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Geometry of stable ruled surface over an elliptic curve

Arame Diaw[‡]

March 1, 2019

Abstract

We consider the stable ruled surface S_1 over an elliptic curve. There is a unique foliation on S_1 transverse to the fibration. The minimal self-intersection sections also define a 2-web. We prove that the 4-web defined by the fibration, the foliation and the 2-web is locally parallelizable.

Keywords: elliptic curve, ruled surface, Riccati foliation and singular web.

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1 Introduction

Let C be an elliptic curve on \mathbb{C} . In 1955, Atiyah proved in [2] that, up to isomorphism, there are only two indecomposable ruled surfaces over C : the semi-stable ruled surface $S_0 \mapsto C$ and the stable ruled surface $S_1 \mapsto C$. In this article, we study the geometry of the stable ruled surface. In fact, the surface S_1 can be seen as the suspension over C of the unique representation onto the dihedral group $\langle -z, \frac{1}{z} \rangle$ (see [6], page 23). Thus, we have a Riccati foliation Ric on S_1 such that the generic leaf is a cover of degree 4 over C and it is the unique foliation transverse to the fibration. On the other hand, the holomorphic section $\sigma: C \mapsto S_1$ have self-intersection $\sigma.\sigma \geq 1$ and those having exactly $\sigma.\sigma = 1$ form a singular holomorphic 2-web \mathcal{W} . Finally, taking into account the fibration, we have a singular holomorphic 4-web on S_1 . The aim of this article is to study the geometry of this 4-web composed by the Riccati foliation, the 2-web \mathcal{W} and the \mathbb{P}^1 -fibration $\pi: S_1 \mapsto C$.

Our first result is the following :

Proposition 1.1. *The discriminant Δ of the 2-web \mathcal{W} defined by the +1 self-intersection sections on S_1 is a leaf of the foliation Ric .*

Using the isomorphism between the curve C and its jacobian, we have the main result :

Theorem 1.2. *There exists a double cover $\varphi: C \times C \mapsto S_1$ ramified on Δ on which the lifted 4-web \mathcal{W} is parallelizable.*

This 4-web is locally comprised of pairwise parallel straight lines: its curvature is zero.

Firstly, we show these results using only the properties of an elliptic curve and its jacobian and after, we use the theory of birational geometry to illustrate our results with computations on a trivialization $S_1 \dashrightarrow C \times \mathbb{P}^1$.

This paper is part of my thesis work under the direction of Frank Loray and Frédéric Touzet.

2 Preliminaries

2.1 Some properties on an elliptic curve

Let $C = \{(x, y) \in \mathbb{C}^2, y^2 = x(x-1)(x-t)\} \cup \{p_\infty\}$, where $t \in \mathbb{C} \setminus \{0, 1\}$ be an elliptic curve. Throughout this article, we use the following background of an elliptic curve.

Proposition 2.1. *The set of points of C forms an abelian group, with p_∞ as the 0 element and with addition characterized for any couple of points $p = (x_1, y_1)$, $q = (x_2, y_2)$ in C by:*

1. $-p = (x_1, -y_1)$;
2. if $p \neq q, -q$, then $p + q = (x_3, y_3)$ where $x_3 = \lambda^2 + (1+t) - x_1 - x_2$, $y_3 = \lambda(x_1 - x_3) - y_1$ and $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$;
3. if $p = q$ and $y_1 \neq 0$, then $2p = (\tilde{x}, \tilde{y})$ where $\tilde{x} = \lambda^2 - (1+t) - 2x_1$, $\tilde{y} = \lambda x_1 - \tilde{x} - y_1$ and $\lambda = \frac{3x_1^2 - 2(1+t)x_1 + t}{2y_1}$;
4. if $p = q$ and $y_1 = 0$ then $2p = p_\infty$.

Remark 2.2. *The points $p_i = (i, 0)$, where $i = 0, 1, t$ are the points of order 2 on C and the map*

$$\begin{cases} I: & C & \mapsto & C \\ & (x, y) & \mapsto & (x, -y) \end{cases}$$

is an automorphism of C which fixes the points of order 2: it is the standard involution of the curve C .

If we denote $Jac(C)$, the jacobian of C , we have:

Lemma 2.3. *There exists a bijection between C and its jacobian defined by this following map:*

$$\begin{cases} C & \mapsto & Jac(C) \\ p & \mapsto & [p] - [p_\infty] \end{cases}$$

From now on, we will use the isomorphism between the additive group structure (C, p_∞) and the group structure on C induced by its jacobian.

2.2 Ruled surface over an elliptic curve

Let C be a smooth curve on \mathbb{C} .

Definition 2.4. A ruled surface over C is a holomorphic map of two dimensional complex variety S onto C $\pi: S \rightarrow C$ which makes S a \mathbb{P}^1 -fibration over C .

Example 2.5. The fiber bundle associated to a vector bundle of rank 2 over C is a ruled surface. We denote it, $\mathbb{P}(E)$.

Conversely, we have the following theorem proved by Tsen in [5] :

Theorem 2.6. Let $\pi: S \rightarrow C$ be a ruled surface over C :

1. there exists a vector bundle E of rank 2 over C such that $S = \mathbb{P}(E)$;
2. there exists a section, i.e a map $\sigma: C \rightarrow S$ such that $\pi \circ \sigma = id$;
3. $\mathbb{P}(E) \cong \mathbb{P}(E')$ if and only if there is a holomorphic line bundle L over C such that $E \cong E' \otimes L$.

Definition 2.7. A ruled surface $\mathbb{P}(E)$ is decomposable if it has two disjoint sections.

The following lemma whose proof is in ([7], page 16) shows the relationship between the ruled surface $S = \mathbb{P}(E)$ and the vector bundle E .

Lemma 2.8. There exists a one-to-one correspondance between the line subbundles of E and the sections of S . Futhermore, if σ_L is the section related to the line subbundle L then:

$$\sigma_L \cdot \sigma_L = \deg E - 2 \deg L$$

where $\deg(E)$ is the degree of the determinant bundle of E .

Notation 2.9. We recall that the notation $\sigma_L \cdot \sigma_L$ means the self-intersection of the section σ_L .

Remark 2.10. By the lemma 2.8, $\mathbb{P}(E)$ is decomposable if and only if E is decomposable, i.e $E = L_1 \oplus L_2$ for line subbundles $L_i \subset E$.

Consider $\kappa = \min \{ \sigma \cdot \sigma, \sigma: C \rightarrow S / \pi \circ \sigma = id \}$. This number only depends on the ruled surface $S = \mathbb{P}(E)$. Indeed, it does not change when we replace E by $E \otimes L$ for a line bundle L on C .

Definition 2.11. The ruled surface $\mathbb{P}(E)$ is stable if $\kappa > 0$.

