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# Output regulation for a cascaded network of $2 \times 2$ hyperbolic systems with PI controller

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## Abstract

We are concerned with the PI control/regulation design for a cascaded network of multi systems governed by hyperbolic partial differential equations. The PI controllers have both control inputs and measured outputs situated on the junctions. The stability analysis of closed-loop network is carried out in the  $L^2$  topology by using the Lyapunov direct method. Then the output regulation is proven based on the stability of closed-loop systems. We finally work out detailed PI controller design for a practical cascaded network of  $n$  hydraulic Saint-Venant models as well as numerical simulations to validate theoretical results.

*Key words:* Hyperbolic PDE systems, Lyapunov functional, exponential stability, PI controllers, Saint-Venant equations, cascaded network, output regulation.

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## 1 Introduction

In this paper, we consider a cascaded network of  $n$  systems described by the following two hyperbolic partial differential equations (PDE)

$$\begin{cases} \partial_t \psi_{i1}(x, t) + \lambda_{i1} \partial_x \psi_{i1}(x, t) = 0 \\ \partial_t \psi_{i2}(x, t) - \lambda_{i2} \partial_x \psi_{i2}(x, t) = 0 \end{cases} \quad (1)$$

where  $x$  is in  $[0, L]$ ,  $t$  in  $[0, \infty)$ ,  $i = \overline{1, n}$ ,  $\psi_{i1}, \psi_{i2} : [0, L] \times [0, \infty) \rightarrow \mathbb{R}$  are two states and  $\lambda_{i1}, \lambda_{i2}$  are two positive constants.

Each systems (1) is a  $2 \times 2$  linear hyperbolic system. This type of dynamics may be used to describe various physical phenomena. Indeed, a large number of models can be transformed into the form (1) by some change of coordinates as shown for example in [4,2,8,16,19].

We consider in this work a network of systems (1) since each systems are connected to some other via boundary conditions. The network topology considered in this work is the case of a line (also named the cascaded networks). Networks of hyperbolic systems have already been studied in the literature. The case of a line has been

studied in [12,3]. The case of a tree-shaped ones can be found in [11,14,19].

One of the main research directions for the control of hyperbolic systems is the boundary stabilization control. In this case, control laws are designed from boundary conditions to ensure asymptotic stability of an equilibrium point. Regarding various studies in the literature, we distinguish two types of control input: one is static control law (see [13,10,5,6]) and the other is dynamic control law that is more complex to design but more robust (for instance faced to constant disturbances), see [17,19,24,4,21] to cite but a few.

In this work, our goal is to design a dynamic control law under the proportional- integral (PI) form in order to guarantee the stability of closed-loop hyperbolic systems and meanwhile to regulate output measurements to a desired set-point.

In industrial applications, the PI controller is considered as one of the most effective approaches to remove steady state errors and oscillations of the controlled system, see in [1]. The idea of designing PI control for hyperbolic systems has been inspired from the seminal work in [18]. Following the seminal work, many results have been developed by using operator and semi-group methods (see [23,15,24]) or frequency domain methods combined with Laplace transformation (see [4,7]). More recently the Lyapunov direct method has been exploited

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to prove the stability of closed-loop systems controlled by PI controllers, see [8,7,21,22]. The Lyapunov method has the advantage of giving direct analysis to stability. Moreover, by the Lyapunov direct method it is possible to deal with stabilisation of nonlinear systems as shown in [5]. Nevertheless it seems challenging to apply Lyapunov techniques for cascaded networks of hyperbolic systems because finding a Lyapunov functional for the global network of closed-loop systems controlled by PI controllers is complex. The main contribution of the present paper is therefore threefold: (i) proposing a systematic design of PI controllers at junctions for the cascaded network of  $n$  hyperbolic systems; (ii) constructing a new strict Lyapunov functional to prove the exponential stability of the cascaded network of closed-loop systems controlled by PI controllers; (iii) applying PI controller design for a practical cascaded network of  $n$  fluid flow Saint-Venant models. Meanwhile numerical simulations are carried out to confirm the robustness performance of PI controllers against constant disturbances.

The paper is organized as follows. Section 2 is devoted to presenting the PI controller structure used throughout the paper and stating our main control design result. In Section 3, by using the Lyapunov direct method we give a proof of the main result. Then we explain step by step in Section 4 how to apply our general PI controller design for a cascaded network of  $n$  fluid flow Saint-Venant models. Meanwhile numerical simulations of closed-loop network are presented to show effectiveness of our designed PI controllers. Finally conclusions are given in Section 5.

## 2 Statement of the problem and main result

### 2.1 Statement of the problem and PI controller structure

We consider  $n$  PDE hyperbolic systems (1) ( see Figure 1) under the initial conditions:  $\psi_{i1}(x, 0) = \psi_{i1}^0(x)$ ,  $\psi_{i2}(x, 0) = \psi_{i2}^0(x)$ . In addition, we suppose that the boundary conditions define a cascaded network. More precisely, we consider the following boundary conditions defined at the junctions

- For the first junction,

$$\psi_{11}(0, t) = R_{11}\psi_{12}(0, t) \quad (2)$$

- For  $n-1$  intermediate junctions for  $i = \overline{1, n-1}$

$$\begin{aligned} \psi_{i2}(L, t) &= R_{i2}\psi_{i1}(L, t) + u_i(t) \\ \psi_{(i+1)1}(0, t) &= R_{(i+1)1}\psi_{(i+1)2}(0, t) + \alpha_i\psi_{i1}(L, t) + \delta_i\psi_{i2}(L, t) \end{aligned} \quad (3)$$

- For the last junction,

$$\psi_{n2}(L, t) = R_{n2}\psi_{n1}(L, t) + u_n(t) \quad (4)$$

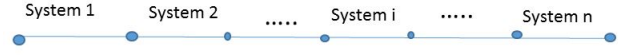


Fig. 1. Cascaded network of  $n$  systems

where  $u_i$  are control inputs located at  $x = L$  of each junction;  $R_{i1}$ ,  $R_{i2}$ ,  $\alpha_i$  and  $\delta_i$  are real constants, with  $i = \overline{1, n}$ .

The measured outputs that need to be regulated are located at the junctions and disturbed by some unknown output perturbations, i.e

$$y_i(t) = a_i\psi_{i1}(L, t) + b_i\psi_{i2}(L, t) + w_{io} \quad (5)$$

where  $a_i$ ,  $b_i$  are unknown constants dependant on chosen outputs and  $w_{io} \in \mathbb{R}$  are unknown constant perturbations. Our purpose in the paper is to propose  $n$  dynamic feedback control laws  $u_i(t)$  with structure of PI controllers such that the cascaded network of closed-loop systems is exponentially stable and the  $n$  disturbed output measurements  $y_i(t)$  are regulated to the desired references  $y_{ir}$ . To be more detail, we must design  $n$  PI controllers located at the junctions with real gain parameters  $K_{iP}$ ,  $K_{iI}$ , and corrupted by some unknown constant control disturbances  $w_{ic}$ , i.e

$$u_i(t) = K_{iP}(y_i(t) - y_{ir}) + K_{iI} \int_0^t (y_i(s) - y_{ir})ds + w_{ic} \quad (6)$$

in order to ensure the exponential stability of the network of closed-loop systems (1)-(6) and guarantee the regulation of measured outputs  $y_i(t)$  to the desired values  $y_{ir}$ .

