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# A generating polynomial for the two-bridge knot with Conway's notation $\mathcal{C}(n, r)$

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## Abstract

We construct an integer polynomial whose coefficients enumerate the Kauffman states of the two-bridge knot with Conway's notation  $\mathcal{C}(n, r)$ .

Keywords: generating polynomial, shadow diagram, Kauffman state.

## 1 Introduction

A *state* of a knot shadow diagram is a choice of splitting its crossings [2, Section 1]. There are two ways of splitting a crossing:

$$(A) \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \Rightarrow \begin{array}{c} \frown \\ \smile \end{array}, \quad (B) \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \Rightarrow \left. \begin{array}{c} \frown \\ \smile \end{array} \right) \left(.$$

By *state of a crossing* we understand either of the split of type (A) or (B). An example for the *figure-eight knot* is shown in [Figure 1](#).

Let  $K$  be a knot diagram. If  $m$  denotes the initial number of crossings, then the final states form a collection of  $2^m$  diagrams of nonintersecting curves. We can enumerate those states with respect to the number of their components – called *circles* – by introducing the sum

$$K(x) := \sum_S x^{|S|}, \tag{1}$$

where  $S$  browses the collection of states, and  $|S|$  gives the number of circles in  $S$ . Here,  $K(x)$  is an integer polynomial which we referred to as generating polynomial [6, 7] (in fact, it is a simplified formulation of what Kauffman calls “state polynomial” [2, Section 1–2] or

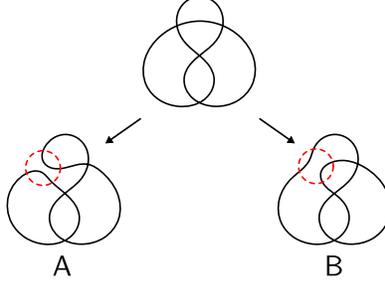


Figure 1: The states of a crossing.

“bracket polynomial” [3]). For instance, if  $K$  is the figure-eight knot diagram, then we have  $K(x) = 5x + 8x^2 + 3x^3$  (the states are illustrated in Figure 2).

In this note, we establish the generating polynomial for the *two-bridge knot* with Conway’s notation  $\mathcal{C}(n, r)$  [4, 5]. We refer to the associated knot diagram as  $B_{n,r}$ , where  $n$  and  $r$  denote the number of half-twists. For example, the figure-eight knot has Conway’s notation  $\mathcal{C}(2, 2)$ . Owing to the property of the shadow diagram which we draw on the sphere [1], we can continuously deform the diagram  $B_{n,r}$  into  $B_{r,n}$  without altering the crossings configuration. We let  $B_{n,r} \rightleftharpoons B_{r,n}$  express such transformation (see Figure 3 (a)). Besides, we let  $B_{n,0}$  and  $B_{n,\infty}$  denote the diagrams in Figure 3 (b) and (c), respectively. Here, “0” and “ $\infty$ ” are symbolic notations – borrowed from tangle theory [2, p. 88] – that express the absence of half-twists. If  $r = \infty$  and  $n \geq 1$ , then  $B_{n,\infty}$  represents the diagram of a  $(2, n)$ -torus knot ( $\rightleftharpoons B_{n-1,1}$ ). Correspondingly, we let  $B_{0,r}$  and  $B_{\infty,r}$  denote the diagrams pictured in Figure 3 (d) and (e), respectively.

## 2 Generating polynomial

Let  $K, K'$  and  $\bigcirc$  be knot diagrams, where  $\bigcirc$  is the trivial knot, and let  $\#$  and  $\sqcup$  denote the connected sum and the disjoint union, respectively. The generating polynomial defined in (1) verifies the following basic properties:

- (i)  $\bigcirc(x) = x$ ;
- (ii)  $(K \sqcup K')(x) = K(x)K'(x)$ ;
- (iii)  $(K \# K')(x) = \frac{1}{x}K(x)K'(x)$ .