Definition 2.12. *A minimal section of S is a section $\sigma: C \rightarrow S$ such that, the self-intersection is minimal. That is to say, $\sigma.\sigma = \kappa$.*

Using lemma 2.8, we notice that a minimal section corresponds to a line subbundle of E with maximal degree. Thus, the invariant κ can be written as:

$$\kappa = \max \{ \deg(E) - 2 \deg(L), L \hookrightarrow E_1 \}$$

Now, we are interested in indecomposable ruled surfaces over an elliptic curve. Let $\mathcal{O}_C(p_\infty)$ be the line bundle related to the divisor $[p_\infty]$. There are unique nontrivial extensions of invertible sheaves:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_0 \longrightarrow \mathcal{O}_C \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_1 \longrightarrow \mathcal{O}_C(p_\infty) \longrightarrow 0$$

Recall the following Atiyah's theorem as proved in ([1], Th.6.1):

Theorem 2.13. *Up to isomorphism, the unique indecomposable ruled surfaces over C are $S_0 = \mathbb{P}(E_0)$ and $S_1 = \mathbb{P}(E_1)$.*

Remark 2.14. *Equivalency, any indecomposable vector bundle E of rank 2 on C takes the form $E = E_i \otimes L$, for $i = 0, 1$ and L a line bundle.*

As our aim in this paper is the study of the ruled surface S_1 , we will show firstly some important properties of E_1 .

Lemma 2.15. *The degree of the maximal line subbundles of E_1 is zero.*

Proof. Let L be a subbundle of E_1 and consider the quotient $M := E_1/L$. By the following exact sequence $0 \longrightarrow L \longrightarrow E_1 \longrightarrow M \longrightarrow 0$ and the fact that E_1 is indecomposable, we have $H^1(M^{-1} \otimes L) \neq 0$. Thus, due to Serre's duality, we have $2 \deg(L) \leq \deg(E_1)$ and then $\deg(L) \leq 0$ because $\deg(E_1) = 1$. Since the trivial line bundle \mathcal{O}_C is a line subbundle over E_1 , we have the result. \square

Remark 2.16. *By this lemma, we can deduce that the ruled surface S_1 is stable. More precisely, up to isomorphism, it is the unique stable ruled surface over an elliptic curve.*

If we consider $\max_{E_1} = \{L \hookrightarrow E_1, \deg L = 0\}$ the set of line subbundles of E_1 having a maximal degree, we have:

Lemma 2.17. *The jacobian of C sets the parameters of the set \max_{E_1} . More precisely, the map*

$$\left\{ \begin{array}{l} M: \max_{E_1} \longrightarrow \text{Jac}(C) \\ L \longmapsto [L] \end{array} \right.$$

is a bijection.

To prove this lemma, we have to use a key lemma of Maruyama in ([7], page 8):

Lemma 2.18. *Let E be a vector bundle of rank 2 over a curve. If L_1 and L_2 are distinct maximal line subbundles of E such that L_1 and L_2 are isomorphic, then $E = L_1 \oplus L_1$.*

Now, we can prove lemma 2.17:

Proof. • Let L_1 and L_2 be two elements in \max_{E_1} such that $L_1 \cong L_2$. We have two possibilities, either $L_1 = L_2$ or they are both distinct. According to the lemma 2.18, the last case cannot occur because E_1 is not decomposable. Thus, the map M is injective.

- Let $L \in \text{Jac}(C)$ be distinct from the trivial line bundle. If we apply the functor $\text{Hom}(L, -)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{f} E_1 \xrightarrow{g} \mathcal{O}_C(p_\infty) \longrightarrow 0$$

and we use Riemann Roch's theorem, we obtain $\dim \text{Hom}(L, E_1) = 1$. There exists a non zero morphism $\tau: L \mapsto E_1$. Thus, if we denote D the effective divisor of zeros of τ , then $L \otimes \mathcal{O}_C(D)$ is a line subbundle of E_1 . Since $\deg(L) = 0$, D is a effective divisor of zero degree, that is to say $\mathcal{O}_C(D) = \mathcal{O}_C$. Hence, L is a line subbundle of E_1 . □

Remark 2.19. *The minimal sections of S_1 have self-intersection equal to 1 and they are parametrised by the jacobian which is isomorphic to C . Hence for every point $\epsilon \in C$, we denote σ_ϵ the minimal section corresponding via lemma 2.17 to the subbundle isomorphic to $\mathcal{O}_C([\epsilon] - [p_\infty])$.*

Lemma 2.20. *Let σ_ϵ and $\sigma_{\epsilon'}$ be two minimal sections of S_1 . If we consider D their intersection divisor, we have:*

$$\pi(D) = [-\epsilon - \epsilon']$$

where $\pi: S_1 \mapsto C$ is the projection map.

Proof. Intuitively, the divisor D is defined by the points above at which the line bundles related to σ_ϵ and $\sigma_{\epsilon'}$ coincide. More precisely, $\pi(D)$ is an effective divisor equivalent to divisor $\det(E_1) \otimes \mathcal{O}_C([p_\infty] - [\epsilon]) \otimes \mathcal{O}_C([p_\infty] - [\epsilon'])$ which itself is equivalent to divisor $\mathcal{O}_C([-\epsilon - \epsilon'])$. Since the degree of D is equal to 1, we obtain the result. \square

Remark 2.21. Let Q be a point of S_1 belonging to the fiber $\pi^{-1}(p)$. If the minimal section σ_ϵ passes through the point Q , then the unique other minimal section passing through the same point Q is the section $\sigma_{-p-\epsilon}$. They might be the same for some Q .

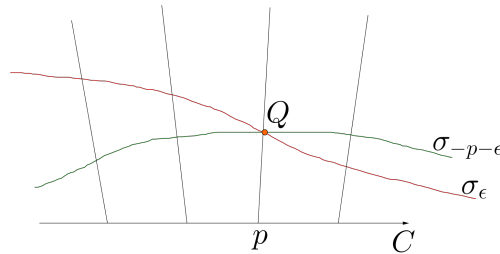


Figure 1: intersection of two minimal sections on S_1

We also have the following theorem proved by André Weil in [11] :

Theorem 2.22. *A holomorphic vector bundle on a compact Riemann surface is flat if and only if it is the direct sum of indecomposable vector bundles of degree 0.*

By this theorem, the Atiyah's bundle E_1 is not flat because $\deg E_1 = 1$. However what can we say about its associated ruled surface ? The answer of this question is given by Frank Loray and David Marin in [6].

Consider C as a torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and let $\rho: \mathbb{Z} + \tau\mathbb{Z} \rightarrow \mathbb{P}SL_2(\mathbb{C})$ be the representation of the fundamental group of C defined by $\rho(1) = -z$ and $\rho(\tau) = \frac{1}{z}$. Up to conjugacy in $\mathbb{P}SL_2(\mathbb{C})$, it is the unique representation onto the 4-order group $\Gamma = \langle -z, \frac{1}{z} \rangle$.

