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► **To cite this version:**

Yang Yu, Hai-Long Pei, Cheng-Zhong Xu. Identification of Water Depth and Velocity Potential for Water Waves. *Systems and Control Letters*, 2019, 10.1016/j.sysconle.2018.12.010 . hal-02056751

HAL Id: hal-02056751

<https://hal.science/hal-02056751v1>

Submitted on 4 Mar 2019

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Identification of Water Depth and Velocity Potential for Water Waves

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Abstract

The purpose of this paper is to investigate the identification of the water depth and the water velocity potential in a coastal region by using the linearized water wave equation (LWWE). Existence and uniqueness of the solutions to the partial differential equation LWWE are shown by using the semigroup theory. Moreover the analytical solution is found by the separation of variables method. We assume that the surface wave elevation is measurable. We like to recover the water depth and the water velocity potential from the measurement. This identification problem is shown to be well-posed by proving the parameters' identifiability by the surface elevation. Based on the classical gradient descent method we elaborate an identification algorithm to recover simultaneously both the water depth and the velocity potential. Numerical simulations are carried out to illustrate effectiveness of the algorithm.

Keywords: Water wave equation, infinite dimensional system, water depth, velocity potential, parametric identification, gradient descent method.

1. Introduction

Detailed knowledge of the ocean bottom topography and hydrological characteristics are of great importance in various coastal engineering problems. Previous studies have shown that the bathymetry problem in coastal region can be solved by direct and indirect approaches. In [1] and [2], the water depth is measured by the infrared and the blue-green scanning LIDAR (light detection and ranging). However, sometimes the bottom topography is not visible due to turbidity or bubbles. Moreover the energy consumption of blue-green laser is important. The indirect approach proposed in [3] and in [4] was based on the dispersion relation, written as $\omega^2 = gk \tanh(kh)$, where h is the water depth, ω the angular frequency defined by $2\pi/T$, with T the wave time period, k the wave number defined by $2\pi/L$ with L the wavelength, and g the gravity acceleration constant. Once we have k and ω we compute directly h by the dispersion relation. However, this approach is useful only for the case of single wave.

Our objective consists of surveying nearshore bathymetry by using the infrared scanning LIDAR mounted on a UAV (unmanned aerial vehicle). We expect to use the system identification technology [5] [16] to recover the water depth and water velocity potential in the coastal region. Parametric identification of a dynamical system is an algorithm based on the mathematical model of the system that allows to recover the parameters by processing available measurement records. In our case we propose to identify the water depth and velocity potential

through the mathematical model of water waves and measuring the surface wave elevation.

The water wave equation consists in describing the motion of the water waves occupying a domain delimited below by a fixed bottom and by a free surface above. To write down the water wave equation, let $\Omega_t = \{(x, z) \in \mathbb{R}^+ \times (-h, \eta(x, t))\}$ denote the water domain in 2-dimensional euclidian framework as illustrated in Fig.??, where $\mathbb{R}^+ = [0, \infty)$, $\eta(x, t)$ is the surface elevation of water wave at position x and time t , and h the water depth. We assume that the water and the water waves satisfy the following assumptions: (A1) The water is incompressible; (A2) There is no surface tension and the water is inviscid; (A3) The water particles do not cross the bottom and the surface; (A4) The external pressure is constant; (A5) The seabed is flat, so that h is positive constant; (A6) The water wave is irrotational. Since the irrotational assumption has been made, consequently there exists a flow potential $\phi = \phi(x, z, t)$ such that the velocity field \mathbf{V} is written by $\mathbf{V} = (\phi_x, \phi_z)^T$, where ϕ_x, ϕ_z denote the partial derivative of ϕ with respect to x and to z , respectively. Thus, the mass conservation is expressed by the Laplace equation

$$\Delta\phi = \phi_{xx} + \phi_{zz} = 0, \quad \forall(x, z) \in \Omega_t, \quad (1)$$

with the boundary condition at the bottom

$$\phi_z = 0, \quad \text{on } z = -h. \quad (2)$$

The Neumann condition (2) means that at the bottom of seabed, the normal component of the velocity is zero. The dynamical and kinematical boundary conditions on the free surface are

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given by

$$\begin{cases} \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0 \\ \eta_t + \phi_x \eta_x - \phi_z = 0 \end{cases} \quad \forall x \in \mathbb{R}^+, z = \eta(x, t), t > 0. \quad (3)$$

In previous studies such as [7, 8], the water wave is assumed to be at rest at infinity. Alternatively, we choose the periodical condition for water waves

$$\phi(x, z, t) = \phi(x + L, z, t), \quad \forall (x, z) \in \Omega_t, t > 0. \quad (4)$$

The detailed development of water wave equations (1)-(4) can be found in [6]. Indeed, following Craig and Sulem [10, 11], the water wave dynamic can be described by the state representation equation with ξ and η as states, where $\xi(x, t) = \phi(x, \eta(x, t), t)$. This fact was noticed by Zakharov [9] in 1968. Indeed where $\xi(x, t)$ is the evaluation of the velocity potential on the free surface, i.e., $\xi(x, t) = \phi(x, \eta(x, t), t)$. Once η fixed, the Dirichlet-Neumann operator G_η maps each ξ nonlinearly onto $(\phi_z - \phi_x \eta_x)|_{z=\eta(x, t)}$, where ϕ is the solution of the following problem

$$\begin{cases} \Delta \phi = 0, \quad \forall (x, z) \in \Omega_t, \quad \phi|_{z=\eta} = \xi, \\ \phi_z|_{z=-h} = 0, \quad \phi(x, z, t) = \phi(x + L, z, t), \quad \forall x \in \mathbb{R}. \end{cases} \quad (5)$$

The major breakthrough of Lannes [12] was to prove the existence of local solutions of system (3), without restrictions on the size of the initial data and without the periodical boundary condition. Recently the bathymetry problem based on the water wave equations (3) has been studied in [13], in which $\eta(x, t)$, $\eta_t(x, t)$ and $\xi(x, t)$ are assumed to be measurable at a fixed time $t_0 > 0$. However, from the point of view of applications, the measurement of the velocity potential is difficult. And we expect to extract simultaneously the water depth and the velocity potential by measuring only the surface elevation. Notice that, the LWWE is often used in oceanography as an alternative choice of the water wave equation, where the former corresponds to the linearized version of the latter around the equilibrium state $(\xi, \eta)^T = (0, 0)^T$.

