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SEMI-CLASSICAL GREEN FUNCTIONS

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Abstract: Let $H(x,p) \sim H_0(x,p) + hH_1(x,p) + \cdots$ be a semi-classical Hamiltonian on $T^*\mathbf{R}^n$, and $\Sigma_E = \{H_0(x,p) = E\}$ a non critical energy surface. Consider f_h a semi-classical distribution (the "source") microlocalized on a Lagrangian manifold Λ which intersects cleanly the flow-out Λ_+ of the Hamilton vector field X_{H_0} in Σ_E. Using Maslov canonical operator, we look for a semi-classical distribution u_h satisfying the limiting absorption principle and $H^w(x, hD_x)u_h = f_h$ (semi-classical Green function). In this report, we elaborate (still at an early stage) on some results announced in [AnDoNaRo1] and provide some examples, in particular from the theory of wave beams.

1. Introduction

Our general motivation is to solve equations like $(H - E)u_h = f_h$, with $f_h(x) = h^{-n} f\big(\frac{x}{h}\big)$, $f \in L^2(\mathbf{R}^n)$, a "localized function" at $x_0 = 0$. In case $H = -h^2\Delta$, we apply h-Fourier transform

$$\mathcal{F}_h u_h(\xi) = \int e^{-ix\xi/h} u_h(x)\, dx$$

to get $\mathcal{F}_h u_h(\xi) = \frac{\mathcal{F}_1 f(\xi)}{\xi^2 - E}$. If $E < 0$, this is an elliptic equation, and u_h has the same form as f_h. If $E > 0$ however,

$$u_+(x; E, h) = (2\pi h)^{-n} \int e^{ix\xi/h} \frac{\mathcal{F}_1 f(\xi)}{\xi^2 - E - i0}\, d\xi \tag{1}$$

is defined through regularisation, so to satisfy the limiting absorption principle. Actually,

$$u_+(x; E, h) = E_+ f_h, \quad E_+ = \frac{i}{h} \int_0^\infty e^{-it(H-E)/h}\, dt$$

is the forward parametrix. When $f_h(x)$ is replaced by $\delta(x - x_0)$, u_+ is the Green function. Such a distribution is very singular since there are infinitely many classical trajectories linking the "source" x_0 to x in time t; so it is suitable to consider instead $\mathcal{F}_1 f$ with compact support.

This problem has of course received considerable attention [Ke], [Ba], [Ku]; our special purpose is to reformulate it in terms of Maslov canonical operator, and describe the solution globally, including unfolding of Lagrangian singularities; this is of special importance in the context of wave propagation.

More generally, let $f_h = f(\cdot; h)$ be a semi-classical Lagrangian distribution (or oscillatory integral) defined on the manifold M (for all practical purposes, we shall assume here $M = \mathbf{R}^n$), i.e. locally

$$f(x; h) = (2\pi h)^{-n/2} \int e^{i\varphi(x,\theta)/h} a(x, \theta; h)\, d\theta$$

where $\varphi(x,\theta)$ is a non-degenerate phase function in the sense of [Hö], and $a(x,\theta;h) = a_0(x,\theta) + ha_1(x,\theta) + \cdots$ an amplitude. With f we associate the critical set $C_\varphi = \{(x,\theta) \in M \times \mathbf{R}^N : \partial_\theta \varphi = 0\}$ and $\iota_\varphi : C_\varphi \to T^*M$ with image the Lagrangian submanifold $\Lambda_\varphi = \{(x, \partial_x \varphi) : (x,\theta) \in C_\varphi\}$. Choose local coordinates $\xi \in \mathbf{R}^d$ on Λ_φ and define the half-density in the local chart $(C_\varphi, \iota_\varphi)$ by $\sqrt{d\mu_\varphi} = |\det \varphi''|^{-1/2} |d\xi|^{1/2}$. The (oscillating) principal symbol of f in Λ_φ is then defined (up to the factor $\sqrt{d\mu_\varphi}$) as

$$e^{i\phi(\xi)/h} A_0(\xi) = e^{i\phi(\xi)/h} e^{i\pi \operatorname{sgn} \varphi''/4} a_0(x(\xi), \theta(\xi))$$