The orbits of the elements of $\mathbb{C} \times \mathbb{P}^1$ modulo the action given by the representation and the universal cover of C form a ruled surface over C , denoted by \overline{E} . It is obvious to see, the horizontal foliation of $\mathbb{C} \times \mathbb{P}^1$ lifts to a regular foliation \mathcal{R} transverse to the fibration of \overline{E} . The foliated surface $(\overline{E}, \mathcal{R})$ is

called the suspension of C . Let $\mathcal{R}_{[x_0, z_0]}$ be a leaf passing through a the point $[x_0, z_0]$ of \overline{E} . Then, by definition, we have a isomorphism between $\mathcal{R}_{[x_0, z_0]}$ and the quotient \mathbb{C}/G where $G = \{\alpha \in \mathbb{Z} + \tau\mathbb{Z} / \varrho_\alpha(z_0) = z_0\}$. Thus, we can deduce that every leaf of foliation \mathcal{R} is a cover of C . It is not difficult to show that the intersection of any fiber of \overline{E} and the leaf $\mathcal{R}_{[x_0, z_0]}$ is given by the set $\{\varrho_\alpha(z_0), \alpha \in \mathbb{Z} + \tau\mathbb{Z}\}$. Using the finitude of the representation, we have every leaf of the foliation is a cover of finite degree. Moreover, the monodromy of the foliation \mathcal{R} on any fiber is the representation ϱ .

Note that, the action of $\langle -z, \frac{1}{z} \rangle$ in \mathbb{P}^1 gives two kind of orbits: orbits of order 4 and three special orbits of order 2 given by $(-1, 1)$, $(-i, i)$ and $(0, \infty)$. Therefore, the foliation \mathcal{R} has a generic leaf which is a cover of order 4 of C and three special leaves, which is a cover of order 2.

Proposition 2.23. *The ruled surface \overline{E} over C is indecomposable such that its invariant $\kappa = 1$. It is the ruled surface S_1 .*

Proof. If \overline{E} is a decomposable ruled surface then its invariant $\kappa = 0$. Indeed, let F be a leaf of \mathcal{R} and σ_0 a minimal section of \overline{E} , then we have: $F \equiv 4\sigma_0 + bf$ or $F \equiv 2\sigma_0 + b'f$, where f represents a fiber. Using the fact that \overline{E} is decomposable, we can find a section σ such that $\sigma \cdot \sigma_0 = 0$. Since $F \cdot \sigma \geq 0$ and $F^2 = 0$, we obtain that $\sigma_0 \cdot \sigma_0 = 0$.

Eventually, if we assume that \overline{E} is a decomposable ruled surface, we have $F \equiv 4\sigma_0$ or $F \equiv 2\sigma_0$ and then $F \cdot \sigma = 0$. The section σ does not meet any leaf of \mathcal{R} , which does not make a sense because the foliation is regular.

The ruled surface \overline{E} is then indecomposable. Hence, it is either isomorphic to S_0 or S_1 .

By the same arguments above, \overline{E} is not isomorphic to S_0 , if not there would be a section which not intersects any leaf of \mathcal{R} . Thus, the ruled surface \overline{E} is isomorphic to S_1 by Atiyah's theorem.

As up to conjugacy the representation ϱ is the unique representation onto the 4-order group $\Gamma = \langle -z, \frac{1}{z} \rangle$, we deduce that the isomorphism between \overline{E} and S_1 is the identity. \square

In summary, we have:

Theorem 2.24. *The ruled surface S_1 has a Riccati foliation Ric with irreducible monodromy group $\langle -z, \frac{1}{z} \rangle$.*

Remark 2.25. *The foliation Ric has a generic leaf which is cover of degree 4 over C and three special leaves which are covers of degree 2 over C .*

3 Geometry of the ruled surface S_1

Let $\pi: S_1 \mapsto C$ be the stable ruled surface over C .

Proposition 3.1. *The automorphism group of S_1 is a group of order 4 which is isomorphic to the 2-torsion group in C .*

Proof. Let $\psi: S_1 \mapsto S_1$ be a non trivial automorphism of S_1 . Since the self-intersection is invariant by automorphism ψ preserves the set of +1 self-intersection sections on S_1 . More precisely, for any $\epsilon \in C$, there exists a unique point $r_\epsilon \in C$ such that $\psi(\sigma_\epsilon) = \sigma_{r_\epsilon}$. The automorphism ψ induces an automorphism $\bar{\psi}$ of C such that for any point $\epsilon \in C$ we have $\bar{\psi}(\epsilon) = r_\epsilon$. If we define C as the complex torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, we can write for any $z \in \mathbb{C}$, $\bar{\psi}(\bar{z}) = a\bar{z} + b$, where $a, b \in \mathbb{C}$ and $a(\mathbb{Z} + \tau\mathbb{Z}) = \mathbb{Z} + \tau\mathbb{Z}$.

If we assume this automorphism has a fix point ϵ_0 , then by definition we have $\psi(\sigma_{\epsilon_0}) = \sigma_{\epsilon_0}$. Hence, using the lemma 2.20, we obtain that for any $p \in C$, $\psi(\sigma_{-p-\epsilon_0}) = \sigma_{-p-\epsilon_0}$. For any fiber, the automorphism ψ is Moebius map which fixes at least three points: it is the trivial automorphism, which does not make sense by hypothesis.

Therefore, the automorphism $\bar{\psi}$ has no fixed points, it is a translation like $\bar{\psi}(\bar{z}) = \bar{z} + b$. As by definition we have: $\bar{\psi}(-p - \bar{z}) = -p - \bar{\psi}(\bar{z})$, the point b is a point of order 2 of C .