Without ambiguity we keep the same notations in the LWWE as in the nonlinear water wave equation. Let $\Omega = \{(x, z) \in \mathbb{R}^+ \times (-h, 0)\}$ denote the linearized water domain, where $z = 0$ corresponds to the still water surface. The summary of LWWE is given by

$$\begin{cases} \Delta \phi = 0, \quad \forall (x, z) \in \Omega, \\ \phi_z|_{z=-h} = 0, \quad \phi_t|_{z=0} = -g\eta, \quad \eta_t = \phi_z|_{z=0}, \\ \phi(x, z, t) = \phi(x + L, z, t), \quad \forall (x, z) \in \Omega, t > 0. \end{cases} \quad (6)$$

In the literature, several mathematical models for the motion of water waves have been established, see [6, 14]. In our paper, we consider the LWWE as the mathematical model of water waves in the coastal region. In some sense dynamics of water waves that we study here are similar to those of mechanical systems governed by PDE, see [21, 17]. The methodology for identification and observation of finite dimensional nonlinear systems has been developed in [15] and successfully applied to

deal with some interesting practical problems. However, there is no general methodology for the identification and observation of infinite-dimensional systems. Nevertheless the works of [19], [20] and [18] have inspired us a lot for the present study.

Different from previous studies in the literatures, we are interested in elaborating an algorithm on the bathymetry which enables us to identify simultaneously both the velocity potential and the water depth by measuring the water surface elevation only. The contribution of our paper is threefold: (i) we prove that the direct problem of periodical (w.r.t. space) water wave system described by LWWE (6) is well-posed in some Hilbert state space and that the corresponding identification problem has a unique solution; (ii) we design an algorithm of identification for both the water depth and the water velocity potential; (iii) we validate our proposed algorithm by numerical simulations.

The rest of the paper is organized as follows. In Section 2, we give an analytical solution of the LWWE (6), which is obtained by the separation of variables method. There, we prove the well-posedness of the LWWE in some Hilbert state space. In Section 3, we show that the water depth and the water velocity potential are simultaneously identifiable by measuring the surface elevation. Section 4 is devoted to presenting our identification algorithm. We use the classical gradient descent method to seek the minimum of a cost function. Numerical simulations are provided in Section 5 to show effectiveness of the algorithm. Our conclusion and future perspective are given in Section 6.

2. Well-posedness of LWWE

Let us consider the PDE (6). We use the method of analysis of Craig and Sulem in [11] to write down the dynamical system of the linearized water waves. That is to say that we will define a similar linear Dirichlet-Neumann operator G . Let $\xi(x, t)$ denote the water velocity potential on the surface, i.e., $\xi(x, t) = \phi(x, 0, t)$. Let G be the linear operator which maps each ξ onto $\phi_z|_{z=0}$ by solving the following elliptic boundary value problem

$$\begin{cases} \Delta \phi = 0, \quad \forall (x, z) \in \Omega, \\ \phi_z|_{z=-h} = 0, \\ \phi(x, 0, t) = \xi(x, t), \quad \forall x \in \mathbb{R}^+, t > 0, \\ \phi(x, z, t) = \phi(x + L, z, t), \quad \forall z \in (-h, 0), t > 0. \end{cases} \quad (7)$$

Then we write LWWE (6) as a dynamical system in the following form:

$$\begin{cases} \xi_t = -g\eta, \\ \eta_t = G(\xi), \\ \xi(x, 0) = \xi^0(x), \quad \eta(x, 0) = \eta^0(x), \\ y(x, t) = \eta(x, t), \quad (x, t) \in (0, L) \times (0, \infty), \end{cases} \quad (8)$$

where $y(x, t)$ is the output measurement. As the water wave is periodic in x with period L , we can restrict the observation field

to $[0, L]$ that corresponds to the information in one period. Obviously the operator G plays a central role in the well-posedness problem of (8). We will show that, given ξ , the Laplace equation (7) has a unique solution, and so that the operator G is well defined. Then the well-posedness of the dynamical system (8) will be proven by the semigroup theory.

2.1. Laplace equation

We define the set $C_p^\infty(0, L)$ as follows

$$C_p^\infty(0, L) = \left\{ f \in C^\infty(\mathbb{R}^+) \mid f(x+L) = f(x), \forall x \in \mathbb{R}^+ \right\}.$$

Let $L^2(0, L)$ denote the Hilbert space of square summable functions defined in $(0, L)$ that is equipped with the inner product $\langle f, g \rangle = \int_0^L f(x)g(x)dx$ and the induced norm $\|f\|^2 = \langle f, f \rangle$. Let $L_p^2(0, L)$ denote the set of square summable and periodic functions with period L that is obtained by the completion of $C_p^\infty(0, L)$ in the $L^2(0, L)$ norm. It is easy to see that $L_p^2(0, L) = L^2(0, L)$. As usual $H^n(0, L)$ denotes the Sobolev space normed by $\|f\|_{H^n}^2 = \sum_{m=0}^n \|f^{(m)}\|^2$ where $f^{(0)} = f$. Thus, for each $n \in \mathbb{N}^*$, $H_p^n(0, L)$ is the Hilbert space by the completion of $C_p^\infty(0, L)$ in the $H^n(0, L)$ norm.

We solve the Laplace equation (7) by the separation of variables method. The main result is given by the following theorem. And the corresponding proof is presented in Appendix A.

Theorem 2.1 *Let $\xi \in C_p^\infty(0, L)$ such that*

$$\xi(x) = \frac{a_0}{\sqrt{L}} + \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(a_m \cos(k_m x) + b_m \sin(k_m x) \right).$$

Then the Laplace equation (7) has a unique solution $\phi \in C^\infty(\Omega)$ given by

$$\phi(x, z) = \frac{a_0}{\sqrt{L}} + \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(a_m \cos(k_m x) + b_m \sin(k_m x) \right) \frac{\cosh(k_m(h+z))}{\cosh(k_m h)}, \quad (9)$$

where $k_m = \frac{2m\pi}{L} \forall m \in \mathbb{N}^*$ ($\mathbb{N}^* = \mathbb{N} \setminus \{0\}$) and

$$\begin{cases} a_0 = \langle \xi, \sqrt{L^{-1}} \rangle_{L_p^2}, \\ a_m = \langle \xi, \sqrt{2L^{-1}} \cos(k_m x) \rangle_{L_p^2}, \quad b_m = \langle \xi, \sqrt{2L^{-1}} \sin(k_m x) \rangle_{L_p^2}. \end{cases}$$

Indeed, the unique solution $\phi(x, z)$ belongs to $C^\infty([0, L] \times [-h, 0])$, and it is periodic with respect to x . From Theorem 2.1, if $\xi \in C_p^\infty(0, L)$, then $G(\xi) = \phi_{z|z=0}$ is well defined, and we have

$$G(\xi) = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(a_m \cos(k_m x) + b_m \sin(k_m x) \right) \frac{\omega_m^2}{g} \quad (10)$$

where $\omega_m^2 = gk_m \tanh(k_m h)$.

From (10), we get

$$\|G(\xi)\|_{L_p^2}^2 = \sum_{m=1}^{\infty} \left(a_m^2 + b_m^2 \right) \frac{\omega_m^4}{g^2}. \quad (11)$$

Recall that

$$\|\xi\|_{H_p^1}^2 = a_0^2 + \sum_{m=1}^{\infty} \left(a_m^2 + b_m^2 \right) \left(1 + k_m^2 \right). \quad (12)$$

By Theorem 2.1 and (11)-(12), the operator G is continuous from $H_p^1(0, L)$ to $L_p^2(0, L)$. Hence it admits a unique extension onto the whole space $H_p^1(0, L)$, as $C_p^\infty(0, L)$ is dense in $H_p^1(0, L)$.

