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**Nonlinear vibrations of thin plates with variable thickness : application to sound
synthesis of cymbals**

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1 Geometrically nonlinear vibrations of thin plates and shells with variable thickness
2 are investigated numerically with the purpose of synthesizing the sound of cymbals.
3 In cymbal making, taper refers to the gradual change in thickness from the centre to
4 the rim, and is known to be a key feature that determines the tone of the instrument.
5 It is generally used in conjunction with shape variations in order to enable the cymbal
6 to play a bell-like sound when hit near its centre, or a crash sound when struck close
7 to the edge. The von Kármán equations for thin plates with thickness and shape
8 variations are derived, and a numerical method combining a Rayleigh-Ritz approach
9 together with a Störmer-Verlet scheme for advancing the problem in time, is detailed.
10 One main advantage of the method is its ability to implement easily any frequency-
11 dependent loss mechanism which is a key property for sound synthesis. Also, the
12 accuracy of the computation of the nonlinear restoring force is especially preserved.
13 The method is employed to synthesize the sounds of cymbal-like instruments. The
14 impact of taper is addressed and the relative effects of both thickness and shape
15 variations, are contrasted.

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16 **I. INTRODUCTION**

17 Cymbals belong to the category of percussion instruments having a long history, and
18 used in various contexts, from orchestral symphonic music to pop-rock and jazz music^{1,2}.
19 Having different shapes, names and tones and a wide range of playability, sounds and colors,
20 they serve as a basis of all drum kits in modern amplified music, jazz bands and percussion
21 ensembles. The usual terminologies to designate their roles and sounds are known as: *ride*,
22 *splash* and *crash*, the last two names of which are evidently onomatopoeic. Ride cymbals
23 are the largest ones, generally used to play a steady rhythmic pattern, whereas crash and
24 splash are smaller and used to play dramatic accents.

25 In cymbal making, taper refers to the thickness variation encountered from the centre to
26 the edge of a cymbal. Indeed, for most of the cymbals, taper is very important in order to
27 confer the cymbals the ability to have two distinct tones. On the one hand, cymbals shall
28 play a clear tone with a dominant pitch when hit near the centre, in the region known as
29 the bell. This bell-like sound is essential to mark the beat, and is favoured by two different
30 physical characteristics : the protruding dome manufactured at centre, and the thickness
31 which is more important in this area. On the other hand, a non-tonal, bright glittering
32 sound is awaited when strongly striking crash and splash cymbals at the edge.

33 A series of measurements have been realized in two different drum stores in Paris with
34 an electronic vernier calliper in order to quantify more precisely the thickness at centre
35 and edge. It has been found that for crash cymbals, the thickness at edge h_e ranges from
36 0.9 mm to 0.5 mm, the minimal values being obtained for series known for their explosive

37 sounds: The Paiste fast crash 14" (35 cm diameter) and the Sabian AA thin crash (30 cm
38 diameter), and also for the Paiste "paperthin formula 602", generally used in hi-hat. The
39 mean measured value for crash is 0.65 - 0.7 mm. The thickness at centre h_c is larger, with
40 measured values ranging from 1.4 mm to 0.9 mm. For ride cymbals, h_c can go up to 1.5 mm
41 while h_e ranges between 1.4 mm to 0.8 mm. Splash cymbals are the smallest ones, and h_e
42 has been found to be of the order of 0.55 mm to 0.65 mm, with a minimal value $h_e=0.4$ mm
43 for a Sabian AAX splash 8" (20 cm diameter).

44 The bright shimmering sound of gongs and cymbals is obtained thanks to strongly nonlin-
45 ear vibrations^{1,3} occurring because the amplitude of the vibrations is larger than the thickness.
46 Recent studies revealed that the phenomenon at hand is wave turbulence, resulting from the
47 geometric nonlinearity, and also at work in gongs and thunder plates⁴⁻⁸. For the gongs, the
48 energy cascade driven by the wave turbulence can be relatively slow and take up to half a
49 second to fully develop and attain the highest frequencies. This results in the characteristic
50 sound of gongs with a blow-up of higher frequencies occurring shortly after the strike. In
51 the case of cymbals (and more specifically for crash and splash), taper strongly favours the
52 nonlinearity by reducing the thickness close to the edge. Consequently, the cascade of energy
53 and the excitation of high frequencies is very fast, so that the maximal frequency is obtained
54 right after the strike, without an audible delay.

55 Numerical simulations and sound synthesis based on physical models face a number of dif-
56 ficulties while solving for the vibrations, the nonlinearity and the presence of the wave turbu-
57 lence effect being the most important ones. However, successful results have been obtained in
58 the last years, where a key component has been the derivation of energy-conserving schemes

59 for von Kármán problems describing geometrically nonlinear vibrations of thin plates^{9,10}.
60 The temporal scheme has then been used together with finite difference methods¹¹, or with
61 a modal approach⁴, the advantage of which is to offer an ability to easily implement losses
62 with any frequency-dependent law, and to give a better control on the accuracy of the
63 computation of the nonlinear terms. These two points are key to the realism of synthe-
64 sized sounds. Other methods have been used where the kinematics of plates and shells is
65 simplified by using ad-hoc bending models that are easier to implement for finite element
66 approaches, hence opening the doors to create more complex geometries for a wide variety
67 of shells¹². This last work has been more deeply investigated recently¹³, with an interesting
68 improvement to efficiently simulate the energy cascade due to wave turbulence thanks to
69 the phenomenological model proposed¹⁴.

70 Even though the sound of gongs has been recovered with accuracy and realism by using
71 the modal approach⁴, simulations of cymbal-like circular plates failed to reproduce the very
72 fast cascade and the immediate emergence of the broadband Fourier spectrum, which gives
73 to crash and splash cymbals their specific sparkling and explosive sound. A main reason is
74 the absence of taper in earlier simulations, thus not covering the ability of cymbals to play
75 two different sounds when hit near the centre or close to the edge. The goal of this article
76 is thus to introduce a nonlinear model for plate vibrations including both taper (thickness
77 variations) and curvature (also denoted as bow in cymbal description terminology, *i.e.* shape
78 variations), and to develop a numerical method for solving the problem in time, based on
79 earlier works^{4,15–17}. Once the model is established, numerical simulations are drawn in order
80 to show the important effect of taper on the nonlinear vibrations, and to contrast both

81 effects of shape and thickness variations, as well as to distinguish the bell sound from the
 82 crash sound obtained with strikes at the edge.

83 II. MODEL AND METHODS

84 In this section, the equations of motion for a thin plate with variable thickness are first
 85 recalled. A linear analysis is then performed for the two unknowns, namely the transverse
 86 displacement w and the Airy stress function F . The problem is subsequently discretized
 87 using a Rayleigh-Ritz approach and the time integration method is detailed. Finally a com-
 88 plete model including thickness together with shape variations, modeled as an imperfection
 89 of the plate at rest, is highlighted.

90 A. Von Kármán model for thin plates with variable thickness

In this paper a circular plate of radius R_d , made of a homogeneous material of volume density ρ , Young's modulus E and Poisson's ratio ν , is considered. The surface of the plate is denoted as $\mathcal{S} = \{(r, \theta) \in [0, R_d] \times [0, 2\pi]\}$. In order to simplify the presentation, the thickness of the plate h is assumed to depend only on the radius coordinate r . The von Kármán model for geometrically nonlinear vibrations of thin plates is used to compute the solutions for the transverse displacement $w(r, \theta, t)$ and the Airy stress function $F(r, \theta, t)$. It

reads¹⁸⁻²¹

$$\begin{aligned} \rho h(r)\ddot{w} + \Delta(D(r)\Delta w) - (1 - \nu) \mathcal{L}(D(r), w) \\ = p(r, \theta, t) - R(\dot{w}) + \mathcal{L}(w, F), \end{aligned} \quad (1a)$$

$$\Delta(B(r)\Delta F) - (1 + \nu) \mathcal{L}(B(r), F) = -\frac{1}{2} \mathcal{L}(w, w). \quad (1b)$$

In these expressions, $D(r) = Eh^3(r)/12(1 - \nu^2)$ is the flexural rigidity, $B(r) = 1/Eh(r)$, p stands for the normal pressure loading and $R(\dot{w})$ is a generic expression accounting for the losses. Δ is the laplacian operator, and \mathcal{L} is the von Kármán bilinear operator, which reads, in polar coordinates for two arbitrary functions $f(r, \theta)$ and $g(r, \theta)$:

$$\begin{aligned} \mathcal{L}(f, g) = f_{,rr} \left(\frac{g_{,r}}{r} + \frac{g_{,\theta\theta}}{r^2} \right) + g_{,rr} \left(\frac{f_{,r}}{r} + \frac{f_{,\theta\theta}}{r^2} \right) \\ - 2 \left(\frac{f_{,r\theta}}{r} + \frac{f_{,\theta}}{r^2} \right) \left(\frac{g_{,r\theta}}{r} + \frac{g_{,\theta}}{r^2} \right). \end{aligned} \quad (2)$$