Conversely, for any point p_i of order 2 on C , we can define an automorphism Φ_i , on S_1 such that for any point $p \in C$, Φ_i restricts to the fiber $\pi^{-1}(p)$ is the unique Moebius map which associates the points of the sections $(\sigma_{p_\infty}, \sigma_{p_0}, \sigma_{p_1}, \sigma_{p_t})$ to the points of the sections $(\sigma_{p_\infty+p_i}, \sigma_{p_0+p_i}, \sigma_{p_1+p_i}, \sigma_{p_t+p_i})$ respectively. It is defined by:

$$\left\{ \begin{array}{l} \Phi_i: \quad S_1 \quad \longrightarrow \quad S_1 \\ \quad \quad z = \sigma_\omega(p) \quad \longmapsto \quad z' = \sigma_{\omega+p_i}(p) \end{array} \right.$$

There exists a one-to-one correspondance between the automorphisms of the fiber bundle S_1 and the points of order 2 in C which preserves the group structure. Hence we have :

$$Aut_C(S_1) = \{\Phi_0, \Phi_1, \Phi_t, \Phi_\infty = Id\}$$

□

Proposition 3.2. *The automorphism group of S_1 preserves the foliation Ric.*

Proof. Using the fact that the fundamental group of C is abelian, we can extend the monodromy map over every fiber and regard it as automorphism on S_1 which fixes the basis C . Thus, we obtain that the monodromy group of the foliation Ric is a subgroup of order 4 of $Aut_C(S_1)$: they are isomorphic. The group $Aut_C(S_1)$ preserves the Riccati foliation on S_1 . □

Corollary 3.3. *The Riccati foliation Ric is the unique Riccati foliation on the ruled surface S_1 .*

Proof. Let \mathcal{F}_1 be a Riccati foliation on S_1 . As its monodromy group is an abelian subgroup of $PGL(\mathbb{C}, 2)$, we have three possibilities for its monodromy representation :

- If the conjugacy class of the monodromy is the linear class defined by the group $\langle az, bz \rangle$, there exists two disjoint invariant sections of S_1 . Hence S_1 is a ruled surface related to the direct sum of two line bundles over C . It does not make sense because S_1 is indecomposable.
- If the conjugacy class of the monodromy is the euclidian class defined by the group $\langle z + 1, z + s \rangle$, there exists an invariant section on S_1 with zero self-intersection. In fact by the Camacho Sad's theorem (in [3]), any invariant curve of regular foliation has a zero self-intersection. This monodromy representation does not make sense in S_1 because we have $\min \{ \sigma.\sigma, \sigma: C \mapsto S_1 / \pi \circ \sigma = id \} = 1$.

The only remaining possibility is that the monodromy has image the group $\langle -z, \frac{1}{z} \rangle$. Thus, the foliation \mathcal{F}_1 is conjugated to Ric by an element in $Aut_C(\tilde{S}_1)$. As this automorphism group of the fibration S_1 preserves the foliation Ric, we have Ric = \mathcal{F}_1 . □

Lemma 3.4. *There exists a ramified double cover of the ruled surface S_1 defined by the map :*

$$\begin{cases} \varphi: & C \times Jac(C) & \longrightarrow & S_1 \\ & (p, \epsilon) & \longmapsto & z = \sigma_\epsilon(p) \end{cases}$$

such that its involution is defined by :

$$\begin{cases} i: & C \times Jac(C) & \longrightarrow & C \times Jac(C) \\ & (p, \epsilon) & \longmapsto & (p, -p - \epsilon) \end{cases}$$

Proof. According to the lemma 2.20, three minimal sections cannot meet at the same point, then we deduce for any $p \in C$, the morphism

$$\begin{cases} \varphi_p: & Jac(C) & \longrightarrow & \pi^{-1}(p) \\ & \epsilon & \longmapsto & \sigma_\epsilon(p) \end{cases}$$

is not constant: it is a ramified cover between Riemann surfaces. Furthermore, by the remark 2.21, we know that at most two minimal sections can pass through a given point, then the map φ is a ramified double cover. □

The immediate consequence of this lemma is the following :

Theorem 3.5. *There exists a singular holomorphic 2-web \mathcal{W} on S_1 defined by the minimal sections whose discriminant Δ is a leaf of the foliation Ric.*

Proof. By lemma 3.4, for any point $P \in \pi^{-1}(p)$ there exists a minimal section σ_r passing through this point. Likewise, by lemma 2.20 the minimal section σ_{-p-r} intersects transversally σ_r at the point P . As the sections σ_r and σ_{-p-r} are distinct if and only if $2r \neq -p$, we deduce that there exists a singular holomorphic 2-web on S_1 such that its discriminant is defined by :

$$\Delta = \cup_{p \in C} \{P \in \pi^{-1}(p) / P \in \sigma_r, 2r = -p\}$$

In order to prove that Δ is a leaf of the Riccati, we need the following :

Lemma 3.6. *There exists a linear foliation \mathcal{F} on $C \times Jac(C)$ such that $\varphi_*\mathcal{F} = Ric$.*

Proof. Assume that $C \times Jac(C) \simeq (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}) \times (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z})$, and let be (x, y) its local coordinates . If we consider the linear foliation $\tilde{\mathcal{F}} := dx + 2dy$ on $\mathbb{C}_x \times \mathbb{C}_y$, then $\tilde{\mathcal{F}}$ is invariant by the action of the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Thus we can lift the foliation $\tilde{\mathcal{F}}$ to a foliation, \mathcal{F} on $C \times Jac(C)$ such that the monodromy is defined by :

$$\left\{ \begin{array}{l} \xi: \Lambda \longrightarrow Aut(C) \\ \lambda \longrightarrow z \mapsto z - \frac{1}{2}\lambda \end{array} \right.$$

where Λ is the lattice $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

The foliation \mathcal{F} is transverse to the first projection on $C \times Jac(C)$ with a monodromy group isomorphic to the group of points of order 2 $\{p_\infty, p_0, p_1, p_t\}$. Moreover, if $\mathcal{F}_{(p,\omega)}$ is the leaf passing through the point (p, ω) , then by definition we have :

$$i(\mathcal{F}_{(p,\omega)}) = \mathcal{F}_{(p,-p-\omega)}$$

where i is the involution of the ramified double cover φ . Hence, $\varphi_*\mathcal{F}$ the direct image of the foliation \mathcal{F} by φ is a Riccati foliation on S_1 having the same monodromy group than Ric. Using the uniqueness of Ric by the corollary 3.3, we obtain the result. \square

As by definition the curve $G = \{(2p, -p) / p \in C\}$ is a leaf of the foliation \mathcal{F} , using the foregoing lemma we can deduce that $\varphi(G) = \Delta$ is a leaf of Ric. Which completes the proof of theorem 3.5. \square

3.1 Study of special leaves of the Riccati foliation Ric

According to lemma 3.6, if $P = \sigma_\omega(p) \in S_1$ then the Riccati leaf passing through at this point is given by

$$Ric_P = \{(q, z) / z = \sigma_{\omega'}(q), 2\omega' = 2\omega + p - q\}$$

Thus, if we use this characterisation of the Riccati leaves on S_1 , we have the following lemma :

Lemma 3.7. *There exists three special leaves Ric_0 , Ric_1 and Ric_t of the foliation Ric which are double cover of C . More precisely, they are respectively the set of fixed points of the automorphisms Φ_0 , Φ_1 and Φ_t .*

Proof. We just give the proof for the leaf Ric_0 because it is the same process for the other special leaves.