We introduce the Hilbert space $H_{p,0}^n(0, L) \subset H_p^n(0, L)$ defined by $H_{p,0}^n(0, L) = \left\{ f \in H_p^n(0, L); \int_0^L f dx = 0 \right\}, \forall n \in \mathbb{N}$. We claim that G is continuously invertible from $H_{p,0}^1(0, L)$ onto $L_{p,0}^2(0, L)$. Indeed, let f_1, f_2 be two functions in $H_{p,0}^1(0, L)$. For $j = 1, 2$, f_j is written as

$$f_j(x) = \sum_{m=1}^{\infty} a_{j,m} \cos(k_m x) + b_{j,m} \sin(k_m x).$$

Then the image of f_j under the mapping G is given by

$$G(f_j) = \sum_{m=1}^{\infty} \left(a_{j,m} \cos(k_m x) + b_{j,m} \sin(k_m x) \right) \frac{\omega_m^2}{g}.$$

Obviously, $G(f_1) = G(f_2)$ implies that $a_{1,m} = a_{2,m}$ and $b_{1,m} = b_{2,m} \forall m \in \mathbb{N}^*$. It is implied that $f_1 = f_2$. Thus the operator G is one-to-one. The onto part can be proved similarly. Hence G is continuous and bijective from $H_{p,0}^1(0, L)$ onto $L_{p,0}^2(0, L)$. It is implied that G is invertible with G^{-1} continuous from $L_{p,0}^2(0, L)$ to $H_{p,0}^1(0, L)$ (see [22, p.19]). So the claim is proved.

2.2. Well-posedness of the dynamical system

Now we prove the well-posedness of the dynamical system (8). First we define the state space for the system. For each $f \in C_p^\infty(0, L)$ such that

$$f(x) = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(a_m \cos(k_m x) + b_m \sin(k_m x) \right),$$

we define the norm $\|f\|_{H^{\frac{1}{2}}}^2$ by

$$\|f\|_{H^{\frac{1}{2}}}^2 = \sum_{m=1}^{\infty} \left(a_m^2 + b_m^2 \right) \frac{\omega_m^2}{g^2}. \quad (13)$$

Let $H_{p,0}^{\frac{1}{2}}(0, L)$ be the completion of $C_p^\infty(0, L)$ in the $H^{\frac{1}{2}}$ norm.

A more general definition of $H_{p,0}^{\frac{1}{2}}$ can be found in [24, section 3.4]. Then the state space of the system (8) is the Hilbert space $X = H_{p,0}^{\frac{1}{2}}(0, L) \times L_{p,0}^2(0, L)$ equipped with the following norm

$$\|(u, v)^T\|_X^2 = \|u\|_{H^{\frac{1}{2}}}^2 + \|v\|_{L^2}^2. \quad (14)$$

We define the system operator $A : D(A) \subset X \rightarrow X$ by

$$D(A) = \left\{ (f_1, f_2)^T \in H_{p,0}^1(0, L) \times H_{p,0}^{\frac{1}{2}}(0, L) \right\}$$

and for all $(f_1, f_2)^T \in D(A)$,

$$A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -gf_2 \\ G(f_1) \end{pmatrix}.$$

The main result about the well-posedness problem to the dynamical system (8) is given by the following theorem.

Theorem 2.2 *The operator A is the generator of a C_0 semigroup on X noted by $(S_A(t))_{t \geq 0}$. For each $(\xi^0, \eta^0)^T \in D(A)$, there exists a unique solution $(\xi(t), \eta(t))$ to the system (8) such that $(\xi, \eta)^T \in C^1([0, \infty), X) \cap C([0, \infty), D(A))$. Moreover, the solution is given by $(\xi(\cdot, t), \eta(\cdot, t))^T = S_A(t)(\xi^0, \eta^0)^T$.*

Proof It is sufficient to apply the Hille-Yosida theorem (Theorem 7.4, p.105, [22]). We first show that A is dissipative. Indeed, let $f = (f_1, f_2)^T \in D(A)$. The inner product between Af and f gives us the following

$$\langle Af, f \rangle_X = \left\langle \begin{pmatrix} -gf_2 \\ G(f_1) \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle_X = -\langle gf_2, f_1 \rangle_{H^{\frac{1}{2}}} + \langle G(f_1), f_2 \rangle_{L^2}.$$

Since $(f_1, f_2) \in H_{p,0}^1(0, L) \times H_{p,0}^{\frac{1}{2}}(0, L)$, we can express f_j , $j = 1, 2$, as Fourier series

$$f_j = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(a_{j,m} \cos(k_m x) + b_{j,m} \sin(k_m x) \right),$$

where the sequences $(k_m a_{1,m})$, $(k_m b_{1,m})$, $(\omega_m a_{2,m})$, and $(\omega_m b_{2,m})$ are in ℓ^2 . It is easy to check that

$$\langle gf_2, f_1 \rangle_{H^{\frac{1}{2}}} = \langle G(f_1), f_2 \rangle_{L^2} = \sum_{m=1}^{\infty} \frac{\omega_m^2}{g} \left(a_{1,m} a_{2,m} + b_{1,m} b_{2,m} \right).$$

Hence $\langle Af, f \rangle_X = 0$, which implies that A is dissipative.

Next we show that A is maximal. To that end, let $u = (u_1, u_2)^T \in X$ written as

$$u_j = \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(\alpha_{j,m} \cos(k_m x) + \beta_{j,m} \sin(k_m x) \right), \quad j = 1, 2,$$

where $(\omega_m \alpha_{1,m})$, $(\omega_m \beta_{1,m})$, $(\alpha_{2,m})$, $(\beta_{2,m}) \in \ell^2$. We want to solve the resolvent equation for $f \in D(A)$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 + gf_2 \\ f_2 - G(f_1) \end{pmatrix} = u. \quad (15)$$

Simple computations allow us to find that for each $m \in \mathbb{N}^*$, the Fourier coefficients of f_1 and f_2 are given by

$$\begin{aligned} a_{1,m} &= \frac{\alpha_{1,m} - g\alpha_{2,m}}{1 + \omega_m^2}, & b_{1,m} &= \frac{\beta_{1,m} - g\beta_{2,m}}{1 + \omega_m^2}, \\ a_{2,m} &= \frac{\alpha_{2,m} + \alpha_{1,m}\omega_m^2 g^{-1}}{1 + \omega_m^2}, & b_{2,m} &= \frac{\beta_{2,m} + \beta_{1,m}\omega_m^2 g^{-1}}{1 + \omega_m^2}. \end{aligned}$$

From these coefficients and by $\omega_m^2 = gk_m \tanh(k_m h)$, we conclude that $(f_1, f_2)^T \in D(A)$. Therefore A is maximal dissipative. By the Hille-Yosida theorem, A is the generator of a C_0 semigroup and there exists a unique solution $(\xi, \eta)^T \in C^1([0, \infty); X) \cap C([0, \infty); D(A))$. \blacksquare

