A NUMERICAL PROCEDURE BASED ON CROSS-ENTROPY METHOD FOR STIFFNESS IDENTIFICATION

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ABSTRACT
The identification of physical parameters in a computational model is a crucial task so that it can be used as a prediction tool. Particularly, in structural dynamics, the identification process often seeks to determine the elastic properties of the system, materialized in the form of a stiffness coefficient. Depending on certain characteristics of the structural system (geometry complexity, vibration regime, material nature, etc.), the optimization problem underlying the identification process becomes nonlinear and nonconvex, and its solution becomes an extremely challenging task, which requires the use of extremely robust numerical methods. In this context, this work deals with the problem of identifying the stiffness coefficient of a torsional spring coupled to an Euler-Bernoulli beam with an inertial element along its axis. A mechanical-mathematical model for the physical system is presented and the underlying natural frequencies are calculated through the solution of an eigenvalue problem. The identification problem is formulated in terms of the minimization of a discrepancy function, which measures the mismatch between natural frequencies calculated by the mathematical model and reference values obtained in laboratory experiments. A numerical strategy based on the Cross-Entropy method, a stochastic metaheuristic used in simulation of rare events and combinatorial optimization, is used to solve the minimization problem. Numerical results demonstrate the robustness and accuracy of the proposed numerical procedure. The method is compared with results from a Genetic Algorithm to appreciate the reduction in number of required function evaluations brought by the cross-entropy strategy.

KEYWORDS: stiffness identification, Euler-Bernoulli beam, inverse problem, cross-entropy method.

1. INTRODUCTION

Elastic boundary conditions are an important subject in beam theory and structural design, where its precise parametric identification remains an ongoing challenge. The correct assessment of the stiffness values that define these supports becomes increasingly computational demanding for complex arrangements. The investigation of different numerical procedures applied to this context and the evaluation of its effectiveness are thus a relevant practical issue, since the system's vibrations are highly influenced by the end supports, especially for the lower natural frequencies [1].

Extensive literature has covered different kinds of methods on the determination of the stiffness coefficients for spring-hinged ends in beam dynamics, with exact solutions for particular cases [2–4] being generalized through approximation methods to multiple scenarios, ranging from deterministic computations with polynomial [5] and trigonometric [6] shape functions to Bayesian strategies for updating and uncertainty quantification [7–9], enhanced by global sensitivity analysis [10], and also treatments over nonlinear updating of the model through non-parametric methods [11, 12]. This work studies the cross-entropy (CE) method on the associated inverse problem scheme for this context, which uses model evaluations in the classic sense but with the main difference that it employs a robust global search for the optimal parameters. The CE method originated considerably recent in rare-event simulation and combinatorial optimization [13, 14], showing promising results and adaptability also to continuous optimization [15].

The vertical bending vibration of an experimental structure is modeled in this work as a slender beam under Euler-Bernoulli theory elastically restrained at its ends, and an inverse problem is formulated to determine the torsional stiffness coefficients that match prescribed natural frequencies to those measured in impact tests of the rotor of Figure 1, which presents an attached disc between the supports. In previous works, this experimental data was used to investigate the efficiency of direct Rayleigh's quotient iterative calculation of the frequencies [16] and of a population-based algorithm aided by a surrogate description [17]. The CE method is appreciated here, observing its accuracy in the calibration and contrasting its efficiency with a genetic algorithm. The eigenfrequencies are accessed through finite element formalism of the approximated solution with polynomial mode shapes.

The rest of this paper is organized as follows. Section 2 defines the theoretical model, the inverse problem and further assumptions of the approximated forward solution. The cross-entropy method is briefly exposed in section 3, where its expected interesting features are commented. The experimental data and numerical results are presented in section 3, where the performance of the CE method is examined and compared to results from a genetic algorithm. Finally, the contributions of this work are emphasized in section 5.

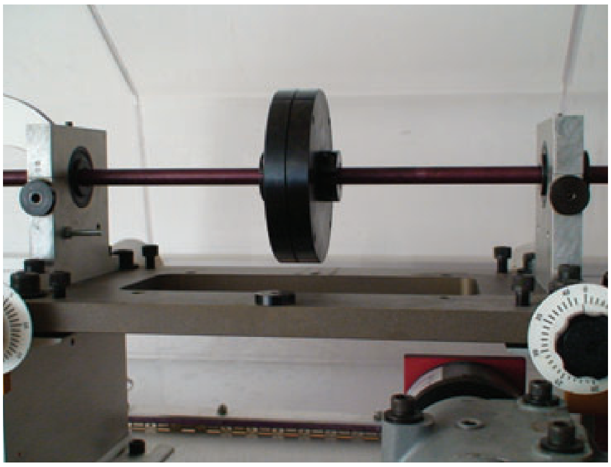

Figure 1 – Experimental structure (*MFS*, SpectraQuest, Inc.).