- Let Ric_0 be the Riccati leaf passing through the point $z_0 = \sigma_{p_0}(p_0)$, then by definition, we have :

$$Ric_0 = \{(p, z) / z = \sigma_\omega(p), 2\omega = p_0 - p\}$$

According to the monodromy of Ric, if the leaf Ric_0 passes through the point $z = \sigma_\omega(p)$ then it passes through the points $\sigma_{\omega+p_i}(p)$, where p_i is a point of C of order 2. Since $2\omega = p_0 - p$, we deduce from lemma 2.20 that: $\sigma_{\omega+p_0}(p) = \sigma_\omega(p)$ and $\sigma_{\omega+p_1}(p) = \sigma_{\omega+p_1}(p)$ thus, Ric_0 meets any fiber of S_1 twice. It is a double cover over C .

- Let Φ_0 be the automorphism of S_1 related to the point p_0 defined by:

$$\left\{ \begin{array}{ccc} \Phi_0: & S_1 & \mapsto & S_1 \\ & z = \sigma_\omega(p) & \longrightarrow & z' = \sigma_{\omega+p_0}(p) \end{array} \right.$$

and consider the set of its fix points

$$Fix_0 = \{(p, z) / \Phi_0(p, z) = (p, z)\}$$

If $z = \sigma_\omega(p)$ is the fix point of Φ_0 , then by lemma 2.20, we have $2\omega = p_0 - p$, and therefore $z \in Ric_0$. Conversely, if $z \in Ric_0$, we have $2\omega = p_0 - p$, and according to lemma 2.20, we have, $\sigma_{\omega+p_0}(p) = \sigma_\omega(p)$. Thus, we deduce that:

$$Ric_0 = \{(p, z) / \Phi_0(p, z) = (p, z)\}$$

□

According to all the foregoing, we have:

Remark 3.8. *The 2-web given by the +1 self-intersection sections, the Riccati foliation and the \mathbb{P}^1 -bundle $\pi: S_1 \mapsto C$ form a singular holomorphic 4-web \mathbf{W} on S_1 .*

3.2 The geometry of the 4-web \mathbf{W}

Let (x, y) be the local coordinates of \mathbb{C}^2 . As the linear foliations \mathcal{G} and \mathcal{H} respectively defined by $dy = 0$ and by $dy + dx = 0$ are invariant by the action of the lattice $\mathbb{Z} + \tau\mathbb{Z}$, we can lift them to a decomposable 2-web \mathcal{W}' on $C \times \text{Jac}(C)$.

Proposition 3.9. *The direct image $\varphi_*(\mathcal{W}')$ of the 2-web \mathcal{W}' by the ramified cover φ is the 2-web \mathcal{W} on S_1 defined by the minimal sections.*

Proof. As by definition the 2-web \mathcal{W}' is invariant by the involution of the ramified double cover φ , its direct image is also a singular holomorphic 2-web on S_1 . Let $(p', \epsilon') \in C \times \text{Jac}(C)$ and consider

- $A_{\epsilon'} = \{(p, \epsilon) \in C \times \text{Jac}(C) / \epsilon = \epsilon'\}$
- $B_{\epsilon'} = \{(p, \epsilon) \in C \times \text{Jac}(C) / \epsilon = -p + (p' + \epsilon')\}$,

the leaves of the 2-web \mathcal{W}' passing through this point. Since, using lemma 2.20, we have: $\varphi(A_{\epsilon'}) = \sigma_{\epsilon'}$ and $\varphi(B_{\epsilon'}) = \sigma_{-p'-\epsilon'}$, then the leaves of $\varphi_*(\mathcal{W}')$ are the minimal sections of S_1 which are the same along the discriminant Δ . □

The local study of the 4-web \mathbf{W} on S_1 is the same as the 4-web on $C \times \text{Jac}(C)$ given by the 2-web \mathcal{W}' , the foliation \mathcal{F} and the $\text{Jac}(C)$ -bundle defined by the first projection on $C \times \text{Jac}(C)$.

Theorem 3.10. *Outside the discriminant locus Δ , the 4-web \mathbf{W} is locally parallelizable .*

Proof. According to the foregoing, the pull-back of 4-web \mathbf{W} by the ramified cover φ is locally the 4-web defined by $W(x, y, y + x, y + 2x)$ on \mathbb{C}^2 . It is a holomorphic parallelizable web. □

Remark 3.11. *An immediate consequence of theorem 3.10 is that the curvature of the 4-web \mathbf{W} is zero.*

The second part of this paper aims to use the theory of birational geometry in order to find the theoretic results of the first part by computations on the birational trivialisation $C \times \mathbb{P}^1$.

4 Geometry of 4-web W after elementary transformations

Let $\pi: S_1 \mapsto C$ be the \mathbb{P}^1 -bundle and $\{p_0, p_1, p_t, p_\infty\}$, the set of points of order 2 in C .

Definition 4.1. *An elementary transformation at the point $P \in \pi^{-1}(p)$ is the birational map given by the composition of the blow-up of the point P , followed by the contraction of the proper transform of the fiber $\pi^{-1}(p)$.*

Remark 4.2. *After elementary transformation at the point P , we obtain a new ruled surface with a point \tilde{P} which is the contraction of the proper transform of the fiber $\pi^{-1}(p)$.*

How many elementary transformations do you need to trivialize the ruled surface S_1 ?

Lemma 4.3. *The ruled surface S_1 is obtained after three elementary transformations at the points $\tilde{P}_0 = (p_0, 0)$, $\tilde{P}_1 = (p_1, 1)$ and $\tilde{P}_\infty = (p_\infty, \infty)$ on the trivial bundle $C \times \mathbb{P}^1$.*

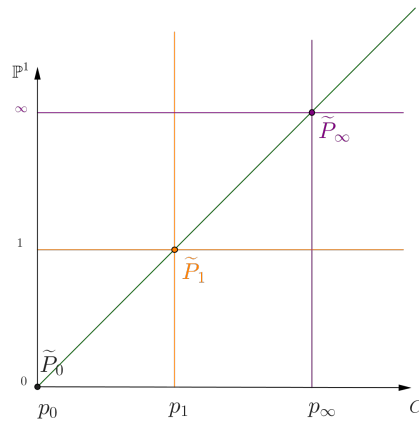


Figure 2: Special points of the trivial \mathbb{P}^1 -bundle over C

In fact, if we perform the elementary transformations of the three special points \tilde{P}_0 , \tilde{P}_1 and \tilde{P}_∞ of $C \times \mathbb{P}^1$: (see figure 2), we have a ruled surface S with three special points P_0 , P_1 and P_∞ (see figure 3).