Next, we develop the exact solution to the PDE (8). First, the orthonormal bases for $H_{p,0}^{\frac{1}{2}}(0, L)$ and for $L_{p,0}^2(0, L)$ are given by, respectively,

$$\sqrt{\frac{2}{L}} \frac{g}{\omega_m} \left(\cos(k_m x), \sin(k_m x) \right)_{m \in \mathbb{N}^*}, \quad \sqrt{\frac{2}{L}} \left(\cos(k_m x), \sin(k_m x) \right)_{m \in \mathbb{N}^*}.$$

Then we express the solutions $\xi(x, t)$ and $\eta(x, t)$ as

$$\begin{aligned} \xi(x, t) &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \frac{g}{\omega_m} \left(A_m(t) \cos(k_m x) + B_m(t) \sin(k_m x) \right), \\ \eta(x, t) &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \left(C_m(t) \cos(k_m x) + D_m(t) \sin(k_m x) \right). \end{aligned} \quad (16)$$

Differentiate $A_m(t)$, $B_m(t)$, $C_m(t)$ and $D_m(t)$ with respect to t to obtain the following ODEs

$$\begin{aligned} \dot{A}_m(t) &= -\omega_m C_m(t); & \dot{B}_m(t) &= -\omega_m D_m(t) \\ \dot{C}_m(t) &= \omega_m A_m(t); & \dot{D}_m(t) &= \omega_m B_m(t). \end{aligned} \quad (17)$$

The initial values for (17) are found to be

$$\begin{aligned} A_m(0) &= \left\langle \xi^0, \sqrt{2L^{-1}} g \omega_m^{-1} \cos(k_m x) \right\rangle_{H_{p,0}^{\frac{1}{2}}}, \\ B_m(0) &= \left\langle \xi^0, \sqrt{2L^{-1}} g \omega_m^{-1} \sin(k_m x) \right\rangle_{H_{p,0}^{\frac{1}{2}}}, \\ C_m(0) &= \left\langle \eta^0, \sqrt{2L^{-1}} \cos(k_m x) \right\rangle_{L_{p,0}^2}, \\ D_m(0) &= \left\langle \eta^0, \sqrt{2L^{-1}} \sin(k_m x) \right\rangle_{L_{p,0}^2}. \end{aligned} \quad (18)$$

The ODE (17) together with the initial values gives

$$\begin{cases} A_m(t) = A_m(0) \cos(\omega_m t) - C_m(0) \sin(\omega_m t), \\ B_m(t) = B_m(0) \cos(\omega_m t) - D_m(0) \sin(\omega_m t), \\ C_m(t) = A_m(0) \sin(\omega_m t) + C_m(0) \cos(\omega_m t), \\ D_m(t) = B_m(0) \sin(\omega_m t) + D_m(0) \cos(\omega_m t). \end{cases} \quad (19)$$

3. Identifiability

In this section, we show that the water depth and the water velocity potential can be simultaneously extracted from the surface elevation measurement. That is to show that the mapping $(h, \xi^0) \mapsto \eta(x, t)$ is one-to-one. We suppose that the wavelength L is known. Our result concerning the identifiability is described as follows.

Theorem 3.1 *Let $h_j > 0$ be water depths and let $(\xi_j(\cdot, t), \eta_j(\cdot, t)) \in X$ be the solution of the dynamical system (8) from the initial value $(\xi_j^0, \eta_j^0) \in X$, $j = 1, 2$. Let $T > 0$ and let $(\xi_j^0, \eta_j^0) \neq 0 \forall j = 1, 2$. Then $\eta_1(\cdot, t) = \eta_2(\cdot, t)$ in $L_{p,0}^2(0, L) \forall t \in [0, T]$ implies that $h_1 = h_2$ and $\xi_1^0 = \xi_2^0$ in $H_{p,0}^{\frac{1}{2}}$.*

Remark 1 For non-zero initial conditions, the water depth h and the velocity potential $\xi(x, t)$ are simultaneously identifiable from measuring $\eta(x, t)$ on any positive interval $[0, T]$. The case $(\xi_j^0, \eta_j^0) = (0, 0)$, which corresponds to the still water, is excluded. Clearly, in this case it is not possible to identify the water depth and the velocity potential.

Proof Assume that for $j = 1, 2$, the initial value $(\xi_j^0, \eta_j^0) \in X$ of system (8) are written as

$$\begin{aligned}\xi_j^0(x) &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} \frac{g}{\omega_{j,m}} (a_{j,m} \cos(k_m x) + b_{j,m} \sin(k_m x)) \\ \eta_j^0(x) &= \sum_{m=1}^{\infty} \sqrt{\frac{2}{L}} (c_{j,m} \cos(k_m x) + d_{j,m} \sin(k_m x)),\end{aligned}\quad (20)$$

where $\omega_{j,m} = \sqrt{gk_m \tanh(k_m h_j)}$. By using the exact solution (16)-(19), the velocity potential and the surface elevation are given by

$$\begin{aligned}\xi_j(x, t) &= \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} \frac{g}{\omega_{j,m}} \left[(a_{j,m} \cos(\omega_{j,m} t) - c_{j,m} \sin(\omega_{j,m} t)) \right. \\ &\quad \left. \cos(k_m x) + (b_{j,m} \cos(\omega_{j,m} t) - d_{j,m} \sin(\omega_{j,m} t)) \sin(k_m x) \right], \quad (21) \\ \eta_j(x, t) &= \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} \left[(a_{j,m} \sin(\omega_{j,m} t) + c_{j,m} \cos(\omega_{j,m} t)) \cos(k_m x) \right. \\ &\quad \left. + (b_{j,m} \sin(\omega_{j,m} t) + d_{j,m} \cos(\omega_{j,m} t)) \sin(k_m x) \right].\end{aligned}$$

If $\eta_1(x, t) = \eta_2(x, t) \forall t \in [0, T]$, we have that for all $m \in \mathbb{N}^*$

$$\begin{pmatrix} a_{1,m} & c_{1,m} \\ b_{1,m} & d_{1,m} \end{pmatrix} \begin{pmatrix} \sin(\omega_{1,m} t) \\ \cos(\omega_{1,m} t) \end{pmatrix} = \begin{pmatrix} a_{2,m} & c_{2,m} \\ b_{2,m} & d_{2,m} \end{pmatrix} \begin{pmatrix} \sin(\omega_{2,m} t) \\ \cos(\omega_{2,m} t) \end{pmatrix}. \quad (22)$$

In particular, at $t = 0$, we have

$$c_{1,m} = c_{2,m}, \quad d_{1,m} = d_{2,m} \quad \forall m \in \mathbb{N}^*. \quad (23)$$