where ϕ is a "reduced phase function". Conversely, assume $\iota : \Lambda \to T^*M$ is a smooth Lagrangian immersion, with a smooth positive half-density $\sqrt{d\mu}$, it can be parametrized locally by phase functions φ in canonical charts $\iota_\varphi : U_\varphi \to T^*M$. These phases can be chosen coherently, and define a class of "reduced phase functions" ϕ, parametrizing ι locally. This gives the fibre bundle of phases $\mathcal{L}_h$, including Maslov indices, equipped with transition functions. We are also given local smooth half-densities $|d\mu_\varphi|^{1/2}$ on Λ, defining the fibre bundle of half-densities $\Omega^{1/2}$, equipped with transition functions. The collection of these objects make a fibre bundle $\Omega^{1/2} \otimes \mathcal{L}_h$ over Λ. A section of $\Omega^{1/2} \otimes \mathcal{L}_h$ will be written as

$$f(x;h) = \big[K^h_{(\Lambda,\mu)} A\big](x;h)$$

where $K^h_{(\Lambda,\mu)}$ is called Maslov canonical operator. The "reduced phase function" and the "principal symbol" of f are defined invariantly. See [M], [Du], [Iv], [DoZh], [DNS] for details.

So our general aim is to construct, in term of Maslov canonical operators, a representation of $E_+ f$. Here are some examples of f (expressed in a single chart):

(1) $\Lambda = \{(x, \partial_x \varphi) : x \in M\}$. WKB functions $f(x,h) = e^{i\varphi(x)/h} a(x,h)$ or in Fourier representation

$$f(x;h) = \frac{e^{i\pi n/4}}{(2\pi h)^{n/2}} \int e^{i(xp + S(p))/h} A(p;h)\, dp \tag{2}$$

(such an integral conveniently normalized including a phase factor, will be written $\int^*(\cdots)$; here $\operatorname{sgn}\big(x \cdot p\big)'' = -n$).

(2) Gaussian functions $f(x;h) = \frac{1}{h^n} \exp(-\frac{\omega^2 \cdot x^2}{2h})$ or more general (superposition of) coherent states.

(3) A conormal distribution with $\Lambda = T^*_N \mathbf{R}^n$, $N = \{x_n = 0\}$

$$f(x;h) = \int^* e^{ix_n p_n/h} A(x', p_n)\, dp_n \tag{3}$$

(4) $\Lambda = \{x = X(\varphi,\psi) = \varphi\omega(\psi), p = P(\varphi,\psi) = \omega(\psi), \varphi \in \mathbf{R}\}$, $\omega \in \mathbf{S}^{n-1}$, f identifies with a "Bessel beam", see Sect. 4.

2. Hypotheses and main result

Let $\Lambda_0 \to T^*M$, $\Lambda_1 \to T^*M$ be smooth embedded Lagrangian manifolds, Λ_1 with a smooth boundary $\partial\Lambda_1$ (isotropic manifold). Following [MelUh], we say that (Λ_0, Λ_1) is an *intersecting pair* iff $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$ and the intersection is *clean*, i.e.

$$\forall \rho \in \partial\Lambda_1 \quad T_\rho \Lambda_0 \cap T_\rho \Lambda_1 = T_\rho \partial\Lambda_1$$

In particular, near each $\rho \in T^*M$, there is a canonical transform mapping Λ_0 to $T_0^*\mathbf{R}^n = \{x = 0\}$ and Λ_1 to

$$\Lambda_+^0 = \{(x,\xi) : x = (0,x_n), \xi = (\xi',0), x_n \geq 0\}$$

i.e. the flow-out of $T_0^*\mathbf{R}^n$ by the Hamilton vector field $X_{\xi_n} = ((0,\cdots,0,1),0)$ of ξ_n, passing through $x = 0, \xi_0 = (\xi',0)$.

Let now $H(x,p;h) = H_0(x,p) + hH_1(x,p) + \cdots$ be a symbol of class $S^0(m)$ and $H^w(x,hD_x;h)$ its Weyl quantization, as a bounded operator on $L^2(M)$. Assume

(A.1) The energy surface $\Sigma_E = H_0^{-1}(E)$ is non critical.

