ON THE WEAK CONSISTENCY OF FINITE VOLUMES SCHEMES FOR CONSERVATION LAWS ON GENERAL MESHES

Thierry Gallouët, R. Herbin, J.-C Latché

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T. GALLOUËT *, R. HERBIN †, AND J.-C. LATCHÉ ‡

Abstract. The aim of this paper is to develop some tools in order to obtain the weak consistency of (in other words, analogues of the Lax-Wendroff theorem for) finite volume schemes for balance laws in the multi-dimensional case and under minimal regularity assumptions for the mesh. As in the seminal Lax-Wendroff paper, our approach relies on a discrete integration by parts of the weak formulation of the scheme. This makes a discrete gradient of the test function appear, and the central argument for the scheme consistency is to remark that this discrete gradient is convergent in L^∞ weak $\star$.

Key words. finite volumes, consistency

1. Introduction. In the early sixties, P.D. Lax and B. Wendroff established that, on uniform 1D grids, a flux-consistent and conservative cell-centered finite-volume scheme for a conservation law is weakly consistent, in the sense that the limit of any a.e. convergent sequence of L^∞-bounded numerical solutions, obtained with a sequence of grids with mesh and time steps tending to zero, is a weak solution of the conservation law [11]; this result is known as the Lax-Wendroff theorem, and is reported in many textbooks with some variants: see e.g. [12, Section 12.10] with a BV bound assumption on the scheme, and [7, Theorem 21.2] for a generalisation to non-uniform meshes. However, convergence proofs on unstructured meshes which were obtained for nonlinear scalar conservation laws in the 90's do not use the Lax-Wendroff theorem; indeed, finite volume schemes on unstructured meshes are known to be in general not TVD (see e.g. an example in [4]), so that a compactness property in L^1_{loc} is not easy to obtain, although it does hold in fact but results from the proof of uniqueness of the so-called entropy process solution, see e.g. [7, chapter VI] and references therein.

Nevertheless, from a practical point of view, the Lax-Wendroff theorem, even if weaker than a full convergence proof, may be fundamental for the design of numerical schemes. In particular, for many hyperbolic systems (especially in the multi-dimensional case), it represents the essential part of the theoretical foundations, since provable estimates on numerical solutions are too weak to provide sufficient compactness to undertake any convergence study.

The seminal Lax-Wendroff paper has been the starting point for several research works, aimed at relaxing the original assumptions: for instance, an extension to schemes which do not admit a local conservative formulation may be found in [3, 2, 1, 14]; the derivation of analogue consistency results for non-conservative hyperbolic systems is presented in [8, 13]. In the multi-dimensional case and for standard finite volumes schemes, Lax-Wendroff type results have been progressively extended to general (and, in particular, unstructured) discretizations [10], in [9, Section 4.2.2] for two-dimensional simplicial meshes and in [5] for 2D and 3D general quasi-uniform grids.

The present paper follows the same route, in the sense that we still extend the application range of the Lax-Wendroff theorem in terms on constraints on the mesh, in particular relaxing the quasi-uniformity assumption. However, compared to [5], the technique of proof is different, and we return to a strategy close to the original work: the scheme is multiplied by a test function and integrated over space and time, and a discrete integration by parts of the convection term yields the integral of the product of the numerical flux by a discrete gradient of the test function (this latter seems to appear first in [6], where its convergence properties are shown for specific

*Université d'Aix-Marseille (thierry.gallouet@univ-amu.fr)
†Université d'Aix-Marseille (raphaele.herbin@univ-amu.fr)
‡Institut de Radioprotection et de Sûreté Nucléaire (IRSN), PSN-RES/SA2I, Cadarache, St-Paul-lez-Durance, 13115, France (jean-claude.latche@irsn.fr)

meshes and norms). Then the passage to the limit of vanishing space and time steps in the scheme requires two ingredients:

- a convergence result for the discrete gradient; contrary to what happens in the 1D case or for Cartesian grids, this convergence is only weak, namely in L^∞ weak $\star$, which is however sufficient to conclude,
- the control of some residual terms, which basically consists in the difference between the numerical solution and a space or time translate of this latter; the difficulty here lies in the fact that the translation amplitude is (locally) mesh-dependent (for instance, the function is translated from one cell to its neighbour), so that standard results for converging sequence of functions in L^1 may no longer be applied.

The presentation is organized as follows: after a definition of the considered space discretizations (Section 2), we address successively the two above-mentioned issues (Section 3 and 4 respectively), carefully clarifying in these two sections the regularity requirements for the mesh. Then we show in Section 5 how to use the obtained results to obtain a weak consistency result for a standard finite volume discretization of a balance law; consistency requirements for the numerical flux appear in this step.

2. Space discretization. Let Ω be an open bounded polyhedral set of $\mathbb{R}^d$, $d \geq 1$. A polyhedral partition $\mathcal{M}$ of Ω is a finite partition of Ω such that each element K of this partition is measurable and has a boundary ∂K that is composed of a finite union of part of hyperplanes (the faces of K) denoted by σ, so that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \sigma$ where $\mathcal{E}_K$ is the set of the faces of K. Such a polyhedral partition is called a "mesh". We denote by $\mathcal{E}$ the set of all the faces, namely $\mathcal{E} = \cup_{K \in \mathcal{M}} \mathcal{E}_K$. If $\sigma \in \mathcal{E}$ is a face of this partition, then one denotes by $|\sigma|$ the $(d-1)$-Lebesgue measure of σ. We denote by $\mathcal{E}_{\rm int}$ the set of elements σ of $\mathcal{E}$ such that there exist K and L in $\mathcal{M}$ $(K \neq L)$ such that $\sigma \in \mathcal{E}_K \cap \mathcal{E}_L$; such a face σ is denoted by $\sigma = K|L$. The set of faces located on the boundary of Ω, *i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm int}$, is denoted by $\mathcal{E}_{\rm ext}$. For $K \in \mathcal{M}$ we denote by h_K the diameter of K. The size of the mesh $\mathcal{M}$ is $h_\mathcal{M} = \max\{h_K, K \in \mathcal{M}\}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\boldsymbol{n}_{K,\sigma}$ the normal vector of σ outward K.

We also introduce now a dual mesh, that is a new partition of Ω indexed by the elements of $\mathcal{E}$, namely $\Omega = \cup_{\sigma \in \mathcal{E}} D_\sigma$. For $\sigma = K|L$, the set D_σ is supposed to be a subset of $K \cup L$ and we define $D_{K,\sigma} = D_\sigma \cap K$, so $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ (see Figure 2.1).

