $\|\partial_t \varphi_\varepsilon\|_{L^\infty} \leq C$  by Proposition 2.3 and that  $\|u\|_{L^\infty} \leq C$  by Proposition 2.1, the conclusion follows.  $\square$

**Corollary 3.4.** — *For almost every  $x \in \mathbb{R}^2$  we can take  $T^*(x) = T$ , more precisely*

$$\min_j \min_{t \in [0, T]} |X(t, x) - h_j(t)| > 0, \quad \text{for a.e. } x \in \mathbb{R}^2.$$

*Proof.* — We argue by contradiction, assuming that  $T^*(x) = T$  is impossible for some  $x \in \mathbb{R}^2 \setminus \{h_{k,0}\}$  where the flow exists, so that there exist  $k \in \{1, \dots, N\}$  and  $\tilde{T} < T$  such that  $\liminf_{t \rightarrow \tilde{T}} |X(t, x) - h_k(t)| = 0$  and  $\min_j \min_{t \in [0, T^*]} |X(t, x) - h_j(t)| > 0$  for any  $T^* < \tilde{T}$ . We further set  $X(t) = X(t, x)$ . Let  $t_n \rightarrow \tilde{T}$  such that  $|X(t_n) - h_k(t_n)| \rightarrow 0$  as  $n \rightarrow +\infty$ . We recall that  $\rho$  is defined by (3.1). For  $n$  sufficiently large we have  $|X(t_n) - h_k(t_n)| < \rho/K$ , with  $K > 3$  large to be determined later on. We take  $t'_n$  maximal such that on  $[t_n, t'_n]$  we have  $|X(t) - h_k(t)| < \rho/3$ . In particular, by (3.1), for  $j \neq k$  we have  $|X(t) - h_j(t)| \geq 2\rho/3$  on  $[t_n, t'_n]$ . We assume first that  $t'_n < \tilde{T}$ : then we have  $|X(t'_n) - h_k(t'_n)| = \rho/3$ . We fix  $n \in \mathbb{N}$ . For all  $\varepsilon > 0$ , by the definition of  $F_k$ ,

we write

$$\begin{aligned}
(3.5) \quad & \frac{\gamma_k}{2\pi} \ln \left( \frac{|X(t'_n) - h_k(t'_n)|}{|X(t_n) - h_k(t_n)|} \right) \\
&= \sum_{j \neq k}^N \frac{\gamma_j}{2\pi} \ln \left( \frac{|X(t_n) - h_j(t_n)|}{|X(t'_n) - h_j(t'_n)|} \right) + \int_{t_n}^{t'_n} F'_k(\tau) d\tau \\
&\quad + \varphi_\varepsilon(t_n, X(t_n)) - \varphi_\varepsilon(t'_n, X(t'_n)) \\
&\quad + \langle X(t_n), \dot{h}_k^\perp(t_n) \rangle - \langle X(t'_n), \dot{h}_k^\perp(t'_n) \rangle,
\end{aligned}$$

so by (2.4), Proposition 3.3 and the previous estimates we get

$$\frac{|\gamma_k|}{2\pi} \ln \left( \frac{K}{3} \right) \leq C \left( 1 + |x| + \varepsilon \int_{t_n}^{t'_n} |X(\tau) - h_k(\tau)|^{-1} d\tau \right).$$

Letting  $\varepsilon \rightarrow 0$  for fixed  $n$ , we find

$$\ln \left( \frac{K}{3} \right) \leq C,$$

which is a contradiction for  $K$  sufficiently large. So we have

$$(3.6) \quad |X(t, x) - h_k(t)| < \frac{\rho}{3} \text{ and } |X(t, x) - h_j(t)| \geq \frac{2\rho}{3} \text{ on } [t_n, \tilde{T}].$$

We have therefore localized the fluid trajectory  $X(t)$  in the neighborhood of one point vortex trajectory  $h_k(t)$ , namely we have proved that if the trajectory goes too close to  $h_k$ , it stays in a neighborhood of radius  $\rho/3$ . We fix  $n_0 \in \mathbb{N}$  sufficiently large so that  $|X(t_{n_0}) - h_k(t_{n_0})| < \rho/K$ . We come back to (3.5), replacing  $t'_n$  by any  $t \in [t_n, \tilde{T})$ , and we apply again Proposition 3.3:

$$\ln |X(t) - h_k(t)| \geq \ln |X(t_{n_0}) - h_k(t_{n_0})| - C \left( 1 + |x| + \varepsilon \int_{t_{n_0}}^t |X(\tau) - h_k(\tau)|^{-1} d\tau \right).$$

Letting  $\varepsilon \rightarrow 0$  we find

$$|X(t) - h_k(t)| \geq |X(t_{n_0}) - h_k(t_{n_0})| e^{-C(1+|x|)} \text{ on } [t_{n_0}, \tilde{T}),$$

which contradicts the fact that  $\liminf_{t \rightarrow \tilde{T}} |X(t, x) - h_k(t)| = 0$ .