(A.2) The set $L = \Sigma_E \cap \Lambda$ is compact.

(A.3) The Hamiltonian vector field X_{H_0} is transverse to Λ along L.

For $t \geq 0$, let $g_{H_0}^t = \exp tX_{H_0}$, $\Lambda_t = g_{H_0}^t(\Lambda)$, and $\Lambda_+ = \bigcup_{t\geq 0} g_{H_0}^t(L)$. The pair (Λ,Λ_+) define a Lagrangian intersection, and $L = \partial\Lambda_+$. We assume also

(A.4) The Hamilton flow $g_{H_0}^t|_L$ extends for all $t > 0$.

(A.5) $|\pi_x(g_{H_0}^t|_L)| \to \infty$ as $t \to \infty$ (non trapping condition).

Let ψ be a system of coordinates on L, completed (locally near L) to coordinates (ψ,τ) on Λ and to coordinates (ψ,t) on Λ_+, so that the positive measure on Λ (square of half-density) takes the form $d\mu = |d\tau \wedge d\psi|$. This implies the measures $d\mu_t = (g_{H_0}^{t*})^{-1}(d\mu)$ on Λ_t and $d\mu_+ = |dt \wedge d\psi|$ on Λ_+.

We shall consider a partition of unity subordinated to a covering of the pair (Λ,Λ_+), with cut-off functions (of small enough support) $\widetilde{\chi}_0 \in C_0^\infty(\Lambda)$, that depends only on τ (the "radial variable" on Λ), with $\widetilde{\chi}_0(\tau) \equiv 1$ near $L = \{\tau = 0\}$, and $\chi_0 \in C_0^\infty(\Lambda_+)$, that depends only on t, $\chi_0(t) \equiv 1$ near $t = 0$. Let also $\chi_T \in C^\infty(\mathbf{R}^+)$, $\chi_T \equiv 1$ for $t \leq T/2$, $\chi_T \equiv 0$ for $t \geq T/2$ where T is so large that $x \notin \pi_x(g_{H_0}^t|_L)$ if $t \geq T$ (using the non-trapping condition (A.5)).

For simplicity, we state our main result [AnDoNaRo1] in the case $\Lambda = T_0^*M$, so that all solutions of Hamilton equations start from some $(0,p)$, with $p \in \operatorname{supp} A$. Representing E_+ as a sum of Maslov canonical operators associated with the pair (Λ,Λ_+) in particular we shall retrieve the theorem on propagation of singularities for wave-front sets $\mathrm{WF}_h\, u \subset \mathrm{WF}_h(Hu) \cup \operatorname{Char} H$.

Theorem 1: *Let $\Omega \subset M$ be a bounded domain. Under hypotheses (A.1)-(A.5) above, there is $u = u_h$, solution of $H^w(x,hD_x;h)u_h = f_h$, satisfying the limiting absorption principle, which can generically be cast in the following form*

$$\begin{aligned} u(x,h) &= \int^* \frac{(1-\widetilde{\chi}_0(\tau(p))A(p)}{H(0,p)-E} e^{ipx/h}\, dp \\ &+ \int_0^{\varepsilon_0} dt \int^* e^{i\Phi(t,x,\eta)/h} e^{-i\Theta(t,x,\eta)} |\det J(t,x,\eta)|^{-1/2} \chi_T(t)\chi_0(t)\widetilde{\chi}_0\big(\tau(\eta)\big)A(\eta)\, d\eta\, dt \\ &+ K^h_{(\Lambda_+,d\mu_+)}\big[\chi_T(\cdot)(1-\chi_0(\cdot))e^{-i\Theta}A_+\big] \end{aligned} \tag{4}$$

up to $\mathcal{O}(h)$ terms; the summands on the right hand side are boundary, transient and wave part, respectively. In Eq.(4), $\Phi(t,x;\eta)$ solves Hamilton-Jacobi equation (see Sect.3), and $\Theta(t,x,\eta) = \int_0^t H_1 \circ$

$\exp(sX_{H_0}(0,\eta))\,ds$ *is the integral of the sub-principal 1-form, where we recall* H_1 *is the sub-principal symbol of* H*. Moreover* $J(t,x,\eta)$ *is a Jacobian computed from Hamilton-Jacobi equation.*