If A is a measurable set of $\mathbb{R}^d$, we denote by $|A|$ the Lebesgue measure of A.

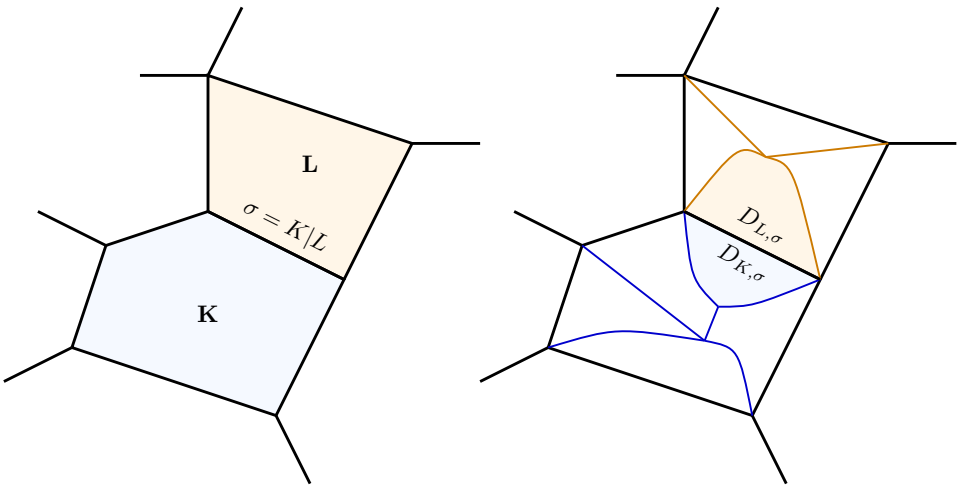

FIG. 2.1. *Mesh and associated notations.*

3. A weakly convergent discrete gradient. Let $\varphi \in C_c^\infty(\Omega)$ and, for $K \in \mathcal{M}$, let $\boldsymbol{x}_K$ be a point of K and $\varphi_K = \varphi(\boldsymbol{x}_K)$. For $\sigma \in \mathcal{E}$, let

$$\begin{aligned} &(\boldsymbol{\nabla}_{\mathcal{E}}\varphi)_\sigma = \frac{|\sigma|}{|D_\sigma|}\,(\varphi_L - \varphi_K)\,\boldsymbol{n}_{K,\sigma} \quad && \text{if } \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L, \\ &(\boldsymbol{\nabla}_{\mathcal{E}}\varphi)_\sigma = 0 && \text{if } \sigma \in \mathcal{E}_{\rm ext}. \end{aligned} \tag{3.1}$$

We characterize the regularity of the discretization by the following parameter:

$$\theta_{\mathcal{M}}^{\nabla} = \max_{\sigma \in \mathcal{E}_{\rm int},\, \sigma = K|L} \frac{|\sigma|\;|\boldsymbol{x}_L - \boldsymbol{x}_K|}{|D_\sigma|}. \tag{3.2}$$

Note that, with this definition, we get, for $\sigma \in \mathcal{E}$,

$$|(\boldsymbol{\nabla}_{\mathcal{E}}\varphi)_\sigma| \le\; \theta_{\mathcal{M}}^{\nabla}\;|\boldsymbol{\nabla}\varphi|_{L^\infty(\Omega)^d}, \tag{3.3}$$

where $\boldsymbol{\nabla}_{\mathcal{E}}\varphi$ is the piecewise constant function equal to $(\boldsymbol{\nabla}_{\mathcal{E}}\varphi)_\sigma$ over D_σ, for $\sigma \in \mathcal{E}$.

LEMMA 3.1. *Let $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes such that the mesh step $h_{\mathcal{M}^{(m)}}$ tends to zero when m tends to $+\infty$. We suppose that there exists a real number θ^{∇} such that $\theta^{\nabla}_{\mathcal{M}^{(m)}} \le \theta^{\nabla}$ for $m \in \mathbb{N}$. Let $\varphi \in C_c^\infty(\Omega)$ and, for $m \in \mathbb{N}$, let $\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi \in L^\infty(\Omega)^d$ be defined by (3.1). Then the sequence $(\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi)_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^d$ uniformly with respect to m and converges to $\boldsymbol{\nabla}\varphi$ in $L^\infty(\Omega)^d$ weak $\star$.*

Proof. The fact that the sequence $(\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi)_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)^d$ is a straightforward consequence of Inequality (3.3) and the assumption $\theta^{\nabla}_{\mathcal{M}^{(m)}} \le \theta^{\nabla}$ for $m \in \mathbb{N}$. Let $\boldsymbol{\psi} \in C_c^\infty(\Omega)^d$. For $m \in \mathbb{N}$ and $\sigma \in \mathcal{E}^{(m)}$, let $\boldsymbol{\psi}_\sigma$ and $\bar{\boldsymbol{\psi}}_\sigma$ be defined by the mean value of $\boldsymbol{\psi}$ over σ and D_σ respectively. Since $\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi$ is piecewise constant over the dual cells D_σ, we have

$$I^{(m)} := \int_\Omega \boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{\sigma\in\mathcal{E}^{(m)}} |D_\sigma|\;(\boldsymbol{\nabla}_{\mathcal{E}_n}\varphi)_\sigma\,\bar{\boldsymbol{\psi}}_\sigma = \tilde{I}^{(m)} + R^{(m)},$$

with

$$\tilde{I}^{(m)} = \sum_{\sigma\in\mathcal{E}^{(m)}} |D_\sigma|\;(\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi)_\sigma\,\boldsymbol{\psi}_\sigma, \quad R^{(m)} = \sum_{\sigma\in\mathcal{E}^{(m)}} |D_\sigma|\;(\boldsymbol{\nabla}_{\mathcal{E}^{(m)}}\varphi)_\sigma\,(\bar{\boldsymbol{\psi}}_\sigma - \boldsymbol{\psi}_\sigma).$$

We first consider $\tilde{I}^{(m)}$. Since ψ has a compact support in Ω, the quantity $\boldsymbol{\psi}_\sigma$ vanishes for any $\sigma \in \mathcal{E}^{(m)}_{\rm ext}$. We thus get, using the definition (3.1) of the discrete gradient and reordering the sums:

$$\tilde{I}^{(m)} = \sum_{\sigma\in\mathcal{E}^{(m)}_{\rm int},\,\sigma=K|L} |\sigma|\;(\varphi_L - \varphi_K)\,\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{\psi}_\sigma = -\sum_{K\in\mathcal{M}^{(m)}} \varphi_K \sum_{\sigma\in\mathcal{E}_K} |\sigma|\;\boldsymbol{\psi}_\sigma\cdot\boldsymbol{n}_{K,\sigma}.$$