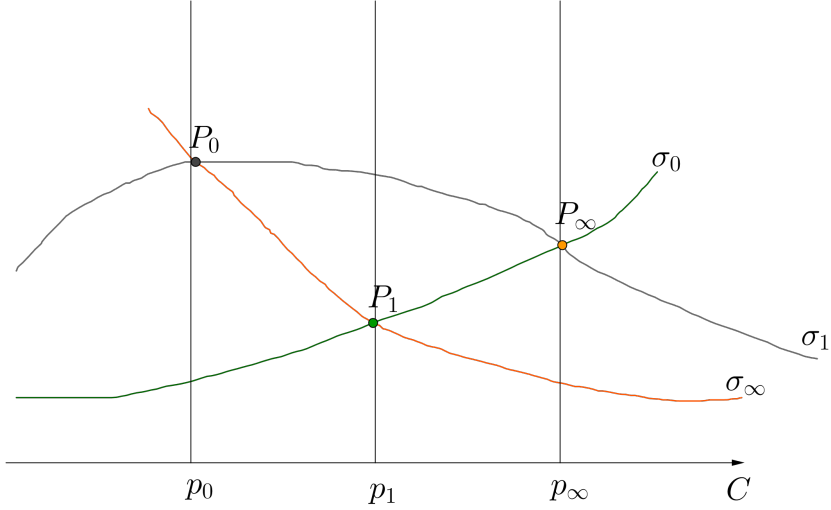


Figure 3: Special points and special +1 self-intersection sections of the ruled surface S_1

Recall that after 3 elementary transformations, if σ is a section on S such that σ' is its strict transform on the trivial bundle, we have: $\sigma.\sigma = \sigma'.\sigma' + r$ where $r = \epsilon_0 + \epsilon_1 + \epsilon_\infty$ such that

$$\epsilon_i = \begin{cases} -1 & \text{if } P_i \in \sigma \\ +1 & \text{if } P_i \notin \sigma \end{cases} \quad (1)$$

in particular, $r \in \{-1, 1, -3, 3\}$. Then, we can deduce that the ruled surface S has a invariant $\kappa \leq 1$.

1. If $\kappa = 0$, then there is $\sigma.\sigma = 0$ on S , then its strict transform $\sigma'.\sigma' = odd$ on $C \times \mathbb{P}^1$. This cannot hold because all the sections of the trivial bundle have even self-intersection;
2. If $\kappa < 0$, then there exists a +2 self-intersection section of $C \times \mathbb{P}^1$ passing through by three points. It is absurd because there exists a

effective divisor equivalent to its normal bundle which contains at least three points.

According to these two cases, after our elementary transformations on the trivial bundle $C \times \mathbb{P}^1$, we obtain a ruled surface such that its invariant $\kappa = 1$. Therefore, it is a stable ruled surface over an elliptic curve.

4.1 The Riccati foliation on S_1 after elementary transformations

Proposition 4.4. *After elementary transformations of the three special points P_0, P_1 and P_∞ on S_1 , the Riccati foliation Ric induces a Riccati foliation $\widetilde{\text{Ric}}$ on the trivial bundle $C \times \mathbb{P}^1$ such that the points $\widetilde{P}_0 = (p_0, 0)$, $\widetilde{P}_1 = (p_1, 1)$ and $\widetilde{P}_\infty = (p_\infty, \infty)$ are radial singularities.*

Proof. As the problem is local, we can prove it on the surface \mathbb{C}^2 . In our context, after elementary transformation at the origin, a regular Riccati foliation becomes the pull-back of this holomorphic foliation $\frac{dz}{dx} = az^2 + bz + c$ where $a, b, c \in \mathbb{C}$ by the birational map $\mathbb{C}^2 \dashrightarrow \mathbb{C}^2 ; (x, z) \mapsto (x, xz)$. Thus, we obtain that a Riccati foliation such that the linear part looks like $xdz - zdx = 0$. Therefore the origin is a radial singularity. Finally, we can say the foliation Ric on S_1 is after elementary transformations a Riccati foliation on $C \times \mathbb{P}^1$ having three radial singularities at the points $\widetilde{P}_0, \widetilde{P}_1$ and \widetilde{P}_∞ . \square

If $((x, y), z)$ are coordinates of the trivial bundle $C \times \mathbb{P}^1$, then the foliation $\widetilde{\text{Ric}}$ is defined by $dz = [a(x, y)z^2 + b(x, y)z + c(x, y)] \frac{dx}{2y}$ where a, b, c are the meromorphic functions with pole of order 1 at the points p_0, p_1 and p_∞ , i.e, $a, b, c \in H^0(\mathcal{O}_C(p_0 + p_1 + p_\infty)) \simeq \mathbb{C} \langle 1, \frac{1}{y}, \frac{x}{y} \rangle$. It means that $a = \frac{a_0 + a_1x + a_2y}{y}$, where a_i are constant. If we write the same relation for the functions b and c , we obtain that the foliation $\widetilde{\text{Ric}}$ is locally defined by the following 1-form:

$$ydz = [(a_0 + a_1x + a_2y)z^2 + (b_0 + b_1x + b_2y)z + (c_0 + c_1x + c_2y)] \frac{dx}{2y}$$

As the foliation is invariant by the involution $(x, y) \mapsto (x, -y)$ on C , the coefficients a_2, b_2 , and c_2 are zero. Furthermore, if we use the relation on an elliptic

curve, $y^2 = x(x-1)(x-t)$, and the fact that the points $(0, 0, 0)$, $(1, 0, 1)$, (p_∞, ∞) are the radial singularities, we have \widetilde{Ric} is defined by the 1-form :

$$w := dz + \left[\frac{-z^2}{4x(x-1)} - \frac{z}{2x} + \frac{1}{4(x-1)} \right] dx$$

Proposition 4.5. *If we fix a point (x_0, y_0) of C , the monodromy group of the foliation \widetilde{Ric} along of the fiber $\pi^{-1}(x_0, y_0)$ is an abelian group isomorphic to a group given by these automorphisms: $\widetilde{\Phi}_0: z \mapsto \frac{z-x_0}{z-1}$, $\widetilde{\Phi}_1: z \mapsto \frac{x_0}{z}$, $\widetilde{\Phi}_t: z \mapsto \frac{x_0(z-1)}{z-x_0}$, $\widetilde{\Phi}_\infty: z \mapsto z$.*

Proof. Let $\widetilde{\sigma}_\infty := \{z = \infty\}$, $\widetilde{\sigma}_0 := \{z = 0\}$, $\widetilde{\sigma}_1 := \{z = 1\}$ and $\widetilde{\sigma}_d := \{z = x\}$ be the four special sections obtained after elementary transformations. By definition of the monodromy group of Ric , we can see that for the point p_0 of order 2, the automorphism $\widetilde{\Phi}_0$ restricted to any fiber is the unique Moebius transformation which relates respectively the points of the sections $(\widetilde{\sigma}_0, \widetilde{\sigma}_1, \widetilde{\sigma}_\infty, \widetilde{\sigma}_d)$ to the points of the sections $(\widetilde{\sigma}_d, \widetilde{\sigma}_\infty, \widetilde{\sigma}_1, \widetilde{\sigma}_0)$. Using the same process for the other automorphisms, we obtain the result. \square