Hence we conclude that  $T^*(x) = T$  is possible.  $\square$

**Corollary 3.5.** — For a.e.  $x \in \mathbb{R}^2$ , the map  $X(\cdot, x)$  is the unique differentiable solution on  $[0, T]$  of the ODE

$$\frac{d}{dt} \gamma(t) = u(t, \gamma(t)) + \sum_{j=1}^N \gamma_j K(\gamma(t) - h_j(t)), \quad \gamma(0) = x,$$

such that  $\min_j \min_{[0, T]} |\gamma(t) - h_j(t)| > 0$ .

*Proof.* — We gather the already mentioned time continuity of  $u$ , the almost-Lipschitz space regularity for  $u$  stated in Proposition 2.1, the no collision property of Corollary 3.4, and the fact that  $K$  is Lipschitz away from the origin. Invoking Osgood's Lemma, we can then conclude.  $\square$

We finish this paragraph with an additional estimate on the Lagrangian trajectories, which can be derived easily from the proof of Corollary 3.4.

**Proposition 3.6.** — *Let  $\omega$  be any weak solution of (1.4) on  $[0, T]$  where  $\{h_k\}$  are given trajectories in  $W^{2,\infty}([0, T])$  satisfying the no collision property (3.1). There exist  $0 < \delta < \min(\rho/3, 1)$  and  $0 < \delta' < 1$ , depending only on  $T$ ,  $\|h_k\|_{W^{2,\infty}([0, T])}$ ,  $\rho$ , and on the initial conditions, satisfying the following property:*

*Let  $x \in \mathbb{R}^2$  such that  $\min_j \min_{t \in [0, T]} |X(t, x) - h_j(t)| > 0$ .*

*If  $|X(t_0, x) - h_k(t_0)| < \delta$  for some  $t_0 \in [0, T]$  and  $k \in \{1, \dots, N\}$ , then*

$$e^{-C(1+|x|)} |x - h_k(0)| \leq |X(t, x) - h_k(t)| < \frac{\rho}{3}, \quad \forall t \in [0, T].$$

*The constant  $C$  depends only on  $T$ , on  $\|h_k\|_{W^{2,\infty}([0, T])}$ , and on the initial conditions.*

*If  $\min_j |X(t_0, x) - h_j(t_0)| > \delta$  for some  $t_0 \in [0, T]$ , then*

$$\delta' \leq \min_j |X(t, x) - h_j(t)|, \quad \forall t \in [0, T].$$

*Proof.* — We start with the first estimate. We come back to the proof of Corollary 3.4 above, with  $t_n$  replaced by  $t_0$  and  $t'_n$  replaced by  $T$ . Recalling that  $K > 3$  is a sufficiently large number appearing in that proof, we set  $\delta = \rho/K$ . By (3.6) we obtain:

$$|X(t, x) - h_k(t)| < \frac{\rho}{3}, \quad |X(t, x) - h_j(t)| \geq \frac{2\rho}{3} \text{ for } j \neq k \text{ on } [t_0, T].$$

We can invoke the same arguments to obtain the same estimates on  $[0, t_0]$ .

Therefore, by Proposition 3.3, this yields:

$$\lim_{\varepsilon \rightarrow 0} \int_0^T |F'_k(\tau)| d\tau \leq C,$$

so that, using again (3.5), we get

$$|X(t, x) - h_k(t)| \geq |x - h_k(0)| e^{-C(1+|x|)} \text{ on } [0, T],$$

for a constant  $C$ , so the first part is proved.