2. MATHEMATICAL FORMULATION

2.1. Forward Problem

Consider the free vibration transversal dynamics of the Euler-Bernoulli beam, pictographically depicted in Figure 2, which evolves according

$$\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 w}{\partial x^2}\right) = 0\,, \tag{1}$$

where ρ is the mass density of the shaft, A its cross-sectional area, E the Young modulus, I the second moment of area, while $w(x,t)$ denotes the vertical deflection, and x and t are the spatial and time coordinates. This model is supplemented with continuity conditions where the disc of mass m and inertia J is localized (position a over the beam length l), and spring-modeled torsional supports in the ends of the beam geometry, that are defined by their stiffness coefficients k_{t_1} (left) and k_{t_2} (right).

Modal analysis is conducted with aid of a finite element numerical procedure, employed to solve the generalized eigenvalue problem

$$\mathbf{K}\phi = \omega^2 \mathbf{M}\phi \tag{2}$$

where ω is a natural frequency of vibration, that depends on the construction of a corresponding mode shape ϕ. The stiffness matrix $\mathbf{K}$ arises from the elastic properties of Eq. (2) (second term) and is modified by the springed-boundary conditions, while the mass matrix $\mathbf{M}$ comes from the dynamical part (first term) of the partial differential equation and is modified by the continuity conditions brought by the disc. In this work, the eigenvectors are assembled via element-wise interpolation of hermit cubic polynomials [18].

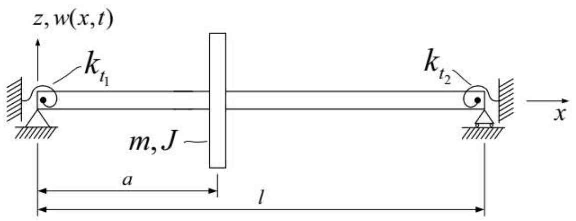

Figure 2 – Euler-Bernoulli beam with torsional spring end supports and rigid disc.

2.2. Inverse Problem

In this setting, the inverse problem underlying the stiffness identification process consist in finding the parameters $\mathbf{x} = (k_{t_1}, k_{t_2})$ that minimize the discrepancy (misfit) function

$$\mathsf{S}(\mathbf{x}) = ||\Omega^{exp} - \Omega(\mathbf{x})||^2 + \lambda\, ||\mathbf{x}||^2 \,, \tag{3}$$

where Ω^{exp} and Ω are the vectors of experimental and predicted (by solving the forward problem) natural frequencies, respectively; and the regularization term, with coefficient λ and vector $\mathbf{x}$ of parameters, is necessary to ensure uniqueness of solution [19].

3. NUMERICAL PROCEDURE

The main idea of the cross-entropy method for this context consists in recasting the optimization of Eq. (3) as a problem of estimating the probability of a rare-event, i.e, given a performance function $\mathsf{S}(\mathbf{X})$ for the random vector of system parameters $\mathbf{X}$ and some level $\gamma \approx \gamma^* = \max S(\mathbf{x}^*)$, then $\mathsf{S}(\mathbf{X}) \geq \gamma$ is a rare event, and the theory for importance sampling and variance minimization can be used [20]. Given some probability density function (PDF) for the system parameters, the algorithm samples them and calculates the performance function, selecting an elite group that provides its highest values. The information is used to update the PDF and minimize the Kullback–Leibler divergence between the PDF and a Dirac delta distribution centered at their optimal value, thus the procedure is repeated until the density is concentrated around the optimal parameters. The steps are summarized in Algorithm 1.

Algorithm 1 Cross-entropy method for optimization

1: Set initial PDF $f(\cdot;\mathbf{v})$ under hyperparameter vector $\mathbf{v}_0$. Set counter $t = 1$.
2: Generate samples of system parameters $\mathbf{X}_1, ..., \mathbf{X}_N \sim f(\cdot;\mathbf{v}_0)$. Compute the performance function $\mathsf{S}(\mathbf{X})$ for each sample and select the elite samples for the set ε_t
3: Solve the stochastic program $\max \sum_{\mathbb{X}_i \in \varepsilon_t} \ln f(\mathbf{X}_i;\mathbf{v})$ to obtain $\mathbf{v}_t$. Increase counter by 1.
4: Repeat from step 2 until some stop criterion is met.
5: The final elite set contains an estimation of optimal system parameters.