We can also describe the special leaves of the foliation \widetilde{Ric} . In fact, if we consider $\phi_i: C \times \mathbb{P}^1 \mapsto C \times \mathbb{P}^1$; $(x, y, z) \mapsto (x, y, \widetilde{\Phi}_i(z))$, then according to lemma 3.7, the special leaves are defined by :

1. $\widetilde{Ric}_0 := \{(x, y, z), \phi_0(x, y, z) = (x, y, z)\} = \{(x, z), -z^2 - x + 2z = 0\}$
2. $\widetilde{Ric}_1 := \{(x, y, z), \phi_1(x, y, z) = (x, y, z)\} = \{(x, z), -z^2 + x = 0\}$
3. $\widetilde{Ric}_t := \{(x, y, z), \phi_t(x, y, z) = (x, y, z)\} = \{(x, z), z^2 - 2xz + x = 0\}$

Now it is natural to ask if we can find the expression of the leaf of order 4 of \widetilde{Ric} . To do this, we use the special leaves to find a first integral. Let

$$f_0 := -z^2 + 2z - x, \quad f_1 := -z^2 + x, \quad f_t := z^2 - 2xz + x$$

be the polynomials which define respectively the leaves \widetilde{Ric}_0 , \widetilde{Ric}_1 , \widetilde{Ric}_t and consider the function $\gamma: C_x \times \mathbb{P}_z^1 \mapsto C_x \times \mathbb{P}_z^1$; $(x, z) \mapsto (x, F(x, z))$, where

$$F(x, z) = \frac{xf_0^2}{xf_0^2 - (x-1)f_1^2}$$

The pull-back γ^*dy of the 1-form dy by γ is a foliation on $C_x \times \mathbb{P}_z^1$ having the function $F(x, z)$ as a first integral and such that the curves \widetilde{Ric}_0 , \widetilde{Ric}_1 and \widetilde{Ric}_t are invariant. Hence, it is the foliation \widetilde{Ric} . we deduce that :

Lemma 4.6. *The foliation $\widetilde{\text{Ric}}$ on $C \times \mathbb{P}^1$ has a rational first integral defined by the following function :*

$$F(x, z) = \frac{x(z^2 - 2z - x)^2}{(-z^2 + 2xz - x)^2}$$

4.2 The generic 2-web after elementary transformations

After elementary transformations at the three special points on S_1 , the generic +1 self-intersection sections (i.e not passing through the three special points) become the +4 self-intersection sections of $C \times \mathbb{P}^1$ passing through the points $(0, 0, 0)$, $(1, 0, 1)$ and (p_∞, ∞) : see figure 4.

Lemma 4.7. *A +4 self-intersection section passing through the points \tilde{P}_0 , \tilde{P}_1 and \tilde{P}_∞ is either given by the graph $z = \frac{(1-x_0)(y_0x - x_0y)}{y_0(x-x_0)}$, or the graph $z = x$.*

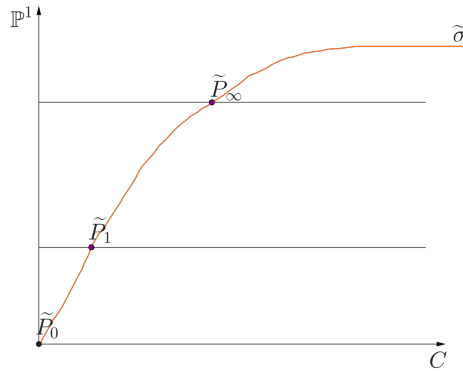


Figure 4: Generic +4 self-intersection section

Proof. If $\sigma: C \mapsto \mathbb{P}^1$ is a +4 self-intersection section on the trivial bundle, then it defines a rational map of degree 2 generated by two sections σ_1 and σ_2 of a line bundle of degree 2 over C ; more precisely, for any point $(x, y) \in C$, $\sigma(x, y) = (\sigma_1(x, y) : \sigma_2(x, y))$. Since for any line bundle of degree 2 over C , there exists a point $p = (x_0, y_0) \in C$ such that $L = [p] + [p_\infty]$, we have two cases:

- if $p \neq p_\infty$, according to the Riemann Roch's theorem, $H^0(L) = \mathbb{C} \langle y - y_0, x - x_0 \rangle$ and then, σ is a graph given by

$$z = \frac{a(y - y_0) + b(x - x_0)}{c(y - y_0) + d(x - x_0)}, \quad a, b, c, d \in C$$

Using the fact that the section passes through the points \tilde{P}_0, \tilde{P}_1 and \tilde{P}_∞ and the puseux parametrisation of elliptic curve at the infinity point is given by $t \mapsto (\frac{1}{t^2}, \frac{1}{t^3})$, we obtain a system of equations which solutions are $\left\{ a = d \frac{x_0(x_0 - 1)}{y_0}, b = -d(x_0 - 1), c = 0, d = d \right\}$ where $d \neq 0$;

- if $p = p_\infty$ then $H^0(L) = \mathbb{C} \langle 1, \frac{1}{x} \rangle$, likewise using the fact that the section passes through the points $(0, 0, 0), (1, 0, 1)$ and (p_∞, ∞) , we obtain that σ is the graph $z = x$.

□

From now on, unless otherwise mentionned, we will assume that a $+4$ self-intersection section is a section which passes through the points \tilde{P}_0, \tilde{P}_1 and \tilde{P}_∞ .

Now using the birational trivialisation of S_1 , we can give another proof to show that the minimal sections of S_1 define a singular 2-web and its discriminant is a leaf of the Riccati foliation on S_1 .

Proposition 4.8. *For any point of $C \times \mathbb{P}^1$, there exists a $+4$ self-intersection section which passes through this point.*

Proof. Let $(u, v, z) \in C \times \mathbb{P}^1$ such that $v \neq 0$, we have to find the points $(x_0, y_0) \neq (u, v)$ of C such that $z = \frac{(1 - x_0)(y_0 u - x_0 v)}{y_0(u - x_0)}$.