We turn now to the second part. Let  $K$  be a number to be determined later on. Let  $(t_1, t_2) \in [0, T]$  be maximal such that  $\min_j |X(t, x) - h_j(t)| > \delta/K$  on  $(t_1, t_2)$ , and let  $k \in \{1, \dots, N\}$  such that  $|X(t_1, x) - h_k(t_1)| = \delta/K$  (or  $|X(t_2, x) - h_k(t_2)| = \delta/K$ ). Repeating the proof of Corollary 3.4 with  $t_n = t_0$

and  $t'_n = t_1$  (or  $t'_n = t_2$ ), we find  $|\ln K| \leq C$  which is a contradiction provided  $K$  is sufficiently large. So, setting  $\delta' = \delta/K$ , the conclusion follows.  $\square$

**3.3. Decomposition of the vorticity and reduction to the case of one point vortex.**— In all this subsection we assume that  $\omega_0$  is constant in a neighborhood of  $\{h_k(0)\}$ , namely that (1.5) holds.

Let  $\delta$  and  $\delta'$  be the constants introduced in Proposition 3.6.

We decompose  $\omega_0$  as

$$\omega_0 = \sum_{k=1}^N \omega_{0,k} + \omega_{0,r},$$

where

$$\omega_{0,k} = \omega_0 \mathbb{1}_{B(h_k(0), \delta)}, \quad k = 1, \dots, N$$

and

$$\omega_{0,r} \text{ is supported in } \mathbb{R}^2 \setminus \cup_{j=1}^N B(h_j(0), \delta).$$

By uniqueness of the weak solution to the linear transport equation associated to the field

$$b(t, x) = u(t, x) + \sum_{j=1}^N \gamma_j K(x - h_j(t)),$$

(see the beginning of Subsection 3.1),  $\omega$  may then be decomposed as

$$\omega(t, \cdot) = \sum_{k=1}^N X(t, \cdot) \# \omega_{0,k} + X(t, \cdot) \# \omega_{0,r} = \sum_{k=1}^N \omega_k(t, \cdot) + \omega_r(t, \cdot).$$

Let  $K_\delta = 1/(2\pi) \nabla^\perp \ln_\delta$ , where  $\ln_\varepsilon$  is defined in Subsection 2.2. So  $K_\delta$  is a smooth, divergence-free map coinciding with  $K$  on  $\mathbb{R}^2 \setminus B(0, \delta)$  such that  $\|K_\delta\|_{L^\infty} \leq C\delta^{-1}$ .

Let  $k = 1, \dots, N$ . By the first part of Proposition 3.6, by definition of  $\delta$ , we have

$$(3.7) \quad |X(t, x) - h_k(t)| < \frac{\rho}{3}, \quad \forall t \in [0, T], \quad \text{for a.e. } x \in \text{supp}(\omega_{0,k}).$$

Therefore,

$$\min_{j \neq k} |X(t, x) - h_j(t)| \geq \frac{2\rho}{3} > \delta, \quad \forall t \in [0, T] \quad \text{for a.e. } x \in \text{supp}(\omega_{0,k}).$$

So by Corollary 3.5, we have

$$(3.8) \quad X(t, x) = \widetilde{X}_k(t, x), \quad \text{for a.e. } x \in \text{supp}(\omega_{0,k}),$$

where  $\widetilde{X}_k$  is the unique regular Lagrangian flow associated to the field

$$b_k(t, x) = u(t, x) + \sum_{j \neq k} \gamma_j K_\delta(x - h_j(t)) + \gamma_k K(x - h_k(t)).$$

In particular,

$$(3.9) \quad \omega_k(t, \cdot) = X(t, \cdot) \# \omega_{0,k} = \widetilde{X}_k(t, \cdot) \# \omega_{0,k}.$$

We observe here for later use that the same argument applied to  $\omega^2$  (noting that it is also a distributional solution of (1.4) with initial datum  $\omega_0^2$ ) yields

$$(3.10) \quad \omega_k^2(t, \cdot) = X(t, \cdot) \# \omega_{0,k}^2 = \widetilde{X}_k(t, \cdot) \# \omega_{0,k}^2, \quad k = 1, \dots, N.$$

So we are left with the case of a linear transport equation with field  $b_k$  given by the superposition of a regular part

$$(3.11) \quad v_k(t, x) = u(t, x) + \sum_{j \neq k} \gamma_j K_\delta(x - h_j(t))$$

and a singular part generated by one point vortex:

$$\gamma_k K(x - h_k(t)).$$

The analysis of this case was performed in [4]. It was proved in particular that for all  $t \in [0, T]$ , the regular Lagrangian flow  $\widetilde{X}_k$  associated to  $b_k$  is the limit in  $L^1_{\text{loc}}(\mathbb{R}^2)$  of the sequence  $\widetilde{X}_{k,n}(t, \cdot)$ , where  $\widetilde{X}_{k,n}$  is the flow associated to any regularization of  $b_k$ :

