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ON AN ARITHMETICAL QUESTION RELATED TO PERFECT NUMBERS

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

For $n \geq 1$ an integer, we shall denote by $\sigma(n)$ the sum of the (positive) divisors of n .

The line of thought leading to this note started with [3, Exemple III, p. 380], where two integers k such that $\sigma(k^3)$ is a perfect square are given.

This suggests that we might look for numbers n such that $\sigma(n)$ be a square. One rapidly notices that $\sigma(66) = 144 = 12^2$; furthermore

$$\sigma(66) - 2.66 = 144 - 132 = 12,$$

whence

$$\sigma(66) = (\sigma(66) - 2.66)^2.$$

We were thus led to the following definition :

Definition. An integer n is termed **quadratically perfect** if

$$\sigma(n) = (\sigma(n) - 2n)^2.$$

As we have just established, 66 is such a number. A quick search yields three others : 1, 3 and 491536.

As a matter of fact, one has

Lemma 1. *The only quadratically perfect primary integers are 1 and 3.*

Proof. Let $n = p^k$ (p prime) be quadratically perfect ; if $k = 0$, $n = 1$. Let us then assume $k \geq 1$; we have

$$\sigma(n) = 1 + p + \dots + p^k,$$

whence, setting $S := \sum_{l=0}^{k-1} p^l$, we find

$$\sigma(n) - 2n = S - p^k$$

and

$$\begin{aligned} (S - p^k)^2 &= (\sigma(n) - 2n)^2 \\ &= \sigma(n) \\ &= S + p^k. \quad (*) \end{aligned}$$

If $k \geq 2$, (*) gives us, by working modulo p^2 , that

$$S^2 \equiv S \pmod{p^2};$$

as $S \equiv 1 + p \pmod{p^2}$, it follows that

$$(1 + p)^2 \equiv 1 + p \pmod{p^2},$$

i.e. p^2 divides p , a contradiction. Therefore $k = 1$, $S = 1$ and (*) becomes

$$(1 - p)^2 = 1 + p,$$

that is $p^2 = 3p$, $p = 3$ and $n = p^k = 3^1 = 3$. □

2. THE MAIN THEOREM

On the positive side, one has

Proposition 1. *If m is a perfect number such that $2m - 1$ is prime, then*

$$n := m(2m - 1)$$

is quadratically perfect.

Remark. $m = 6$ ($2m - 1 = 11$) yields $n = 66$, and $m = 496$ ($2m - 1 = 991$) yields $n = 491536$.

Proof. Seeing that m and $2m - 1$ are coprime, one has

$$\begin{aligned} \sigma(n) &= \sigma(m(2m - 1)) \\ &= \sigma(m)\sigma(2m - 1) \\ &= 2m(1 + (2m - 1)) \\ &\quad \text{[as } m \text{ is perfect and } 2m - 1 \text{ is prime]} \\ &= 4m^2. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma(n) - 2n &= 4m^2 - 2m(2m - 1) \\ &= 2m, \end{aligned}$$

and

$$(\sigma(n) - 2n)^2 = (2m)^2 = 4m^2 = \sigma(n);$$

n is quadratically perfect. □

A partial converse holds :

Theorem 1. *Let n be an even quadratically perfect number ; then there exists an even perfect number m such that $n = m(2m - 1)$.*

Remark. It is well-known (see e.g. [1,Theorem 277], or [2,pp.33-34]) that m can then be written as

$$m = 2^{p-1}(2^p - 1)$$

with $2^p - 1$ (hence also p) prime ; $m = 6$ corresponds to $p = 2$, and $m = 496$ to $p = 5$.

3. THE PROOF OF THEOREM 1

For convenience, this will be broken up into four lemmas.

Let us write $n = 2^a x$ ($a \geq 1$, x odd) ; then

$$\begin{aligned}\sigma(n) &= \sigma(2^a)\sigma(x) \\ &= (2^{a+1} - 1)\sigma(x).\end{aligned}$$

Lemma 2. *There are integers b , e and f such that $\sigma(x) = be^2$ and*

$$bce^2 - (bc^2 + 1)f - e = 0.$$

Proof. Let us define b as the *square-free part* of $2^{a+1} - 1$; then

$$2^{a+1} - 1 = bc^2,$$

for some $c \in \mathbf{N}$, $c \geq 1$. As $2^{a+1} - 1 \equiv -1 \pmod{4}$, $2^{a+1} - 1$ is not a square, therefore $b \geq 2$; in fact, $b \geq 3$ as b is odd.

As $(\sigma(n) - 2n)^2 = \sigma(n)$, we have

$$\begin{aligned}bc^2\sigma(x) &= (2^{a+1} - 1)\sigma(x) \\ &= \sigma(n) \\ &= (\sigma(n) - 2n)^2\end{aligned}$$

whence c divides $\sigma(n) - 2n$:

$$\sigma(n) - 2n = \epsilon cd,$$

with $\epsilon \in \{-1, 1\}$ and $d \geq 1$. It follows that

$$bc^2\sigma(x) = c^2d^2,$$

i.e. $b\sigma(x) = d^2$, and b divides d^2 ; b being square-free, b divides d : $d = be$ ($e \geq 1$). Therefore $b\sigma(x) = b^2e^2$, *i.e.* $\sigma(x) = be^2$. It now appears that

$$\begin{aligned}b^2c^2e^2 - (bc^2 + 1)x &= bc^2.be^2 - 2^{a+1}x \\ &= (2^{a+1} - 1)\sigma(x) - 2n \\ &= \sigma(n) - 2n \\ &= \epsilon cd \\ &= \epsilon cbe;\end{aligned}$$

therefore, bc divides x .