As compared to the equations of motion with constant thickness¹⁶, the main incidence of the variable thickness onto the equations of motion is an added complexity to derive the linear terms. On the other hand, the nonlinear terms are not affected by the thickness variations, which is in line with the definition of the geometric nonlinearity. As applications to cymbals are targeted, a free-edge boundary condition is selected. It reads¹⁶, $\forall t, \forall \theta \in [0, 2\pi]$, and for

$r = R_d$:

$$w_{,rr} + \frac{\nu}{R_d}w_{,r} + \frac{\nu}{R_d^2}w_{,\theta\theta} = 0, \quad (3a)$$

$$w_{,rrr} + \frac{1}{R_d}w_{,rr} - \frac{1}{R_d^2}w_{,r} + \frac{2-\nu}{R_d^2}w_{,r\theta\theta} - \frac{3-\nu}{R_d^3}w_{,\theta\theta} = 0, \quad (3b)$$

$$F_{,r} + \frac{1}{R_d}F_{,\theta\theta} = 0, \quad F_{,r\theta} + \frac{1}{R_d}F_{,\theta} = 0. \quad (3c)$$

91 **B. Linear analysis**

92 The aim of this section is to analyse the linear part of Eqs. (1). As already underlined,
 93 the linear terms are deeply modified when taking the variable thickness into account, hence
 94 most of the added work focuses on calculating these new terms. In the course of this section,
 95 the results will be derived without referring to a particular case of boundary conditions, for
 96 the sake of generality. However when turning to numerical results, the free-edge boundary
 97 condition presented in the previous section, is used. The linear parts of Eqs. (1a) and
 98 (1b) correspond to two distinct linear problems in terms of the two unknowns w and F ,
 99 respectively. Each of these two problems is tackled separately, by using the Rayleigh-Ritz
 100 method with the appropriate boundary conditions.

101 **1. Transverse vibration**

102 The linear solution for the transverse vibration is first derived. The problem at hand
 103 reads :

$$\rho h(r)\ddot{w} + \Delta(D(r)\Delta w) - (1 - \nu) \mathcal{L}(D(r), w) = 0. \quad (4)$$

104 Let us assume that the eigenmodes of this problem with constant thickness (*i.e.* a perfect flat
 105 plate with h_0 a reference thickness,) are known and denoted as $\Phi_p(r, \theta)$ for the eigenfunctions
 106 and ω_p for the eigenfrequencies, meaning that we have :

$$\Delta\Delta\Phi_p(r, \theta) = \frac{\rho h_0}{\omega_p^2} \omega_p^2 \Phi_p(r, \theta). \quad (5)$$

107 In the computations, h_0 is defined from the thickness variation profile as the value at the
 108 plate centre. The eigenfunctions $\Phi_p(r, \theta)$ verify the boundary conditions for the transverse
 109 displacement w at the edge, and they are assumed to be normalized such that $\int_{\mathcal{S}} \Phi_p \Phi_q d\mathcal{S} =$
 110 δ_{pq} , with δ_{pq} the Kronecker symbol. Following the Rayleigh-Ritz approach, the problem (4)
 111 is discretized in space with the expansion $w(r, \theta) = \sum^{N_\Phi} X_p(t) \Phi_p(r, \theta)$. Substituting this
 112 last expression in (4), multiplying by another basis function Φ_j and integrating over the
 113 surface \mathcal{S} , the problem can be rewritten as :

$$\mathbf{M}_\Phi \ddot{\mathbf{X}} + \mathbf{K}_\Phi \mathbf{X} = \mathbf{0}, \quad (6)$$

114 with $\mathbf{X} = [X_1, X_2, \dots, X_{N_\Phi}]^T$ the vector of generalized coordinates and N_Φ the number
 115 of transverse modes retained in the truncation. The complete expressions for the mass
 116 and stiffness matrices \mathbf{M}_Φ and \mathbf{K}_Φ are given in Appendix A. The advantage of using the
 117 Rayleigh-Ritz method in the case of a circular plate where the modes of the perfect plate
 118 $\Phi_p(x)$ are analytic is that the resulting coefficients appearing in \mathbf{M}_Φ and \mathbf{K}_Φ , such as the
 119 finite-difference scheme where the convergence is more difficult to be achieved up to high
 120 frequencies²¹.

121 **2. Airy stress function**

122 The same methodology is applied to the second linear equation (1b), which reads:

$$\Delta(B(r)\Delta F) - (1 + \nu)\mathcal{L}(B(r), F) = 0. \quad (7)$$

123 This problem in case of constant thickness is associated with in-plane eigenmodes Ψ_j which
 124 verify $\Delta\Delta\Psi_j = \zeta_j^4\Psi_j$, together with the associated boundary conditions. Expanding the
 125 unknown $F(r, \theta) = \sum^{N_\Psi} Y_p\Psi_p(r, \theta)$, the Rayleigh-Ritz approach applied to (7) leads to

$$\mathbf{K}_\Psi\mathbf{Y} = \mathbf{0}, \quad (8)$$

126 with $\mathbf{Y} = [Y_1, Y_2, \dots, Y_{N_\Psi}]^T$ the vector of generalized coordinates and N_Ψ the number of
 127 in-plane modes retained in the truncation. Note that no mass matrix is present here as the
 128 longitudinal inertia is neglected in the von Kármán model. The complete expressions of the
 129 entries of \mathbf{K}_Ψ are given in appendix B.

130 **3. Comparison with a finite element computation**

131 In order to validate eigenmode computation with the Rayleigh-Ritz method, a comparison
 132 is drawn with the results obtained from a commercial finite element (FE) software, namely
 133 ANSYS. The case study is that of a free edge circular plate, with radius $R_d = 0.2$ m, and
 134 material parameters as $E = 2.10^{11}$ Pa, $\nu = 0.38$ and $\rho = 7860$ kg/m³. Two thickness
 135 variations are selected and represented in Fig. 1(a-b). In the first case the thickness is
 136 constant and equal to 1 mm from the centre to $r = 0.04$ m, then it decreases linearly from

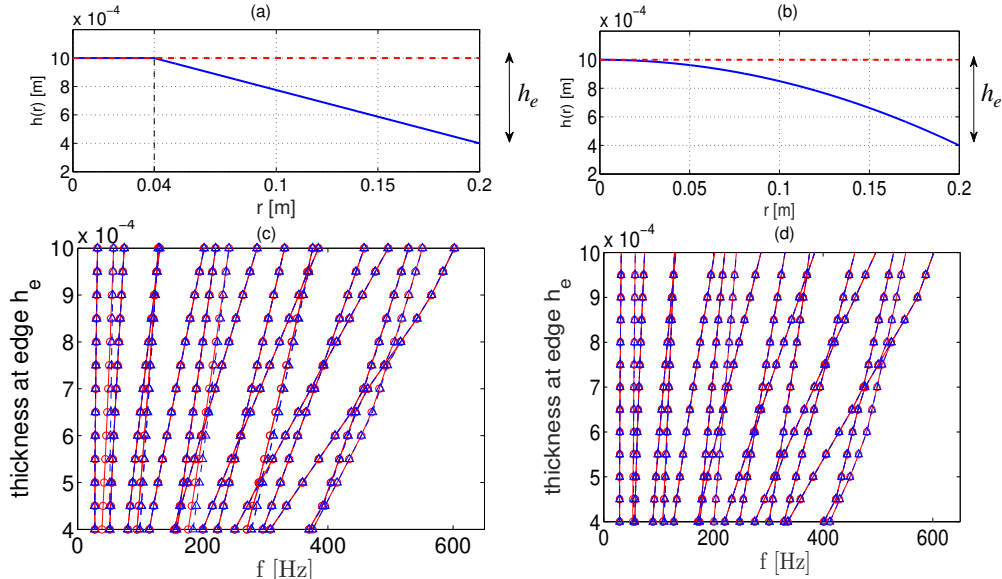


FIG. 1. Comparison of eigenfrequency computation for two different cases of thickness variation. (a) thickness constant near centre then linearly decreasing. (b) parabolic thickness. (c-d) eigenfrequency variations for $h_e \in [0.4, 1]$ mm, corresponding respectively to cases (a) and (b). Red circles: Rayleigh-Ritz method, Blue triangles: Finite-element model.

137 that point to the edge, with a final value at edge h_e which is a parameter ranging from $h_e = 1$
 138 mm (case of a plate with constant thickness) to $h_e = 0.4$ mm. In the second case shown
 139 in 1(b), a parabolic dependence of the thickness on the radius is selected, with the same
 140 range of variation for the thickness at edge h_e . In this numerical test, only the transverse
 141 eigenfrequencies corresponding to (6) can be easily computed as no direct output relative
 142 to the in-plane problem can be found. However since the coding of the two problems share
 143 numerous similarities, it has been found sufficient to test only the transverse problem. In
 144 the Rayleigh-Ritz method, the convergence is controlled by the number of modes N_Φ used as
 145 a projection space to build the matrices \mathbf{M}_Φ and \mathbf{K}_Φ . It has been found that $N_\Phi = 150$ was
 146 sufficient to obtain converged values for the first 31 eigenfrequencies shown in Fig. 1(c-d). .