Hence,

$$\tilde{I}^{(m)} = -\sum_{K\in\mathcal{M}^{(m)}} \varphi(\boldsymbol{x}_K)\;\int_K \mathrm{div}\boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \varphi(\boldsymbol{x})\,\mathrm{div}\boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \tilde{R}^{(m)},$$

with

$$\tilde{R}^{(m)} = \sum_{K\in\mathcal{M}^{(m)}} \int_K \big(\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}_K)\big)\;\mathrm{div}\boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The remainder term $\tilde{R}^{(m)}$ may be bounded as follows

$$|\tilde{R}^{(m)}| \le |\boldsymbol{\nabla}\varphi|_{L^\infty(\Omega)^d}\;|\boldsymbol{\nabla}\boldsymbol{\psi}|_{L^\infty(\Omega)^{d\times d}}\;|\Omega|\;h_{\mathcal{M}^{(m)}}, \tag{3.4}$$

and thus

$$\lim_{m\to+\infty} \tilde{I}^{(m)} = -\int_\Omega \varphi(\boldsymbol{x})\;\mathrm{div}\boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\nabla}\varphi(\boldsymbol{x})\cdot\boldsymbol{\psi}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The term $R^{(m)}$ reads, once again thanks to the definition (3.1) of the discrete gradient:

$$R^{(m)} = \sum_{\sigma \in \mathcal{E}_{\rm int}^{(m)},\, \sigma = K|L} |D_\sigma| \; (\boldsymbol{\nabla}_{\mathcal{E}^{(m)}} \varphi)_\sigma \, \boldsymbol{n}_{K,\sigma} \cdot (\bar{\boldsymbol{\psi}}_\sigma - \boldsymbol{\psi}_\sigma).$$

Since $\boldsymbol{\psi}$ is regular, there exists $\boldsymbol{x}_\sigma$ and $\boldsymbol{x}_{D_\sigma}$ such that $\boldsymbol{\psi}_\sigma = \boldsymbol{\psi}(\boldsymbol{x}_\sigma)$ and $\bar{\boldsymbol{\psi}}_\sigma = \boldsymbol{\psi}(\boldsymbol{x}_{D_\sigma})$ respectively. Hence, since $|\boldsymbol{x}_\sigma - \boldsymbol{x}_{D_\sigma}| \leq \max(h_K, h_L)$, we have thanks to (3.3):

$$|R^{(m)}| \leq |\boldsymbol{\nabla}\varphi|_{L^\infty(\Omega)^d} \; |\boldsymbol{\nabla}\boldsymbol{\psi}|_{L^\infty(\Omega)^{d\times d}} \; |\Omega| \; \theta^\nabla_{\mathcal{M}^{(m)}} \; h_{\mathcal{M}^{(m)}}, \tag{3.5}$$

and thus

$$\lim_{m \to +\infty} |R^{(m)}| = 0.$$

The conclusion of the proof is obtained by invoking the density of the functions of $C_c^\infty(\Omega)^d$ in $L^1(\Omega)^d$.
□

REMARK 3.1. *Estimates* (3.4) *and* (3.5)*, show that the difference* $D^{(m)}$ *defined by*

$$D^{(m)} = \int_\Omega \left| \left(\boldsymbol{\nabla}_{\mathcal{E}^{(m)}} \varphi(\boldsymbol{x}) - \boldsymbol{\nabla}\varphi(\boldsymbol{x}) \right) \cdot \boldsymbol{\psi}(\boldsymbol{x}) \right| \mathrm{d}\boldsymbol{x}$$

only depends on the sequence of meshes and on φ, $|\boldsymbol{\nabla}\varphi|_{L^\infty(\Omega)^d}$ *and* $|\boldsymbol{\nabla}\boldsymbol{\psi}|_{L^\infty(\Omega)^{d\times d}}$. *This point is used hereafter to extend the present convergence result to time depending functions.*

REMARK 3.2 (Choice of $\boldsymbol{x}_K$ and D_σ). *Note that, apart from the need to ensure the regularity of the sequence of meshes (i.e.* $\theta^\nabla_{\mathcal{M}^{(m)}} \leq \theta^\nabla$ *for* $m \in \mathbb{N}$*), there is almost no constraint on the choice of* $\boldsymbol{x}_K$ *in* K *and on* D_σ*, which is just a volume associated to the face* σ *which, for the proof of Lemma 3.1, does not even need to contain* σ *itself.*

REMARK 3.3 (On the mesh regularity assumption). *Some regularity constraints for the sequence of meshes may be shown to be strong enough to control the parameter* $\theta^\nabla_{\mathcal{M}}$. *First, let us suppose that the cells are not allowed to be too flat, i.e. that there exists* $C > 0$ *such that*

$$|K| \geq C\, h_K^d, \quad \forall K \in \mathcal{M}^{(m)}, \; \forall m \in \mathbb{N}. \tag{3.6}$$

Then, let us denote by $\tau_{\mathcal{M}}$ *the parameter:*

$$\tau_{\mathcal{M}} = \max_{K \in \mathcal{M},\; \sigma \in \mathcal{E}_K} \frac{|K|}{|D_{K,\sigma}|}.$$

Since the definition of D_σ *is almost arbitrary,* $\tau_{\mathcal{M}}$ *may be kept bounded away from zero for any sequence of meshes, provided that the number of faces of the cells is bounded; for instance, in Section 5, we choose* D_σ *is such a way that* $\tau_{\mathcal{M}}$ *is equal to the inverse of the maximum number of cell faces. We then have:*

$$\theta^\nabla_{\mathcal{M}} \leq \max_{\sigma \in \mathcal{E}_{\rm int},\, \sigma = K|L} \frac{(h_K + h_L) \; \min(h_K^{d-1}, h_L^{d-1})}{\tau_{\mathcal{M}} \; \max(|K|, |L|)} \leq 2\, \frac{C}{\tau_{\mathcal{M}}}.$$

The case of "flat cells" is more intricate. However, let us suppose that, for $m \in \mathbb{N}$ *and* $\sigma \in \mathcal{E}_{\rm int}^{(m)}$, $\sigma = K|L$,

- $\boldsymbol{x}_K$ *and* $\boldsymbol{x}_L$ *may be chosen such that* K *and* L *are star-shaped with respect to* $\boldsymbol{x}_K$ *and* $\boldsymbol{x}_L$ *respectively,*
- *we choose for* $D_{K,\sigma}$ *and* $D_{L,\sigma}$ *the cones of basis* σ *and vertex* $\boldsymbol{x}_K$ *and* $\boldsymbol{x}_L$ *respectively,*

FIG. 3.1. *Choice of $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$ for "flat cells": (a) convenient choice, (b) quasi-orthogonality is lost when cells become flatter and flatter.*

- *there exists $C > 0$ such that $|(\boldsymbol{x}_K - \boldsymbol{x}_L) \cdot \boldsymbol{n}_{K,\sigma}| \geq C\,|\boldsymbol{x}_K - \boldsymbol{x}_L|$ (which may be referred to as a "uniform orthogonality condition").*

Then $|\boldsymbol{x}_K - \boldsymbol{x}_L|\ |\sigma| \leq d\,|D_\sigma|/C$ (see Fig. 3.1).