The case when Λ is slightly "tilted" with respect to the vertical plane, i.e. has generating function $\phi(x,\xi) = x\xi + S(\xi)$ can be formulated in a very similar way (see Sect.3). For more general f's, the boundary part should be written as $\big[K^h_{(\Lambda,\mu)}\frac{(1-\widetilde{\chi_0})A}{H}\big](x;h)$, and the transient part as the integral over t of a phase factor times $\big[K^h_{(\Lambda_t,\mu_t)}\chi_T(\cdot)\widetilde{\chi}_0(\cdot)\chi_0(\cdot)A_t\big](x;h)$, A_t being the solution of the transport equation along the projection of $\exp tX_{H_0}$ with initial data $A(\eta)$.

The boundary part is microlocalized on Λ, the wave part on Λ_+. Let us make first a few comments:

1) (4) can be interpreted as "integrated" Van Vleck Formula [CdV], which expresses the semi-classical propagator $e^{-itH/h}$ acting on a localized function.

2) The wave part contributes generally at $x \in \Omega$ only if $t \mapsto \Phi(t,x,\eta)$ has a non-degenerate critical point for all $\eta \in \operatorname{supp} f$: let hD_{x_n} be the "model" operator, and $f(x;h) = \int^* e^{ixp/h}A(p)\,dp$. Then [MelUh]

$$u_h(x) = \frac{i}{h}\int_0^\infty \chi_T(t)\,dt \int^* e^{i(x'\xi'+(x_n-t)\xi_n)/h}A(\xi)\,d\xi$$

verifies the limiting absorption principle and $hD_{x_n}u(x,h) = f(x,h) + \mathcal{O}(h^\infty)$ for $x_n \le T/2$; still u_h has no "wave part".

3) Formula (4) is also valid near a focal point $x \in \Omega$, or more generally when x is linked to x_0 through a trajectory containing several focal points. The wave-part simplifies outside the focal points to a WKB form, involving non trivial Maslov indices passing the first focal point.

4) The phase function $\Phi(t,x,\eta)$ can be also replaced by a Lagrangian action, in the spirit of [DNS]. This will be discussed in detail in a future work.

We illustrate Theorem 1 by computing u_h explicitely in the 2-D case for Helmholtz operator with constant coefficient as in (1), but f with compact support. Let f also be radially symmetric; its Fourier transform $g = \mathcal{F}_1 f$ is again of the form $g(p) = g(|p|) = g(r)$ and extends holomorphically to $\mathbf{C}^2$. For $E = k^2$, $k > 0$, we rewrite (1) as $u_h(x) = u(x) = u_0(x) + u_1(x)$ with

$$u_0(x) = \frac{k+i\,\varepsilon}{(2\pi h)^2}\int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r\cos\theta/h]\frac{g(r)}{r^2-(k+i\,\varepsilon)^2}\,dr$$
$$u_1(x) = \frac{1}{(2\pi h)^2}\int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r\cos\theta/h]\frac{g(r)}{r+k+i\,\varepsilon}\,dr$$

To compute u_0 we use contour integrals. When $\theta \in]-\frac{\pi}{2},\frac{\pi}{2}[$, we shift the contour of integration to the positive imaginary axis and get by the residues formula

$$\begin{aligned}\int_0^\infty \exp[i|x|r\cos\theta/h]\frac{g(r)}{r^2-(k+i\,\varepsilon)^2}\,dr + \int_0^\infty \exp[-|x|r\cos\theta/h]\frac{g(ir)}{r^2+(k+i\,\varepsilon)^2}\,idr = \\ 2i\pi\frac{g(k+i\,\varepsilon)}{2(k+i\,\varepsilon)}\exp[i|x|(k+i\,\varepsilon)\cos\theta/h]\end{aligned} \tag{5}$$

while for $\theta \in]\frac{\pi}{2},\frac{3\pi}{2}[$,

$$\int_0^\infty \exp[i|x|r\cos\theta/h]\frac{g(r)}{r^2-(k+i\,\varepsilon)^2}\,dr - \int_0^\infty \exp[|x|r\cos\theta/h]\frac{g(-ir)}{r^2+(k+i\,\varepsilon)^2}\,idr = 0 \tag{6}$$