Computing successively the first order, second order and third order derivative of (22) and evaluating the results at $t = 0$, we obtain

$$(a_{1,m}, b_{1,m})^T \omega_{1,m} = (a_{2,m}, b_{2,m})^T \omega_{2,m} \quad \forall m \in \mathbb{N}^*, \quad (24)$$

$$(c_{1,m}, d_{1,m})^T \omega_{1,m}^2 = (c_{2,m}, d_{2,m})^T \omega_{2,m}^2 \quad \forall m \in \mathbb{N}^*, \quad (25)$$

$$(a_{1,m}, b_{1,m})^T \omega_{1,m}^3 = (a_{2,m}, b_{2,m})^T \omega_{2,m}^3 \quad \forall m \in \mathbb{N}^*. \quad (26)$$

If $\eta_j^0 \neq 0$, by (23), there exists some $l \in \mathbb{N}^*$ such that $(c_{j,l}, d_{j,l}) \neq 0$, $j = 1, 2$. From (25), we have $\omega_{1,l} = \omega_{2,l}$. Hence $h_1 = h_2$ and $\omega_{1,m} = \omega_{2,m} \forall m \in \mathbb{N}^*$. From (24), we deduce that $a_{1,m} = a_{2,m}$ and $b_{1,m} = b_{2,m} \forall m \in \mathbb{N}^*$. It is implied that $\xi_1(\cdot, t) = \xi_2(\cdot, t) \forall t \in [0, T]$.

If $\xi_j^0 \neq 0$, there exists some $l \in \mathbb{N}^*$ such that $(a_{1,l}, b_{1,l}) \neq 0$. Multiplying (24) by $\omega_{2,m}^2$ and subtracting the product from (26) give us

$$\omega_{1,m}(\omega_{1,m}^2 - \omega_{2,m}^2) \begin{pmatrix} a_{1,m} \\ b_{1,m} \end{pmatrix} = 0 \quad \forall m \in \mathbb{N}^*. \quad (27)$$

By our assumption, we deduce that $\omega_{1,l} = \omega_{2,l}$. Hence $h_1 = h_2$ and $\omega_{1,m} = \omega_{2,m} \forall m \in \mathbb{N}^*$. From (24), we have $a_{1,m} = a_{2,m}$ and $b_{1,m} = b_{2,m} \forall m \in \mathbb{N}^*$. It is implied that $\xi_1^0 = \xi_2^0$ or $\xi_1(\cdot, t) = \xi_2(\cdot, t), \forall t \in [0, T]$. ■

4. The algorithm of identification

From the section 3, we see that the identification of $\xi(x, t)$ is equivalent to that of $\xi^0(x)$. Hence, we design an algorithm of identification for $\xi^0(x)$ and for h . Let J be the cost functional defined by

$$\begin{aligned}J: H_{p,0}^1(0, L) \times \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ (\xi^0, h) &\mapsto \frac{1}{2} \int_0^T \int_0^L (\eta(x, t) - \widehat{\eta}(x, t))^2 dx dt,\end{aligned}$$

where $\eta(x, t)$ is the solution of the system (8) from the initial condition (ξ^0, η^0) with the water depth h and $\widehat{\eta}(x, t)$ is the output measurement.

We use the gradient descent method, see [23] and Annexe C of [26], to seek the minimum of J . The key step of this algorithm is to find the descent direction of J at the point (ξ^0, h) . This direction is opposite to the gradient of J , denoted by $dJ(\xi^0, h)$. The expression of $dJ(\xi^0, h)$ is given by the following formulas (30)-(32).

We want to compute the differential $dJ(\xi^0, h) \begin{pmatrix} \Delta \xi^0 \\ \Delta h \end{pmatrix}$. For the purpose, let us put

$$J(\xi^0 + \epsilon \Delta \xi^0, h + \epsilon \Delta h) = \frac{1}{2} \int_0^T \int_0^L (\widehat{\eta}(x, t) - \widetilde{\eta}(x, t))^2 dx dt,$$

where the surface elevation $\widetilde{\eta}(x, t)$ is the solution of the system (8) from the initial value $(\xi^0 + \epsilon \Delta \xi^0, \eta^0)$ with the water depth $(h + \epsilon \Delta h)$. By using (16)-(19), $\widetilde{\eta}(x, t)$ is written by

$$\begin{aligned}\widetilde{\eta}(x, t) &= \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} \left[(\widetilde{A}_m(0) \sin(\widetilde{\omega}_m t) + C_m(0) \cos(\widetilde{\omega}_m t)) \cos(k_m x) \right. \\ &\quad \left. + (\widetilde{B}_m(0) \sin(\widetilde{\omega}_m t) + D_m(0) \cos(\widetilde{\omega}_m t)) \sin(k_m x) \right],\end{aligned}\quad (28)$$

where $(C_m(0), D_m(0))$ is the same as that of (18), $\widetilde{\omega}_m = \sqrt{gk_m \tanh(k_m(h + \epsilon \Delta h))}$ and

$$\begin{cases} \widetilde{A}_m(0) = \left\langle \xi^0 + \epsilon \Delta \xi^0, \sqrt{2L^{-1}} g \widetilde{\omega}_m^{-1} \cos(k_m x) \right\rangle_{H_{p,0}^{\frac{1}{2}}}, \\ \widetilde{B}_m(0) = \left\langle \xi^0 + \epsilon \Delta \xi^0, \sqrt{2L^{-1}} g \widetilde{\omega}_m^{-1} \sin(k_m x) \right\rangle_{H_{p,0}^{\frac{1}{2}}}. \end{cases}\quad (29)$$

Direct computation gives us the following formula

$$dJ(\xi^0, h) \begin{pmatrix} \Delta \xi^0 \\ \Delta h \end{pmatrix} = \int_0^T \int_0^L \left[\eta(x, t) - \widehat{\eta}(x, t) \right] \frac{d\widetilde{\eta}(x, t)}{d\epsilon} \Big|_{\epsilon=0} dx dt.$$

where

$$\begin{aligned} \left. \frac{d\bar{\eta}(x, t)}{d\epsilon} \right|_{\epsilon=0} &= \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} \left\{ \left[\left. \frac{\partial \bar{A}_m(0)}{\partial \epsilon} \right|_{\epsilon=0} \sin(\omega_m t) + \right. \right. \\ &\left. \left. \left(t A_m(0) \cos(\omega_m t) - t C_m(0) \sin(\omega_m t) \right) \frac{\partial \bar{\omega}_m}{\partial \epsilon} \right]_{\epsilon=0} \cos(k_m x) \right. \\ &+ \left[\left. \frac{\partial \bar{B}_m(0)}{\partial \epsilon} \right|_{\epsilon=0} \sin(\omega_m t) + \left(t B_m(0) \cos(\omega_m t) \right. \right. \\ &\left. \left. - t D_m(0) \sin(\omega_m t) \right) \frac{\partial \bar{\omega}_m}{\partial \epsilon} \right]_{\epsilon=0} \left. \sin(k_m x) \right\}, \end{aligned}$$

$$\left. \frac{d\bar{\omega}_m}{d\epsilon} \right|_{\epsilon=0} = \frac{g k_m^2 \operatorname{sech}^2(k_m h) \Delta h}{2\omega_m}.$$