### 2.2 Main result

Denoting the new state variables  $z_i(t)$  where  $\partial_t z_i = y_i(t) - y_{ir}$ , the network of closed-loop systems (1)-(6) becomes a PDE-ODE system and is governed by ( $i = \overline{1, n}$  and  $j = \overline{2, n}$ ):

$$\left\{ \begin{aligned} \partial_t \psi_{i1}(x, t) &= -\lambda_{i1} \partial_x \psi_{i1}, \\ \partial_t \psi_{i2}(x, t) &= \lambda_{i2} \partial_x \psi_{i2}, \\ \partial_t z_i &= y_i(t) - y_{ir}, \\ \psi_{11}(0, t) &= R_{11}\psi_{12}(0, t), \\ \psi_{i2}(L, t) &= R_{i2}\psi_{i1}(L, t) + K_{iP}(y_i(t) - y_{ir}) \\ &\quad + K_{iI}z_i(t) + w_{ic}, \\ \psi_{j1}(0, t) &= R_{j1}\psi_{j2}(0, t) + \alpha_j\psi_{(j-1)1}(L, t) \\ &\quad + \delta_j\psi_{(j-1)2}(L, t), \\ y_i(t) &= a_i\psi_{i1}(L, t) + b_i\psi_{i2}(L, t) + w_{io}. \end{aligned} \right. \quad (7)$$

The closed-loop system (7) is completed by an initial condition in  $((L^2(0, L))^2 \times \mathbb{R})^n$ , which satisfy the  $C^0$  and  $C^1$  compatibility conditions in (7).

Let  $\mathbb{E} = ((L^2(0, L))^2 \times \mathbb{R})^n$  be the state space of the closed-loop system (7) equipped with the following norm:

$$\|Y\|_{\mathbb{E}}^2 = \sum_{i=1}^n \left( \|Y_{3i-2}\|_{L^2(0,L)}^2 + \|Y_{3i-1}\|_{L^2(0,L)}^2 + Y_{3i}^2 \right)$$

where  $Y = (Y_1, Y_2, \dots, Y_{3n}) \in \mathbb{E}$ .

For each smooth initial condition  $(\psi_{11}^0(x), \psi_{12}^0(x), z_1^0, \dots, \psi_{n1}^0(x), \psi_{n2}^0(x), z_n^0)$  in  $\mathbb{E}$  satisfying  $C^0$  and  $C^1$  compatibility conditions, then there exists a unique smooth solution of (7) in  $\mathbb{E}$  for all  $t$  (see in [2,22,20]). We study therefore the exponential stability of the closed-loop system (7) in  $\mathbb{E}$  and the output regulation to the desired references.

Let us define iteratively equilibrium points  $\psi_{i1\infty}$ ,  $\psi_{i2\infty}$  and  $z_{i\infty}$  as follows

$$\begin{aligned} \psi_{i2\infty} &= \frac{y_{ir} - w_{io} + a_i D_i}{b_i + a_i R_{i1}}, \quad \psi_{i1\infty} = R_{i1} \psi_{i2\infty} - D_i, \\ z_{i\infty} &= \frac{\psi_{i2\infty} - R_{i2} \psi_{i1\infty} - w_{ic}}{K_{iI}}. \end{aligned} \quad (8)$$

where  $D_1 = 0$ ,  $D_j = \alpha_j \psi_{(j-1)1\infty} + \delta_j \psi_{(j-1)2\infty}$ ,  $\forall j = \overline{2, n}$ . We suppose the following hypotheses for the output measures:

- H1:  $a_i \neq 0 \forall i = \overline{1, n}$ .
- H2:  $a_i + b_i R_{i2} \neq 0 \forall i = \overline{1, n}$ .
- H3:  $b_i + a_i R_{i1} \neq 0 \forall i = \overline{1, n}$ .

Then the main result of our paper is given in the following theorem.

**Theorem 1** *Assume that the three hypotheses H1, H2 and H3 are satisfied. Then, there exists  $\mu^* > 0$  such that for each initial condition in  $\mathbb{E}$  satisfying the  $C^0$  and  $C^1$  compatibility conditions, each  $\mu \in (0, \mu^*)$  and each PI controller with the following proportional gain  $K_{iP}$  and the integral gain  $K_{iI}$  for all  $i = \overline{1, n}$ :*

$$K_{iP} = \frac{-R_{i2}}{a_i}, \quad K_{iI} = -\mu \frac{(b_i + a_i R_{i1} e^{\mu L})(a_i + b_i R_{i2})}{a_i}, \quad (9)$$

the following two properties hold true:

- The equilibrium state defined in (8) of network of closed-loop systems (7) is exponentially stable in the state space  $\mathbb{E}$ .

- For smooth initial condition in  $((H^1(0, L))^2 \times \mathbb{R})^n$  satisfying the  $C^0$  and  $C^1$  compatibility conditions, the  $n$  measured outputs  $y_i(t)$  are regulated to the desired set-points  $y_{ir}$ , i.e.  $\lim_{t \rightarrow \infty} |y_i(t) - y_{ir}| = 0, \forall i = \overline{1, n}$ .

**Remark :** As shown in the following proof, the design parameter  $\mu$  has to be selected sufficiently small. As a consequence, the integral term  $K_{iI}$  is small and the convergence rate obtained following this approach may be small. This is one of the drawback of this Lyapunov method.

We need to impose Hypothesis H1 for our PI controller to exist. On the contrary, if H2 is not satisfied, then our PI controller can still be implemented. However the integral term  $K_i$  disappears and consequently the PI control law becomes only a proportional control law. In that case, even so stability of the closed loop system may still be obtained, the integral effect of our PI controllers that leads to the output regulation is lost. Up to now, removing these conditions is an open question.

In addition, hypothesis H3 is a necessary condition for the existence of an equilibrium point of the closed-loop system (7) for all values of disturbances  $w_{io}$  and  $w_{ic}$ . Without this assumption, the procedure introduced in (8) is no longer valid. Hence, H3 is a necessary condition for the output regulation by integral action. ◀

### 3 Proof of the main result

To prove Theorem 1, the following coordinate transformation is considered

$$\phi_{i\ell}(x, t) = \psi_{i\ell}(x, t) - \psi_{i\ell\infty}, \quad \ell = \{1, 2\}, \quad (10)$$

$$\xi_i(t) = z_i(t) - z_{i\infty} \quad (11)$$

where  $\psi_{i1\infty}$ ,  $\psi_{i2\infty}$  and  $z_{i\infty}$  are defined in (8). With the new coordinates defined in (10) and applying the PI controller design (9) in Theorem 1, then we obtains the following network of closed-loop systems without perturbations

$$\left\{ \begin{aligned} \partial_t \phi_{i1}(x, t) &= -\lambda_{i1} \partial_x \phi_{i1}, \\ \partial_t \phi_{i2}(x, t) &= \lambda_{i2} \partial_x \phi_{i2}, \\ \partial_t \xi_i &= a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t) \\ \phi_{i2}(L, t) &= -k_i \xi_i(t), \\ \phi_{i1}(0, t) &= R_{i1} \phi_{i2}(0, t) + \alpha_i \phi_{(i-1)1}(L, t) + \beta_i \xi_{i-1}(t). \\ y_i(t) - y_{ir} &= a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t). \end{aligned} \right. \quad (12)$$

completed by the initial conditions in  $\mathbb{E}$ :

$$\phi_{i1}(x, 0) = \phi_{i1}^0(x), \phi_{i2}(x, 0) = \phi_{i2}^0(x), \xi_i(0) = \xi_i^0,$$

where

$$k_i = \mu(b_i + a_i R_{i1} e^{\mu L}) \quad (13)$$

and  $\alpha_1 = 0, \beta_1 = 0, \beta_j = -k_{j-1} \delta_j$  for  $j = \overline{2, n}$ .