The analysis of finite volume schemes for (systems of) conservation laws necessitates the extension of Lemma 3.1 to time-dependent functions. Let $T > 0$ and a time discretization of the interval $(0,T)$, *i.e.* a sequence $\mathcal{T} = (t_n)_{0\leq n\leq N}$ with $0 = t_0 < t_1 \cdots < t_n < t_{n+1} \cdots < t_N = T$, be given; we define $\delta t_n = t_{n+1} - t_n$ and $\delta t_{\mathcal{T}} = \max\{\delta t_n,\ 0 \leq n < N\}$. Let $\varphi \in C_c^\infty(\Omega \times [0,T])$; then the piecewise function $\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\,\varphi$ is defined by:

$$\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\,\varphi = \sum_{n=0}^{N} \sum_{\sigma\in\mathcal{E}} (\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\,\varphi)_\sigma^n \mathbb{1}_{D_\sigma}(x) \mathbb{1}_{[t_n,t_{n+1}[}(t), \text{ with}$$

$$(\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\,\varphi)_\sigma^n = \begin{cases} \frac{|\sigma|}{|D_\sigma|}\ (\varphi_L^n - \varphi_K^n)\,\boldsymbol{n}_{K,\sigma} & \text{if } \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L, \\ 0 & \text{if } \sigma \in \mathcal{E}_{\rm ext}. \end{cases} \tag{3.7}$$

where, for $K \in \mathcal{M}$ and $0 \leq n < N$, $\varphi_K^n = \varphi(\boldsymbol{x}_K, t_n)$, and for a given set A, $\mathbb{1}_A$ is the characteristic function of A, that is $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise. Then the following weak convergence result holds.

LEMMA 3.2. *Let $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes such that the mesh step $h_{\mathcal{M}^{(m)}}$ tends to zero when m tends to $+\infty$. We suppose that there exists a real number θ^∇ such that $\theta^\nabla_{\mathcal{M}^{(m)}} \leq \theta^\nabla$ for $m \in \mathbb{N}$. For $m \in \mathbb{N}$, we suppose given a time discretization $\mathcal{T}^{(m)}$, and suppose that $\delta t_{\mathcal{T}^{(m)}}$ also tends to zero when m tends to $+\infty$.*

Let $\varphi \in C_c^\infty(\Omega \times [0,T])$ and, for $m \in \mathbb{N}$, $\boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\,\varphi \in L^\infty(\Omega \times [0,T])^d$ be defined by (3.7). *Then the sequence $(\boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\,\varphi)_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega\times[0,T])^d$ uniformly with respect to m and converges to $\boldsymbol{\nabla}\varphi$ in $L^\infty(\Omega\times[0,T])^d$ weak $\star$.*

Proof. Let $\boldsymbol{\psi} \in C_c^\infty(\Omega \times [0,T])^d$, and let us define, for $m \in \mathbb{N}$, the following functions of time:

$$I(t) = \int_\Omega \boldsymbol{\nabla}\varphi(\boldsymbol{x},t) \cdot \boldsymbol{\psi}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}, \quad I^{(m)}(t) = \int_\Omega \boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\,\varphi(\boldsymbol{x},t) \cdot \boldsymbol{\psi}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}.$$

Thanks to remark 3.1 and the regularity of φ and $\boldsymbol{\psi}$, $I^{(m)}(t)$ converges to $I(t)$ uniformly with respect to t, with $I \in C_c^\infty(0,T)$. The integral

$$\int_0^T \int_\Omega (\boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\,\varphi(\boldsymbol{x},t) \cdot \boldsymbol{\psi}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \sum_{n=0}^{N^{(m)}-1} (t_{n+1} - t_n)\ I^{(m)}(t_n) + R,$$

where R is a remainder term tending to zero as $\delta t_{\mathcal{T}^{(m)}}$ thanks to the regularity of $\boldsymbol{\psi}$. We have

$$\sum_{n=0}^{N^{(m)}-1} (t_{n+1} - t_n)\ I^{(m)}(t_n) = \sum_{n=0}^{N^{(m)}-1} (t_{n+1} - t_n)\ I(t_n) + \sum_{n=0}^{N^{(m)}-1} (t_{n+1} - t_n)\ \big(I^{(m)}(t_n) - I(t_n)\big)$$

The second term tends to zero when m tends to $+\infty$ thanks to the uniform convergence of $I^{(m)}$ to I. By the regularity of I, the first one converges to

$$\int_0^T I(t)\,\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{\nabla}\varphi(\boldsymbol{x},t)\cdot\boldsymbol{\psi}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

The conclusion follows by density of $C_c^\infty(\Omega\times[0,T])^d$ in $L^1(\Omega\times[0,T])^d$.
□

4. Convergence of "discrete translations" of functions of L^1. The proof of the original Lax-Wendroff theorem (extended to non uniform meshes in [7, Theorem 12.2]) relies on the mean continuity of integrable functions, which is used to prove that for a sequence $(u_p)_{p\in\mathbb{N}}$ of converging functions in $L^1(\Omega)$,

$$\int_\Omega |u_p(\cdot+h)-u_p|\,\mathrm{d}\boldsymbol{x}\to 0 \text{ as } h\to 0 \text{ uniformly w.r.t. } p. \tag{4.1}$$

This proof may be extended to Cartesian meshes; however, on an unstructured mesh, the notion of translation is no longer clear and the problem must be reformulated as follows. For $u\in L^1(\Omega)$ and $K\in\mathcal{M}$, we denote by u_K the mean value of u and we set for $\sigma\in\mathcal{E}_{\rm int}$, $\sigma=K|L$, $[u]_\sigma=|u_K-u_L|$. We now introduce the quantity

$$T_{\mathcal{M}}u=\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L}|D_\sigma|[u]_\sigma. \tag{4.2}$$

The objective of this section is to prove that, for a sequence $(u_p)_{p\in\mathbb{N}}$ of functions of $L^1(\Omega)$ and a sequence of increasingly refined meshes $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$, supposed to be regular in a sense to be defined, the quantity $T_{\mathcal{M}^{(m)}}u_p$ tends to zero as $m\to+\infty$ uniformly with respect to p. Note that in the case of a uniform 1D mesh such that $|D_\sigma|=h$ for any $\sigma\in\mathcal{E}$, this is equivalent to showing (4.1).