Hence, the differential of J at (ξ^0, h) is given by

$$dJ(\xi^0, h) (\Delta \xi^0, \Delta h)^T = \langle \Delta \xi^0, F_1 \rangle_{H_{p,0}^{\frac{1}{2}}} + \Delta h F_2 \quad (30)$$

where

$$F_1(x) = \sum_{m=1}^{\infty} \frac{2g}{L\omega_m} \int_0^T \int_0^L \left(\eta(s, t) - \bar{\eta}(s, t) \right) \sin(\omega_m t) \cos(k_m(s-x)) ds dt, \quad (31)$$

$$\begin{aligned} F_2 = & \sum_{m=1}^{\infty} \frac{g k_m^2 \operatorname{sech}^2(k_m h)}{\omega_m \sqrt{2L}} \int_0^T \int_0^L \left(\eta(x, t) - \bar{\eta}(x, t) \right) \times \\ & \left\{ \left[- \left(A_m(0) \omega_m^{-1} + C_m(0) t \right) \sin(\omega_m t) + A_m(0) t \cos(\omega_m t) \right] \cos(k_m x) \right. \\ & + \left[- \left(B_m(0) \omega_m^{-1} + D_m(0) t \right) \sin(\omega_m t) + B_m(0) t \cos(\omega_m t) \right] \times \\ & \left. \sin(k_m x) \right\} dx dt. \end{aligned} \quad (32)$$

Based on the gradient descent method, our designed algorithm of identification is provided by the following algorithm.

Algorithm of identification

- (1) Choose an initial data $(\xi^{0,1}, h^1)$ and a small parameter ϵ . Initialize the counter $j = 1$.
- (2) Realize a convergence test: if $|J(\xi^{0,j}, h^j)| < \epsilon$, stop the algorithm.
- (3) Determine a descent direction. In the paper, this direction is opposite to the differential of J at the point $(\xi^{0,j}, h^j)$.
- (4) Determine two steps α_1 and α_2 sufficiently small such that the functional J decreases sufficiently.
- (5) Determine a new iteration

$$\xi^{0,j+1}(x) = \xi^{0,j}(x) - \alpha_1 F_1^j(x), \quad h^{j+1} = h^j - \alpha_2 F_2^j,$$
 where $F_1^j(x)$ and F_2^j are defined by (31)-(32).
- (6) Put $j = j + 1$ and return to step 2.

5. Numerical Simulations

In this section, we present an example to show the effectiveness of the proposed algorithm, where the implementation of the algorithm is carried out by using the software Matlab.

Due to lack of actual data, we use the exact solution $\eta(x, t)$ in (16) to generate the measurement records by adding some random noise. First we give the relevant coefficients as

$$g = 9.8 \text{ m/s}^2; \quad L = 3 \text{ m}; \quad T = 2.047 \text{ s},$$

where g is the gravitational acceleration, L the wavelength and T the duration of measurement. The initial values are chosen as

$$\xi^0(x) = \sum_{m=1}^3 0.1 \cos(k_m x); \quad \eta^0(x) = \sum_{m=1}^3 0.1 \cos(k_m x), \quad (33)$$

where $k_m = 2m\pi/L$, for $m = 1, 2, 3$. Then the surface elevation $\eta(x, t)$ is expressed by

$$\eta(x, t) = \sum_{m=1}^3 \left[0.1 \frac{\omega_m}{g} \sin(\omega_m t) + 0.1 \cos(\omega_m t) \right] \cos(k_m x), \quad (34)$$

where $\omega_m = \sqrt{g k_m \tanh(k_m h)}$ with the water depth $h = 2$. In the computational process, the surface elevation $\eta_i^n = \eta(x_i, t_n)$ is obtained at the points of mesh grid: $\Delta x = 0.1 \text{ m}$, $\Delta t = 0.01 \text{ s}$. Then the measurement records are given by

$$\begin{aligned} \bar{\eta}_i^n &= \eta_i^n + \delta \max(\eta) \cdot \text{rand}(\text{size}(\eta_i^n)), \\ &\text{for } i = 0, 1, \dots, L/\Delta x; \quad n = 0, 1, \dots, T/\Delta t, \end{aligned}$$

where $\text{rand}(\text{size}(\eta_i^n))$ is a random matrix in $[0, 1]$ to indicate the measurement noise, δ denotes the percentage error level.

The initial iteration for the algorithm of identification is chosen as

$$\xi^{0,1}(x) = 0.5 \cos\left(\frac{2\pi x}{L}\right), \quad h^1 = 1. \quad (35)$$

The step sizes for the algorithm are $\alpha_1 = 0.01$ and $\alpha_2 = 0.3853$. The convergence criterion is $\epsilon = 10^{-6}$.

The true value of $\xi^0(x)$ in (33) is displayed in Fig.1 in solid line and $\xi^{0,1}(x)$ is shown in dash line. It can be seen, from the figure, that $\xi^0(x)$ and $\xi^{0,1}(x)$ are quite different. Before we apply the algorithm, the value of cost functional is $J(\xi^{0,1}, h^1) = 0.5839$. The difference between the water depth h and h^1 is $J_2 = |h - h^1| = 1$. And the difference between $\xi^0(x)$ and $\xi^{0,1}(x)$ is computed by the norm $J_3 = \frac{1}{2} \int_0^L (\xi^0(x) - \xi^{0,1}(x))^2 dx = 0.27$.

Fig.2 presents the identification results without measurement noise, i.e., $\delta = 0$. The convergence is achieved by using 280 iterations of descent. The comparison between the true value $\xi^0(x)$ and identified $\xi^{0,280}(x)$ is shown in Fig.2(a), where the exact values are given by the solid line (red) and the estimated results are shown in dash line (blue). Fig.2(a) shows that the result of identification is quite satisfactory. The evolution of the identified water depth is shown in Fig.2(c). The identified water depth (in dash line) quickly converges to the

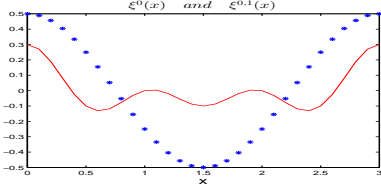


Figure 1: Initial values of $\xi^0(x)$ in solid line and $\xi^{0,1}(x)$ in dash line

true value ($h = 2$), though their initial values are quite different. The evolution of J_2 (with $J_2 = |h - h^j|$) and evolution of J_3 (with $J_3 = \frac{1}{2} \int_0^L (\xi^0(x) - \xi^{0,j}(x))^2 dx$) are presented in Fig.2(d) and Fig.2(b), respectively. The minimum obtained is $J_{min} = 9.99 \cdot 10^{-7}$, $J_{2min} = 2.3 \cdot 10^{-3}$ and $J_{3min} = 8.62 \cdot 10^{-7}$.