$$b_{k,n}(t, x) = v_{k,n}(t, x) + \frac{\gamma_k}{2\pi} \frac{(x - h_k(t))^\perp}{|x - h_k(t)|^2 + n^{-2}},$$

with  $v_{k,n}$  a smooth and divergence-free approximation of  $v_k$ . By Liouville's theorem,  $\widetilde{X}_{k,n}(t, \cdot)$  thus preserves Lebesgue's measure. Moreover, Proposition 3.1 also applies to  $\widetilde{X}_{k,n}$  (with a constant independent of  $n$ ). Therefore, passing to the limit, we conclude that  $\widetilde{X}_k(t, \cdot)$  preserves Lebesgue's measure:

$$(3.12) \quad \widetilde{X}_k(t, \cdot) \# dx = dx, \quad \forall t \in [0, T].$$

We next derive a localization property for  $\widetilde{X}_k(t, \cdot)$ .

**Proposition 3.7.** — *For all  $R > 0$ , there exists  $C_R$  depending only on  $R, T, \|h_k\|_{W^{2,\infty}([0,T])}, \rho$  and the initial data, but not on  $\delta_0$ , such that*

$$C_R |x - h_k(0)| \leq |\widetilde{X}_k(t, x) - h_k(t)|, \\ \forall t \in [0, T], \quad \text{for all } x \in B(0, R) \setminus \{h_k(0)\}, \quad k = 1, \dots, N.$$

**Remark 3.8.** — By (3.8), we already know that this holds for a.e.  $x$  in  $\text{supp}(\omega_{0,k})$ .

*Proof.* — As long as  $\widetilde{X}_k(t, x) \neq h_k(t)$ , we introduce the new energy

$$\widetilde{F}_k(t) = \frac{\gamma_k}{2\pi} \ln |\widetilde{X}_k(t, x) - h_k(t)| + \varphi_\varepsilon(t, \widetilde{X}_k(t, x)) + \psi_\delta(t, \widetilde{X}_k(t, x)) + \langle \widetilde{X}_k(t, x), \dot{h}_k(t)^\perp \rangle,$$

where

$$\psi_\delta(t, x) = \sum_{j \neq k} \frac{\gamma_j}{2\pi} \ln_\delta |x - h_j(t)|,$$

so that

$$\nabla^\perp \psi_\delta(t, x) = \sum_{j \neq k} \gamma_j K_\delta(x - h_j(t)).$$

Exactly as in the proof of Proposition 3.3, we compute, recalling the definition (3.11) of  $v_k$ ,

$$\begin{aligned} \widetilde{F}_k'(t) &= - \left\langle v_k^\perp(t, \widetilde{X}_k(t, x)), \dot{h}_k(t) \right\rangle + \partial_t \varphi_\varepsilon(t, \widetilde{X}_k(t, x)) + \partial_t \psi_\delta(t, \widetilde{X}_k(t, x)) \\ &\quad + \left\langle \dot{\widetilde{X}}_k(t, x)^\perp, R_\varepsilon(t, \widetilde{X}_k(t, x)) \right\rangle + \left\langle \widetilde{X}_k(t, x), \ddot{h}_k(t)^\perp \right\rangle. \end{aligned}$$

Using the uniform bounds on  $v_k$ ,  $\nabla \psi_\delta$ ,  $\partial_t \psi_\delta$ ,  $h_j$ ,  $\dot{h}_j$  and  $\ddot{h}_j$  for  $j = 1, \dots, N$ , and using the previous bounds for  $\partial_t \varphi_\varepsilon$  and  $R_\varepsilon$  we therefore get for all  $\varepsilon > 0$

$$|\widetilde{F}_k'(t)| \leq C(|x| + \varepsilon |\widetilde{X}_k(t, x) - h_k(t)|^{-1} + 1).$$

We may now conclude exactly as in the proof of the first part of Proposition 3.6: as long as  $\widetilde{X}_k(t, x) \neq h_k(t)$ , letting  $\varepsilon$  tend to zero after integrating the inequality above on  $[0, t]$ , we get

$$\left| \ln \left( \frac{|\widetilde{X}_k(t, x) - h_k(t)|}{|x - h_{k,0}|} \right) \right| \leq C(1 + |x|),$$

where  $C$  depends on  $\delta$ ,  $T$ ,  $\|h_k\|_{W^{2,\infty}([0,T])}$ ,  $\rho$  and on the initial data. So, setting  $C_R = e^{-C(1+R)}$ , the conclusion follows.  $\square$

**Proposition 3.9.** — *We have*

$$|x - h_k(0)| \leq C(|\widetilde{X}_k(t, x) - h_k(t)| + 1),$$

where  $C$  depends only on  $T$ ,  $\|h_k\|_{W^{2,\infty}([0,T])}$ ,  $\rho$  and on the initial data, but not on  $\delta_0$ .