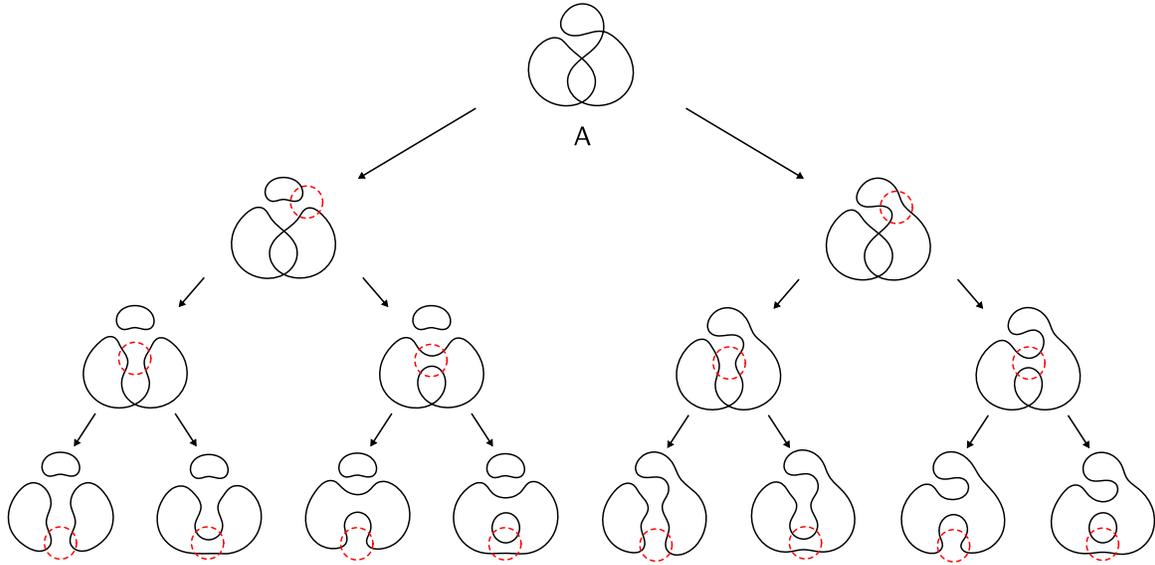
Furthermore, if  $K \rightleftharpoons K'$ , then  $K(x) = K'(x)$  [6].

**Lemma 1.** *The generating polynomial for the knots  $B_{n,0}$  and  $B_{n,\infty}$  are given by*

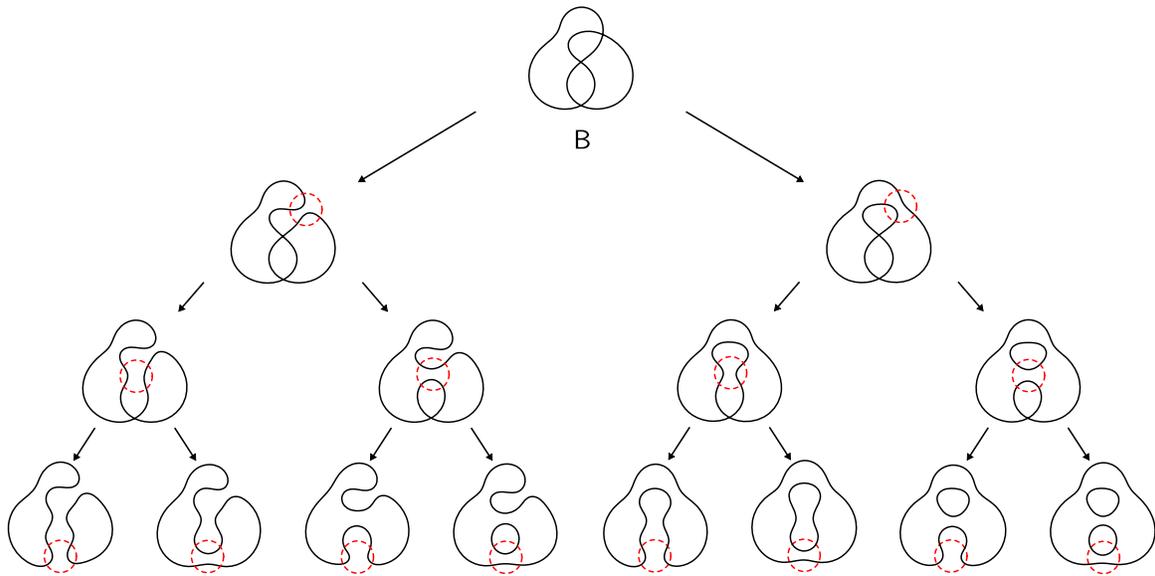
$$B_{n,0}(x) = x(x+1)^n \tag{2}$$

and

$$B_{n,\infty}(x) = (x+1)^n + x^2 - 1. \tag{3}$$



(a) The states of the figure-eight knot following the initial “A” split.