For the case of $\delta \neq 0$, we give the simulation results in Table 1. According to the values of δ , we list the iterations needed to realise the convergence criterion $\epsilon < 10^{-6}$ in the second line, and we list J_{min} , J_{2min} and J_{3min} in the third, the fourth, and the fifth line of Table 1. From this table, we can see that the algorithm is robust if $\delta \leq 18\%$. If $\delta > 18\%$, then the term of measurement noise is important. Consequently, the algorithm cannot converge to the desired values.

6. Conclusions

Three tasks have been carried out in this paper. First, we have proven the well-posedness of LWWE and found out the exact solution by the separation of variables method. Second, we have shown that the water depth and the water velocity potential are identifiable from the water surface elevation records. Third, the numerical simulations based on the gradient descent method have been done to show the effectiveness of the proposed algorithm.

Future work is focused on improving the convergence rate of the identification algorithm by combining the classic gradient method with the Newton method [25][5]. As we have pointed out in the Introduction, our initial objective is to realize the bathymetry in the coastal region by using the infrared scanning LIDAR. The water depth identification algorithm that we have developed here will be tested with the field real measurement data.

Appendix A. Proof of Theorem 2.1

Consider the following Laplace equation

$$\begin{cases} \Delta \phi = 0, & \forall (x, z) \in \Omega, \quad \phi_z|_{z=-h} = 0, \quad \forall x \in \mathbb{R}^+, \\ \phi(x, 0) = \xi(x), & \phi(x, z) = \phi(x + L, z), \quad \forall x \in \mathbb{R}^+. \end{cases} \quad (\text{A.1})$$

We compute the nontrivial solution of (A.1) by separation of variables method and then we prove this solution is unique by variational method.

Assume that the **velocity potential** of (A.1) is written as $\phi(x, z) = \phi_1(x)\phi_2(z) \neq 0$, where ϕ_1 and ϕ_2 only depend on x and

z , respectively. Substituting this expression into the Laplace equation, we have

$$\phi_1^{-1}(x)\phi_1''(x) + \phi_2^{-1}(z)\phi_2''(z) = 0. \quad (\text{A.2})$$

Clearly, the first term of (A.2) depends on x alone, while the second term depends only on z . The only way that the equation holds is that each term is equal to some constant $-k^2 \in \mathbb{C}$, that is

$$\phi_1^{-1}(x)\phi_1''(x) = -\phi_2^{-1}(z)\phi_2''(z) = -k^2. \quad (\text{A.3})$$

The equations (A.3) are now ordinary differential equations and can be solved separately.

If $k \in \mathbb{C} \setminus \{0\}$, the general solution of ODE (A.3) is given by

$$\phi(x, z) = (C_1 \cos(kx) + C_2 \sin(kx))(C_3 e^{kz} + C_4 e^{-kz}), \quad (\text{A.4})$$

where C_1, C_2, C_3 and C_4 are any constants in \mathbb{C} . We show that the velocity potential can be expressed by a Fourier series. For (A.4) to satisfy the Neumann condition ($\phi_z|_{z=-h} = 0$), we must have $C_3 = C_4 e^{2kh}$. Thus the velocity potential is expressed as

$$\phi(x, z) = (\tilde{C}_1 \cos(kx) + \tilde{C}_2 \sin(kx)) \cosh k(h+z), \quad (\text{A.5})$$

where $\tilde{C}_1 = 2C_1 C_4 e^{kh}$ and $\tilde{C}_2 = 2C_2 C_4 e^{kh}$. For (A.5) to satisfy the periodic condition ($\phi(x, z) = \phi(x + L, z)$), we need the following:

$$\tilde{C}_1 \cos(kx) + \tilde{C}_2 \sin(kx) = \tilde{C}_1 \cos(kx + kL) + \tilde{C}_2 \sin(kx + kL).$$

It implies that k is uniquely determined by $k = k_m = \frac{2m\pi}{L}$, for $m \in \mathbb{N}^*$. Finally, consider the Dirichlet boundary condition ($\phi(x, 0) = \xi(x)$). If $\xi(x) \in C_p^\infty(0, L)$ is written as

$$\xi(x) = \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} (a_m \cos(k_m x) + b_m \sin(k_m x)),$$

simple computations allow us to find that the Dirichlet condition holds if and only if the velocity potential is written by the following series

$$\phi(x, z) = \sqrt{\frac{2}{L}} \sum_{m=1}^{\infty} (a_m \cos(k_m x) + b_m \sin(k_m x)) \frac{\cosh k_m(h+z)}{\cosh k_m h}. \quad (\text{A.6})$$

If $k = 0$, we can prove that $\phi(x, z) = \frac{a_0}{\sqrt{L}}$ if and only if

$$\xi(x) = \frac{a_0}{\sqrt{L}}.$$

Next we show that this solution is unique. Let $\tilde{\phi}$ be another solution of the Laplace equation (A.1). Obviously the error $\epsilon = \phi(x, z) - \tilde{\phi}(x, z)$ satisfy the following Laplace equation

$$\begin{cases} \Delta \epsilon = 0, & \forall (x, z) \in \Omega, \quad \epsilon_z|_{z=-h} = 0, \quad \forall x \in \mathbb{R}^+, \\ \epsilon(x, 0) = 0, & \forall x \in \mathbb{R}^+, \quad \epsilon(x, z) = \epsilon(x + L, z), \quad \forall x \in \mathbb{R}^+. \end{cases} \quad (\text{A.7})$$

By using the Green formula, we can see that

$$\begin{aligned} 0 &= \int_{\Omega} \epsilon \Delta \epsilon d\Omega = \int_{-h}^0 \left[\epsilon \frac{\partial \epsilon}{\partial x} \Big|_{x=L} - \epsilon \frac{\partial \epsilon}{\partial x} \Big|_{x=0} \right] dz - \int_0^L \epsilon \frac{\partial \epsilon}{\partial z} \Big|_{z=-h} dx \\ &\quad + \int_0^L \epsilon \frac{\partial \epsilon}{\partial z} \Big|_{z=0} dx - \int_{\Omega} \|\nabla \epsilon\|^2 d\Omega = - \int_{\Omega} \|\nabla \epsilon\|^2 dx dz. \end{aligned}$$

Hence we have $\int_{\Omega} \|\nabla \epsilon\|^2 d\Omega = 0$. It is implied that ϵ is a constant. Furthermore, we have $\epsilon = 0$ since $\epsilon(x, 0) = 0$. This complete the proof of Theorem 2.1. ■

δ	0	2%	4%	6%	8%	10%	12%	14%	16%	18%
Iteration	280	281	277	279	296	323	270	291	300	293
$J_{min} [10^{-7}]$	9.99	9.96	9.86	9.94	9.92	9.91	9.79	9.93	9.89	9.99
$J_{2min} [10^{-3}]$	2.3	2.3	2.6	1.5	1.9	3	1.3	1.3	5.2	3.7
$J_{3min} [10^{-7}]$	8.62	8.56	8.39	8.66	7.81	7.93	8.36	8.34	6.22	6.82

Table 1: The iterations required to satisfy the convergence criterion $\epsilon < 10^{-6}$ for different δ .