Summing up (5) and (6), integrating over $\theta \in]0, 2\pi[$ and letting $\varepsilon \to 0$, we obtain

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k\cos\theta/h]\, d\theta +$$
$$\int_0^\infty \frac{dr}{r^2+k^2} \Big[\int_{-\pi/2}^{\pi/2} g(ir) - \int_{\pi/2}^{3\pi/2} g(-ir) \Big] \exp[-|x||\cos\theta|/h]\, d\theta$$

Since $g(ir) = g(-ir)$, the latter integral vanishes, so we end up with

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k\cos\theta/h]\, d\theta$$

It is readily seen that

$$\mathrm{WF}_h\, u_0 \subset \{x = 0\} \cup \{(x, k\frac{x}{|x|}), x \neq 0\} = \Lambda \cup \Lambda_+$$

Consider now u_1. We let $\varepsilon \to 0$ and set $\widetilde{g}(r) = \frac{g(r)}{r(r+k)}$. Since $\widetilde{g}(r)\sqrt{r} \in L^1(\mathbf{R}_+)$, we have $u_1(x) = H_0(\widetilde{g})(\frac{|x|}{h})$, where H_0 denotes Hankel transform of order 0.

Let $\chi \in C_0^\infty(\mathbf{R}^2)$ be radially symmetric, and equal to 1 near 0, since $\mathrm{WF}_h\, f_h = \{x = 0\}$, we have

$$g = \mathcal{F}_h(\chi f_h) + \mathcal{O}(h^\infty) = (2\pi h)^{-2} \mathcal{F}_h(\chi) * g + \mathcal{O}(h^\infty)$$

so in the expression for u_1 we may replace mod $\mathcal{O}(h^\infty)$, $\widetilde{g}(r)$ by a constant times $\widehat{g}(r) = \frac{(\mathcal{F}_h(\chi)*g)(r)}{r(r+m)}$ (see [Bad] for 2-D convolution and Fourier transform in polar coordinates). To estimate $\mathrm{WF}_h\, u_1$, we compute again the Fourier transform of $(1-\widetilde{\chi})\widehat{g}$ where $\widetilde{\chi}$ is a cut-off equal to 1 near 0, and we find it is again $\mathcal{O}(h^\infty)$ if $\chi \equiv 1$ on $\operatorname{supp}\widetilde{\chi}$. This shows that $\mathrm{WF}_h\, u_1 \subset \{x = 0\}$. See also [MelUh], Prop. 2.3. Note that the decomposition $u_0 + u_1$ is directly related with the corresponding one in Theorem 1 as the sum of the boundary (u_1), and the wave part (u_0). We conjecture that the transient part in Theorem 1 can be removed (taking a limit $\operatorname{supp}\chi_0 \to \emptyset$) when the Hamiltonian flow X_{H_0} enjoys some non-degeneracy properties (see Sect.3), as in the case of a geodesic flow.

3. Maslov canonical operators associated with (Λ, Λ_+)

Proof of Theorem 1 consists first in looking at the propagator $e^{-itH/h}$ acting on Lagrangian distribution f_h as above, i.e. the solution of the Cauchy problem $\big(hD_t + H^w(x, hD_x)\big)v = 0$, $v|_{t=0} = f_h$. Next step is to integrate with respect to t after introducing the partition of unity above.

In this report, we shall content to construct the phase functions by solving Hamilton-Jacobi equation. First we recall from [Hö],Thm 6.4.5 the following:

Theorem 2: *Denote the variable in $T^*\mathbf{R}^d$ by $(y, \eta) = (y', y_d; \eta', \eta_d)$. Let $\mathcal{H}$ be a real valued, smooth Hamiltonian near $(0, \eta)$ such that $\mathcal{H}(0, \eta) = 0$, $\partial_{\eta_d}\mathcal{H}(0, \eta) \neq 0$, and let ϕ be a real valued, smooth function on $\mathbf{R}^{d-1}$ such that $\partial_{y'}\phi = \eta'$. Then there exists in a neighborhood of $0 \in \mathbf{R}^d$ a unique real valued solution $\Phi(y; \eta)$ of Hamilton-Jacobi equation $\mathcal{H}(y, \partial_y\Phi) = 0$ satisfying the boundary condition*

$$\Phi(y', 0; \eta) = \phi(y'), \quad \frac{\partial\Phi}{\partial y}(0; \eta) = \eta$$

We consider here the case of a "maximally singular" chart U for Λ where $\Lambda = \Lambda_\phi$ has generating function $\phi(x,\xi) = S(\xi) + x\xi$. Even if Λ is a plane, the outgoing manifold Λ_+ may be very complicated far away from x_0, but changing the canonical charts, we can proceed step by step.