*Proof.* —

$$\begin{aligned} &\frac{d}{dt} |\widetilde{X}_k(t, x) - h_k(t)|^2 \\ &= 2 \left\langle \widetilde{X}_k(t, x) - h_k(t), v_k(t, \widetilde{X}_k(t, x)) + \gamma_k K(\widetilde{X}_k(t, x) - h_k(t)) - \dot{h}_k(t) \right\rangle \\ &= 2 \left\langle \widetilde{X}_k(t, x) - h_k(t), v_k(t, \widetilde{X}_k(t, x)) - \dot{h}_k(t) \right\rangle \\ &\geq -2 |\widetilde{X}_k(t, x) - h_k(t)| (\|\dot{h}_k\|_{L^\infty} + \|v_k\|_{L^\infty}), \end{aligned}$$

hence

$$\frac{d}{dt}|\widetilde{X}_k(t, x) - h_k(t)| \geq -C,$$

so the conclusion follows.  $\square$

**3.4. The vorticity remains constant in the neighborhood of the point vortices.** — Let  $C$  be the constant of Proposition 3.9. We set

$$R = 2C + |h_k(0)|,$$

and we consider the corresponding constant  $C_R$  of Proposition 3.7. We may decrease  $\delta_0$  so that

$$\max(\delta_0, C_R\delta_0) < \min\left(\delta, \delta', 1, \frac{2\rho}{3}\right),$$

where we recall  $\delta$  and  $\delta'$  were found in Proposition 3.6.

We fix  $t \in [0, T]$ . We claim that

$$(3.13) \quad \omega_j(t, y) = 0, \quad \text{for a.e. } y \in B(h_k(t), C_R\delta_0), \quad \forall j \neq k.$$

Indeed, by (3.10), considering the  $L^1$  function  $\varphi_k = \mathbf{1}_{B(h_k(t), C_R\delta_0)}$ , we find

$$\int_{\mathbb{R}^2} \omega_j^2(t, y) \varphi_k(y) dy = \int_{\mathbb{R}^2} \omega_{0,j}^2(x) \varphi_k(X(t, x)) dx.$$

On the other hand, for  $x \in \text{supp}(\omega_{0,j})$ , we have by (3.7)  $|X(t, x) - h_j(t)| < \rho/3$  and therefore  $|X(t, x) - h_k(t)| > 2\rho/3 > C_R\delta_0$ . So the right hand side above vanishes, which establishes (3.13).

Using that  $\delta' > C_R\delta_0$ , by the same arguments as above, the second part of Proposition 3.6 yields that

$$(3.14) \quad \omega_r(t, y) = 0, \quad \text{for a.e. } y \in \bigcup_{k=1}^N B(h_k(t), C_R\delta_0).$$

Finally, we show that

$$(3.15) \quad \omega_k(t, y) = \alpha_k, \quad \text{for a.e. } y \in B(h_k(t), C_R\delta_0).$$

Indeed, since  $\omega_k(t, \cdot) = \widetilde{X}_k(t, \cdot) \# \omega_{0,k}$ ,  $\omega_k^2(t, \cdot) = \widetilde{X}_k(t, \cdot) \# \omega_{0,k}^2$ , and  $\widetilde{X}_k(t, \cdot) \# dx = dx$  by (3.9), (3.10) and (3.12), we compute

$$\begin{aligned} & \int_{\mathbb{R}^2} (\omega_k(t, y) - \alpha_k)^2 \varphi_k(y) dy \\ &= \int_{\mathbb{R}^2} \omega_k(t, y)^2 \varphi_k(y) dy - 2\alpha_k \int_{\mathbb{R}^2} \omega_k(t, y) \varphi_k(y) dy + \alpha_k^2 \int_{\mathbb{R}^2} \varphi_k(y) dy \\ &= \int_{\mathbb{R}^2} \omega_{0,k}^2(x) \varphi_k(\widetilde{X}_k(t, x)) dx - 2\alpha_k \int_{\mathbb{R}^2} \omega_{0,k}(x) \varphi_k(\widetilde{X}_k(t, x)) dx \end{aligned}$$

$$\begin{aligned}
& + \alpha_k^2 \int_{\mathbb{R}^2} \varphi_k(\widetilde{X}_k(t, x)) dx \\
& = \int_{\mathbb{R}^2} (\omega_{0,k}(x) - \alpha_k)^2 \varphi_k(\widetilde{X}_k(t, x)) dx \\
& = \int_{\widetilde{X}_k(t, \cdot)^{-1}(B(h_k(t), C_R \delta_0))} (\omega_{0,k}(x) - \alpha_k)^2 dx.
\end{aligned}$$