Acknowledgements

This work was supported by Scientific Instruments Development Program of NSFC [615278010] and by National Key Basic Research Program of China (973 program) under grant No. 2014CB845301/2/3.

References

- [1] J.L. Irish and W.J. Lillycrop, Scanning laser mapping of the coastal zone: The shoals system, *ISPRS Journal of Photogrammetry and Remote Sensing* 54, pp.123-129,1999.
- [2] C.K. Wang and W.D. Philpot, Using airborne bathymetric lidar to detect bottom type variation in shallow waters, *Remote Sensing of Environment*, vol. 106, pp.123-135, 2007.
- [3] S.T. Grilli, Depth inversion in shallow water based on nonlinear properties of shoaling periodic waves, *Coastal Engineering*, vol. 35, pp.185-209, 1998.
- [4] T.K. Holland, Application of the linear dispersion relation with respect to depth inversion and remotely sensed imagery, *IEEE Transactions on Geoscience and Remote Sensing*, vol.39, pp.2060-2072, 2001.
- [5] T. Söderström and P. Stoica, System identification, *Prentice-Hall, Inc*, 1988.
- [6] R.G. Dean and R.A. Dalrymple, Water wave mechanics for engineers and scientists, *World scientific publishing Co Inc*, Vol.2, 1991.
- [7] D. Lannes, The water waves problem: mathematical analysis and asymptotics, *In: Mathematical Surveys and Monographs*, American Mathematical Society Providence, RI vol.188, 2013.
- [8] G.B. Whitham, Linear and nonlinear waves, *John Wiley, New York: Wiley & Sons*, Vol. 42, 1974.
- [9] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *Journal of Applied Mechanics and Technical Physics*, vol.9, pp.190-194, 1968.
- [10] W. Craig, U. Schanz, and C. Sulem, The modulational regime of three-dimensional water waves and the davey-stewartson system, *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, vol.14, pp.615-667, 1997.
- [11] W. Craig and C. Sulem, Numerical simulation of gravity waves, *Journal of Computational Physics*, vol.108, pp.73-83, 1993.
- [12] D. Lannes, Well-posedness of the water-waves equations, *Journal of the American Mathematical Society*, vol.18, pp.605-654, 2005.
- [13] M. A. Fontelos, R. Lecaros, J. C. Lopez-Rios and J. H. Ortega, Bottom detection through surface measurements on water waves, *SIAM Journal on control and optimization*, vol. 55, pp. 3890-3907, 2017.
- [14] R.S. Johnson, A modern introduction to the mathematical theory of water waves, *Cambridge University Press*, 1997.
- [15] E. Busvelle, and J.P. Gauthier, Observation and identification tools for non-linear systems: application to a fluid catalytic cracker, *International Journal of Control*, vol.78, pp.208-234, 2005.
- [16] G. Sallet, Introduction à l'identification des paramètres, *Réunion RTP Mathématique et décision pour le développement durable*, Institut d'études Scientifiques de Cargèse, du 6 au 10 avril 2009.
- [17] A.M. Bloch, and E.S. Titi, On the dynamics of rotating elastic beams, *Proc. Conf. New Trends System Theory*, July-9-11-1990.
- [18] Z.X. Zhao, M.K. Banda, B.Z. Guo, Simultaneous identification of diffusion coefficient, spacewise dependent source and initial value for one-dimensional heat equation, *Mathematical Methods in the Applied Sciences*, vol.40, pp.3552-3565, 2017.
- [19] K. Ramdani, M. Tucsnak, G. Weiss, Recovering the initial state of an infinite-dimensional system using observers, *Automatica*, vol. 46, pp. 1616-1625, 2010.

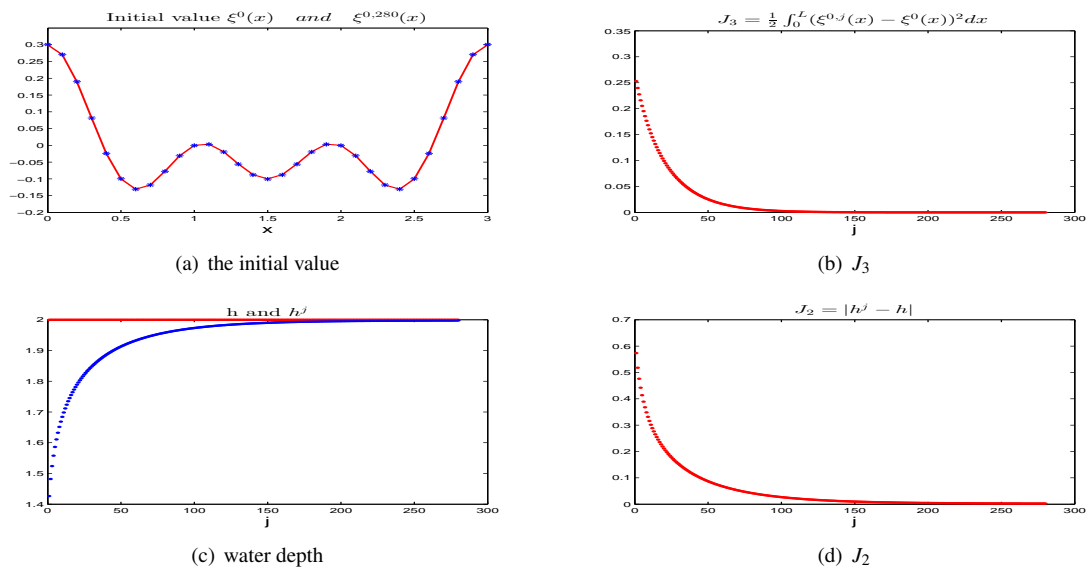


Figure 2: Reconstruction of initial value $\xi^0(x)$ and water depth h by using measurement data without random error

- [20] X.D. Li, and C.Z.Xu. Infinite-dimensional Luenberger-like observers for a rotating body-beam system, *Systems & Control Letters*, vol. 60, pp.138-145, 2011.
- [21] J. Baillieul, and M. Levi, Rotational elastic dynamics, *Physica D: Non-linear Phenomena*, vol. 27, pp. 43-62, 1987.
- [22] H. Brezis, Analyse fonctionnelle-théorie et applications, Collection Mathématiques appliquées pour la maîtrise sous la direction de P.G.Ciarlet et J.L.Lions, Masson, Paris, 1983.
- [23] G. Zimmer, A new algorithm for approximating the state of nonlinear systems, *International journal of systems science* vol.24, pp.777-788, 1993.
- [24] M. Tucsnak, G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Basel, 2009.
- [25] A. Ghaffari, M. Krstic and D. Nestic, Multivariable newton-based extremum seeking, *Automatica*, vol.48, pp.1759-1767, 2012.
- [26] M. Bergmann, Optimisation aérodynamique par réduction de modèle POD et contrôle optimal: application au sillage laminaire d'un cylindre circulaire, PhD thesis, Institut National Polytechnique de Lorraine (France), 2004.