Ramsey for complete graphs with a dropped edge or a triangle

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Abstract

Let $K_{[k,t]}$ be the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped (note that $K_{[k,1]}$ is just K_k). In this paper we study $R(K_{[k_1,t_1]},\ldots,K_{[k_r,t_r]})$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i).

We first present a general upper bound (containing the well-known Graham-Rödl upper bound for complete graphs in the particular case when $t_i = 1$ for all i). We then focus our attention when $r = 2$ and dropped cliques of order 2 and 3 (edges and triangles). We give the exact value for $R(K_{[n,2]}, K_{[4,3]})$ and $R(K_{[n,3]}, K_{[4,3]})$ for all $n \geq 2$.

Keywords: Ramsey number, recursive formula.

1 Introduction

Let K_n be a complete graph and let $r \geq 2$ be an integer. A r-edge coloring of a graph is a surjection from $E(G)$ to $\{0,\ldots,r-1\}$ (and thus each color class is not empty). Let $k \geq t \geq 1$ be positive integers. We denote by $K_{[k,t]}$ the complete graph on k vertices from which a set of edges, induced by a clique of order t, has been dropped, see Figure 1.

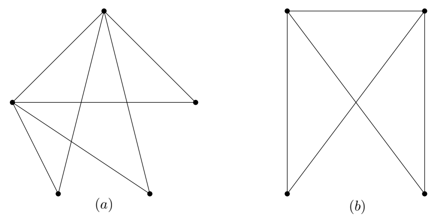

Fig. 1. (a) $K_{[5,3]}$ and (b) $K_{[4,2]}$

Let $k_1,\ldots,k_r$ and $t_1,\ldots,t_r$ be positive integers with $k_i \geq t_i$ for all $i \in \{1,\ldots,r\}$. Let $R([k_1,t_1],\ldots,[k_r,t_r])$ be the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic $K_{[k_i,t_i]}$ for some i. In the case when $k_i = t_i$ for some i, we set

$$R([k_1,t_1],\ldots,[k_{i-1},t_{i-1}],[t_i,t_i],[k_{i+1},t_{i+1}],\ldots,[k_r,t_r]) \leq t_i.$$

We note that equality is reached at $\min\limits_{1\leq i\leq r}\{t_i | t_i = k_i\}$. Since the set of all the edges of $K_{[t_i,t_i]}$ (which is empty) can always be colored with color i. We also notice that the case $R([k_1,1],\ldots,[k_r,1])$ is exactly the classical Ramsey number $r(k_1,\ldots,k_r)$ (the smallest integer n such that for any r-edge coloring of K_n there always occurs a monochromatic K_{k_i} for some i). We refer the reader to the excellent survey [6] on Ramsey numbers for small values. In this paper, we investigate $R([k_1,t_1],\ldots,[k_r,t_r])$.

2 General upper bound

In this section we present a recursive formula (Lemma 2.1) that yields to an explicit general upper bound (Theorem 2.2). The latter contains the well-known explicit general upper bound for $R([k_1,1],\ldots,[k_r,1])$ due to Graham and Rödl [3] (see Equation (4)).

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The following recursive inequality is classical in Ramsey theory

$$(1)\quad r(k_1,k_2,\dots,k_r) \le r(k_1-1,k_2,\dots,k_r)+r(k_1,k_2-1,\dots,k_r)+\cdots+ \\ +r(k_1,k_2,\dots,k_r-1)-(r-2)$$

In the same spirit, we have the following.

Lemma 2.1 *Let $r \ge 2$ and let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers with $k_i \ge t_i+1 \ge 2$ for all i. Then,*

$$\begin{aligned} R([k_1,t_1],\dots,[k_r,t_r]) \le\ & R([k_1-1,t_1],[k_2,t_2],\dots,[k_r,t_r]) \\ & +R([k_1,t_1],[k_2-1,t_2],\dots,[k_r,t_r]) \\ & \vdots \\ & +R([k_1,t_1],[k_2,t_2],\dots,[k_r-1,t_r])-(r-2). \end{aligned}$$

A similar recursive inequality has been treated in [7] in a much more general setting in which a family of graphs are intrinsically constructed via two operations *disjoin unions* and *joins* (see also [4] for the case $r=2$). However, it is not clear how the latter could be used to obtain Lemma 2.1 that allows us to give the following general upper bound for $R([k_1,t_1],\dots,[k_r,t_r])$ (which was not considered in [7]).