In the new coordinates,  $y_i(t) - y_{ir} = a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t)$ , the output regulation is obtained if

$$\lim_{t \rightarrow \infty} |a_i \phi_{i1}(L, t) + b_i \phi_{i2}(L, t)| = 0, \forall i = \overline{1, n}.$$

Hence, to prove the asymptotic stability and the output regulation for the disturbed network systems in (7) with the PI control design in (6), we must naturally prove the stability of equivalent system (12) to the origin.

In the following, the Lyapunov function candidate is given in Section 3.1 and then the proof of the Theorem 1 by using direct Lyapunov method is presented in Section 3.2.

**Remark :** In this paper, we have considered cascaded systems of conservation laws without the presence of source terms and dissipative terms. Our approach follows a Lyapunov design. It would be very difficult to extend our result to systems of balance laws with source terms. In that case, we should probably restrict ourselves to some very particular class of systems and lose our level of generality. For instance, authors in [3] have constructed Lyapunov function for a special class of cascaded network of hyperbolic systems of balance laws, but only with the static control laws, i.e without the integral terms. There are other methods, for instance operator and semi-group [24] or predictive control in [17], studying the hyperbolic systems with source terms. However, these techniques are very difficult to extend to nonlinear case compared to the Lyapunov method used in our approach.  $\blacktriangleleft$

### 3.1 Lyapunov candidate functional

In the paper, we construct the following Lyapunov candidate functional:

$$\mathbb{V}(\phi_{11}, \phi_{12}, \xi_1, \dots, \phi_{n1}, \phi_{n2}, \xi_n) = \sum_{i=1}^n p_i V_i \quad (14)$$

where  $V_i$  is defined by

$$V_i(\phi_{i1}, \phi_{i2}, \xi_i) = \int_0^L F_i^T P_i F_i dx$$

$$\text{with } F_i = \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix} \text{ and } P_i = \begin{pmatrix} 1 & 0 & q_{i3} \\ 0 & q_{i1} & q_{i4} \\ q_{i3} & q_{i4} & q_{i2} \end{pmatrix}.$$

Here  $p_i$  and  $q_{i1}, \dots, q_{i4}$  are positive real number that be designed later on.

To begin with, consider the set  $\mathbb{X} = (L^2(0, L))^2 \times \mathbb{R}$ . The following lemma for the constructing of each sub-function  $V_i$  can be obtained.

**Lemma 1** *Let  $k_i$  be defined in (13) and  $q_{i1}, q_{i2}, q_{i3}, q_{i4}$  be defined as follows:*

$$\begin{aligned} q_{i1} &> \frac{3\lambda_{i1} R_{i1}^2}{\lambda_{i2}}, \quad q_{i2} = \mu e^{\mu L} \lambda_{i2} q_{i1}, \\ q_{i3} &= \mu e^{\frac{3\mu L}{2}} \frac{a_i \lambda_{i2} q_{i1}}{\lambda_{i1}}, \quad q_{i4} = \mu e^{\frac{3\mu L}{2}} a_i R_{i1} q_{i1}. \end{aligned} \quad (15)$$

Then there exists  $\mu^* > 0$  such that for every  $\mu \in (0, \mu^*)$ , we have

(1) *There exists  $M_i > 0$  such that  $\forall (\phi_{i1}, \phi_{i2}, \xi_i)$  in  $\mathbb{X}$ :*

$$\begin{aligned} &\frac{1}{M_i} V_i(\phi_{i1}, \phi_{i2}, \xi_i) \\ &\leq \|\phi_{i1}(\cdot, t)\|_{L^2(0, L)}^2 + \|\phi_{i2}(\cdot, t)\|_{L^2(0, L)}^2 + \xi_i^2(t) \\ &\leq M_i V_i(\phi_{i1}, \phi_{i2}, \xi_i) \end{aligned} \quad (16)$$

(2) *There exists  $\gamma_i > 0$  such that along smooth solutions of (12), for all  $t$  such that the solution is well defined*

$$\begin{aligned} \dot{V}_i(t) &\leq -\gamma_i V_i(t) - \frac{1}{4} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ &- \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \phi_{(i-1)1}^2(L, t) A_i + \xi_{i-1}^2(t) B_i, \end{aligned} \quad (17)$$

where  $A_i = \lambda_{i1} \alpha_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right)$ ,  $B_i = \lambda_{i1} \beta_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right)$  and where we have used the slight abuse of notation  $V_i(t) = V_i(\phi_{i1}(\cdot, t), \phi_{i2}(\cdot, t), \xi_i(t))$ .

The proof of Lemma 1 is given in the Appendix.

Now, by employing Lemma (1), the following lemma for the design of Lyapunov candidate function  $\mathbb{V}$  is given,

**Lemma 2** *Let  $k_i$  and  $q_{i1}, q_{i2}, \dots, q_{i4}$  be defined in Lemma 1 and  $p_i$  be defined as follows*

$$p_1 > 0, \quad p_{i+1} = \epsilon p_i. \quad (18)$$

Then there exists  $\epsilon > 0$  and  $\mu^* > 0$  such that for every  $\mu \in (0, \mu^*)$ , we have :

(1) There exists  $M > 0$  such that for all  $Z = (\phi_{11}, \phi_{12}, \xi_1, \dots, \phi_{n1}, \phi_{n2}, \xi_n)$  in  $\mathbb{E}$ :

$$\frac{1}{M} \mathbb{V}(Z) \leq \|Z\|_{\mathbb{E}}^2 \leq M \mathbb{V}(Z). \quad (19)$$

(2) There exists  $\gamma > 0$  such that along smooth solutions of (7), for all  $t$  at which the solution is well defined

$$\dot{\mathbb{V}}(t) \leq -\gamma \mathbb{V}(t) - \delta \sum_{i=1}^n (\xi_i^2(t) + \phi_{i1}^2(L, t)), \quad (20)$$

we have used the notation  $\mathbb{V}(t) = \mathbb{V}(\phi_{11}(\cdot, t), \phi_{12}(\cdot, t), \xi_1(t), \dots, \phi_{n1}(\cdot, t), \phi_{n2}(\cdot, t), \xi_n(t))$ .

**Proof :** To begin with, applying the property (16) of Lemma 1, with  $\mu$  small enough, one easily finds that

$$\begin{aligned} \sum_{i=1}^n \frac{p_i}{M_i} V_i &\leq \|(\phi_{11}, \phi_{12}, \xi_1, \dots, \phi_{n1}, \phi_{n2}, \xi_n)\|_{\mathbb{E}}^2 \\ &\leq \sum_{i=1}^n p_i M_i V_i. \end{aligned} \quad (21)$$

This leads to the existence of  $M > 0$  such that (19) holds.