(b) The states of the figure-eight knot following the initial “B” split.

Figure 2: The states of the figure-eight knot.

The key ingredient for establishing (2) and (3) consists of the states of specific crossings which produce the recurrences

$$B_{n,0}(x) = (\bigcirc \sqcup B_{n-1,0})(x) + B_{n-1,0}(x)$$

and

$$B_{n,\infty}(x) = B_{n-1,0}(x) + B_{n-1,\infty}(x),$$

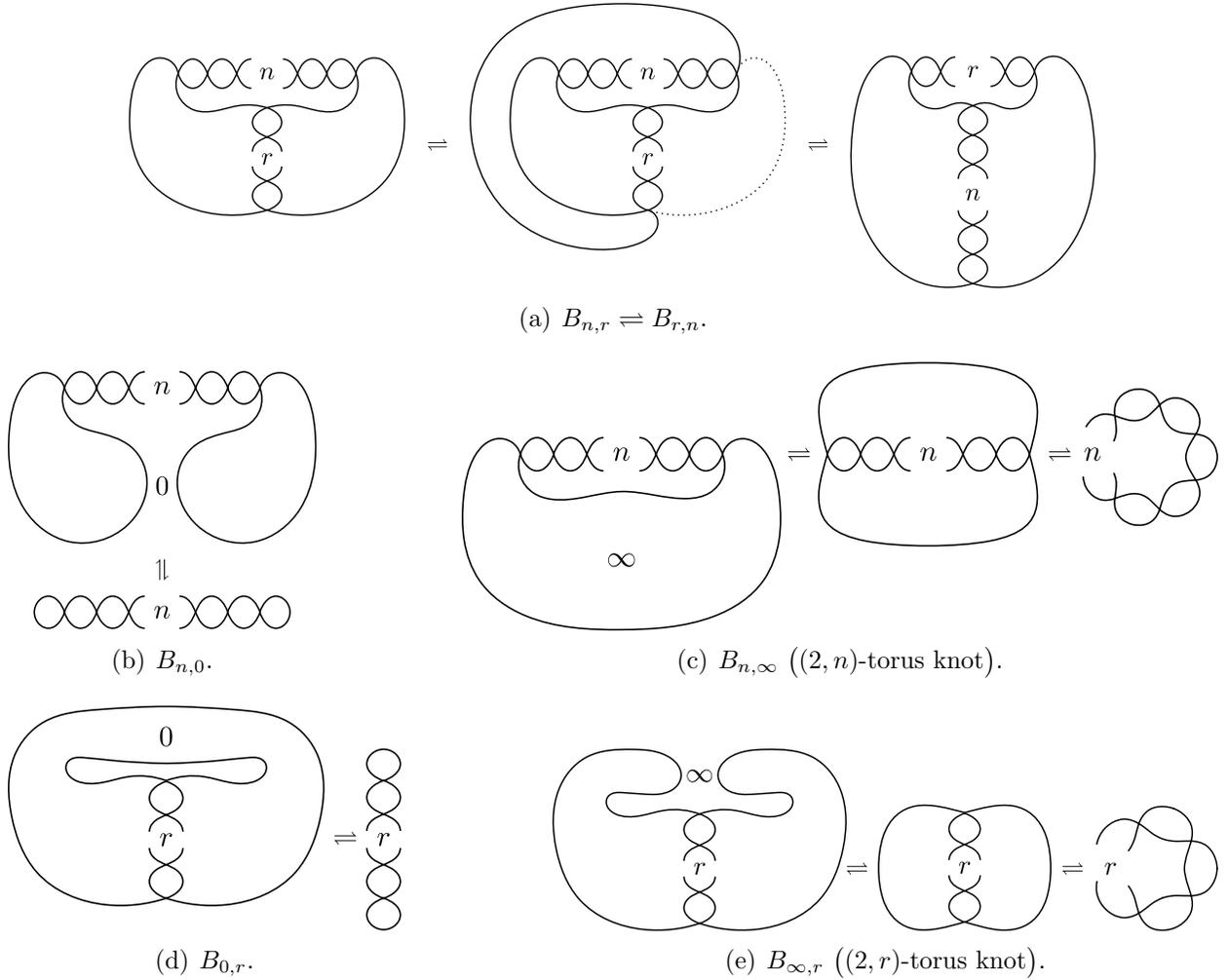


Figure 3: Two-bridge knots with Conway's notation  $\mathcal{C}(n, r)$ .