147 Regarding the computations realised with ANSYS, the FE solution has been obtained
 148 by using the element SHELL181, a four-node element with six degrees of freedom at each
 149 node implementing a Reissner-Mindlin kinematics. These computations have reached a fine
 150 convergence by using up to 33387 nodes (33091 elements).

151 Fig. 1(c-d) shows the results obtained by varying h_e . When $h_e = 1$ mm, the thickness
 152 is uniform and one retrieves the eigenfrequencies of a perfect circular plate. Decreasing h_e
 153 leads to the reduction of the local stiffness of the plate, so that the eigenfrequencies are
 154 also lessened, explaining the general trend observed on the curves. Finally one observes a
 155 very good match between the eigenfrequencies calculated with the two methods. Whereas
 156 the results are perfectly similar in the case 2 (parabolic variation of thickness), case 1
 157 (linear variation with two different slopes) shows some slight discrepancies between the two
 158 methods. They are attributed to the modeling difference and in particular to the fact that
 159 transverse shear is taken into account in the FE model.

160 C. Nonlinear analysis and time integration

We now address the time integration of the full problem given by Eqs. (1), including the nonlinear terms and the losses. Following the Rayleigh-Ritz procedure started in the previous section to solve the linear part, *i.e.* using the expansions $w(r, \theta) = \sum^{N_\Phi} X_p(t)\Phi_p(r, \theta)$ and $F(r, \theta) = \sum^{N_\Psi} Y_j(t)\Psi_j(r, \theta)$, one can rewrite Eqs. (1) as :

$$\mathbf{M}_\Phi \ddot{\mathbf{X}} + \mathbf{K}_\Phi \mathbf{X} + \mathbf{C}_\Phi \dot{\mathbf{X}} = \mathbf{N}_\Phi(\mathbf{X}, \mathbf{Y}) + \mathbf{P}_f, \quad (9a)$$

$$\mathbf{K}_\Psi \mathbf{Y} = -\frac{1}{2} \mathbf{N}_\Psi(\mathbf{X}, \mathbf{X}). \quad (9b)$$

Note that, as underlined before, the thickness variation modifies only the linear parts of the von Kármán equations of motion. Consequently, in the semi-discrete equations (9), the new terms appearing are : the force vector \mathbf{P}_f , of which the entry k reads : $\mathbf{P}_{f_k} = \int_{\mathcal{S}} p(r, \theta, t) \Phi_k(r, \theta) d\mathcal{S}$, $\forall k \in [0, N_{\Phi}]$; the damping matrix \mathbf{C}_{Φ} which will be detailed later, and the two quadratic nonlinear terms $\mathbf{N}_{\Phi}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{N}_{\Psi}(\mathbf{X}, \mathbf{X})$. These two nonlinear terms are computed following earlier studies on plates with uniform thickness, see *e.g.* ^{4,16,19}.

They are two vectors with the entry k reading as

$$\mathbf{N}_{\Phi}(\mathbf{X}, \mathbf{Y})_k = \sum_{i=1}^{N_{\Phi}} \sum_{j=1}^{N_{\Psi}} E_{ij}^k X_i Y_j, \quad (10a)$$

$$\mathbf{N}_{\Psi}(\mathbf{X}, \mathbf{X})_k = \sum_{p=1}^{N_{\Phi}} \sum_{q=1}^{N_{\Phi}} H_{pq}^k X_p X_q, \quad (10b)$$

where the introduced coefficients E_{ij}^k and H_{pq}^k write

$$E_{ij}^k = \int_{\mathcal{S}} \Phi_k \mathcal{L}(\Phi_i, \Psi_j) d\mathcal{S}, \quad (11a)$$

$$H_{pq}^k = \int_{\mathcal{S}} \Psi_k \mathcal{L}(\Phi_p, \Phi_q) d\mathcal{S}. \quad (11b)$$

161 Note that Eqs. (9) can be condensed by replacing (9b) into (9a), highlighting the fact
 162 that a cubic nonlinearity is at hand for the transverse displacement variable \mathbf{X} . Also, the
 163 von Kármán operator \mathcal{L} has some symmetry properties¹⁹, such that for some particular
 164 boundary conditions, the relationship $E_{m,n}^l = H_{m,l}^n$ is fulfilled⁴. The free-edge boundary
 165 conditions considered in this paper belong to the category where the relationship holds, so
 166 that subsequent gain in pre-computing time can be saved thanks to this symmetry property.

167 The damping matrix \mathbf{C}_Φ has to be populated with ad-hoc values. As underlined in⁴,
 168 one advantage of the present approach is that one can select modal damping factors at
 169 ease, following any frequency dependence. However, as the problem is semi-discretized by
 170 using the eigenmodes of the plate with constant thickness that are not the eigenmodes of the
 171 problem considered, the relationship between the modal damping matrix $\mathbf{C} = \text{diag}(\mu_k)$ (with
 172 $\mu_k = 2\xi_k\omega_k$ the modal loss coefficient associated to the modal loss factor ξ_k) and \mathbf{C}_Φ has to
 173 be used. Even though the equations of motion will be integrated in time using Eqs. (9), the
 174 idea is to use the modal loss coefficients μ_k as input parameters of the simulations, since the
 175 physical meaning of these values is more tractable. Then, using \mathbf{P} the matrix of normalized
 176 eigenvectors computed from (6), the matrix \mathbf{C}_Φ is computed using the relationship $\mathbf{C} =$
 177 $\mathbf{P}^T \mathbf{C}_\Phi \mathbf{P}$.

178 To integrate the problem in time, a Störmer-Verlet scheme is used. This is a symmetric
 179 and symplectic method of order two²². It is simply deduced from Eqs. (9) by replacing the
 180 second-order time derivative by the centred finite difference operator $\delta_{tt} \equiv \frac{1}{k^2}(e_{t+} - 2 + e_{t-})$,
 181 where k is the time step, and e_{t+} and e_{t-} are respectively the forward and backward shift
 182 operators. The first-order time derivative for the damping term is replaced by the centred
 183 finite difference operator $\delta_t \equiv \frac{1}{2k}(e_{t+} - e_{t-})$, and all the other terms are computed at the
 184 current time step. The method has a stability condition reading $f_S > \pi f_{N_\Phi}$, with $f_S = 1/k$
 185 is the sampling rate and f_{N_Φ} the eigenfrequency of the last transverse mode retained in the
 186 truncation.

D. A complete model with shape imperfection

A complete model for the simulation of cymbal vibrations can be derived from the previous one by adding a static, geometric imperfection to the plate model, representing the position of the structure at rest. Let us denote $w_0(r, \theta)$ this geometric imperfection. In order to comply with von Kármán assumptions^{17,19}, w_0 should not be too large so as to obtain a shallow shell. The full model equations can be derived easily following for example^{15,23}, and read

$$\begin{aligned} \rho h(r)\ddot{w} + \Delta(D(r)\Delta w) - (1 - \nu) \mathcal{L}(D(r), w) \\ = \mathcal{L}(w, F) + \mathcal{L}(w_0, F) + p - R(\dot{w}), \end{aligned} \quad (12a)$$

$$\begin{aligned} \Delta(B(r)\Delta F) - (1 + \nu) \mathcal{L}(B(r), F) \\ = -\frac{1}{2} [\mathcal{L}(w, w) + 2 \mathcal{L}(w, w_0)] \end{aligned} \quad (12b)$$

188 Note that taking this shape imperfection into account modifies only the nonlinear terms
 189 appearing in the right-hand side. The complete model can be semi-discretized in the same
 190 manner, following the Rayleigh-Ritz method and using the eigenmodes of the perfect plate
 191 without imperfection nor thickness variation as the expansion basis. The shape imperfection
 192 w_0 has also to be expanded as¹⁵:

$$w_0(r, \theta) = \sum_{k=1}^{N_k} a_k \Phi_k(r, \theta) + z_g, \quad (13)$$

where z_g is the centre of mass offset due to imperfection, and the coefficients a_k are the semi-discrete representation of the shape. They read

$$a_k = \int_{\mathcal{S}} (w_0 - z_g) \Phi_k d\mathcal{S}, \quad (14)$$

$$z_g = \frac{\int_{\mathcal{S}} w_0 d\mathcal{S}}{A_p}, \quad (15)$$

where A_p is the area of the perfect plate. The two unknowns w and F are expanded using the same procedure as in the previous sections. This leads to a semi-discrete problem reading

$$\mathbf{M}_\phi \ddot{\mathbf{X}} + \mathbf{K}_\phi \mathbf{X} + \mathbf{C}_\phi \dot{\mathbf{X}} = \mathbf{N}_\phi(\mathbf{X}, \mathbf{Y}) + \mathbf{N}_\phi(\mathbf{a}, \mathbf{Y}) + \mathbf{P}_f, \quad (16a)$$

$$\mathbf{K}_\psi \mathbf{Y} = -\frac{1}{2} [\mathbf{N}_\psi(\mathbf{X}, \mathbf{X}) + 2\mathbf{N}_\psi(\mathbf{a}, \mathbf{X})]. \quad (16b)$$