To prove (20), we analyze the time derivative of  $\mathbb{V}$  along the solution of the network of closed-loop systems in (12),

$$\dot{\mathbb{V}}(t) = \sum_{i=1}^n p_i \dot{V}_i(t)$$

To simplify the writing in the following, we denote  $p_{n+1}$ ,  $A_{n+1}$  and  $B_{n+1}$  such that  $p_{n+1} = A_{n+1} = B_{n+1} = 0$ . Now, by employing the property (17) of Lemma 1, one finds that

$$\begin{aligned} \dot{\mathbb{V}}(t) &\leq - \sum_{i=1}^n p_i \gamma_i V_i(t) \\ &\quad - \sum_{i=1}^n \xi_i^2(t) \left( p_i \frac{k_i^2 \lambda_{i2} q_{i1} e^{\mu L}}{4} - p_{i+1} B_{i+1} \right) \\ &\quad - \sum_{i=1}^n \phi_{i1}^2(L, t) \left( p_i \frac{\lambda_{i1} e^{-\mu L}}{2} - p_{i+1} A_{i+1} \right). \end{aligned} \quad (22)$$

Since  $p_{i+1} = \epsilon p_i$ , by taking  $\epsilon$  small enough, it is clearly to see that

$$\begin{aligned} p_i \frac{k_i^2 \lambda_{i2} q_{i1} e^{\mu L}}{4} - p_{i+1} B_{i+1} &> 0 \\ p_i \frac{\lambda_{i1} e^{-\mu L}}{2} - p_{i+1} A_{i+1} &> 0 \end{aligned} \quad (23)$$

From (22) and (23), there exists  $\gamma > 0$  and  $\delta > 0$  such that

$$\dot{\mathbb{V}}(t) \leq -\gamma \sum_{i=1}^n p_i V_i(t) - \delta \sum_{i=1}^n (\xi_i^2(t) + \phi_{i1}^2(L, t))$$

This inequality implies that (20) holds.  $\square$

**Remark :** In our Lyapunov functional (14), compared to the one in [3], dynamic feedbacks are added with  $n$  new states  $\xi_i(t)$ . Some Lyapunov functional for dynamic feedback states have been studied for single hyperbolic systems in literature [8,7], but it should be pointed out that our Lyapunov function is different since it contains coupling terms of the states  $\phi_i$  and feedback states  $\xi_i$ . These coupling terms allow to avoid the damping term in PI controller as in [7]. It allows also to consider a larger class of hyperbolic PDE systems than the one considered in [8].

Note that in [21], a Lyapunov functional with the dynamic feedback terms and the coupled terms is considered only for the single hyperbolic system, but cannot be extended for a cascaded network.  $\blacktriangleleft$

### 3.2 Proof of the theorem 1

In this section, by using the result of Lemma 2, we present the proof of Theorem 1.

First of all, we prove the exponential stability of the closed-loop system (7) to the equilibrium state  $(\psi_{11\infty}, \psi_{12\infty}, z_{1\infty}, \dots, \psi_{n1\infty}, \psi_{n2\infty}, z_{n\infty})$ . From Lemma 2, there exists  $\gamma > 0$  such that along smooth solutions

$$\mathbb{V}(t) \leq \mathbb{V}(0) e^{-\gamma t}. \quad (24)$$

With (19), it implies that there exists  $S > 0$  such that for all smooth initial conditions in  $\mathbb{E}$  and satisfying  $C^0$ ,  $C^1$  compatibility conditions, the solution of (12) is defined for all positive time and satisfies that for all  $t \geq 0$

$$\begin{aligned} \|(\phi_{11}(\cdot, t), \phi_{12}(\cdot, t), \xi_1(t), \dots, \phi_{n1}(\cdot, t), \phi_{n2}(\cdot, t), \xi_n(t))\|_{\mathbb{E}}^2 \\ \leq S e^{-\gamma t} \|(\phi_{11}^0, \phi_{12}^0, \xi_1^0, \dots, \phi_{n1}^0, \phi_{n2}^0, \xi_n^0)\|_{\mathbb{E}}^2. \end{aligned} \quad (25)$$

Inequality (25) implies that the origin of the closed-loop system (12) is exponentially stable in  $\mathbb{E}$ . This leads to the proof of the first property in Theorem 1.

Secondly, we prove the output regulation property of Theorem 1. Let denote  $v_{i1} = -\lambda_{i1} \partial_x \phi_{i1}$ ,  $v_{i2} = \lambda_{i2} \partial_x \phi_{i2}$ , and  $s_i(t)$  such that  $\partial_t s_i = a_i v_{i1}(L, t) + b_i v_{i2}(L, t)$ . It can be easily found that the dynamics of  $v_{i1}$ ,  $v_{i2}$  and  $s_i$  are

the same as the ones of  $\phi_{i1}$ ,  $\phi_{i2}$  and  $\xi_i$  in (12):

$$\begin{cases} \partial_t v_{i1}(x, t) = -\lambda_{i1} \partial_x v_{i1}, \\ \partial_t v_{i2}(x, t) = \lambda_{i2} \partial_x v_{i2}, \\ \partial_t s_i = a_i v_{i1}(L, t) + b_i v_{i2}(L, t) \\ v_{i2}(L, t) = -k_i s_i(t), \\ v_{i1}(0, t) = R_{i1} v_{i2}(0, t) + \alpha_i v_{(i-1)1}(L, t) + \beta_i s_{i-1}(t). \end{cases} \quad (26)$$

From the hypothesis in theorem, we deduce that (26) has also smooth initial condition  $(v_{i1}^0(\cdot), v_{i2}^0(\cdot), s_i^0)$  in  $\mathbb{E}$  ( $\forall i = \overline{1, n}$ ) and satisfying the  $C^0$  and  $C^1$  compatibility conditions. Employing similar analysis, one finds that the origin of (26) is also exponentially stable in  $\mathbb{E}$ . This allows us to prove that:

$$\lim_{t \rightarrow \infty} \|\phi_{i1}(\cdot, t)\|_{H^1(0, L)} = 0, \quad \lim_{t \rightarrow \infty} \|\phi_{i2}(\cdot, t)\|_{H^1(0, L)} = 0.$$

By the Sobolev embedding theorem, one gives that

$$\lim_{t \rightarrow \infty} \phi_{i1}(x, t) = 0, \quad \lim_{t \rightarrow \infty} \phi_{i2}(x, t) = 0 \quad \forall x \in [0, L].$$

Therefore, the output regulation is obtained, i.e  $\lim_{t \rightarrow \infty} |y_i(t) - y_{ir}| = 0, \forall i = \overline{1, n}$ .

This completes the proof of the Theorem 1.