respectively, with initial values  $B_{0,0}(x) = x$  and  $B_{0,\infty}(x) = x^2$  [6]. Note that the lemma still holds if we replace index  $n$  by  $r$ .

**Proposition 2.** *The generating polynomial for the two-bridge knot  $B_{n,r}$  is given by the recurrence*

$$B_{n,r}(x) = B_{n-1,r}(x) + (x+1)^{n-1} B_{\infty,r}(x), \quad (4)$$

and has the following closed form:

$$B_{n,r}(x) = \left( \frac{(x+1)^r + x^2 - 1}{x} \right) (x+1)^n + (x^2 - 1) \left( \frac{(x+1)^r - 1}{x} \right). \quad (5)$$

*Proof.* By Figure 4 we have

$$\begin{aligned} B_{n,r}(x) &= B_{n-1,r}(x) + (B_{n-1,0} \# B_{\infty,r})(x) \\ &= B_{n-1,r}(x) + (x+1)^{n-1} B_{\infty,r}(x), \end{aligned}$$

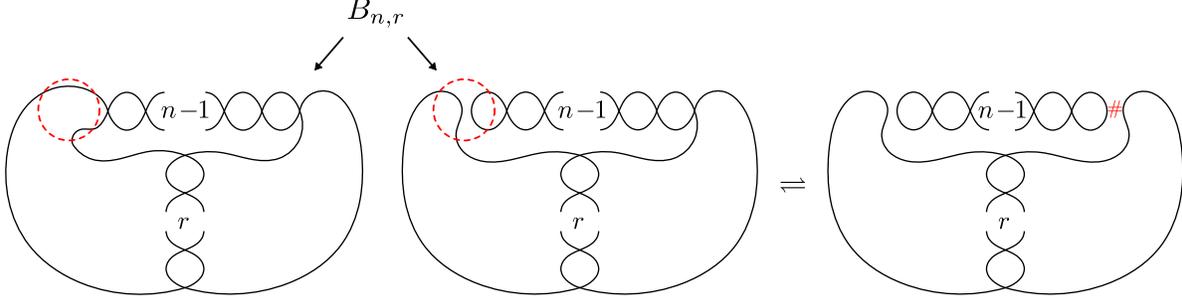


Figure 4: The splits at a crossing allow us to capture  $B_{n-1,r}$ ,  $B_{n-1,0}$  and  $B_{\infty,r}$ .

where the last relation follows from property (iii). Solving the recurrence for  $n$  yields

$$B_{n,r}(x) = B_{0,r}(x) + B_{\infty,r}(x) \left( \frac{(x+1)^n - 1}{x} \right).$$

We conclude by the closed forms in [Lemma 2](#). □

*Remark 3.* We can write

$$B_{n,0}(x) = x^2 \alpha_n(x) + x \tag{6}$$

and

$$B_{n,\infty}(x) = x \alpha_n(x) + x^2, \tag{7}$$

where  $\alpha_n(x) := \frac{(x+1)^n - 1}{x}$ , so that identity (5) becomes

$$B_{n,r}(x) = (x^2 \alpha_n(x) + x) + (x^2 \alpha_r(x) + x \alpha_n(x) \alpha_r(x)). \tag{8}$$