193 In these equations, the nonlinear terms \mathbf{N}_ϕ and \mathbf{N}_ψ have the same expressions as in Eqs. (10),
 194 the only difference being that they are also applied to the imperfection represented by
 195 the vector $\mathbf{a} = [a_1, a_2, \dots, a_{N_\phi}]^T$. Consequently the \mathbf{Y} vector, linked to the Airy stress
 196 function, is a function with linear and quadratic dependence on \mathbf{X} . Substituting (16b)
 197 into (16a), one observes now that the problem from the transverse motion arising from the
 198 right-hand side of (16a) shows a linear, quadratic and cubic dependence on \mathbf{X} . The linear
 199 dependence can be solved alone so as to obtain the eigenmodeshapes and eigenfrequencies
 200 of the complete problem. All these computations follow closely those presented¹⁵, where
 201 only the shape imperfection was taken into account. The method was compared to finite
 202 element simulations to assess the results, and this step has been repeated here, showing a

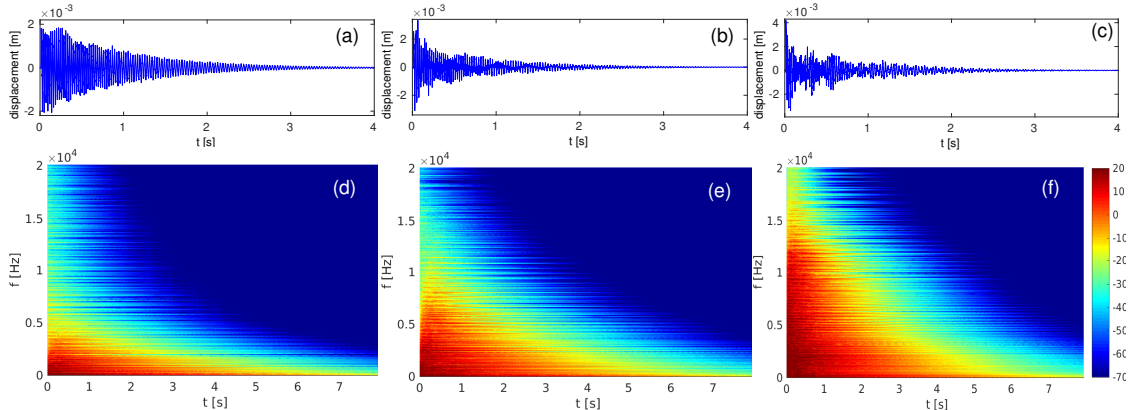


FIG. 2. Output displacements (first row, a-c) and velocity spectrograms (second row, d-f) for the simulations of the nonlinear vibrations of a thin circular plate with a uniform thickness $h=1$ mm (a,d) ; with a linearly varying thickness down to $h_e = 0.7$ mm (b,e), and to $h_e = 0.4$ mm (c,f). Displacement of an arbitrary point located at $r = 0.1792$ in m, and spectrogram of the velocity derived from the same displacement signal, in dB with a 90 dB dynamic.

203 perfect agreement. Finally, to integrate the semi-discrete problem in time, the Störmer-
 204 Verlet scheme is used, as in Section II C.

205 III. NUMERICAL SIMULATIONS

206 A. Effect of variable thickness

207 Numerical simulations are conducted to highlight the effect of the thickness variation
 208 on the nonlinear vibrations of circular plates, and in particular on the explosiveness of the
 209 cymbal sound at the very beginning of a strong hit. Indeed, cymbals are known to produce
 210 on a very short time scale a very rich spectrum with a broadband frequency content, which
 211 is typical of their bright shimmering sound. In order to quantify the effect of thickness
 212 variations, a circular plate of radius $R_d = 0.2$ m has been selected, with material parameters
 213 typical for a metallic alloy : Young's modulus $E = 2.10^{11}$ Pa, Poisson's ratio $\nu = 0.38$ and

214 mass density $\rho=7860 \text{ kg.m}^{-3}$. The thickness variation has been selected with a constant
 215 thickness $h_0 = 1 \text{ mm}$, for $r \in [0, 0.05] \text{ m}$, and then a linear decrease to a value h_e at $r = R_d$.
 216 Two cases, namely $h_e = 0.7 \text{ mm}$ and $h_e = 0.4 \text{ mm}$, are investigated and compared to the
 217 plate with uniform thickness h_0 . These two cases have been selected from the thickness
 218 measurements realized on a number of cymbals: while $h_e = 0.7 \text{ mm}$ corresponds to a mean
 219 value found for crash cymbals, $h_e = 0.4 \text{ mm}$ is taken as the minimal thickness that can be
 220 reasonably attained and which has been measured on a splash cymbal. The modal damping
 221 coefficients have been selected following the rule : $\mu_k = 0.007\omega_k^{0.6} + 2$. The constant term
 222 entails a small constant amount of damping even for the very low frequency modes, while
 223 the power-law form has already been used in⁴; it is based on measured experimental values
 224 identified in very large thin plates²⁴ and has been found to give realistic results for the sound
 225 synthesis of gongs.

226 To assess the explosiveness of the sound, a strike is given as an input force to the plate.
 227 This strike is located at a given point $\mathbf{x}_0 = (r_0, \theta_0)$ so that $p(r, \theta, t) = \delta(\mathbf{x} - \mathbf{x}_0)g(t)$, and the
 228 temporal content is a raised cosine :

$$g(t) = \begin{cases} \frac{p_m}{2} [1 + \cos(\pi(t - t_0)/T_{wid})] & \text{if } |t - t_0| \leq T_{wid}; \\ 0 & \text{if } |t - t_0| > T_{wid}. \end{cases} \quad (17)$$

229 The input parameters of the force are thus the time t_0 , the temporal width of the interaction
 230 T_{wid} and the amplitude of the force p_m in N. All the computations have been implemented
 231 using the software VK-GONG²⁵, an open source code developed to handle nonlinear vibra-
 232 tions of plates and built from previous works by the second author. The first eigenfrequency

233 of the plate with constant thickness $h_0 = 1$ mm is equal to 32 Hz, while the 800th transverse
 234 mode frequency is 18861 Hz. Consequently, all the calculations have been performed with
 235 $N_\Phi = 800$ so as to ensure an almost complete covering of the frequency band of human hear-
 236 ing. For the number of in-plane modes, $N_\varphi = 60$ has been selected following the convergence
 237 guidelines given⁴. Note that this number is given as the cardinal of the subset of in-plane
 238 modes having non-vanishing coupling terms. This is another advantage of the method used
 239 to compute the coupling coefficients, especially for the case of free-edge circular plates where
 240 the modes are analytic and vanishing coefficients can be predicted from analytical formulas
 241 resulting in significant computational time savings. The sampling rate f_S has to be larger
 242 than 59 kHz in order to comply with the stability condition for the Störmer-Verlet scheme,
 243 *i.e.* $f_S > \pi f_{N_\Phi}$ with $N_\Phi = 800$. For all the computations, the sampling rate has been
 244 selected as $f_S = 100$ kHz, well over the stability limit.

245 A first set of simulations is realized with an amplitude of the strike as $p_m = 90$ N
 246 located at the edge of the plate ($r_0 = R_d$), and a short interaction time selected as $T_{wid} =$
 247 1 ms, in order to mimick a strong hit given by a drumstick. Fig. 2 shows the results
 248 obtained for the three cases investigated : the uniform plate, and the two plates with
 249 variable thickness, respectively with $h_e = 0.7$ mm and $h_e = 0.4$ mm. The first row, Fig. 2(a-
 250 c) shows the displacement at an arbitrary point located at $r = 0.1792$ m (close to the
 251 edge), and $\theta = 0.52$ rad, while the second row presents the spectrograms of the velocity
 252 of the same output point. The effect of the thickness variation is huge and can be clearly
 253 assessed both on the temporal displacement signal and the velocity spectrogram. For the
 254 displacement, one can observe that for the plate with uniform thickness, the maximum value

255 of the displacement is reached at the beginning and is equal to 2 mm, while for $h_e = 0.7$
 256 mm the maximum is 3 mm and finally 4 mm for the thinnest plate with $h_e = 0.4$ mm.

257 The fast spectral enrichment can be more evidently assessed with the velocity spectro-
 258 grams shown in Fig. 2(d-f). For the plate with uniform thickness, most of the energy is
 259 concentrated below 5000 Hz, while the rapid generation of high frequencies is much more
 260 pronounced for the two cases with thickness variations. In particular when $h_e = 0.4$ mm,
 261 frequencies up to 15 kHz are created in the very first milliseconds of the vibration. This
 262 results in a more explosive sound which is typical for a cymbal, while the sound produced
 263 by the plate with uniform thickness has not the same properties. The readers are invited
 264 to hear at the associated sound files, corresponding to the velocity resampled at 44.1 kHz,
 265 which are available as supplementary materials. Note that all the sounds corresponding to
 266 the simulations presented in the figures are available as WAV files.