Without loss of generality we can also assume here $E = 0$, $S(0) = 0$, $\partial_\xi S(0) = 0$. Let η, τ such that $\tau + H(0,\eta) = 0$, by Theorem 2 (after slightly changing the notations), there exists $\Phi(x,t;\eta,\tau)$ such that

$$\begin{aligned} &\frac{\partial \Phi}{\partial t} + H(x, \frac{\partial \Phi}{\partial x}) = 0 \\ &\Phi(x,0;\eta,\tau) = S(\eta) + x\eta \\ &\frac{\partial \Phi}{\partial x}(0,0;\eta,\tau) = \eta, \quad \frac{\partial \Phi}{\partial t}(0,0;\eta,\tau) = \tau \end{aligned} \tag{7}$$

Moreover, when τ is small enough, the intersecting pair (Λ, Λ_+) for energy level $H = 0$ extends to a smooth family of intersecting pairs $(\Lambda, \Lambda_+(\tau))$ for energy levels $H = -\tau$

$$\Lambda_+(\tau) = \{(x,\xi) \in T^*M, \exists t \geq 0, \exists (y,\eta) \in \Lambda,\ (x,\xi) = \exp tX_H(y,\eta), \tau + H(y,\eta) = 0\}$$

intersecting along $L(\tau)$ given by $\eta_n = \widetilde{\eta}_n(\eta', \tau)$, with $\widetilde{\eta}_n(\xi_0', 0) = 0$.

So near $x = x_0$ we can assume, possibly after permuting the ξ-coordinates, that $H(0;\xi_0',0) = 0$, $\frac{\partial H}{\partial \xi_n}(0;\xi_0',0) \neq 0$, and for τ small enough, the equation $\tau + H(x,\xi) = 0$ is equivalent to $\xi_n = \widetilde{\xi}_n(x,\xi',\tau)$, with $\widetilde{\xi}_n(0,\xi_0',0) = 0$.

Now we want to set $\tau = 0$ and therefore, given (x,η), solve the equation $\frac{\partial \Phi}{\partial t} = 0$ for $t \geq 0$. As we have seen, uniqueness of solutions doesn't always hold, as shows the "model case" hD_{x_n}. Namely, the phase function given by Hamilton-Jacobi theory is $\Phi(x,t;\eta) = \phi(x', x_n - t, \eta)$, so $\partial_t \Phi = 0$ iff $\eta_n = 0$, for all t. However the phase parametrizing Λ_+ is given for small t by Taylor expansion

$$\Phi(x,t;\eta) = x\eta + S(\eta) - tH(x,\eta) + \frac{t^2}{2}\partial_\xi H(x,\eta)\partial_x H(x,\eta) + \cdots$$

The following assumption in turn ensures existence of a finite number of solutions $t = t(x;\eta) > 0$ of $\frac{\partial \Phi}{\partial t} = 0$. Denote by $\mathrm{Exp}_{x_0}^t\, \eta = \pi_x(\exp tX_{H_0}(x_0,\eta))$ the projection of the bicharacteristic of H starting from (x_0,η) near (x_0,ξ_0) at time 0. Assume:

(A_0') *$\exists \Omega_\xi \subset \mathbf{R}^n$ open (small) neighborhood of ξ_0, such that if $x = \mathrm{Exp}_{x_0}^t\, \eta$ for some $t > 0$, and $\eta \in \Omega_\xi$, then the map $\xi \mapsto \mathrm{Exp}_{x_0}^t\, \xi$ is a local diffeomorphism near η; in other terms, x_0 and x are not conjugated along any trajectory that links them together within time t, with initial momentum ξ.*

The set of such x is an open set Ω_x.