## 4 Application for the cascaded network of $n$ open channels

### 4.1 Modelling of network

In this section, we consider a cascaded network of  $n$  horizontal channels which are described by Saint Venant equations with the neglected friction slope, see in Figure 2, studied for example in [21,8,16] for a single channel and [17] for network of  $n$  channels with  $x$  in  $[0, L]$  and  $t \in [0, \infty)$ ,  $i = \overline{1, n}$ ,

$$\begin{cases} B_i \partial_t H_i(x, t) + \partial_x Q_i(x, t) = 0 \\ \partial_t Q_i(x, t) + \partial_x \left( \frac{Q_i^2(x, t)}{B_i H_i(x, t)} + \frac{1}{2} g B_i H_i^2(x, t) \right) = 0 \\ y_i(t) = H_i(L, t) + w_{io} \end{cases} \quad (27)$$

where  $H_i(x, t)$  denote water level,  $Q_i(x, t)$  water discharge at the position  $x$  and time  $t$  respectively,  $B_i$  the channel width,  $g$  the gravitational constant,  $y_i(t)$  the measured output, and  $w_{io}$  unknown output disturbance for channel  $i$ . The network is controlled by  $n$  underflow

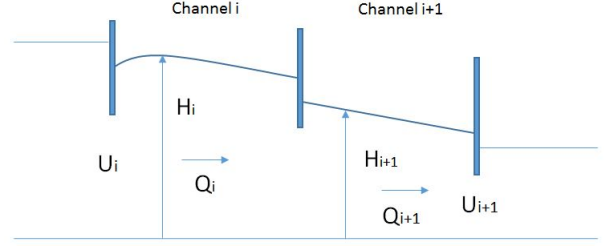


Fig. 2. Cascaded network of  $n$  fluid flow channels

gates at the end of each channel:

$$Q_i^2(L, t) = 2gF_i^2 U_i^2(t) (H_{gi}^u - H_{gi}^d), \quad i = \overline{1, n} \quad (28)$$

where  $F_i$  is the coefficient of gates,  $U_i(t)$  the opening of the gate considering as control inputs,  $H_{gi}^u$  and  $H_{gi}^d$  the water levels at upstream and downstream of gate  $i$  (see in [17]).

Moreover,  $n$  channels are interconnected by  $n - 1$  following discharge conservation constrains

$$Q_i(L, t) = Q_{i+1}(0, t), \quad i = \overline{1, n-1} \quad (29)$$

The last boundary condition comes from the control of the inflow discharge by an appropriate constant value  $Q_0$ , i.e

$$Q_1(0, t) = Q_0. \quad (30)$$

**Remark :** In the present paper, the disturbances  $w_{io}$  and upperstream input  $Q_0$  are supposed constant. In the case that they are not constant, the regulation problem becomes much more difficult. It should be mentioned in [24] a special case where the disturbances are quadratically close to constant, the output regulation is still guaranteed for the following sense:

$$\lim_{t \rightarrow \infty} \int_t^\infty |y_i(t) - y_r|^2 = 0.$$

### 4.2 Linearized model

Let us first consider the set-points  $H_i^*$  for each channels satisfying the subcritical conditions  $gB_i^2(H_i^*)^3 - Q_0^2 > 0 \forall i = \overline{1, n}$ . One can easily see that  $(H_i^*, Q_0)$  is also steady state for each channel of the network (27)-(30).

Now considering the linearization of system (27) around a steady state  $(H_i^*, Q_0)$  with  $i = \overline{1, n}$  with the notation

$h_i = H_i - H_i^*$  and  $q_i = Q_i - Q_0$  as following:

$$\begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} h_i \\ q_i \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{B_i} \\ \frac{-Q_0^2}{B_i(H_i^*)^2} + gB_iH_i^* & \frac{2Q_0}{B_iH_i^*} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} h_i \\ q_i \end{pmatrix} = 0 \\ y_i(t) = h_i(L, t) + H_i^* + w_{i0} \end{cases} \quad (31)$$

Then by using the following change of coordinates

$$h_i = \psi_{i1} + \psi_{i2}, \quad q_i = (B_i\sqrt{gH_i^*} + \frac{Q_0}{H_i^*})\psi_{i1} - (B_i\sqrt{gH_i^*} - \frac{Q_0}{H_i^*})\psi_{i2}$$

One obtains the characteristic form

$$\begin{cases} \partial_t \psi_{i1}(x, t) + \lambda_{i1} \partial_x \psi_{i1}(x, t) = 0 \\ \partial_t \psi_{i2}(x, t) - \lambda_{i2} \partial_x \psi_{i2}(x, t) = 0 \\ y_i(t) = \psi_{i1}(L, t) + \psi_{i2}(L, t) + H_i^* + w_{i0}, \end{cases} \quad (32)$$

where  $\lambda_{i1} = \sqrt{gH_i^*} + \frac{Q_0}{B_iH_i^*} > 0$ ,  $\lambda_{i2} = \sqrt{gH_i^*} - \frac{Q_0}{B_iH_i^*} > 0$ .

By using the gates in (28) in an appropriate way, let us rewrite all boundary conditions in (28)-(30) in the new coordinates

$$\psi_{i2}(L, t) = R_{i2}\psi_{i1}(L, t) + u_i(t)$$

$$\psi_{i1}(0, t) = R_{i1}\psi_{i2}(0, t) + \alpha_i\psi_{(i-1)1}(L, t) + \delta_i\psi_{(i-1)2}(L, t), \quad (33)$$

where

$$R_{i1} = \frac{\lambda_{i2}}{\lambda_{i1}}, \quad R_{i2} = 0, \quad \forall i = \overline{1, n}$$

$$\alpha_1 = \delta_1 = 0, \quad \alpha_k = \frac{\lambda_{(k-1)1}}{\lambda_{k1}}, \quad \delta_k = -\frac{\lambda_{(k-1)2}}{\lambda_{k1}}, \quad \forall k = \overline{2, n}$$

and  $n$  new control inputs  $u_i(t)$  deduced from the opening gates  $U_i(t)$  by

$$U_i(t) = \frac{Q_i^* + B_i\lambda_{i1}(y_i - H_i^*)}{F_i\sqrt{2g(H_{gi}^u - H_{gi}^d)}} + \frac{B_i(\lambda_{i1} + \lambda_{i2})}{F_i\sqrt{2g(H_{gi}^u - H_{gi}^d)}} u_i(t) \quad (34)$$

### 4.3 PI control design

Inspired by the result in Theorem 1, we design  $n$  feedback control laws  $u_i(t)$  at junctions by the form of PI

controllers with unknown control disturbance  $w_{ic}$  as follows

$$u_i(t) = K_{iP}(y_i(t) - H_i^*) + K_{iI} \int_0^t (y_i(s) - H_i^*) ds + w_{ic}$$

where  $K_{iP}$  and  $K_{iI}$  are computed by

$$K_{iP} = 0$$

$$K_{iI} = -2\mu(1 + e^{\mu L} \frac{\sqrt{gH_i^*} - \frac{Q_0}{B_iH_i^*}}{\sqrt{gH_i^*} + \frac{Q_0}{B_iH_i^*}}), \quad \forall i = \overline{1, n} \quad (35)$$

Here, the tuning parameter  $\mu$  is chosen small enough. Then  $n$  PI controllers in (35) stabilize the linearized model (31) of channel network, and  $n$  measured outputs are regulated to the desired references.

**Remark :** To control this network of  $n$  cascaded models, we only need  $n$  control inputs represented by  $n$  underflow gates at junctions. Moreover, to implement controllers, only water levels at the end of each channel are required for the output measurements. This is an advantage because in practical engineering, water level is much simpler to measure than water discharge.