267 Fig. 3 compares the displacement field of the plates with uniform thickness 1 mm (left
 268 column) to that obtained with the variable thickness and $h_e = 0.4$ mm, at different instants.
 269 For these figures, the amplitude of the excitation force p_m has been raised to 150 N in order
 270 to enhance the vibration amplitudes. The temporal width of the strike is kept as $T_{wid} =$
 271 1 ms, which means that, according to the raised cosine formula (17), the input force lasts
 272 from $t = 0$ to 3 ms. The first row shows the displacement field just after the beginning
 273 of the strike, at $t = 0.5$ ms. One can see that the difference in the local stiffness between
 274 the two plates (local thickness 1 mm *vs* 0.4 mm), gives rise to very distinct magnitudes of
 275 displacements at the striking point (0.5 mm for the uniform thickness *vs* 2.2 mm). The
 276 second row shows the displacement fields at $t = 3.3$ ms, *i.e.* right after the end of the input

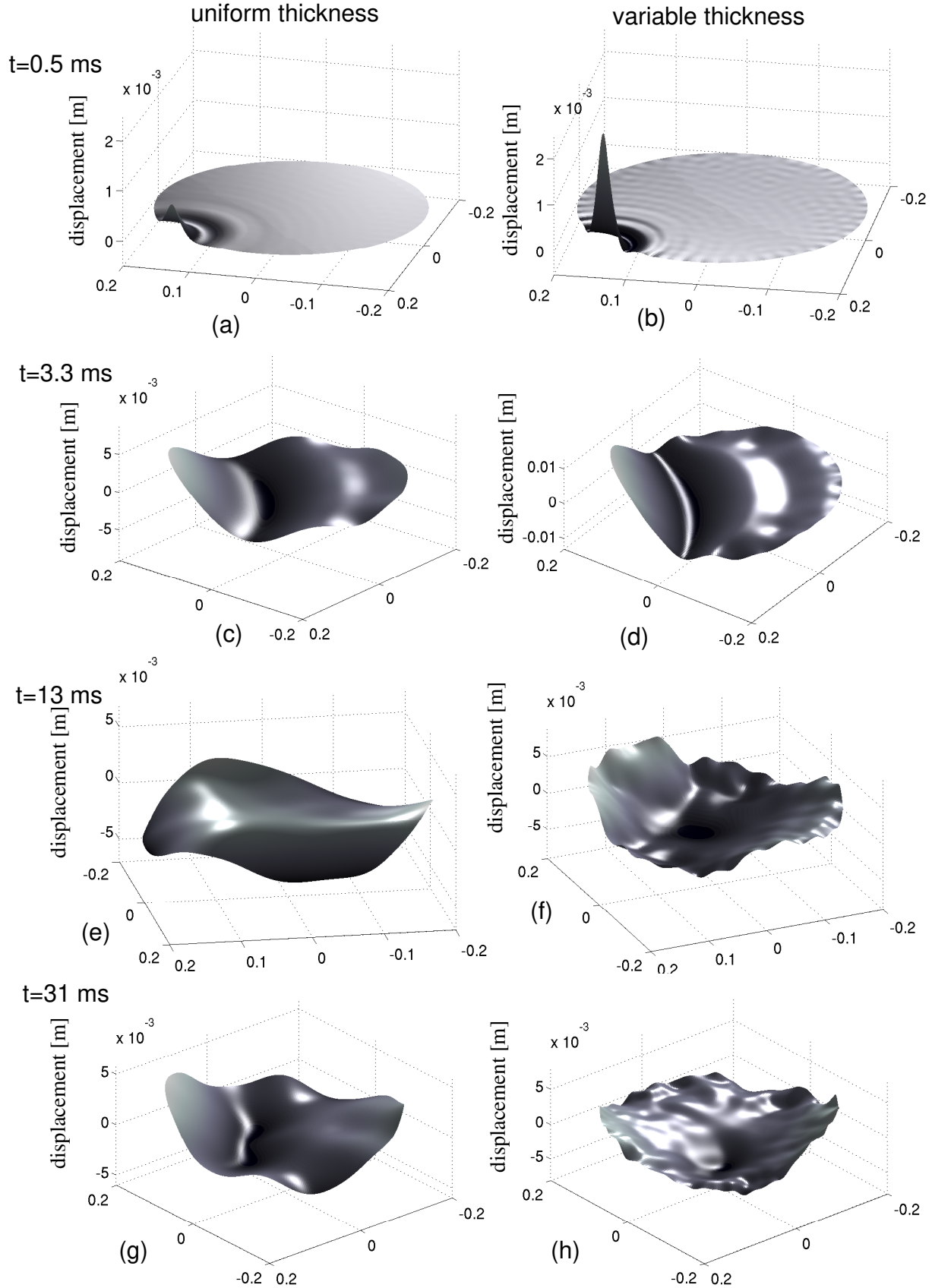


FIG. 3. Snapshots of the displacement fields. Comparison between the plate with uniform thickness $h_0 = 1$ mm (left column, a,c,e,g) and the plate with thickness at edge $h_e = 0.4$ mm (right column, b,d,f,h), and at four different instants: $t = 0.5$ ms (a,b), $t = 3.3$ ms (c,d), $t = 13$ ms (e,f) and $t =$

277 striking force. One can clearly observe that the plate with variable thickness experiences
 278 much more high frequencies, as attested by the important wavelets localized on the opposite
 279 side of the striking point. The last two rows present the displacement fields at $t = 13$ ms
 280 and 31 ms, showing undoubtedly how the high frequency content is very quickly excited in
 281 the vibration and localized in the regions of smaller thickness. The last two rows clearly
 282 evidenced the very nonlinear characteristics of the vibration field when the thickness of the
 283 plate is severely decreased, as compared to the case of the constant thickness.

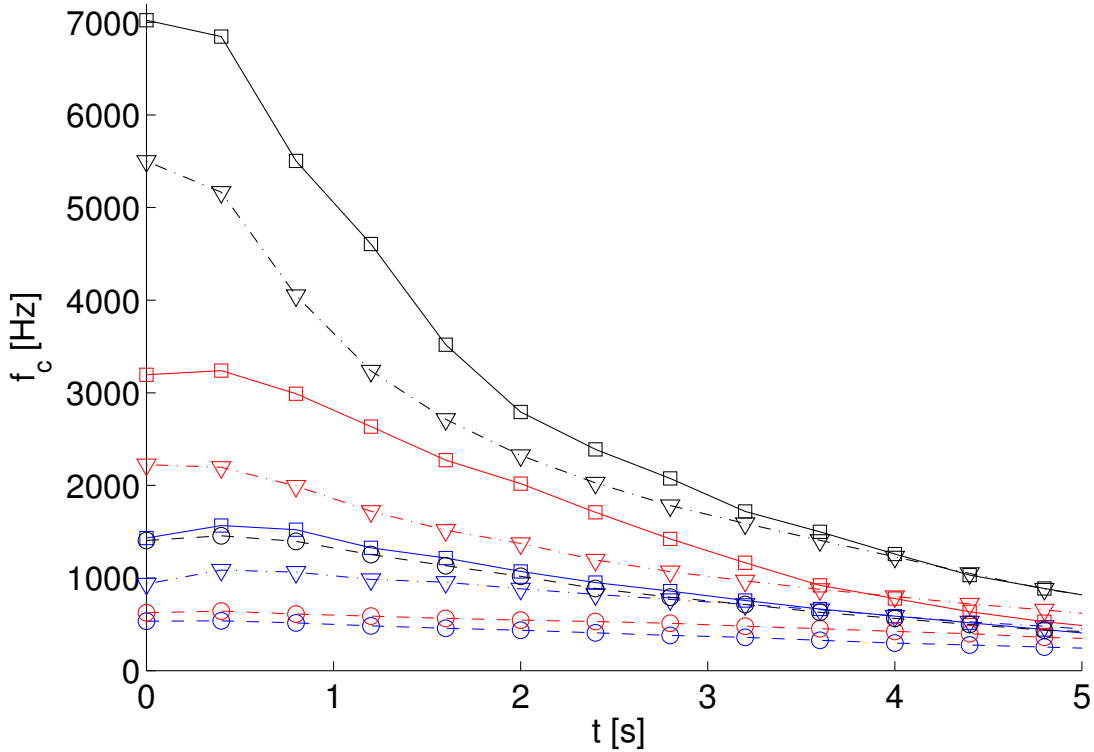


FIG. 4. Characteristic frequency f_c defined in Eq. (18) as a function of the time t of the computed velocities. Blue lines correspond to the case of the plate with uniform thickness $h = 1$ mm, red lines to the plate with linear varying thickness and $h_e = 0.7$ mm, and black lines with $h_e = 0.4$ mm. Three different strike amplitudes are used for each case: $p_m = 20$ N (circles \circ), 90 N (triangles ∇) and 150 N (squares \square).