Now we observe that since  $C_R \delta_0 < 1$ , by Proposition 3.9, we get

$$\widetilde{X}_k(t, \cdot)^{-1}(B(h_k(t), C_R \delta_0)) \subset B(h_k(0), 2C) \subset B(0, R).$$

Thus, we are allowed to use Proposition 3.7, and we have for  $x \in \widetilde{X}_k(t, \cdot)^{-1}(B(h_k(t), C_R \delta_0))$ :

$$C_R |x - h_k(0)| \leq |\widetilde{X}_k(t, x) - h_k(t)| \leq C_R \delta_0.$$

We get therefore

$$\int_{\mathbb{R}^2} (\omega(t, y) - \alpha_k)^2 \varphi_k(y) dy \leq \int_{B(h_k(0), \delta_0)} (\omega_0(x) - \alpha_k)^2 dx = 0,$$

and the conclusion follows.

In view of (3.13), (3.14) and (3.15), we finally conclude that

$$(3.16) \quad \omega(t, \cdot) = \alpha_k, \quad \text{a.e. on } B(h_k(t), C_R \delta_0).$$

#### 4. Proof of Theorem 1.7.

##### Step 1: uniqueness in the case of one point vortex.

We start with the case  $N = 1$ . Let  $(\omega, h)$  and  $(\widetilde{\omega}, \widetilde{h})$  two solutions of (1.1) with initial datum  $(\omega_0, h_0, \ell_0)$  satisfying the assumption of Theorem 1.7. So Theorem 1.6 holds for both solutions:  $\omega$  and  $\widetilde{\omega}$  remain constant in a neighborhood of the trajectories of  $h$  and  $\widetilde{h}$ .

Noting that  $u - \widetilde{u} = K * (\omega - \widetilde{\omega})$  with  $\int(\omega - \widetilde{\omega}) = 0$  and  $\omega, \widetilde{\omega}$  compactly supported, we have  $u - \widetilde{u} \in L^2(\mathbb{R}^2)$  (see [11, Proposition 3.3]) and we may consider the quantity

$$D(t) = \|u(t, \cdot) - \widetilde{u}(t, \cdot)\|_{L^2}^2 + |h(t) - \widetilde{h}(t)|^2 + |\dot{h}(t) - \dot{\widetilde{h}}(t)|^2, \quad t \in [0, T].$$

In what follows we establish a Gronwall inequality for  $D(t)$ .

We remark that the only difference between (1.1) and the vortex-wave system (1.3) is the ODE for the point vortex, since the PDE for the vorticity is the same. Thus we may directly use the estimates derived for (1.3) in [9, Subsection 3.4] for the quantity  $\|u(t, \cdot) - \widetilde{u}(t, \cdot)\|_{L^2}^2$ . More precisely, by the estimate (3.9) in [9] we have for  $t \in [0, T^*)$  and for all  $p \geq 2$

$$\|u(t, \cdot) - \widetilde{u}(t, \cdot)\|_{L^2}^2 \leq C \int_0^t \left( r(\tau) + \sqrt{r(\tau)} \varphi(\sqrt{r(\tau)}) + p r(\tau)^{1-1/p} \right) d\tau,$$



where

$$r(t) = \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^2}^2 + |h(t) - \tilde{h}(t)|^2,$$

$\varphi(\tau) = \tau |\ln \tau|$  and  $T^* \in [0, T]$  is the largest time such that  $|h(t) - \tilde{h}(t)| < \min(1, \delta/2)$  on  $[0, T^*]$ . So using that  $r(t) \leq D(t)$ , and the inequalities  $\tau\varphi(\tau) \leq \varphi(\tau^2)$ ,  $\tau \leq \varphi(\tau)$  for  $\tau \leq 1$  and  $\varphi(\tau) \leq p\tau^{1-1/p}$  (for all  $p \geq 2$ ), we get for  $t \in [0, T^*]$  and for all  $p \geq 2$

$$(4.1) \quad \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{L^2}^2 \leq Cp \int_0^t D(\tau)^{1-1/p} d\tau.$$

We emphasize that the property obtained in Theorem 1.6 is crucial in order to obtain the previous estimate, by implying in particular that  $u - \tilde{u}$  is harmonic in the neighborhood of  $h$  and  $\tilde{h}$ .