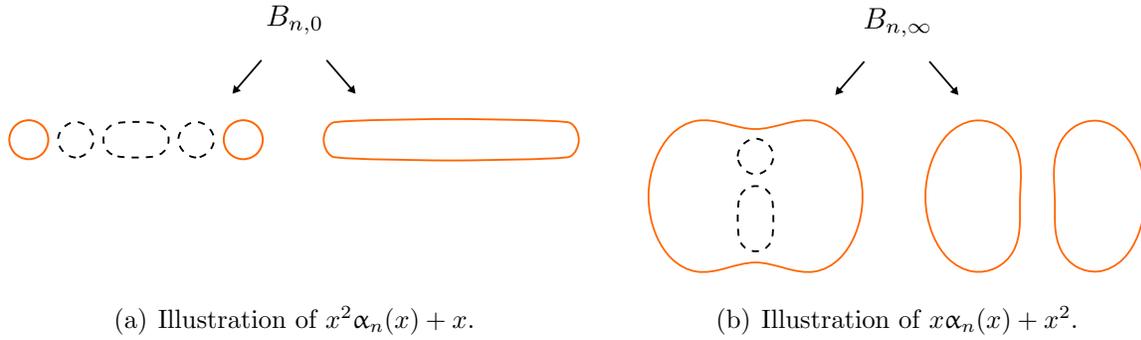
Since the coefficients of  $\alpha_n(x)$  are all nonnegative, it is clear, by (6), that the polynomial  $x^2 \alpha_n(x)$  counts the states of  $B_{n,0}$  that have at least 2 circles. This is illustrated in [Figure 5 \(a\)](#). Likewise, we have an interpretation of (7) in [Figure 5 \(b\)](#). In [Figure 5](#) and [6](#), the dashed diagrams represent all possible disjoint union of  $\ell - 1$  circles ( $\ell = n$  or  $r$ , depending on the context), counted by  $\alpha_\ell(x)$  and eventually empty.

Therefore, for  $n, r \notin \{0, \infty\}$ , identity (8) means that we can classify the states into 4 subset as shown in [Figure 6](#). In these illustrations, there are  $2^n - 1$  and  $2^r - 1$  states of (a) and (b) kind, respectively, and  $\binom{n}{1} \times \binom{r}{1} + 1$  one-component states of (c) and (d) kind. The remaining states are of (c) kind, bringing the total number of states to  $2^{n+r}$ .

### 3 Particular values

Let  $\sum_{k \geq 0} b(n, r; k) x^k := B_{n,r}(x)$ , or  $b(n, r; k) := [x^k] B_{n,r}(x)$ . Then

$$b(n, r; k) = \binom{n+r}{k+1} + \binom{n}{k-1} + \binom{r}{k-1} - \binom{n}{k+1} - \binom{r}{k+1} - \delta_{1,k},$$



(a) Illustration of  $x^2\alpha_n(x) + x$ .

(b) Illustration of  $x\alpha_n(x) + x^2$ .

Figure 5: Illustrations of  $B_{n,0}(x)$  and  $B_{n,\infty}(x)$  as functions of  $\alpha_n(x)$ .

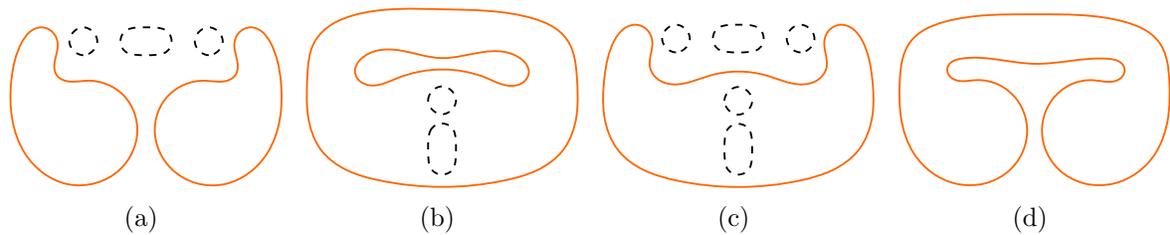


Figure 6: The states of  $B_{n,r}$ : states in (a) are counted by  $x^2\alpha_n(x)$ , those in (b) by  $x^2\alpha_r(x)$ , those in (c) by  $x\alpha_n(x)\alpha_r(x)$ , and state in (d) is simply counted by  $x$ .

where  $\delta_{1,k}$  is the Kronecker symbol. By (1), we recognize  $b(n, r; k)$  as the cardinal of the set  $\{|S| = k : S \text{ is a state of } B_{n,r}\}$ , i.e., the number of states having  $k$  circles. In this section, the coefficients  $b(n, r; k)$  are tabulated for some values of  $n, r$  and  $k$ . We give as well the corresponding A-numbers in the *On-Line Encyclopedia of Integer Sequences* [8].