In addition, in paper [9] the authors give a nice work in experimenting of PI controllers on an open channel. Thus the present Section could be also seen as the possibility to use PI controllers for a more complex network of reaches in experimental aspect. ◀

### 4.4 Numerical simulations

In this Section, in order to validate theoretical results we make numerical simulations for a cascade network of three channels with the following data:

- Length of channels is  $L = 500$  m, with the same width  $B = 4$  m.
- Level set-points  $H_1^* = 10$  m,  $H_2^* = 8$  m,  $H_3^* = 6.5$  m, and constant discharge  $Q_0 = 7$  m<sup>3</sup>/s.
- Output disturbances  $w_{1o} = 0.1$ ,  $w_{2o} = 0.2$ ,  $w_{3o} = 0.15$ ; and control disturbances  $w_{1c} = 0.02$ ,  $w_{2c} = 0.03$ ,  $w_{3c} = 0.01$ .

Obviously, subcritical conditions are satisfied with above data. Tuning parameter  $\mu$  is chosen by 0.001. The simulations are based on Preissmann schema with the weighting coefficient  $\theta = 0.6$  and space discretization  $\Delta x = 1$  m.

Figure 3- Figure 8 verify the stability of linearized closed-loop network and show the evolutions states of the water level and water discharge for each channel controlled by



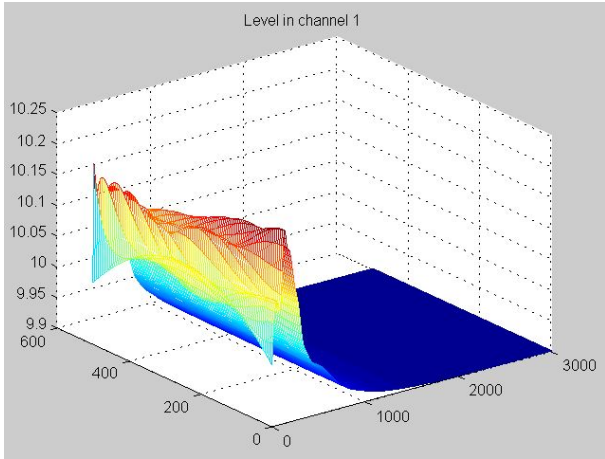


Fig. 3.  $H_1(x, t)$

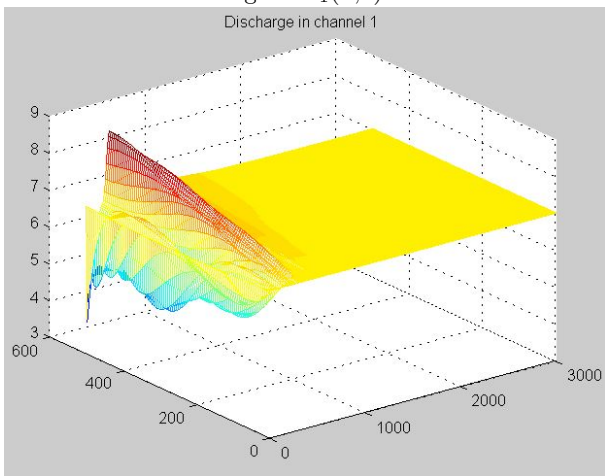


Fig. 4.  $Q_1(x, t)$

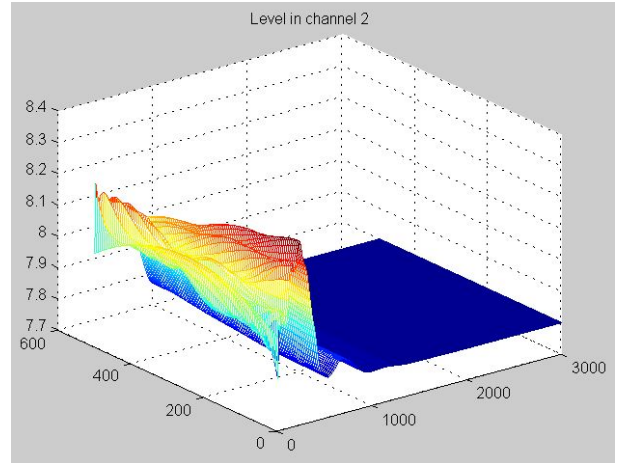


Fig. 5.  $H_2(x, t)$

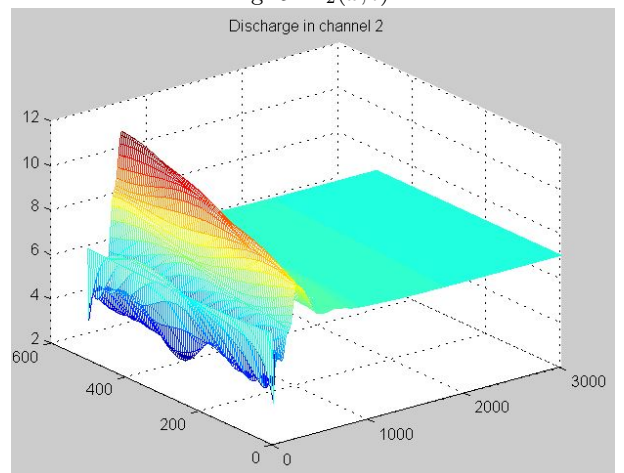


Fig. 6.  $Q_2(x, t)$

three PI controllers at junctions.

In Figure 9, we see that three output measurements  $y_i(t) = H_i(L, t)$  are regulated to the desired references  $H_i^*$  in spite of output disturbances  $w_{io}$  and control disturbances  $w_{ic}$  ( $i = \overline{1, 3}$ ).

Finally, note that with the big value of tuning parameter  $\mu$  (bigger than 0.002) in simulations, the controlled network becomes unstable.

## 5 Conclusions

In the paper, we have addressed the PI controller regulation problem for a class of cascaded network of linear  $2 \times 2$  hyperbolic PDE systems. The PI controllers are designed at junctions and are applied for each subsystem of the network. The exponential stability for the closed-loop systems and the output regulation are proven by using the Lyapunov direct method. The PI controller design can find applications for many networks of processes with the same steps as presented in Section 4. Although the stability analysis in the paper is taken in the  $L^2$  norm for a network of linear systems, it would be an interesting

work to generalize the PI controller design to the nonlinear case by following the works in [22] and [5]. In the future, the objective is to study the PI controller design for other types of networks of PDE hyperbolic systems, such as tree-shaped networks. We believe that the performance of the nonlinear closed-loop systems achieved by the PI controllers deserve more studies.