We turn next to the estimate for the point vortices. We compute

$$\begin{aligned} & \frac{d}{dt}|h - \tilde{h}|^2 + \frac{d}{dt}|\dot{h} - \dot{\tilde{h}}|^2 \\ &= 2\langle h - \tilde{h}, \dot{h} - \dot{\tilde{h}} \rangle - 2\frac{\gamma}{m}\langle \dot{h} - \dot{\tilde{h}}, u(t, h)^\perp - \tilde{u}(t, \tilde{h})^\perp \rangle \\ &\leq D(t) + 2\frac{\gamma}{m}\sqrt{D(t)}|u(h) - u(\tilde{h})| + 2\frac{\gamma}{m}\sqrt{D(t)}|(u - \tilde{u})(\tilde{h})|. \end{aligned}$$

Exactly as in Step 2 in the proof of [9, Proposition 3.10], we use the analyticity of  $u - \tilde{u}$  near  $h$  and  $\tilde{h}$  to get finally that for all  $p \geq 2$ ,

$$(4.2) \quad \frac{d}{dt}|h - \tilde{h}|^2 + \frac{d}{dt}|\dot{h} - \dot{\tilde{h}}|^2 \leq C\varphi(D(t)) \leq CpD(t)^{1-1/p}, \quad \forall t \in [0, T^*].$$

Finally, gathering (4.1) and (4.2), we find

$$D(t) \leq Cp \int_0^t D(\tau)^{1-1/p} d\tau, \quad \forall p \geq 2.$$

So we conclude by usual arguments (see [11, Chapter 8] that  $D \equiv 0$  on  $[0, T^*]$ ). Thus by definition of  $T^*$  we get  $T^* = T$  and uniqueness follows on  $[0, T]$ .

### Step 2: Proof of Theorem 1.7 completed

Once the case of one point is settled, the conclusion of Theorem 1.7 follows easily by adapting the proof above to the case of several points, using (3.16), (2.12) and (2.13). We refer also to the proof of uniqueness in [15, Theorem 2.1, Chapter 2] dealing with several points.

## 5. Some additional properties

We prove in this section that System (1.1) has an energy and a momentum which are conserved.

**Proposition 5.1.** — Let  $\omega_0$  and  $(\{h_{k,0}\}, \{\ell_{k,0}\})$  be as in (1.2) and let  $(\omega, \{h_k\})$  be any corresponding weak solution to (1.1) on  $[0, T]$ . The following quantities are conserved:

– The energy,

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \omega(t,y) \omega(t,x) dx dy + \frac{1}{\pi} \sum_{k=1}^N \gamma_k \int_{\mathbb{R}^2} \ln|x-h_k(t)| \omega(t,x) dx \\ &+ \sum_{j \neq k} \frac{\gamma_k \gamma_j}{2\pi} \ln|h_k(t) - h_j(t)| - \sum_{k=1}^N m_k |\dot{h}_k(t)|^2. \end{aligned}$$

– The momentum,

$$\mathcal{I}_0 = \int_{\mathbb{R}^2} |x|^2 \omega(t,x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 - 2 \sum_{k=1}^N m_k h_k(t)^\perp \cdot \dot{h}_k(t).$$

*Proof.* — (sketch) For  $\varepsilon < \frac{1}{3} \min_{j \neq k} \min_{t \in [0, T]} |h_j(t) - h_k(t)|$ , we replace  $\ln$  by the smooth function  $\ln_\varepsilon$  defined in the first section and we set  $\varphi_\varepsilon = \frac{1}{2\pi} \ln_\varepsilon * \omega$  as in (2.3), so that, setting

$$\begin{aligned} \mathcal{H}_\varepsilon &= \int_{\mathbb{R}^2} \varphi_\varepsilon(t,x) \omega(t,x) dx + 2 \sum_{k=1}^N \gamma_k \varphi_\varepsilon(t, h_k(t)) \\ &+ \sum_{j \neq k} \frac{\gamma_j \gamma_k}{2\pi} \ln_\varepsilon |h_j(t) - h_k(t)| - \sum_{k=1}^N m_k |\dot{h}_k(t)|^2, \end{aligned}$$

we have  $\sup_{[0, T]} |\mathcal{H}_0 - \mathcal{H}_\varepsilon| \leq C\varepsilon$ , with the quantity  $C$  depending only on  $\sup_{[0, T]} \|\omega\|_{L^\infty}$ ,  $\|h_k\|_{L^\infty}$ ,  $m_k$ ,  $\gamma_k$  etc.