- $b(n, 0; k) = [x^k] x(x+1)^n$ , essentially giving entries in Pascal's triangle [A007318](#) (see [Table 1](#)).

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	0	1							
1	0	1	1						
2	0	1	2	1					
3	0	1	3	3	1				
4	0	1	4	6	4	1			
5	0	1	5	10	10	5	1		
6	0	1	6	15	20	15	6	1	
7	0	1	7	21	35	35	21	7	1

Table 1: Values of  $b(n, 0; k)$  for  $0 \leq n \leq 7$  and  $0 \leq k \leq 8$ .

- $b(n, 1; k) = [x^k] ((x + 1)^{n+1} + x^2 - 1)$ , generating a subtriangle in [A300453](#) (see [Table 2](#)).

$n \setminus k$	0	1	2	3	4	5	6	7	8
0	0	1	1						
1	0	2	2						
2	0	3	4	1					
3	0	4	7	4	1				
4	0	5	11	10	5	1			
5	0	6	16	20	15	6	1		
6	0	7	22	35	35	21	7	1	
7	0	8	29	56	70	56	28	8	1

Table 2: Values of  $b(n, 1; k)$  for  $0 \leq n \leq 7$  and  $0 \leq k \leq 8$ .

- $b(n, 2; k) = [x^k] ((2x + 2)(x + 1)^n + (x^2 - 1)(x + 2))$ , giving triangle in [A300454](#) (see [Table 3](#)).

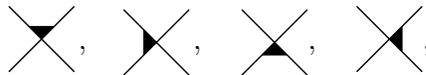
$n \setminus k$	0	1	2	3	4	5	6	7	8
0	0	1	2	1					
1	0	3	4	1					
2	0	5	8	3					
3	0	7	14	9	2				
4	0	9	22	21	10	2			
5	0	11	32	41	30	12	2		
6	0	13	44	71	70	42	14	2	
7	0	15	58	113	140	112	56	16	2

Table 3: Values of  $b(n, 2; k)$  for  $0 \leq n \leq 7$  and  $0 \leq k \leq 8$ .

- $b(n, n; k) = [x^k] \left( \frac{(x + 1)^{2n} + (x^2 - 1)(2(x + 1)^n - 1)}{x} \right)$ , giving triangle in [A321127](#) (see [Table 4](#)).

- $b(n, r; 1) = nr + 1$ , giving [A077028](#), and displayed as square array in [Table 5](#).

In Kauffman's language,  $b(n, r; 1)$  is, for a fixed choice of star region, the number of ways of placing *state markers* at the crossings of the diagram  $B_{n,r}$ , i.e., of the forms



$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1												
1	0	2	2											
2	0	5	8	3										
3	0	10	24	21	8	1								
4	0	17	56	80	64	30	8	1						
5	0	26	110	220	270	220	122	45	10	1				
6	0	37	192	495	820	952	804	497	220	66	12	1		
7	0	50	308	973	2030	3059	3472	3017	2004	1001	364	91	14	1

Table 4: Values of  $b(n, n; k)$  for  $0 \leq n \leq 7$  and  $0 \leq k \leq 13$ .

$n \setminus r$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	5	7	9	11	13	15
3	1	4	7	10	13	16	19	22
4	1	5	9	13	17	21	25	29
5	1	6	11	16	21	26	31	36
6	1	7	13	19	25	31	37	43
7	1	8	15	22	29	36	43	50

Table 5: Values of  $b(n, r; 1)$  for  $0 \leq n \leq 7$  and  $0 \leq r \leq 7$ .

so that the resulting states are “Jordan trails” [2, Section 1–2]. Note that a state marker is interpreted as an instruction to split a crossing as shown below:



The process is illustrated in [Figure 7](#) for the figure-eight knot.

- $b(n, r; 2) = n \left( \binom{r}{2} + 1 \right) + r \left( \binom{n}{2} + 1 \right)$ , giving square array in [A300401](#) (see [Table 6](#)).