Step 3 from Algorithm 1 has extensive literature over analytical formulas for different families of PDFs [21], which in particular are of easy implementation for exponential families. Additionally, an smoothing updating rule is usually employed when calculating $\mathbf{v}_t$, which can be seen in [22] and has relevant impact on convergence. A common stopping criteria, known as normal updating, is to pause the algorithm when the sum of the standard deviations are smaller than some tolerance [20].

The most interesting feature of the CE method is that there is theoretical assurance that the algorithm will find the global optimum, if only one global optimum exists [23]. Besides, it attains easy implementation and has in general very few, intuitive and defined control parameters, namely, sample size, elite set size, tolerance and smoothing parameter, which is not a common feature for other metaheuristic procedures. In the next section, its performance on a computational point of view is verified for the problem.

4. RESULTS

The first and second natural frequencies for the structure in Figure 1 were obtained through impact tests and are presented under ω_1 and ω_2 respectively in the following tables. The experiment's specifications are: $L = 0.362\,\mathrm{m}$, $E = 73\,\mathrm{GPa}$, $I = 1.277 \times 10^{-9}\mathrm{m}^4$, $\rho = 2766\,\mathrm{kg\,m^{-3}}$, $A = 1.2668 \times 10^{-4}\,\mathrm{m}^2$, $m = 3\,\mathrm{kg}$, $J =$

$2.8\times10^{-3}\,\mathrm{kg\,m^2}$. The calculated first and second natural frequency are denoted ω_1 and ω_2 (converted to Hertz), and each pair with the experimental ones were used to minimize Eq. (3) to obtain the optimal k_{t_1} and k_{t_2} presented in Table 1 for different positions a of the disc along the beam. The number of times the eigenvalue problem had to be solved is shown under "*Func. Eval.*". Finally, for each case, a second solution is provided where the additional condition $k_{t_1}=k_{t_2}$ is also enforced.

Table 1 – **CROSS-ENTROPY METHOD**. First and second experimental and model natural frequencies for each position of disc a, with optimal stiffness coefficients and number of function evaluations required. Values in parenthesis are the relative percent error. For each position a, the first set (top) allows $k_{t_1}\neq k_{t_2}$ while the second set (bottom) requires $k_{t_1}=k_{t_2}$.

a (m)	$\frac{\omega_1}{2\pi}\|_{exp}$ (Hz)	$\frac{\omega_2}{2\pi}\|_{exp}$ (Hz)	$\frac{\omega_1}{2\pi}$ (Hz)	$\frac{\omega_2}{2\pi}$ (Hz)	k_{t_1} $(\frac{\mathrm{N\,m}}{\mathrm{rad}})$	k_{t_2} $(\frac{\mathrm{N\,m}}{\mathrm{rad}})$	*Func. Eval.*
$\frac{2}{8}l$	45.454	218.327	45.4538 (4.3E-4 %)	199.5404 (8.6E+0 %)	533.59	317.80	600
			45.4424 (2.6E-2 %)	199.6094 (8.6E+0 %)	464.54		90
$\frac{3}{8}l$	39.764	187.378	39.7664 (5.9E-3 %)	178.0942 (5.0E+0 %)	800.09	617.38	240
			39.7646 (1.5E-3 %)	178.1245 (4.9E+0 %)	727.40		90
$\frac{4}{8}l$	38.907	178.156	38.9090 (5.2E-3 %)	173.6609 (2.5E+0 %)	935.24	935.24	225
			38.9082 (3.1E-3 %)	173.6602 (2.5E+0 %)	935.13		90
$\frac{5}{8}l$	39.833	188.724	39.8291 (9.9E-3 %)	178.1482 (5.6E+0 %)	631.10	802.05	255
			39.8400 (1.7E-2 %)	178.1870 (5.6E+0 %)	735.71		105
$\frac{6}{8}l$	45.918	211.980	45.9203 (5.0E-3 %)	199.8459 (5.7E+0 %)	328.07	574.45	480
			45.9196 (3.5E-3 %)	199.9252 (5.7E+0 %)	496.40		90