The proof of this result is split in two steps; we first consider a fixed function $u\in L^1(\Omega)$ and prove a generalisation of the mean continuity in Lemma 4.1; we then address the case of a converging sequence in $L^1(\Omega)$ in Lemma 4.3. The technique used here is reminiscent of underlying arguments invoked in [9, Section 4.2.2] for two-dimensional triangular meshes; we consider here general meshes, paying a special attention to mesh regularity requirements.

To formulate the regularity assumption of the sequence of meshes considered in this section, we introduce the following parameter:

$$\theta_{\mathcal{M}}=\max_{K\in\mathcal{M}}\max_{\sigma\in\mathcal{E}_K}\frac{|D_\sigma|}{|K|}. \tag{4.3}$$

By definition, we thus have $|D_\sigma|\le\theta_{\mathcal{M}}\,|K|$, for $\sigma\in\mathcal{E}_K$, $K\in\mathcal{M}$.

LEMMA 4.1. *Let $\theta>0$ and $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes such that $\theta_{\mathcal{M}^{(m)}}\le\theta$ for all $m\in\mathbb{N}$ and $\lim_{m\to+\infty}h_{\mathcal{M}^{(m)}}=0$. We suppose that the number of faces of a cell $K\in\mathcal{M}^{(m)}$ is bounded by $\mathcal{N}_{\mathcal{E}}$, for any $m\in\mathbb{N}$. Let $u\in L^1(\Omega)$, let u_K denote the mean value of u on a cell K, and let $T_{\mathcal{M}^{(m)}}u$ be defined by* (4.2)*. Then,*

$$\lim_{m\to+\infty}T_{\mathcal{M}^{(m)}}u=0. \tag{4.4}$$

Proof. We first note that for $u\in L^1(\Omega)$ and a mesh $\mathcal{M}$,

$$\begin{aligned}T_{\mathcal{M}}(u)\le\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L}|D_\sigma|\,\big([u_K|+|u_L|\big)&\le\theta\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L}\big(|K|\,|u_K|+|L|\,|u_L|\big)\\&\le\mathcal{N}_{\mathcal{E}}\,\theta\,||u||_{L^1(\Omega)}.\end{aligned}$$

Then, for any $\varphi\in C_c^\infty(\mathbb{R}^d,\mathbb{R})$,

$$T_{\mathcal{M}^{(m)}}u\le T_{\mathcal{M}^{(m)}}(u-\varphi)+T_{\mathcal{M}^{(m)}}\varphi\le\mathcal{N}_{\mathcal{E}}\,\theta\,||u-\varphi||_{L^1(\Omega)}+T_{\mathcal{M}^{(m)}}\varphi. \tag{4.5}$$

Let $\epsilon > 0$. Since $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$, there exists $\varphi \in C_c^\infty(\Omega)$ such that $\mathcal{N}_\mathcal{E}\,\theta\,||u-\varphi||_{L^1(\Omega)} \le \epsilon$. Then, with this choice of φ,

$$T_{\mathcal{M}^{(m)}} u \le \epsilon + T_{\mathcal{M}^{(m)}} \varphi. \tag{4.6}$$

We now use only the fact that φ is Lipschitz continuous. There exists $M_\varphi > 0$ such that $|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le M_\varphi\,|\boldsymbol{x}-\boldsymbol{y}|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. Then, for any $\sigma = K|L \in \mathcal{E}_{\rm int}$, using $\bar{K} \cap \bar{L} \neq \emptyset$,

$$\begin{aligned} |K||L|[\varphi]_\sigma = |K|\;|L|\;|\varphi_K - \varphi_L| &= \Big|\int_K \int_L \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\Big| \\ &\le M_\varphi \int_K \int_L |\boldsymbol{x}-\boldsymbol{y}|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \le M_\varphi\;|K|\;|L|\;(h_K + h_L). \end{aligned}$$

This yields

$$T_{\mathcal{M}^{(m)}}\varphi = \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |D_\sigma|[\varphi]_\sigma \le 2M_\varphi\, h_{\mathcal{M}^{(m)}} \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |D_\sigma| \le 2M_\varphi\, h_{\mathcal{M}^{(m)}}|\Omega|.$$

Since $\lim_{m\to+\infty} h_{\mathcal{M}^{(m)}} = 0$, there exists m_0 such that $2M_\varphi\, h_{\mathcal{M}^{(m)}}|\Omega| \le \epsilon$ for $m \ge m_0$ and then, with (4.6),

$$T_{\mathcal{M}^{(m)}} u \le 2\epsilon \quad \text{for } m \ge m_0.$$

□

The convergence result which constitutes the aim of this section is stated in the following lemma.

LEMMA 4.2. *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes such that $\theta_{\mathcal{M}^{(m)}} \le \theta$ for all $m \in \mathbb{N}$ and $\lim_{m\to+\infty} h_{\mathcal{M}^{(m)}} = 0$. We suppose that the number of faces of a cell $K \in \mathcal{M}^{(m)}$ is bounded by $\mathcal{N}_\mathcal{E}$, for any $m \in \mathbb{N}$. Let $u \in L^1(\Omega)$ and $(u_p)_{p\in\mathbb{N}}$ be a sequence of functions of $L^1(\Omega)$ such that $u_p \to u$ in $L^1(\Omega)$ as $p \to +\infty$. Then $T_{\mathcal{M}^{(m)}} u_p$ tends to zero when m tends to $+\infty$ uniformly with respect to $p \in \mathbb{N}$.*

Proof. Using (4.5) with $u = u_p$ and $\varphi = u$, we obtain

$$T_{\mathcal{M}^{(m)}} u_p \le T_{\mathcal{M}^{(m)}}(u_p - u) + T_{\mathcal{M}^{(m)}} u \le \mathcal{N}_\mathcal{E}\,\theta\,||u-u_p||_{L^1(\Omega)} + T_{\mathcal{M}^{(m)}} u.$$