We paid a special attention to the case  $k = 2$  because, surprisingly, columns  $(b(n, 1; 2))_n$  and  $(b(n, 2; 2))_n$  match sequences [A000124](#) and [A014206](#), respectively [6]. The former gives the maximal number of regions into which the plane is divided by  $n$  lines, and the latter the maximal number of regions into which the plane is divided by  $(n + 1)$  circles.

- $b(n, r; d(n, r)) = \text{leading coefficient of } B_{n,r}(x)$ , giving square array in [A321125](#) (see [Table 7](#)). Here,  $d(n, r) = \max(n + 1, r + 1, n + r - 1)$  denotes the degree of  $B_{n,r}(x)$ , and gives entries in [A321126](#). We have [Table 8](#) giving the numbers  $d(n, r)$  for  $0 \leq n \leq 7$  and  $0 \leq r \leq 7$ .

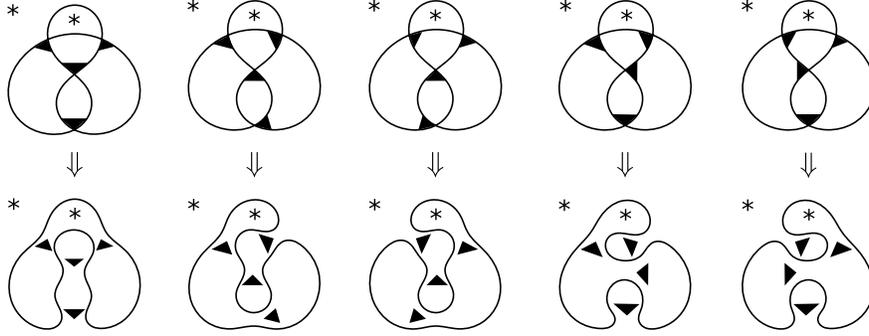


Figure 7: Illustration of  $b(2, 2; 1)$ : mark two adjacent regions by stars (\*), then assign a state marker at each crossing so that no region of  $B_{2,2}$  contains more than one state marker, and regions with stars do not have any.

$n \setminus r$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	4	7	11	16	22	29
2	2	4	8	14	22	32	44	58
3	3	7	14	24	37	53	72	94
4	4	11	22	37	56	79	106	137
5	5	16	32	53	79	110	146	187
6	6	22	44	72	106	146	192	244
7	7	29	58	94	137	187	244	308

Table 6: Values of  $b(n, r; 2)$  for  $0 \leq n \leq 7$  and  $0 \leq r \leq 7$ .

$n \setminus r$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	1	1	1	1	1	1
2	1	1	3	2	2	2	2	1
3	1	1	2	1	1	1	1	1
4	1	1	2	1	1	1	1	1
5	1	1	2	1	1	1	1	1
6	1	1	2	1	1	1	1	1
7	1	1	2	1	1	1	1	1

Table 7: Leading coefficients of  $B_{n,r}(x)$  for  $0 \leq n \leq 7$  and  $0 \leq r \leq 7$ .

We have the following properties:

- $d(n, r) = d(r, n)$ ;
- if  $r = 0$ , then  $d(n, r) = n + 1$ ;
- if  $r = \infty$ , then sequence  $(d(n, r))_n$  begins: 2, 2, 2, 3, 4, 5, 6, 7, 8, ... ([A233583](#) with

$n \setminus r$	0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7	8
1	2	2	3	4	5	6	7	8
2	3	3	3	4	5	6	7	8
3	4	4	4	5	6	7	8	9
4	5	5	5	6	7	8	9	10
5	6	6	6	7	8	9	10	11
6	7	7	7	8	9	10	11	12
7	8	8	8	9	10	11	12	13

Table 8: Values of  $d(n, r)$  for  $0 \leq n \leq 7$  and  $0 \leq r \leq 7$ .

offset 1).

Diagrammatically, we give the corresponding illustration for some values of  $n$  and  $r$  in [Figure 8](#).