Proposition 1: *Under hypothesis (A_0'), for all $(x,\eta) \in \Omega_x \times \Omega_\xi$ there is a finite number of $t_j = t_j(x,\eta) > 0$ $(1 \leq j \leq N)$ solutions of $\partial_t \Phi(x,t;\eta) = 0$ and t_j are non degenerate critical points.*

Moreover (A_0') is generically fulfilled before occurence of the first focal point, as show the following examples.

Let H_0 be a geodesic flow, $E > 0$, f conormal to $N = \{x_0\}$; Ω_x is a small enough ball centered at x_0, (geodesic ball), or Ω_x is a neighborhood of a minimal geodesic $\{y = \mathrm{Exp}_{x_0}^s\, \eta, 0 \leq s \leq t_0\}$ for η

in some small neighborhood of η_0. This applies when f is as in (2) and $\operatorname{supp} A$ is localized near η_0. The same holds with f as in (3) conormal to the hypersurface $N = \{x_n = 0\}$ (or more generally to a surface N of positive codimension in M). For instance when $N = \{x_n = 0\}$, (A_0') holds with Ω_ξ an open (small) neighborhood of $\eta = (0, \cdots, 1)$, and Ω_x a neighborhood of a minimal geodesic from N to some x_1. This follows from a well-known property of minimal geodesics (see e.g. [HeSj], Proposition 6.3). In case of a Schrödinger operator with principal symbol $H_0(x, \xi) = \xi^2 + V(x) - E$ (where E is a scattering energy) we use Maupertuis-Jacobi principle to reduce again to a Riemannian metric d_E given by $ds = (E - V(x))^{1/2}|dx|$. So we arrive at the same conclusions, the metric d_E being conformal to the standard metric. We proved the following:

Proposition 2: *Under the minimality assumptions above, (A_0') holds. For $\eta \in \Omega_\xi$ and $x = Exp^t_{x_0}(\eta) \in \Omega_x$, $0 < t < t_0$, the Lagrangian manifold $\Lambda_t = \exp tX_H(\Lambda_\phi)$ has same rank as Λ, and is of the form $\Lambda_{\Phi(t,\cdot)}$. In particular*

$$\exp tX_{H_0}(\Lambda_\phi) = \{(x, \partial_x \Phi(t, x; \eta)) : \partial_\eta \Phi(t, x; \eta) = 0\} \subset \{\exp tX_{H_0}(x, \eta), (x, \eta) \in T^*N^\perp\}$$

Moreover for all $(x, \eta) \in \Omega_x \times \Omega_\xi$ there is a unique $t(x, \eta) > 0$ solution of $\partial_t \Phi(x, t; \eta) = 0$ and $t(x, \eta)$ is a non degenerate crtical point.

Extending the geodesic γ beyond the first focal point occuring in some Λ_t for some $t = t_0$. we only need another representation of the phase function. Using the mixed representation for Lagrangian manifolds, we know that for any $k = 0, \cdots, n$, ($k = 0$ corresponds to a maximally singular chart, $k = n$ to a regular chart, they are mapped onto each other by Fourier transform), there exists a partition of variables $x = (x', x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ and $\xi = (\xi', \xi'')$, such that if $\widetilde{\pi} : \mathbf{R}^{2n} \to \mathbf{R}^n$, $(x, \xi) \mapsto (x', \xi'')$, then rank $d\widetilde{\pi} = n$. In such a chart, the generating function for Λ_t takes the form $\phi(x, \eta) = x''\eta'' + S(x', \eta'')$. We can reformulate Hamilton-Jacobi equations as in (7) in these coordinates, which has again a unique solution for small $t - t_0$. Generically (i.e. under an assumption (A_k') modeled after (A_0') in the new (x, ξ) coordinates), one still obtains a non degenerate phase function, which contributes to the fibre bundle $\mathcal{L}_h$ over Λ_+ together with Maslov index of γ. More explicit formulae will be given elswewhere. Of course, γ may contribute a finite number of times in the expression for the Green function at x, and a finite number of γ contribute to the wave part of the phase function.