Let $\epsilon > 0$. Since $u_p \to u$ in $L^1(\Omega)$, as $p \to +\infty$, there exists p_0 such that for $p \ge p_0$, $\mathcal{N}_\mathcal{E}\,\theta\,||u-u_p||_{L^1(\Omega)} \le \epsilon$ and then

$$T_{\mathcal{M}^{(m)}} u_p \le \epsilon + T_{\mathcal{M}^{(m)}} u.$$

We use now Lemma 4.1. There exists m_0 such that $T_{\mathcal{M}^{(m)}} u \le \epsilon$ for $m \ge m_0$ and then, for $m \ge m_0$ and $p \ge p_0$,

$$T_{\mathcal{M}^{(m)}} u_p \le 2\epsilon.$$

Using again Lemma 4.1 for $p \in \{0, \ldots, p_0 - 1\}$ gives m_1 such for $m \ge m_1$ and $p \in \mathbb{N}$,

$$T_{\mathcal{M}^{(m)}} u_p \le 2\epsilon.$$

□

REMARK 4.1. *The underlying ideas of the proofs of this section may be summed up as follows. Let $(T^{(m)})_{m\in\mathbb{N}}$ be a sequence of semi-norms acting on a Banach space $\mathcal{B}$ to $\mathbb{R}^+$ satisfying two properties:*

- *uniform boundedness: there exists $C > 0$ such that $T^{(m)}u \le C\;||u||_\mathcal{B}$ for any $u \in \mathcal{B}$ and $m \in \mathbb{N}$,*
- *convergence to zero on a dense subspace: there exists $\mathcal{D} \subset \mathcal{B}$ and dense in $\mathcal{B}$ such that, for any $\bar{u} \in \mathcal{D}$, $\lim_{m\to+\infty} T^{(m)}\bar{u} = 0$.*

Then, for any converging sequence $(u_p)_{p\in\mathbb{N}}$ in $\mathcal{B}$, $T^{(m)}u_p$ tends to zero uniformly with respect to p.

This result extends to time depending functions as follows. Let $T > 0$ and $\mathcal{T}$ be a time discretization of the interval $(0,T)$, as defined in the previous section. For $u \in L^1(\Omega\times(0,T))$, $K \in \mathcal{M}$ and $0 \le n < N$, let u_K^{n+1} be the mean value of u over $K\times(t_n,t_{n+1})$. For $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$ and for $0 < n \le N$, let $[u^n]_\sigma = |u_K^n - u_L^n|$; for $K \in \mathcal{M}$ and $0 < n < N$, we set $[u_K]^n = |u_K^{n+1} - u_K^n|$. We define the following quantity:

$$T_{\mathcal{M},\mathcal{T}}u = \sum_{n=0}^{N-1}(t_{n+1}-t_n)\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L}|D_\sigma|[u^{n+1}]_\sigma + \sum_{n=1}^{N-1}(t_{n+1}-t_n)\sum_{K\in\mathcal{M}}|K|[u_K]^n. \tag{4.7}$$

Then the following lemma results from easy extensions (in fact, reasoning in $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$) of the previous proofs of this section.

LEMMA 4.3. *Let $\theta > 0$ and $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes such that $\theta_{\mathcal{M}^{(m)}} \le \theta$ for all $m \in \mathbb{N}$ and $\lim_{m\to+\infty} h_{\mathcal{M}^{(m)}} = 0$. We suppose that the number of faces of a cell $K \in \mathcal{M}^{(m)}$ is bounded by $\mathcal{N}_\mathcal{E}$, for any $m \in \mathbb{N}$. For $m \in \mathbb{N}$, we suppose given a time discretization $\mathcal{T}^{(m)}$, and suppose that $\delta t_{\mathcal{T}^{(m)}}$ also tends to zero when m tends to $+\infty$. Let $u \in L^1(\Omega\times(0,T))$ and $(u_p)_{p\in\mathbb{N}}$ be a sequence of functions of $L^1(\Omega\times(0,T))$ such that $u_p \to u$ in $L^1(\Omega\times(0,T))$ as $p\to+\infty$.*
Then $T_{\mathcal{M}^{(m)},\mathcal{T}^{(m)}}u_p$ tends to zero when m tends to $+\infty$ uniformly with respect to $p \in \mathbb{N}$.

5. Weak consistency of conservative finite volumes discretizations of conservation laws. Let us consider the following conservation law posed over $\Omega\times(0,T)$:

$$\partial_t(u) + \mathrm{div}(\boldsymbol{F}(u)) = 0, \tag{5.1}$$

where $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^d$ is a regular flux function. This equation is complemented with the initial condition $u(\boldsymbol{x},0) = u_0(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$, where u_0 is a given function of $L^1(\Omega)$, and convenient boundary conditions, see remark 5.1. For a given mesh $\mathcal{M}$ and time discretization $\mathcal{T}$, a finite volume approximation of Equation (5.1) reads

$$\frac{|K|}{t_{n+1}-t_n}(u_K^{n+1}-u_K^n) + \sum_{\sigma\in\mathcal{E}(K)}|\sigma|\,\boldsymbol{F}_\sigma^n\cdot\boldsymbol{n}_{K,\sigma} = 0, \text{ for } K\in\mathcal{M} \text{ and } 0\le n<N, \tag{5.2}$$

where $\boldsymbol{F}_\sigma^n$ is the numerical flux. Equation (5.2) is complemented by the initial condition:

$$u_K^0 = \frac{1}{|K|}\int_K u_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \text{ for } K\in\mathcal{M}, \tag{5.3}$$

and convenient boundary fluxes. The discrete unknowns $(u_K^n)_{K\in\mathcal{M},0\le n\le N}$ are associated to a function of $L^\infty(\Omega)$ as follows:

$$u(\boldsymbol{x},t) = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} u_K^n\ \mathcal{X}_K\ \mathcal{X}_{(t_n,t_{n+1}]}, \tag{5.4}$$

where $\mathcal{X}_K$ and $\mathcal{X}_{(t_n,t_{n+1}]}$ stand for the characteristic function of K and the interval $(t_n,t_{n+1}]$ respectively.