Using the fact that $y_0^2 = x_0(x_0 - 1)(x_0 - t)$ and $v^2 = u(u - 1)(u - t)$, we have the following equation:

$$(\star) : (u - z)^2 x_0^3 - [(2(uz - u))(-z + u) - (-z + u)^2 t - v^2] x_0^2 + [(uz - u)^2 - (2(uz - u))(-z + u)t + v^2] x_0 - (uz - u)^2 t = 0$$

1. if $(u - z) = 0$, then (\star) becomes $(x_0 - u) \left(x_0 - \frac{t(u - 1)}{u - t} \right) = 0$. As by hypothesis $v \neq 0$, we obtain two solutions given by the point (x_0, y_0) such that $x_0 = \frac{t(u - 1)}{u - t}$ and the point p_∞ ;

2. if $(u - z) \neq 0$, then the solutions verify the following second degree equation:

$$(\star) : (u - z)^2 x_0^2 + [(-t - u)z^2 + 2u(t + 1)z - u(t + u)]x_0 + tu(z - 1)^2 = 0$$

□

The +4 self-intersection sections define a singular holomorphic 2-web \mathcal{W} such that the discriminant Δ is the discriminant of the equation \star . Thus we have :

$$\Delta := (t - u)z^4 - 4(t - 1)uz^3 + 2u(2tu + t - u - 2)z^2 - 4u^2(t - 1)z + u^2(t - u) = 0$$

Lemma 4.9. *The discriminant of the 2-web \mathcal{W} is a leaf of order 4 of the Riccati foliation \widetilde{Ric} .*

Proof. In fact, by the definition of the first integral of the foliation \widetilde{Ric} , we have:

$$F(x, z) - t = - \frac{(t - u)z^4 - 4(t - 1)uz^3 + 2u(2tu + t - u - 2)z^2 - 4u^2(t - 1)z + u^2(t - u)}{(2xz - z^2 - x)^2}$$

Therefore, the first integral is constant along of the discriminant Δ . □

According to the foregoing, on the birational trivialisation of S_1 , we have a 4-web \mathcal{W}_4 defined by the 2-web \mathcal{W} , the Riccati foliation \widetilde{Ric} and the trivial fiber bundle.

4.3 Geometry of the 4-web \mathcal{W}_4

We want to find the slopes of the leaves of \mathcal{W}_4 in order to represent it by a differential equation. Let $(x_0, y_0, z_0) \in C \times \mathbb{P}^1$ be a generic point. As the leaves of the 2-web \mathcal{W} passing through this point are respectively the graph $z = \frac{(1 - a_0)(b_0 x - a_0 y)}{b_0(x - a_0)}$ and $z = \frac{(1 - a_0')(b_0' x - a_0' y)}{b_0'(x - a_0')}$ such that the points (a_0, b_0) and (a_0', b_0') verify the equation (\star) , we deduce that their slopes at the point (x_0, y_0, z_0) are respectively given by the following formulas:

1. $Z_1 = \frac{1 - a_0 - z_0}{x_0 - a_0} + \left(z_0 + \frac{(a_0 - 1)x_0}{x_0 - a_0} \right) \left(\frac{3x_0^2 - 2(1 + t)x_0 + t}{2x_0(x_0 - 1)(x_0 - t)} \right);$
2. $Z_2 = \frac{1 - a_0' - z_0}{x_0 - a_0'} + \left(z_0 + \frac{(a_0' - 1)x_0}{x_0 - a_0'} \right) \left(\frac{3x_0^2 - 2(1 + t)x_0 + t}{2x_0(x_0 - 1)(x_0 - t)} \right).$

Thus, the 2-web \mathcal{W} is defined by the following differential equation:

$$\left(\frac{dz}{dx}\right)^2 - \left(\frac{z^2 + 2(x-1)z - x}{2x(x-1)}\right) \frac{dz}{dx} + \frac{z(z-1)((2tx - x^2 - t)z - x^3 + x^2 - tx + 2)}{4x^2(x-1)^2(t-x)}$$

Futhermore, if we consider

$$Z_0 = \frac{1}{4} \left[\frac{(z_0^2 + 2(x_0 - 1)z_0 - x_0)}{x_0(x_0 - 1)} \right],$$

the slope of the foliation \widetilde{Ric} at the point (x_0, y_0, z_0) , then the 4-web \mathcal{W}_4 is locally equivalent to the 4-web on the complex plane given by $\mathcal{W}(\infty, Z_0, Z_1, Z_2)$.

Theorem 4.10. *The 4-web $\mathcal{W}(\infty, Z_0, Z_1, Z_2)$ is locally equivalent to a parallelizable 4-web.*

Proof. The pull-back of the foliation \widetilde{Ric} by the multiplication of order 2 on C is another Riccati foliation \widetilde{Ric}_2 on $C \times \mathbb{P}^1$ with trivial monodromy.

Let $M_2: C \mapsto C$, be the multiplication of order 2 on C then, for any point $(x, y) \in C$ the first projection of $M_2(x, y)$ is given by the following formula :

$$pr_1 \circ M_2(x, y) = \frac{(3x^2 - 2(t+1)x + t)^2}{4x(x-1)(x-t)} + (1+t) - 2x$$

Using the pull-back of the special leaves, we can choose three curves by:

1. $C_0 := \left\{ (x, y, z), z = z_0 = \frac{-x^2 + t}{2(t-x)} \right\}$
2. $C_1 := \left\{ (x, y, z), z = z_1 = \frac{(-x^2 + t)}{2y} \right\}$
3. $C_2 := \left\{ (x, y, z), z = z_2 = -\frac{(-x^2 + t)}{2y} \right\}$

which are the leaves of \widetilde{Ric}_2 . Now, if we consider the map $\psi: C \times \mathbb{P}^1 \mapsto C \times \mathbb{P}^1$ which for any coordinate (X, Z) relates:

$$\psi(X, Z) = \left(\frac{(3X^2 - 2(t+1)X + t)^2}{4X(X-1)(X-t)} + (1+t) - 2X, \frac{Z\mu z_1 - z_0}{Z\mu - 1} \right)$$

where $\mu = \frac{z_3 - z_0}{z_3 - z_1}$, then the pull-back of the first integral of \widetilde{Ric} by ψ is the following meromorphic function :

$$(\psi_* F)(X, Z) = \frac{(Z^2 - 2Z + 2)^2}{Z^2(Z - 2)^2}$$

Finally, the foliation $\psi_*\widetilde{Ric}$ is locally defined by the 1-form $dZ = 0$. Likewise, the pull-back of the slopes Z_1 and Z_2 by ψ defines a 2-web such that the leaves verify the following differential equation :

$$(\star\star) : \left(\frac{dZ}{dX} \right)^2 + \frac{(t-1)Z^4 + (-4t+4)Z^3 + (4t-8)Z^2 + 8Z - 4}{4X(X-1)(X-t)} = 0$$

In summary, the 4-web $\psi_*\mathcal{W}(\infty, Z_0, Z_1, Z_2)$ is locally equivalent to the web $\mathcal{W}(\infty, 0, \beta, -\beta)$, where β is a solution of $(\star\star)$. As the 4-web $\mathcal{W}(\infty, 0, \beta, -\beta)$ has a constant cross-ratios equal to -1 and all the 3 subweb are hexagonal, we can deduce that it is locally parallelizable. \square

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