284 In order to have more quantitative results on the high-frequency content generated by
 285 the vibration, the following characteristic frequency f_c is introduced as

$$f_c = \frac{\int_{f=0}^{f_S/2} a(f)^2 f df}{\int_{f=0}^{f_S/2} a(f)^2 df}, \quad (18)$$

286 where $a(f)$ is the Fourier amplitude (evaluated at frequency f) of the velocity signal obtained
 287 from the output point at $r = 0.1792$ m, and $f_S = 100$ kHz is the sampling rate. This
 288 characteristic frequency has already been used for quantifying the frequency content of
 289 a turbulent cascade of energy²⁶, as well as for the energy transfer observed in the contact
 290 dynamics of strings against frets²⁷. Fig. 4 shows the behaviour of this characteristic frequency
 291 as function of the time t of the computed velocity output, and for 9 simulations, with 3
 292 different plates (uniform thickness, $h_e = 0.7$ mm and $h_e = 0.4$ mm) and 3 different striking
 293 amplitudes ($p_m = 20, 90$ and 150 N); thus giving a quantitative point of view to the frequency
 294 behaviours displayed by the spectrograms shown in Fig. 2(d-f). In particular, one can
 295 clearly see the considerable effect of decreasing the thickness at edge from 1 to 0.4 mm,
 296 as the characteristic frequency is almost equal in the cases $h_e = 1$ mm, $p_m = 150$ N and
 297 $h_e = 0.4$ mm, $p_m = 20$ N. Concentrating on the case $p_m = 150$ N, one can observe that the
 298 characteristic frequency is multiplied by a factor of 2 when h_e is decreased from 1 mm to
 299 0.7 mm, and once again a factor of 2.3 by decreasing from 0.7 to 0.4 mm.

300 B. A complete model for cymbal vibrations

301 In this section, a complete model including both shape and thickness variations, is con-
 302 sidered. The shape of the imperfection has been measured on a real Zildjian "custom rock"

303 crash cymbal having a radius $R_d = 0.2$ m, and shown in Fig. 5. The height at the centre
 304 H_c has been found equal to 3.4 cm, however it will be used as a free parameter in order to
 305 compare cymbals with different shallowness. In particular, simulations with $H_c = 1.7$ cm
 306 (half of the real measurement, shown as a dashed line in Fig. 5), will be addressed.

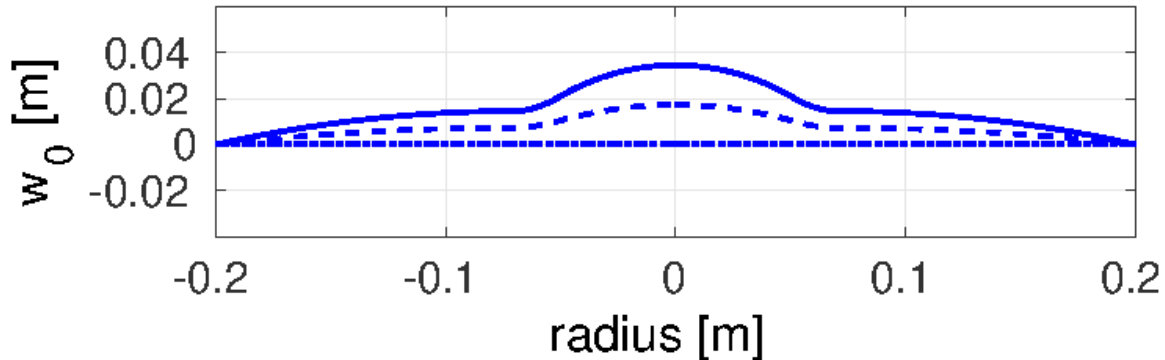


FIG. 5. Profile of the shape imperfection used in the simulations. The shape has been measured from a real cymbal, with a height at centre $H_c = 3.4$ cm. An intermediate case with $H_c = 1.7$ cm (dashed line), is also be considered.

307 The main aim of the section is to compare the relative effects of shape and thickness
 308 variations on the sound produced by cymbals, and in particular the ability of producing two
 309 distinctive sounds when hit on the bell or at the edge.

310 A first comparison is drawn out in Fig. 6, which shows the velocity spectrograms of two
 311 different cases simulated. The first case, Fig. 6(a), is that of a plate having uniform thickness
 312 $h = 1$ mm, and only shape variation following the profile shown in Fig. 5 with a maximum
 313 height at centre $H_c = 3.4$ cm. In the second case, the same shape variation is considered, and
 314 a thickness dependence following the guidelines used in the previous section, is taken into
 315 account: a linear thickness variation, as in Fig. 1(a), with $h_e = 0.4$ mm. The two velocity

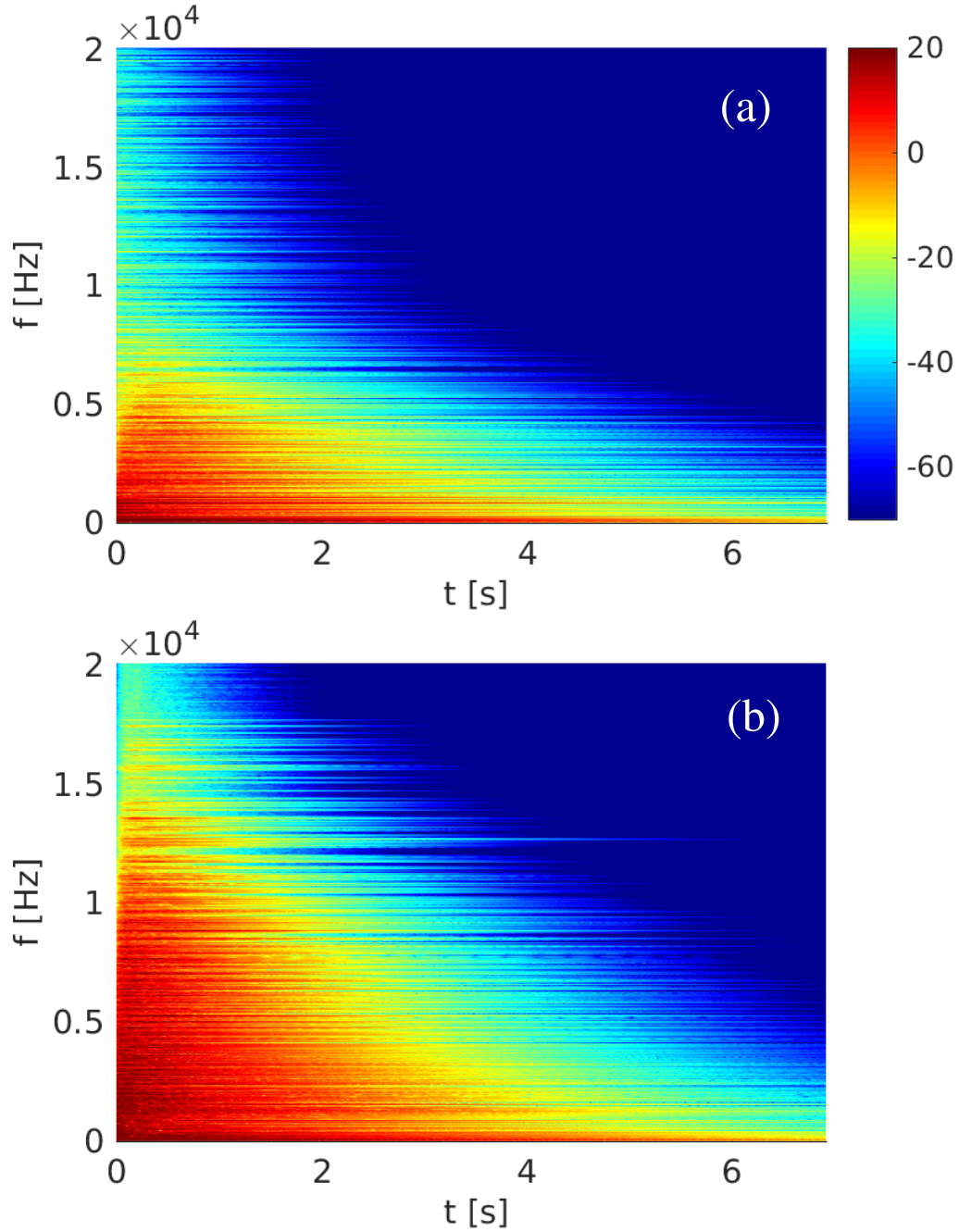


FIG. 6. Spectrogram of velocity with a 90 dB range. Strike at $t = 0$ s, with amplitude $p_m=90$ N, and contact time $T_{wid} = 1$ ms. (a) cymbal with uniform thickness $h = 1$ mm, shape variation only with height at centre $H_c = 3.4$ cm. (b) cymbal with the same shape variation, $H_c = 3.4$ cm, and linear decrease of the thickness, $h_e = 0.4$ mm.

316 spectrograms shown in Fig. 6 are thus complementary to those shown in Fig. 2 and must be

317 compared altogether. In particular, contrasting Fig. 6(a) with 2(d), *i.e.* the two cases with
 318 the uniform thickness $h = 1$ mm and with/without shape variations, one can observe that
 319 the energy transfer is slightly more pronounced in the case with shape variation. Indeed,
 320 the maximum frequency attained in the spectrogram of Fig. 6(a) is around 750-800 Hz
 321 whereas it is only 500 Hz in 2(d) for the flat plate. This phenomenon is easily explained
 322 by the presence of quadratic nonlinearity in the case of shape imperfections, which is key
 323 to generating more easily energy transfers²⁸. This effect is aurally particularly striking (see
 324 companion wav files) : the enrichment brought by the shape and the quadratic nonlinearities
 325 is really noticeable, especially because it happens in the most sensitive frequency band for the
 326 ear. However, shape variation alone is not sufficient in order to create a rich spectrum as that
 327 obtained in Fig. 2(f). Adding the thickness variation allows retrieving the bright spectrum
 328 with frequencies up to 14 kHz, as shown in Fig. 6(b). Hence, variation of thickness seems
 329 to be really needed in order to build the explosive sound and excite very high frequencies
 330 beyond 10 kHz; however it is undoubted that shape variation is already important and adds
 332 a brightness to the sound produced by adding frequencies in the sensitive band of the ear.