We now suppose given a sequence of meshes $(\mathcal{M}^{(m)})_{m\in\mathbb{N}}$ and time discretizations $(\mathcal{T}^{(m)})_{m\in\mathbb{N}}$, with $h_{\mathcal{M}^{(m)}}$ and $\delta t_{\mathcal{T}^{(m)}}$ tending to zero as m tends to $+\infty$, and, for $m\in\mathbb{N}$, denote by $u^{(m)}$ the function obtained with $\mathcal{M}^{(m)}$ and $\mathcal{T}^{(m)}$. Let us suppose that the sequence $(u^{(m)})_{m\in\mathbb{N}}$ converge in $L^1(\Omega\times(0,T))$ to a function $\bar{u}$. The aim of this section

is to determine sufficient conditions for $\bar{u}$ to satisfy the weak formulation of Equation (5.1), *i.e.*

$$-\int_0^T \int_\Omega \Big(\bar{u}\ \partial_t \varphi + \boldsymbol{F}(\bar{u}) \cdot \boldsymbol{\nabla} \varphi \Big)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = \int_\Omega u_0(\boldsymbol{x}) \varphi(\boldsymbol{x}, 0)\, \mathrm{d}\boldsymbol{x}, \quad \text{for any } \varphi \in C_c^\infty(\Omega \times [0,T)). \quad (5.5)$$

This is obtained by letting the space and time step tend to zero and passing to the limit in the scheme, and this issue is referred to as a weak consistency property of the scheme.

REMARK 5.1 (Weak formulation and boundary conditions). *Note that the weak formulation* (5.5) *is incomplete, in the sense that it does not imply anything on boundary conditions, since test functions are supposed to have a support compact in* $\Omega \times [0,T)$*; in fact, weak formulation of boundary conditions is a difficult problem for hyperbolic problems, still essentially open for systems, and out of scope of the present paper.*

To this purpose, let $\varphi \in C_c^\infty(\Omega \times [0,T))$; then there exists $\varepsilon_\varphi > 0$ such that $\varphi(x,t) = 0$ if the distance $d(x, \partial\Omega)$ from x to the boundary $\partial\Omega$ of Ω is such that $d(x,\partial\Omega) \leq \varepsilon_\varphi$, and similarly $\varphi(x,t) = 0$ if $|T - t| \leq \varepsilon_\varphi$. For a given mesh $\mathcal{M}^{(m)}$ whose size $h^{(m)}$ is such that $h^{(m)} \leq \varepsilon_\varphi$ and time discretization $(t_n)_{0 \leq n \leq N^{(m)}}$ such that $t^{N^{(m)}} - t^{N^{(m)}} \leq \varepsilon_\varphi$, let us define φ_K^n by

$$\varphi_K^n = \varphi(\boldsymbol{x}_K, t_n), \text{ for } K \in \mathcal{M}^{(m)} \text{ and } 0 \leq n \leq N^{(m)},$$

where $\boldsymbol{x}_K$ stands for a point of K. Let us multiply Equation (5.2) by $(t_{n+1} - t_n)\, \varphi_K^n$ and sum over the cells and time steps, to obtain $T_1^{(m)} + T_2^{(m)} = 0$, with

$$T_1^{(m)} = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} |K|\ (u_K^{n+1} - u_K^n)\ \varphi_K^n,$$

$$T_2^{(m)} = \sum_{n=0}^{N^{(m)}-1} (t_{n+1} - t_n) \sum_{K \in \mathcal{M}^{(m)}} \varphi_K^n \sum_{\sigma = K|L} |\sigma|\ \boldsymbol{F}_\sigma^n \cdot \boldsymbol{n}_{K,\sigma},$$

where the notation $\sum_{\sigma = K|L}$ means that the summation is performed over the internal faces of the cell K, each internal face separating K from an adjacent cell denoted by L. Reordering the sums (this may be seen as a discrete integration by parts with respect to time), the term $T_1^{(m)}$ may be recast as $T_1^{(m)} = \tilde{T}_{1,1}^{(m)} + \tilde{T}_{1,2} + R_1^{(m)}$, with

$$\tilde{T}_{1,1}^{(m)} = -\sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} (t_{n+1} - t_n)\ |K|\ \frac{\varphi_K^{n+1} - \varphi_K^n}{t_{n+1} - t_n}\ u_K^n,$$

$$\tilde{T}_{1,2}^{(m)} = -\sum_{K \in \mathcal{M}^{(m)}} |K|\ u_K^0\ \varphi_K^0,$$

$$R_1^{(m)} = -\sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} (t_{n+1} - t_n)\ |K|\ \frac{\varphi_K^{n+1} - \varphi_K^n}{t_{n+1} - t_n}\ (u_K^{n+1} - u_K^n).$$

Let us denote by $\eth_t^{(m)} \varphi$ the following time discrete derivative of φ:

$$\eth_t^{(m)} \varphi = \sum_{n=0}^{N^{(m)}-1} \sum_{K \in \mathcal{M}^{(m)}} \frac{\varphi_K^{n+1} - \varphi_K^n}{t_{n+1} - t_n}\ \mathcal{X}_K\ \mathcal{X}_{(t_n, t_{n+1}]}.$$

Thanks to the regularity of φ, $\eth_t^{(m)}\varphi$ tends to $\partial_t\varphi$ when m tends to $+\infty$ in $L^\infty(\Omega\times(0,T))$. Since

$$\tilde{T}_{1,1}^{(m)} = -\int_0^T \int_\Omega u^{(m)}\ \eth_t^{(m)}\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

we thus get

$$\lim_{m\to+\infty} \tilde{T}_{1,1}^{(m)} = -\int_0^T \int_\Omega \bar{u}\ \partial_t\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Thanks to Equation (5.3) and the regularity of φ, we also get

$$\lim_{m\to+\infty} \tilde{T}_{1,2}^{(m)} = -\int_\Omega u_0(\boldsymbol{x})\,\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}.$$

Finally,

$$|R_1^{(m)}| \le C_\varphi \sum_{n=0}^{N^{(m)}-1} (t_{n+1}-t_n) \sum_{K\in\mathcal{M}^{(m)}} |K|\ [u_K^{n+1}-u_K]^n$$

with C_φ the Lipschitz-continuity constant of φ, and $R_1^{(m)}$ tends to zero thanks to Lemma 4.3.