Theorem 2.2 *Let $r \ge 2$ be a positive integer and let $k_1,\dots,k_r$ and $t_1,\dots,t_r$ be positive integers such that $k_i \ge t_i$ for all $i \in \{1,\dots,r\}$. Then,*

$$R\left([k_1,t_1],\dots,[k_r,t_r]\right) \le \max_{1\le i\le r}\{t_i\}\binom{k_1+\cdots+k_r-(t_1+\cdots+t_r)}{k_1-t_1,k_2-t_2,\dots\dots,k_r-t_r}$$

where $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\dots\dots,n_r}$ is the multinomial coefficient defined by $\binom{n_1+n_2+\cdots+n_r}{n_1,n_2,\dots\dots,n_r} = \frac{(n_1+\cdots+n_r)!}{n_1!n_2!\cdots n_r!}$, for all nonnegative integers $n_1,\dots,n_r$.

Theorem 2.2 is a natural generalization of the well-known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of Theorem 2.2 (by taking $t_i=1$ for all i) is the following classical upper bound due to Graham and Rödl [3, (2.48)]

$$(2)\qquad R\left([k_1,1],\dots,[k_r,1]\right) \le \binom{k_1+\cdots+k_r-r}{k_1-1,\dots,k_r-1}.$$

Let $k \ge t \ge 2$ and $r \ge 2$ be integers and let $R_r([k,t]) = R(\underbrace{[k,t],\dots,[k,t]}_{r})$.

An immediate consequence of Theorem 2.2 (by taking $k=k_1=\cdots=k_n$ and

$t = t_1 = \cdots = t_n$) is the following inequality

$$R_r([k,t]) \leq t\binom{r(k-t)}{k-t,\ldots,k-t} \tag{3}$$

Moreover, if $t = 1$ then

$$R_r([k,1]) \leq \frac{(rk-r)!}{((k-1)!)^r}. \tag{4}$$

3 Exact values

By the so-called Chvátal's result [2], we know that the exact value of the Ramsey number of $K_{[4,3]}$ (a star) versus cliques is given by $R([n,1],[4,3]) = 3n-2$ for all $n \geq 1$. We then naturally focus our attention to the Ramsey number of $K_{[4,3]}$ versus cliques with either a dropped edge or a dropped triangle, see [1] where $R([m,1],[n,2])$ has been computed for numerous cases. We provide the new following exact values of Ramsey numbers.

Theorem 3.1 *Let $n \geq 2$ be an integer. Then,*

- *$R([n,2],[4,3]) = 2$ for $n = 2$,*
- *$R([n,2],[4,3]) = 5$ for $n = 3$,*
- *$R([n,2],[4,3]) = 3n-5$ for $n \geq 4$.*

Theorem 3.2 *Let $n \geq 2$ be an integer. Then,*

- *$R([n,3],[4,3]) = 3$ for $n = 3$,*
- *$R([n,3],[4,3]) = 6$ for $n = 4$,*
- *$R([n,3],[4,3]) = 8$ for $n = 5$,*
- *$R([n,3],[4,3]) = 11$ for $n = 6$,*
- *$R([n,3],[4,3]) = 3n-8$ for $n \geq 7$.*

3.1 An estimation for $R([n,2],[5,3])$

By considering $K_{[5,3]}$ as the book graph B_3, it was proved in [5,8] that

$$R([n,1],[5,3]) \leq \frac{3n^2}{\log(n/e)},$$

for all positive integers n.

The following result is a first estimation for the value $R([n,2],[5,3])$.

Theorem 3.3 *Let $n \geq 2$ be an integer. Then,*

- $R([n,2],[5,3]) = 2$ *for* $n = 2$,
- $R([n,2],[5,3]) = 7$ *for* $n = 3$,
- $R([n,2],[5,3]) \leq 3\binom{n+1}{2} - 5n + 4$ *for* $n \geq 4$.

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