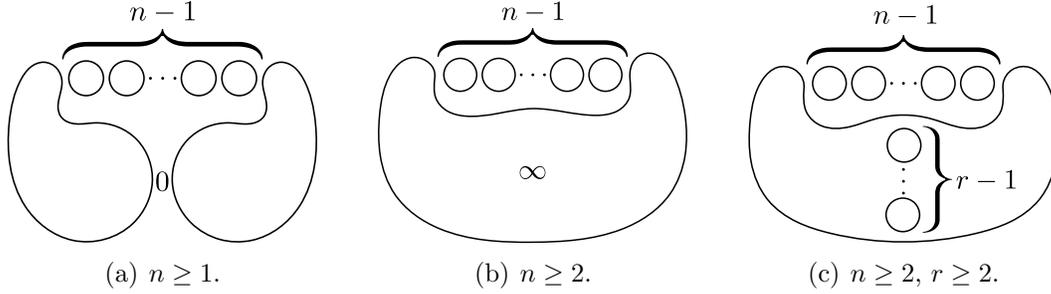


Figure 8: Illustration of the numbers  $d(n, r)$ .

Correspondingly, we have

- $b(n, r; d(n, r)) = b(r, n; d(r, n))$ ;
- if  $r = 0$ , then  $b(n, r; d(n, r)) = 1$ ;
- if  $r = \infty$ , then sequence  $(b(n, r; d(n, r)))_n$  begins:  $1, 1, 2, 1, 1, 1, 1, 1, \dots$  ([A294619](#) with initial term equals to 0).

Remarkable values in [Table 7](#) correspond to knots  $B_{1,1}$  (“Hopf link”, see [Figure 9](#)),  $B_{2,2}$  (figure-eight knot, see [Figure 1, 2](#)) and  $B_{n,2}$  (“twist knot” [6]) for  $n \geq 3$ . The latter case can be observed from identity (8) for which the leading coefficient is larger than 1 when  $n + 1 = n + r - 1$  is satisfied. Also, consider the identity below:

$$B_{n,2}(x) = B_{n,0}(x) + B_{n,\infty}(x) + B_{n,\infty}(x) + (\bigcirc \sqcup B_{n,\infty})(x).$$

We notice that the leading coefficient is deduced from the contribution of the summands  $B_{n,0}(x)$  and  $(\bigcirc \sqcup B_{n,\infty})(x)$  [6].

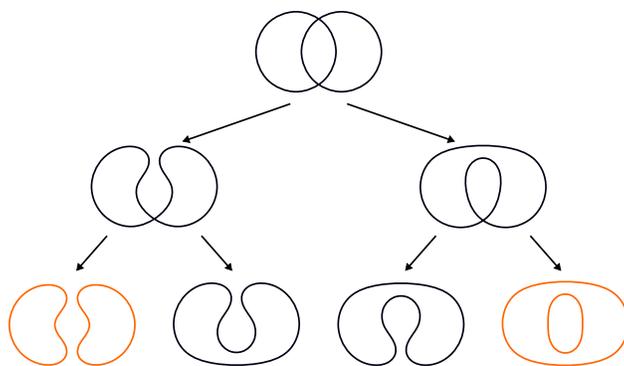


Figure 9: The states of the knot  $B_{1,1}$ :  $d(1, 1) = 2$  and  $b(1, 1, d(1, 1)) = 2$ .

## References

- [1] Daniel Denton and Peter Doyle, Shadow movies not arising from knots, arXiv preprint, 2011, <https://arxiv.org/abs/1106.3545>.
- [2] Louis H. Kauffman, *Formal Knot Theory*, Princeton University Press, 1983.
- [3] Louis H. Kauffman, State models and the Jones polynomial, *Topology* **26** (1987), 95–107.
- [4] Kelsey Lafferty, The three-variable bracket polynomial for reduced, alternating links, *Rose-Hulman Undergraduate Mathematics Journal* **14** (2013), 98–113.
- [5] Matthew Overduin, The three-variable bracket polynomial for two-bridge knots, California State University REU, 2013, <https://www.math.csusb.edu/reu/OverduinPaper.pdf>.
- [6] Franck Ramaharo, Enumerating the states of the twist knot, arXiv preprint, 2017, <https://arxiv.org/abs/1712.06543>.
- [7] Franck Ramaharo, Statistics on some classes of knot shadows, arXiv preprint, 2018, <https://arxiv.org/abs/1802.07701>.
- [8] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at <http://oeis.org>, accessed 2019.

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(Concerned with sequences [A000124](#), [A007318](#), [A014206](#), [A077028](#), [A233583](#), [A294619](#), [A300401](#), [A300453](#), [A300454](#), [A321125](#), [A321126](#) and [A321127](#).)

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