Let us now turn to term $T_2^{(m)}$. Reordering the sums (which, now, may be seen as a discrete integration by part with respect to space), we get:

$$T_2^{(m)} = -\sum_{n=0}^{N^{(m)}-1} (t_{n+1}-t_n) \sum_{\sigma\in\mathcal{E}_{\rm int}^{(m)},\,\sigma=K|L} |\sigma|\ \boldsymbol{F}_\sigma^n\cdot\boldsymbol{n}_{K,\sigma}(\varphi_L^n-\varphi_K^n).$$

For $\sigma\in\mathcal{E}_{\rm int}$, $\sigma=K|L$, we must now define a volume D_σ, and this choice may be to some extent tuned according to the mesh at hand (see Remark 3.2). Let us for instance suppose here that $D_\sigma = D_{K,\sigma}\cup D_{L,\sigma}$, with $D_{K,\sigma}=K\cap D_\sigma$ (respectively $D_{L,\sigma}=L\cap D_\sigma$) and $|D_{K,\sigma}|=|K|/\mathcal{N}_K$, where $\mathcal{N}_K$ stands for the number of edges of K (it is easy to check that such a partition exists, noting that $D_{K,\sigma}$ needs not be a polyhedron). With this definition, we may write $T_2^{(m)}=\tilde{T}_2^{(m)}+R^{(m)}$ with

$$\tilde{T}_2^{(m)} = -\sum_{n=0}^{N^{(m)}-1} (t_{n+1}-t_n) \sum_{\sigma\in\mathcal{E}_{\rm int}^{(m)},\,\sigma=K|L} \Big(|D_{K,\sigma}|\ \boldsymbol{F}(u_K^n)+|D_{L,\sigma}|\ \boldsymbol{F}(u_L^n)\Big)\cdot\Big(\frac{|\sigma|}{|D_\sigma|}\ (\varphi_L^n-\varphi_K^n)\,\boldsymbol{n}_{K,\sigma}\Big).$$

We identify

$$\tilde{T}_2^{(m)} = \int_0^T\int_\Omega \boldsymbol{F}(u^{(m)})\cdot\boldsymbol{\nabla}_{\mathcal{E}^{(m)},\mathcal{T}^{(m)}}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

with the definition (3.7) of $\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\varphi$. Let us suppose that (i) the sequence $(\boldsymbol{F}(u^{(m)}))_{m\in\mathbb{N}}$ converges in $L^1(\Omega\times(0,T))$ to $\boldsymbol{F}(\bar{u})$ and (ii) that the sequence of meshes satisfies the regularity assumptions of Lemma 3.2. Then

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T\int_\Omega \boldsymbol{F}(\bar{u})\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Let us now suppose (iii) that there exists a real number C_F depending only on F, such that

$$|\boldsymbol{F}_\sigma^n-\boldsymbol{F}(u_K^n)|\le C_F\ |u_K^n-u_L^n|,\quad |\boldsymbol{F}_\sigma^n-\boldsymbol{F}(u_L^n)|\le C_F\ |u_K^n-u_L^n|. \tag{5.6}$$

Then we get, thanks to the regularity of φ which implies that $\boldsymbol{\nabla}_{\mathcal{E},\mathcal{T}}\varphi$ is bounded in $L^\infty(\Omega\times(0,T))^d$ under the mesh regularity assumptions of Lemma 3.2,

$$|R^{(m)}| \leq C_F\ C_\varphi \sum_{n=0}^{N^{(m)}-1} (t_{n+1}-t_n) \sum_{\sigma\in\mathcal{E}_{\rm int}^{(m)},\,\sigma=K|L} |D_\sigma|\ [u^{(m)}]_\sigma,$$

where $C_\varphi \in \mathbb{R}_+$ depends only on φ, and thus, assuming (iv) that the regularity assumptions for the mesh of Lemma 4.3 are satisfied, $R^{(m)}$ tends to zero as M tends to $+\infty$.

Let us now collect the assumptions used in this computation:

- Assumptions (ii) and (iv) are regularity assumptions for the sequence of meshes. Hypothesis (iv) amounts to suppose that the number of cell faces is bounded and, with the definition of the volumes D_σ chosen in this section, that the ratio $|K|/|L|$, for any pair (K,L) of neighbouring cells, is bounded. The constraints associated to Assumption (ii) are discussed in Remark 3.3 (for instance, we may assume that the cells do not becomes too flat, in the sense of Inequality (3.6)).
- Assumption (i) states the convergence of the sequence $(\boldsymbol{F}(u^{(m)}))_{m\in\mathbb{N}}$ to $\boldsymbol{F}(\bar u)$ in $L^1(\Omega\times(0,T))$. It is just a consequence of the convergence of $(u^{(m)})_{m\in\mathbb{N}}$ to $\bar u$ in $L^1(\Omega\times(0,T))$ if the function $\boldsymbol{F}$ is bounded (thanks to the Lebesgue dominated convergence theorem); in the other cases, it implies stronger convergence assumptions (for instance, with $F(u)=u^2$, convergence in $L^2(\Omega\times(0,T))$ is required).
- Assumption (iii) (*i.e.* Inequalities (5.6)) is a constraint over the numerical flux. For instance, with a two-point flux $\boldsymbol{F}_\sigma = \boldsymbol{F}_\sigma(u_K,u_L)$, it is implied by the usual assumptions:

  $$\boldsymbol{F}_\sigma(u,u) = \boldsymbol{F}(u)\ \forall u\in\mathbb{R},\ \boldsymbol{F}_\sigma \text{ Lipschitz continuous w.r.t. its arguments.}$$

  The Lipschitz continuity assumption may in fact be replied by the following weaker "Lip-diag" condition:

  $$|F(a,b)-F(a,a)| \leq C_F|a-b| \tag{5.7}$$
  $$|F(b,a)-F(a,a)| \leq C_F|a-b|. \tag{5.8}$$

  In the case of a MUSCL scheme for the convection equation by a regular velocity $\boldsymbol{b}$, the discretization of the flux on $\sigma=K|L$ is given by $u_\sigma\,\boldsymbol{b}\cdot\boldsymbol{n}_{K,\sigma}$ with u_σ depending on u_K and u_L but also on another upwind cell; however, if the flux is assumed to be Lip-diag with respect to the value u_K, u_σ is always a convex combination of u_K and u_L and Assumption (iii) holds.

Finally, let us conclude by mentioning a possible generalization of (iii) for multiple point fluxes. For instance, if, on $\sigma=K|L$, $\boldsymbol{F}_\sigma = \boldsymbol{F}_\sigma(u_K,u_L,u_M)$ with M a neighbour of L, we may replace the first inequality of (5.6) by

$$|\boldsymbol{F}_\sigma^n - \boldsymbol{F}(u_K^n)| \leq C_F\ \Big(|u_K^n-u_L^n| + |u_K^n-u_M^n|\Big),$$

and then use $|u_K^n-u_M^n| \leq |u_K^n-u_L^n| + |u_L^n-u_M^n|$. We obtain that $R^{(m)}$ still reads as a summation of jumps across the faces; however, the weight of $[u]_\sigma$ is now more complex, and its control by $|D_\sigma|$ needs a stronger regularity assumption on the mesh. Such a situation is faced, for instance, when using a second order Runge-Kutta scheme for the time discretization instead of the Euler scheme.

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