Let us then set $x = bcf$ ($f \geq 1$); as $b > 1$, $bc > 1$, $be^2 = \sigma(x) \geq x + f$ and

$$\begin{aligned}b^2c^2e^2 &= bc^2\sigma(x) \\ &\geq bc^2x + bc^2f \\ &\geq bc^2x + bcf \\ &= bc^2x + x,\end{aligned}$$

thus $\epsilon = 1$.

We now get

$$b^2c^2e^2 - (bc^2 + 1)bcf = cbe,$$

that is

$$bce^2 - (bc^2 + 1)f - e = 0.$$

□

Lemma 3. $c = 1$, i.e. $2^{a+1} - 1$ is square-free.

Proof. We know that $b > 1$; then

$$be^2 = \sigma(x) = \sigma(bcf) \geq bcf + cf,$$

and, using Lemma 3.1 :

$$\begin{aligned} 0 &= bce^2 - (bc^2 + 1)f - e \\ &\geq c(bcf + cf) - (bc^2 + 1)f - e \\ &= (c^2 - 1)f - e. \end{aligned}$$

Assuming that $c > 1$, it would follow that $f \leq \frac{e}{c^2 - 1}$, whence that

$$\begin{aligned} bce^2 &= (bc^2 + 1)f + e \\ &\leq (bc^2 + 1)\frac{e}{c^2 - 1} + e \\ &= \frac{e}{c^2 - 1}(bc^2 + c^2) \\ &= e(b + 1)\frac{c^2}{c^2 - 1}. \end{aligned}$$

It would now appear that

$$\begin{aligned} e &\leq \frac{b + 1}{b} \frac{c}{c^2 - 1} \\ &\leq \frac{4}{3} \frac{3}{8} = \frac{1}{2} < 1, \end{aligned}$$

a contradiction.

Therefore $c = 1$. □

Lemma 4. b is prime.

Proof. From Lemmas 3.1 and 3.2 follows that

$$be^2 - e - (b + 1)f = 0.$$

As $x = bcf = bf$, one has $\sigma(bf) = \sigma(x) = be^2$.

We know that $b \geq 3$; let us assume that b is not prime, and let p denote the smallest prime factor of b . Then

$$\sigma(b) \geq b + p + 1 \geq b + 4$$

and

$$f(b + 4) \leq f\sigma(b) \leq \sigma(bf) = be^2,$$

whence

$$f(b + 4) \leq be^2 = e + (b + 1)f,$$

and $3f \leq e$. It follows that

$$\begin{aligned}
3be^2 &= 3e + (3b + 3)f \\
&\leq 3e + (b + 1)e \\
&= e(b + 4),
\end{aligned}$$

hence $3b \leq 3be \leq b + 4$, $2b \leq 4$ and $b \leq 2$, a contradiction.
Therefore b is prime. □

Lemma 5. *b does not divide f .*

Proof. Let us assume for a moment that b divides f ; then, as

$$be^2 = e + bf + f,$$

b divides e . It follows that

$$be^2 = \sigma(bf) \geq bf + f + \frac{f}{b},$$

whence

$$\frac{f}{b} \leq be^2 - bf - f = e,$$

thus $f \leq be$.

Now

$$\begin{aligned}
be^2 &= e + bf + f \\
&\leq e + b^2e + be
\end{aligned}$$

and

$$\begin{aligned}
be &\leq b^2 + b + 1 \\
&= b(b + 1) + 1 \\
&< b(b + 2).
\end{aligned}$$

It appears that $e < b + 2$, hence $e \leq b + 1 < 2b$; as b divides e , $e = b$ and

$$(b + 1)f = be^2 - e = b^3 - b,$$

thus $f = b(b - 1)$.

Now we have

$$\sigma(b^2(b - 1)) = \sigma(bf) = be^2 = b^3 = b^2(b - 1) + b^2,$$

an obvious contradiction as

$$\sigma(b^2(b - 1)) \geq b^2(b - 1) + b^2 + b + 1.$$

We conclude that b does not divide f . □

Conclusion

We may now write

$$\begin{aligned}
 (b+1)\sigma(f) &= \sigma(b)\sigma(f) \\
 &\quad \text{[by Lemma 3.3]} \\
 &= \sigma(bf) \\
 &\quad \text{[as } b \text{ and } f \text{ are coprime, by Lemmas 3.3 and 3.4]} \\
 &= be^2 \\
 &= bf + e + f ;
 \end{aligned}$$

thus $b+1$ divides e : $e = (b+1)g$, hence

$$\begin{aligned}
 (b+1)f &= e(be-1) \\
 &= (b+1)g(b(b+1)g-1)
 \end{aligned}$$

and $f = g(b(b+1)g-1)$. In particular, g divides f , and $g \neq f$. As

$$(b+1)\sigma(f) = bf + (b+1)g + f,$$

we obtain $\sigma(f) = f + g$; therefore $g = 1$ and f is prime.

Furthermore, $f = b(b+1) - 1$.

Now let $m := 2^a(2^{a+1} - 1) = 2^a b$; then m is perfect ([1, Theorem 276], or [2, p.33]) and even, $2m - 1 = 2^{a+1}b - 1 = (b+1)b - 1 = f$ is prime, and

$$m(2m - 1) = 2^a b f = 2^a x = n.$$

4. FINAL COMMENTS

It is reasonable to expect the converse of Proposition 2.1 to still hold for all odd $m \geq 5$; combined with the proof of the long-standing conjecture that there is no odd perfect number, this would imply the nonexistence of odd quadratically perfect numbers other than 1 and 3.

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