333 For a more quantitative comparison, Fig. 7 shows the characteristic frequencies for a
 334 number of test cases, mixing both shape and thickness variations. Interestingly, it is found
 335 that when the thickness is uniform, $h = 1$ mm, and with the maximum height of the shape
 336 imperfection being half the measured one, $H_c = 1.7$ cm; a better energy build-up is obtained
 337 than with $H_c = 3.4$ cm. This can be explained that this case offers a better compromise
 338 between the appearance of quadratic nonlinearity and the stiffening effect produced by
 339 the curvature. Increasing H_c results in a more and more stiff structure with increasing

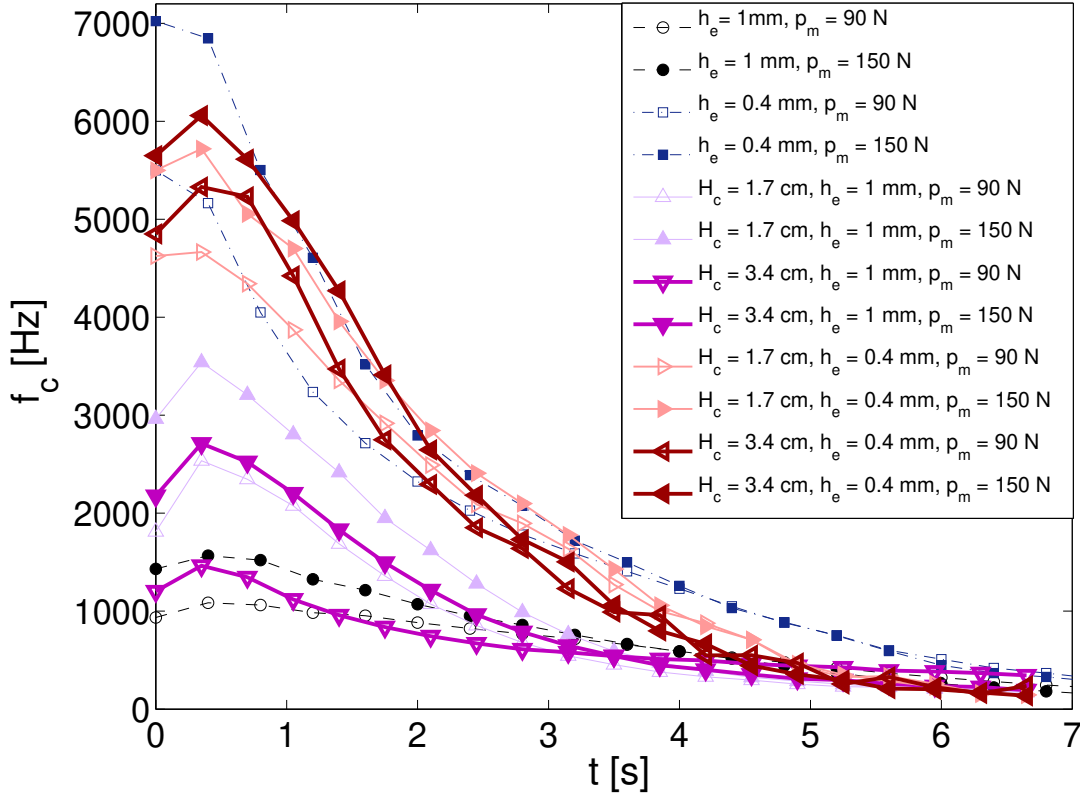


FIG. 7. Characteristic frequency f_c as a function of the time t for twelve different cases reported in inset, mixing shape and thickness variations, amplitudes of strike at the edge 90 and 150 N.

340 eigenfrequencies, making the couplings between modes and energy transfer more difficult.
 341 Anyhow Fig. 7 highlights that adding a shape imperfection always favours the energy build-
 342 up, but the thickness variation is also needed in order to attain very high values for this
 343 characteristic frequency. Indeed, all the curves with $h_e = 0.4$ mm are always the upper one
 344 for the characteristic frequency f_c .

345 Another aspect not investigated yet with the thickness decrease is related to the pitch
 346 glide. Indeed, decreasing the thickness at the edge results in geometric nonlinearities more
 347 easily excited since they are proportional to the ratio between the transverse displacement

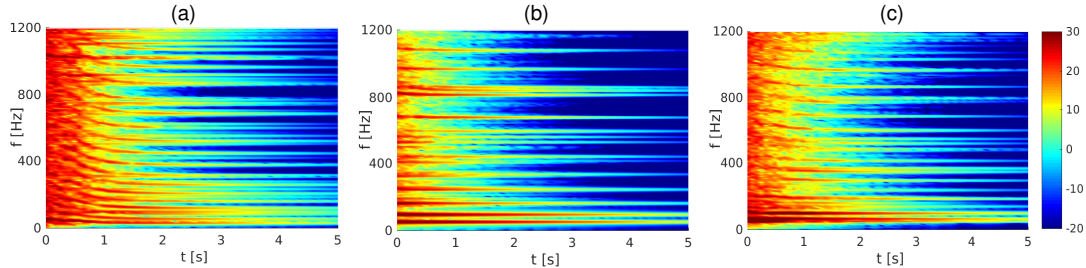


FIG. 8. Velocity spectrograms in the low frequency range for a strike at the edge of magnitude $p_m = 150$ N, and for three different cymbal configurations. Restrained range of 50 dB in order to highlight the frequency peaks. (a) only linear thickness variation down to $h_e = 0.4$ mm, flat plate without shape imperfection. (b) shape variation only with height at centre $H_c = 3.4$ cm, and uniform thickness $h=1$ mm. (c) shape and thickness variation with $H_c=3.4$ cm and $h_e = 0.4$ mm.

348 and the thickness. The pitch glide is related to the backbone curve of the modes of the
 349 structure, and is more pronounced if the transverse displacement is large as compared to
 350 $h^{3,11,29-31}$. problem one may encounter if the plate is too thin, is that the pitch glide will be
 351 too important, resulting in a sound which is not pleasant to the ear and will more resemble
 352 that of a membrane. This effect can be easily simulated with the methods used in this
 353 paper, showing the need to stiffen the structure and explaining why very thin flat plates are
 354 not used as cymbals. This is illustrated in Fig. 8(a) concerning the case of the plate with
 355 linear thickness variation and $h_e = 0.4$ mm, for a vigorous strike with $p_m = 150$ N. The
 356 attention is paid to the low frequency range, below 1200 Hz, and the very important pitch
 357 glides of all frequency peaks are clearly visible. On the other hand, Fig. 8(b) is concerned
 358 with a cymbal with uniform thickness and shape variation only, with a maximum height at
 359 centre $H_c = 3.4$ cm. In this case, the cymbal is so stiff that almost no pitch glide is present,
 360 even though an important strike is given to the structure. Fig. 8(c) displays the case with
 361 $h_e = 0.4$ mm and $H_c = 3.4$ cm, where a good compromise is found between creating a strong

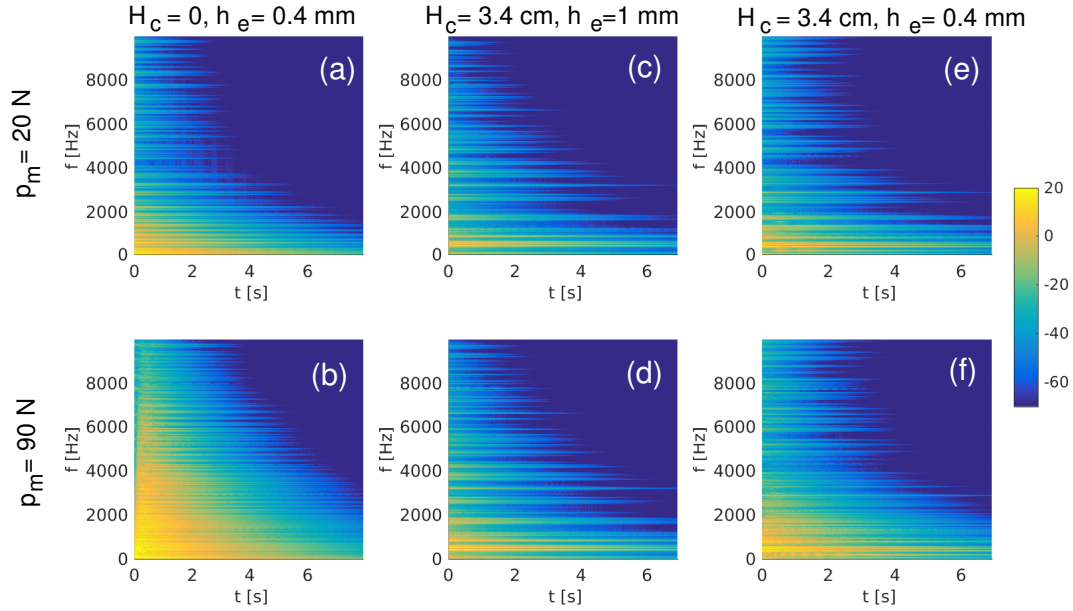


FIG. 9. Velocity spectrograms, cymbal hit at $r=4$ cm from the centre. First row : soft strike, $p_m=20$ N, second row, hard strike, $p_m=90$ N. First column (a-b) : variable thickness only, $h_e = 0.4$ mm. Second column, (c-d) : Shape variation only, uniform thickness $h=1$ mm and height at centre $H_c=3.4$ cm. Third column (e-f) : $H_c=3.4$ cm and linear thickness variation down to $h_e = 0.4$ mm.