It suffices then to compute the time derivative of  $\mathcal{H}_\varepsilon$  using the weak formulation for  $\omega$  and the ODE for the  $h'_k$ 's, which yields  $\sup_{[0, T]} |\dot{\mathcal{H}}_\varepsilon| \leq C\varepsilon$ . Letting  $\varepsilon$  tend to zero, the conclusion follows.

For  $\mathcal{I}_0$  we compute directly the time derivative using the weak formulation for  $\omega$  and the ODE for the  $h'_k$ 's and we show that it vanishes, which yields the result.  $\square$

With these conservations, we can prove that the gyroscopic point vortices are confined if  $\omega$  and  $\{\gamma_k\}$  have the same sign.

**Corollary 5.2.** — Assume moreover that

$$\omega_0 \geq 0, \text{ a.e. on } \mathbb{R}^2, \quad \gamma_k > 0, \quad k = 1, \dots, N.$$

Let  $(\omega, \{h_k\})$  be any corresponding weak solution to (1.1) on  $[0, T]$ . Then there exists  $C > 0$  and  $d > 0$ , depending only on  $\mathcal{H}_0$ ,  $\mathcal{I}_0$ ,  $m_k$ ,  $\gamma_k$  and  $\|\omega_0\|_{L^\infty}$  and

$R_0$ , but not on  $T$ , such that

$$\sup_{t \in [0, T]} \left( |\dot{h}_k(t)|^2 + |h_k(t)|^2 \right) \leq C$$

and

$$\inf_{t \in [0, T]} \min_{j \neq k} |h_j(t) - h_k(t)| \geq d.$$

*Proof.* — Since  $\omega$  is transported by the flow, we have  $\omega(t, \cdot) \geq 0$  almost everywhere for  $t \in [0, T]$ . Similarly to the proof of Proposition 2.5, picking  $m \neq n$ , we have, using that  $\ln(|x - y|) \leq |x| + |y|$ ,

$$\begin{aligned} \frac{\gamma_m \gamma_n}{2\pi} \ln |h_m(t) - h_n(t)| &\geq \mathcal{H}_0 - \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (|x| + |y|) \omega(t, y) \omega(t, x) dx dy \\ &\quad - \frac{1}{\pi} \sum_{k=1}^N \gamma_k \int_{\mathbb{R}^2} (|x| + |h_k(t)|) \omega(t, x) dx - \sum_{j \neq k} \frac{\gamma_k \gamma_j}{2\pi} (|h_k(t)| + |h_j(t)|) \end{aligned}$$

therefore by Cauchy-Schwartz inequality:

(5.1)

$$\frac{\gamma_m \gamma_n}{2\pi} \ln |h_m(t) - h_n(t)| \geq \mathcal{H}_0 - C \left( \int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{1/2},$$

where  $C$  depends only on  $\|\omega(t, \cdot)\|_{L^1} = \|\omega_0\|_{L^1}$  and on  $\gamma_k$ .

By the same estimates we also obtain

$$(5.2) \quad \sum_{k=1}^N m_k |\dot{h}_k(t)|^2 \leq -\mathcal{H}_0 + C \left( \int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{1/2}.$$

On the other hand, by Cauchy-Schwartz inequality:

$$\begin{aligned} \int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 &\leq \mathcal{I}_0 + C \left( \sum_{k=1}^N m_k |\dot{h}_k(t)|^2 \right)^{1/2} \left( \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{1/2} \\ &\leq \mathcal{I}_0 + C \left( -\mathcal{H}_0 + C \left( \int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{1/2} \right)^{1/2} \left( \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{1/2} \\ &\leq \mathcal{I}_0 + C \left( 1 + \int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 \right)^{3/4} \end{aligned}$$

where we have used (5.2). We conclude that

$$\int_{\mathbb{R}^2} |x|^2 \omega(t, x) dx + \sum_{k=1}^N \gamma_k |h_k(t)|^2 \leq C$$

with  $C$  depending only  $\mathcal{I}_0, \mathcal{H}_0$  and  $\|\omega_0\|_{L^1}$ . Coming back to (5.1) and (5.2), the conclusion follows.  $\square$

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March 4, 2019

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