The precision of the CE method for the given problem is attested in Table 1 for negligible errors in the fundamental frequency for all cases, showcasing its potential at estimating such quantity. While the errors for the second frequency have a relevant degree, the method was successful at attaining at least the same level of precision encompassed by previous procedures on these data [16, 17], allowing estimation of w_2 up to unity precision. The results also respect geometric expected behavior: same result for $a=4/8\,l$ and close mirrored results for the symmetric cases. Comparison with the results under the additional constraint $k_{t_1}=k_{t_2}$ reveal great gain in efficiency (less function evaluations required) but minor precision variation for ω_1, while providing effectively no change for ω_2. This unexpected behavior contrasts the gain in precision that was observed in the previous works using other methods when allowing the stiffness to be different, and suggests a possible lesser degree of sensitivity in the CE method to variations in ω_2, for which further investigation in future works is required to ascertain the impact of the regularization and stopping criteria in such measure. Figure (3) illustrates the mode shapes obtained in the formulation for $a = 4/8\,l$ up to $5th$, using hermit cubic interpolation.

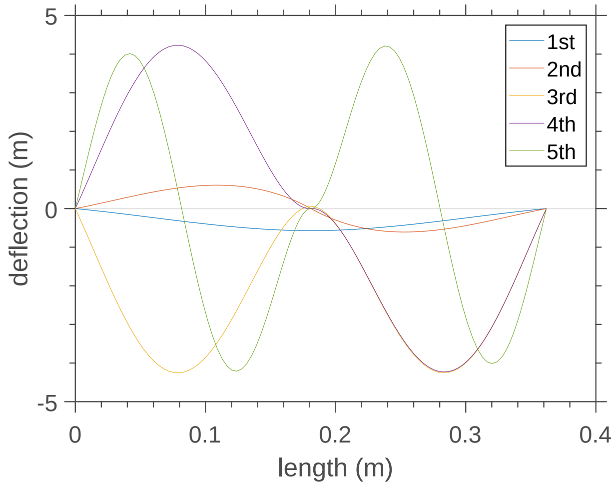

Figure 3 – First to fifth mode shape for the case $a = 4/8\,l$.

Table 2 compiles the calibration solution for each case using the standard Genetic Algorithm (GA) from MATLAB. Population size and elite group of GA were set equal to the sample size and elite set size of CE method in each instance, to allow comparison between the method efficiency. It is clear how the global search method is way faster for obtaining the solutions with increased precision by noticing the difference in thousands from the required number of evaluations. Even so, the few hundred evaluations required in the CE method may be computational costly for more complex problems settings, which advocates research of allowable modifications in the CE method that have been known to increase the rate of convergence, as dynamic smoothing and Gaussian mixture update for example [15, 20], in the specific context of structural design calibration, as well as the application of metamodels to alleviate computational cost, as proposed in the framework of [17].

Table 2 – **GENETIC ALGORITHM**. First and second experimental and model natural frequencies for each position of disc a, with optimal stiffness coefficients and number of function evaluations required. Values in parenthesis are the relative percent error.

a (m)	$\frac{\omega_1}{2\pi}\vert_{exp}$ (Hz)	$\frac{\omega_2}{2\pi}\vert_{exp}$ (Hz)	$\frac{\omega_1}{2\pi}$ (Hz)	$\frac{\omega_2}{2\pi}$ (Hz)	k_{t_1} $(\frac{\mathrm{N\,m}}{\mathrm{rad}})$	k_{t_2} $(\frac{\mathrm{N\,m}}{\mathrm{rad}})$	*Func. Eval.*
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$\frac{4}{8}l$	38.907	178.156	38.9088 (4.7E-3 %)	173.6608 (2.5E+0 %)	935.28	935.15	14 115
$\frac{5}{8}l$	39.833	188.724	39.8814 (1.2E-1 %)	178.1866 (5.6E+0 %)	614.73	823.65	19 200
$\frac{6}{8}l$	45.918	211.980	45.8267 (2.0E-1 %)	199.7872 (5.8E+0 %)	332.39	562.80	35 940

5. CONCLUDING REMARKS

The cross-entropy method is applied on the calibration of a Euler-Bernoulli beam vertical deflection, assessing its efficiency on prediction of torsional stiffness coefficients for the end supports and comparing predicted natural frequencies with values obtained experimentally. The results show remarkable precision for the first frequency and sustainable estimations of the second at low cost of function evaluations, at least when compared with Genetic Algorithm solutions of the inverse problem, which are also presented, suggesting a promising numerical procedure for this context of structural parameters identification that may be enhanced further. Considerations about previous works that used different numerical methods for the same set of data were also commented. In future works, the authors intend to implement modifications in the Cross-entropy method to evaluate the rate of convergence, pair the procedure with surrogate modeling and ascertain the model efficiency under different regularization schemes.

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