362 nonlinear effect resulting in rapid energy build-up, and minimizing the pitch glide which
 363 shall not be too pronounced and unpleasant to the ear.

364 The last simulation investigates the behaviour of the cymbal when hit at centre. Indeed,
 365 as stated before, a bell sound has to be obtained when striking the cymbal near the centre.
 366 Fig. 9 shows the velocity spectrograms for three different cymbal configurations and two
 367 amplitudes of strikes, $p_m = 20$ N and $p_m = 90$ N. The strike is imposed near the centre, at
 368 $r = 4$ cm. Fig. 9(a-b) considers the case of only thickness variation, with linear variation
 369 and $h_e = 0.4$ mm. It shows that even with moderate strikes the nonlinear regime is too
 370 rapidly excited, and the broadband spectrum is at hand, even though the cymbal is hit
 371 at centre. This shows undoubtedly that this configuration is not and cannot be used for

372 a cymbal since the two distinctive sounds (bell and crash) are out of reach. Fig. 9(c-d)
 373 shows the results when only shape variation is considered, with $H_c = 3.4$ cm, and a constant
 374 thickness $h = 1$ mm. In this case the spectrograms are almost unchanged and the sounds
 375 are completely equivalent to the ear. This is always the case if the strike is increased up
 376 to $p_m = 150$ N, meaning that linear vibrations are at hand up to very large amplitudes of
 377 input forces. This case appears to be too stiff, preventing the drummer some colours to the
 378 sound found when striking the bell. Finally, Fig. 9(e-f) shows the results obtained with both
 379 shape and thickness variations, where $H_c = 3.4$ cm and $h_e = 0.4$ mm. This case seems to
 380 show a correct compromise, the sound produced when striking at centre being almost linear
 381 with a small excitation of nonlinearity for $p_m=90$ N, giving a nice colour to the sound.

382 IV. CONCLUSION

383 This paper investigates the nonlinear vibrations of cymbals with a particular emphasis on
 384 the presence of the thickness variations in a physical model. While the model of flat plates
 385 with uniform thickness derived has been shown to be effective for simulating the sound of
 386 gongs⁴, the explosive sound of cymbal was more difficult to obtain. Indeed, thickness and
 387 shape variations play important roles in cymbal making, particularly in order to obtain two
 388 different behaviours when the excitation is at the centre or at the edge. When a cymbal
 389 is hit at its centre, a bell-like sound is awaited so that nonlinearities should not be too
 390 much excited. On the contrary, a scintillating sound with a rapid build-up of energy and
 391 generation of a large broadband Fourier spectrum in the first milliseconds is expected from
 392 vigorous strikes at the edge. One of the main outcomes of the present study is to derive such a

393 model, based on the von Kármán assumptions for thin plates, and accounting for both effects
 394 of thickness and shape variations. An ad-hoc numerical method is adopted, from earlier
 395 studies, to handle the complexity brought by these new terms for a time-domain simulation,
 396 using the Störmer-Verlet scheme for time integration and the Rayleigh-Ritz method, which
 397 allows one to give frequency-dependent modal damping factors as inputs. The substantial
 398 effect of thickness variations has been underlined and a set of simulations has been conducted
 399 in order to get a comprehensive view of the two different effects, resulting from shape and
 400 thickness variations. Simulations and synthesized sounds show that shape variations are
 401 very important the energy build-up, a result that was not expected but could easily be
 402 understood thanks to the quadratic nonlinearity. However, imposing only shape variations
 403 results in too stiff cymbals, so that only linear behaviour is observed when hit at centre,
 404 and the energy build-up is less important than that obtained with thickness variations. On
 405 the other hand, thickness variations alone make the plate too compliant in general, causing
 406 too high pitch glides that are not pleasant to the ear. Both of the physical features are
 407 thus needed, but the fact that shape variations already have an important effect, which is
 408 especially prominent in the frequency band where the ear is the most sensitive, shows that
 409 the thickness variations may be at a moderate level, a conclusion that is confirmed by the
 410 values of the edge thickness found in most of the cymbals.

411 APPENDIX A: LINEAR ANALYSIS FOR THE TRANSVERSE PROBLEM

412 In this section the complete formula for computing the mass and stiffness matrices \mathbf{M}_Φ
 413 and \mathbf{K}_Φ , related to the linear problem for the transverse vibrations, are given. The mass

414 matrix comes from the inertia term in (4). A simple calculation shows that the entries $\mathbf{M}_{\Phi_{ij}}$
 415 read:

$$\mathbf{M}_{\Phi_{ij}} = \int_S \rho h(r) \Phi_i \Phi_j r dr d\theta. \quad (\text{A1})$$

416 In order to compute easily the current term in the stiffness matrix, the following identity is
 417 used:

$$\Delta(D(r)\Delta\Phi_p) = \Delta D \Delta\Phi_p + D \Delta\Delta\Phi_p + 2\nabla D \cdot \nabla(\Delta\Phi_p). \quad (\text{A2})$$

Consequently the stiffness matrix can be decomposed as $\mathbf{K}_{\Phi} = \mathbf{K}_{\Phi}^1 + \mathbf{K}_{\Phi}^2 + \mathbf{K}_{\Phi}^3 + \mathbf{K}_{\Phi}^4$, where
 the entries of each term read:

$$\mathbf{K}_{\Phi_{ij}}^1 = \int_S (\Delta D) (\Delta\Phi_j) \Phi_i r dr d\theta, \quad (\text{A3})$$

$$\mathbf{K}_{\Phi_{ij}}^2 = \int_S \omega_j^2 D(r) \Phi_i \Phi_j r dr d\theta, \quad (\text{A4})$$

$$\mathbf{K}_{\Phi_{ij}}^3 = 2 \int_S \nabla D \cdot \nabla(\Delta\Phi_j) \Phi_i r dr d\theta, \quad (\text{A5})$$

$$\mathbf{K}_{\Phi_{ij}}^4 = -(1 - \nu) \int_S \mathcal{L}(D, \Phi_j) \Phi_i r dr d\theta. \quad (\text{A6})$$

418 These equations can be used for an arbitrary boundary conditions once the eigenmodes
 419 Φ_i and eigenfrequencies ω_i are known. In the case of a free edge, the eigenmodes write
 420 $\Phi_{kn}(r, \theta) = R_{kn}(r) \cos k\theta$ (first configuration) and $\Phi_{kn}(r, \theta) = R_{kn}(r) \sin k\theta$ (second configu-
 421 ration), where the index k refers to the number of nodal diameters and n to the number of
 422 nodal circles. When $k = 0$ the modes are axisymmetric and no dependence on θ is found.
 423 When $k \neq 0$, the eigenfrequency is degenerated with a multiplicity of 2, and the two con-
 424 figurations, one in sine and the other one in cosine, are present. $R_{kn}(r)$ is a combination of

425 Bessel functions, the expression of which is given¹⁶. Plugging these expressions of Φ_i into
 426 the previous equations, one can separate the radial part from the angular part, and all the
 427 expressions are analytic. The integrals are then computed numerically from the analytic
 428 expressions, leading to a perfect control of the accuracy of the results.

429 **APPENDIX B: LINEAR ANALYSIS FOR THE IN-PLANE PROBLEM**

Using the same identity (A2) to develop the terms arising from (7), one can write $\mathbf{K}_\Psi = \mathbf{K}_\Psi^1 + \mathbf{K}_\Psi^2 + \mathbf{K}_\Psi^3 + \mathbf{K}_\Psi^4$, where the elements of each matrix read:

$$\mathbf{K}_{\Psi ij}^1 = \int_S \Delta B \Delta \Psi_j \Psi_i r dr d\theta, \quad (\text{B1})$$

$$\mathbf{K}_{\Psi ij}^2 = \int_S \zeta_j^4 D(r) \Psi_j \Psi_i r dr d\theta, \quad (\text{B2})$$

$$\mathbf{K}_{\Psi ij}^3 = 2 \int_S \nabla B \cdot \nabla (\Delta \Psi_j) \Psi_i r dr d\theta, \quad (\text{B3})$$

$$\mathbf{K}_{\Psi ij}^4 = - (1 + \nu) \int_S \mathcal{L}(B, \Psi_j) \Psi_i r dr d\theta. \quad (\text{B4})$$

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