## 6 Appendix

### 6.1 Proof of Lemma 1

#### Proof of the first property 16

To begin with, we prove that the matrix  $P_i$  is positive definite. Making use of the Sylvester criterion, matrix  $P_i$  is positive definite if and only if:

$$q_{i1} > 0, \quad q_{i2} > q_{i3}^2 + \frac{q_{i4}^2}{q_{i1}}.$$

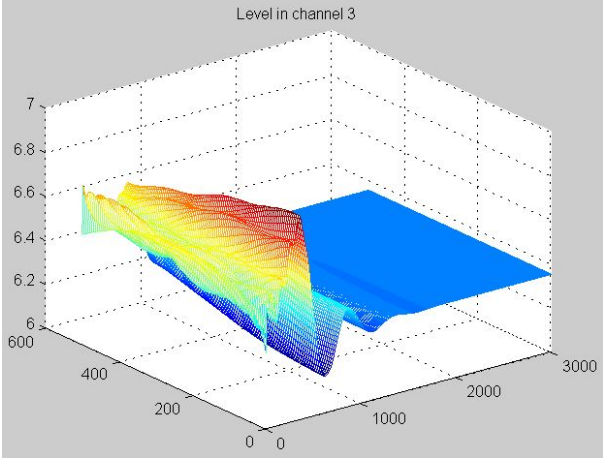


Fig. 7.  $H_3(x, t)$

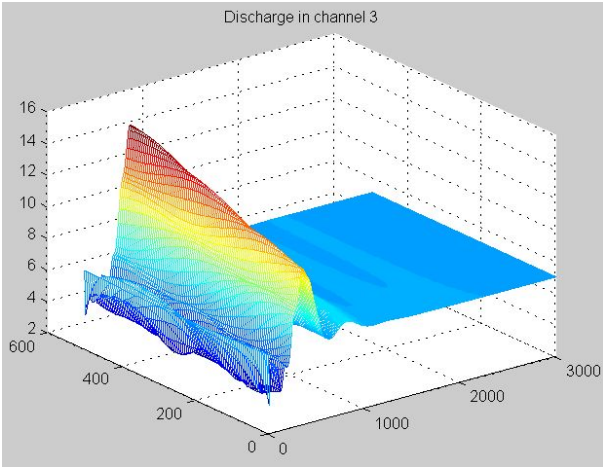


Fig. 8.  $Q_3(x, t)$

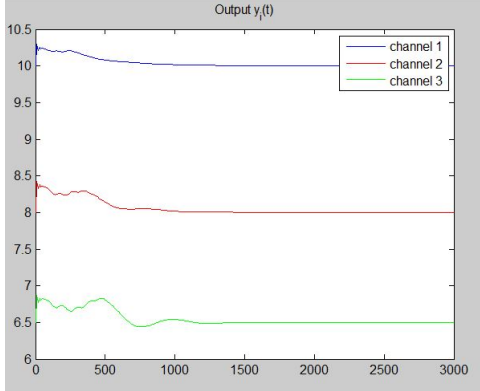


Fig. 9. Output measurements  $y_i(t)$

Employing (15), one can find that:

$$q_{i1} > 0, \quad q_{i2} - q_{i3}^2 - \frac{q_{i4}^2}{q_{i1}} = \mu e^{\mu L} q_{i1} \left( \lambda_{i2} - \mu e^{2\mu L} \frac{a_i^2 \lambda_{i2}^2 q_{i1}}{\lambda_{i1}^2} - \mu e^{2\mu L} a_i^2 R_{i1}^2 \right).$$

It is clearly that if  $\mu$  is small enough,  $q_{i2} - q_{i3}^2 - \frac{q_{i4}^2}{q_{i1}} > 0$ . It therefore yields that the matrix  $P_i$  is symmetric positive definite. Hence, there exists  $\sigma_{i1}, \sigma_{i2} > 0$  such that

$$\begin{aligned} \sigma_{i1} \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix}^T \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix} dx &\leq V_i(\phi_{i1}, \phi_{i2}, \xi_i) \\ &\leq \sigma_{i2} \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix}^T \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix} dx. \end{aligned}$$

As a result, there exists  $M_i > 0$  such that (16) holds.

*Proof of the second property 17*

The time derivative of  $V$  along the solution of the system (12) has the following form:

$$\begin{aligned} \dot{V}_i(t) = & - \int_0^L \begin{pmatrix} \phi_{i1}(x, t) e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x, t) e^{\frac{\mu x}{2}} \\ \xi_i(t) \\ \phi_{i1}(L, t) \end{pmatrix}^T Q_i \begin{pmatrix} \phi_{i1}(x, t) e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x, t) e^{\frac{\mu x}{2}} \\ \xi_i(t) \\ \phi_{i1}(L, t) \end{pmatrix} dx - F(t) \end{aligned}$$

where

$$\begin{aligned} F(t) = & \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \phi_{i2}^2(0, t) \lambda_{i2} q_{i1} \\ & + \frac{3}{4} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ & - \lambda_{i1} (R_{i1} \phi_{i2}(0, t) + \alpha_i \phi_{(i-1)1}(L, t) + \beta_i \xi_{i-1}(t))^2 \\ & - 2 \xi_i(t) q_{i3} \lambda_{i1} (\alpha_i \phi_{(i-1)1}(L, t) + \beta_i \xi_{i-1}(t)), \quad (36) \end{aligned}$$

and,

$$Q_i = \begin{pmatrix} \mu \lambda_{i1} & 0 & T_{\phi_{i1}, \xi_i} & T_{\phi_{i1}, \phi_{i1}L} \\ 0 & \mu q_{i1} \lambda_{i2} & T_{\phi_{i2}, \xi_i} & T_{\phi_{i2}, \phi_{i1}L} \\ T_{\phi_{i1}, \xi_i} & T_{\phi_{i2}, \xi_i} & T_{\xi_i} & 0 \\ T_{\phi_{i1}, \phi_{i1}L} & T_{\phi_{i2}, \phi_{i1}L} & 0 & \frac{\lambda_{i1} e^{-\mu L}}{2L} \end{pmatrix}$$

with

$$\begin{aligned} T_{\xi_i} &= \mu^2 \frac{c^2 \lambda_{i2} q_{i1} e^{\mu L}}{4L}, T_{\phi_{i1}, \phi_{i1L}} = -\mu e^{\frac{3\mu L}{2}} \frac{a_i^2 \lambda_{i2} q_{i1}}{\lambda_{i1}}, \\ T_{\phi_{i2}, \phi_{i1L}} &= -\mu e^{\frac{3\mu L}{2}} a_i^2 R_{i1} q_{i1}, \\ T_{\phi_{i1}, \xi_i} &= \mu^2 e^{\frac{3\mu L}{2}} \frac{a_i \lambda_{i2} q_{i1} (2b_i c + \lambda_{i1})}{2\lambda_{i1}}, \\ T_{\phi_{i2}, \xi_i} &= \mu^2 e^{\frac{3\mu L}{2}} a_i R_{i1} q_{i1} \frac{2b_i c + \lambda_{i2}}{2\lambda_{i2}}, \end{aligned}$$

and  $c = b_i + a_i R_{i1} e^{\mu L}$ .

At first, we consider the boundary terms  $F(t)$  in (36). Applying the Cauchy-Schwarz inequality, one can find that

$$\begin{aligned} (R_{i1} \phi_{i2}(0, t) + \alpha_i \phi_{(i-1)1}(L, t) + \beta_i \xi_{i-1}(t))^2 &\leq \\ 3 \left( R_{i1}^2 \phi_{i2}^2(0, t) + \alpha_i^2 \phi_{(i-1)1}^2(L, t) + \beta_i^2 \xi_{i-1}^2(t) \right) &\quad (37) \end{aligned}$$

$$\begin{aligned} 2\xi_i(t) q_{i3} \lambda_{i1} (\alpha_i \phi_{(i-1)1}(L, t) + \beta_i \xi_{i-1}(t)) &\leq \frac{1}{2} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} (\alpha_i^2 \phi_{(i-1)1}^2(L, t) + \beta_i^2 \xi_{i-1}^2(t)) &\quad (38) \end{aligned}$$

From (36), (37), (38) and with the choice of  $q_{i1}$  in (15) it can be deduced that

$$\begin{aligned} F(t) \geq \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \frac{1}{4} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ - \phi_{(i-1)1}^2(L, t) A_i - \xi_{i-1}^2(t) B_i, \quad (39) \end{aligned}$$

where  $A_i = \lambda_{i1} \alpha_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right)$ , and  $B_i = \lambda_{i1} \beta_i^2 \left( 3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right)$ .

In the following, we prove that by picking  $\mu$  small enough, matrix  $Q$  is symmetric positive definite. From the Sylvester criterion,  $Q$  is symmetric positive definite if and only if

$$D_1 = \det \begin{pmatrix} \mu \lambda_{i1} & 0 & T_{\phi_{i1}, \xi_i} \\ 0 & \mu q_{i1} \lambda_{i2} & T_{\phi_{i2}, \xi_i} \\ T_{\phi_{i1}, \xi_i} & T_{\phi_{i2}, \xi_i} & T_{\xi_i} \end{pmatrix} > 0,$$

$$D_2 = \det(Q_i) > 0.$$

By the direct computing, we have that

$$D_1 = \mu^4 f_1(\mu), \quad D_2 = \mu^4 f_2(\mu),$$

with  $\lim_{\mu \rightarrow 0} f_1(\mu) = \frac{1}{L} \lambda_{i1} \lambda_{i2}^2 q_{i1}^2 c^2 e^{\mu L} > 0$  and  $\lim_{\mu \rightarrow 0} f_2(\mu) = \frac{1}{2L} \lambda_{i1}^2 \lambda_{i2}^2 q_{i1}^2 c^2 > 0$ .

Taking  $\mu$  small enough, it yields that the two terms  $D_1, D_2$  are both positive. Consequently, the matrix  $Q_i$  is symmetric positive definite.

Therefore, with the adaptive choice of  $\mu$ , there exists a positive real number  $K_i > 0$  such that for all  $t$  at which the solution is well defined we have:

$$\begin{aligned} \dot{V}_i(t) \leq -K_i \|(\phi_{i1}, \phi_{i2}, \xi_i)\|_{\mathbb{X}}^2 - \frac{1}{4} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ - \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \phi_{(i-1)1}^2(L, t) A_i + \xi_{i-1}^2(t) B_i. \end{aligned}$$

With (16) the former inequality implies that we can find  $\gamma_i > 0$  such that (17) holds. This completes the proof of Lemma 1.

## References

- [1] K.J. Astrom and T. Hagglund, *Advanced PID Control*, ISA - Instrumentation, Systems, and Automation Society, 2006.
- [2] G. Bastin and J.M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*, PNLDE (88), Birkhauser, 2016.
- [3] G. Bastin, J.M. Coron and B. d'Andréa Novel, *On Lyapunov stability of linearised Saint-Venant equations for a sloping channel*, Networks and Heterogeneous Media, vol.4, pp. 177-187, 2009.
- [4] G. Bastin, J.M. Coron and S.O. Tamasoiu, *Stability of linear density-flow hyperbolic systems under PI boundary control*, Automatica 53, pp. 37-42, 2015.
- [5] J.-M. Coron, B. d'Andréa Novel and G. Bastin, *A strict lyapunov function for boundary control of hyperbolic systems of conservation laws*, IEEE Transactions on Automatic Control, vol.52, pp. 2-11, 2007.
- [6] J.M. Coron, G. Bastin and B. d'Andréa Novel, *Dissipative Boundary Conditions for One-Dimensional Nonlinear Hyperbolic Systems*, SIAM Journal on Control and Optimization 47, No.3, pp. 1460-1498, 2008.
- [7] J.M. Coron and S.O. Tamasoiu, *Feedback stabilization for a scalar conservation law with PID boundary control*, Chinese Annals of Mathematics, Series B 36, no. 5, pp. 763-776, 2015.
- [8] V. Dos Santos, G. Bastin, J.M. Coron and B. d'Andréa Novel, *Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments*, Automatica, IFAC 44(5), pp. 1310-1318, 2008.
- [9] V. Dos Santos and C. Prieur, *Boundary control of open channels with numerical and experimental validations*, IEEE Transactions on Control Systems Technology, 16(6), pp. 1252-1264, 2008.
- [10] J. Greenberg and T.T Li, *The effect of boundary damping for the quasilinear wave equation*, Journal of Differential Equations 52, pp.66-75, 1984.
- [11] M. Gugat, M. Dick and G. Leugering, *Stabilization of the Gas Flow in Star-Shaped Networks by Feedback Controls with Varying Delay*, IFIP Advances in Information and Communication Technology, vol. 391, pp. 255-265, 2011.

- [12] J. de Halleux, C. Prieur, J.M. Coron and G. Bastin, *Boundary control in networks of open channels*, Automatica 39, pp. 1365-1376, 2003.
- [13] J.K Hale and S.M Verduyn Lunel. *Introduction to functional differential equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.
- [14] G. Leugering and J. P. Georg Schmidt, *On the modelling and stabilization of flows in networks of open canals*, SIAM Journal on Control and Optimization, vol.41, No.1, pp. 164-180, 2002.
- [15] H. Logemann and S. Townley, *Low gain control of uncertain regular linear systems*, SIAM Journal on Control and Optimization 35, pp. 78-116, 1997.
- [16] V.T. Pham, D. Georges and G. Besancon, *Predictive control with guaranteed stability for hyperbolic systems of conservation laws*, 49th IEEE Conference on Decision and Control, pp. 6932-6937, Atlanta, USA, 2010.
- [17] V.T. Pham, D. Georges and G. Besancon, *Infinite-dimensional predictive control for hyperbolic systems*, SIAM Journal on Control and Optimization, 52 (6), pp. 3592-3617, 2014.
- [18] S.A. Pohjolainen, *Robust multivariable PI controllers for infinite dimensional systems*, IEEE Transactions on Automatic Control 27, pp. 17-30, 1982.
- [19] V. Perrollaz and L. Rosier, *Finite time stabilization of  $2 \times 2$  hyperbolic systems on tree-shaped networks*, SIAM Journal on Control and Optimization, vol. 52, No. 1, pp. 143-163, 2014.
- [20] M. E. Taylor, *Partial Differential Equation III Nonlinear Equation*, Applied Mathematical Sciences, volume 117, Springer, 1996.
- [21] N.T. Trinh, V. Andrieu and C.Z. Xu. *Multivariable PI controller design for  $2 \times 2$  systems governed by hyperbolic partial differential equations with Lyapunov techniques*, Proceeding of 55th IEEE Conference on Decision and Control, pp. 5654-5659, Las Vegas, USA, 2016.
- [22] N.T. Trinh, V. Andrieu and C.Z. Xu. *Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations*, Full paper in IEEE Transactions on Automatic Control, Vol. PP, Issue 99, February 2017.
- [23] C.Z. Xu and H. Jerbi, *A robust PI controller for infinite dimensional systems*, International Journal of Control 61, pp. 33-45, 1995.
- [24] C.Z. Xu and G. Sallet, *Multivariable boundary PI control and regulation of a fluid flow system*, Mathematical Control and Related Fields, pp. 1-20, 2014.