4. Using eikonal coordinates.

The computations above can be simplified using special coordinates adapted to Λ, called *eikonal coordinates.*

Let $\iota : \Lambda \to T^*M$ be a smooth embedded Lagrangian manifold. The 1-form $p\,dx$ is closed on Λ, so locally exact, and $p\,dx = dS$ on any simply connected domain U. Such a S is called an *eikonal* (or action) and is defined up to a constant. Assume $dS \neq 0$ on Λ, then S can be chosen as a coordinate on U, which we complete by smooth functions $\psi \in \mathbf{R}^{n-1}$.

We use here eikonal coordinates to construct a phase function solving Hamilton-Jacobi equation in case of a positively homogeneous Hamiltonian of degree m with respect to p. To fix ideas, we take

$n = 2$ (for simplicity) $m = 1$ and

$$H(x,p) = \frac{|p|}{n(x)} \tag{8}$$

Example 1: $\Lambda = \{x = 0\}$ intersects the energy surface $H = 1$ along L. Let us compute the eikonal S on Λ_+. Integrating Hamilton equations we have $x = X(t,\psi), p = P(t,\psi)$ where $\psi \in \mathbf{R}^2$, hence $dS = p\,dx|_{\Lambda_+} = \langle P(t,\psi), dX(t,\psi)\rangle$. Since $dx = 0$ on Λ, we have $S(0,\psi) = \text{Const.} = S_0$, and

$$S(t,\psi) = S(0,\psi) + \int_{(0,\psi)}^{(t,\psi)} p\,dx|_{\Lambda_+} = S_0 + \int_0^t \langle P(s,\psi), \dot{X}(s,\psi)\rangle\,ds \tag{9}$$

By Hamilton equations and Euler identity, we have on $H = 1$.

$$\langle P(s,\psi), \dot{X}(s,\psi)\rangle = \langle P(s,\psi), \partial_p H(X,P\rangle = mH(X,P) = m$$

and $S(t,\psi) = S_0 + mt$ is the action on Λ_+. So

$$m\,dt = \langle P(t,\psi), dX(t,\psi)\rangle = \langle P(t,\psi), \dot{X}(t,\psi)\rangle\,dt + \langle P(t,\psi), X_\psi(t,\psi)\rangle\,d\psi$$

It follows that $\langle P(t,\psi), \dot{X}(t,\psi)\rangle = m$ and $\langle P(t,\psi), \partial_\psi X(t,\psi)\rangle = 0$. Now we complete the coordinate system ψ on Λ by a smooth function r such that L is given by $r = 1$, and set

$$\Phi(x,(t,\psi,r)) = mt + r\langle P(t,\psi), x - X(t,\psi)\rangle$$

where r can be interpreted as a Lagrange multiplier. Let us check that Φ satisfies Hamilton-Jacobi equation. We have

$$\begin{aligned}
&\partial_t\Phi = \dot{\Phi} = m + r\langle \dot{P}, x - X(t,\psi)\rangle - r\langle P, \dot{X}\rangle = \\
&m(1-r) + r\langle \dot{P}(t,\psi), x - X(t,\psi)\rangle \\
&\partial_r\Phi = \langle P(t,\psi), x - X(t,\psi)\rangle \\
&\partial_\psi\Phi = r\langle \partial_\psi P(t,\psi), x - X(t,\psi)\rangle
\end{aligned} \tag{10}$$

Last 2 equations in $(\dot{\Phi}, \partial_r\Phi, \partial_\psi\Phi) = 0$ give an homogeneous linear system with determinant $\det(P, P_\psi)$. On $x = 0$ we get $|p| = n(0) > 0$, and in dimension $n = 2$, $p = n(0)^t(\cos\psi, \sin\psi)$. It follows that det $(P, P_\psi) = |n(0)|^2$ so for small t, we get $x - X(t,\psi) = 0$, so the phase is critical with respect to (ψ, r) for $x = X(t,\psi)$. Substituting into the last equation (10) we get $\dot{\Phi} = 0$ when $r = 1$. We complete the proof that Φ is a generating function for Λ_+ by checking $d\partial_t\phi, d\partial_\psi\Phi, d\partial_r\Phi$ are linearly independent on the set $x = X(t,\psi)$. This can be done by examining the variational system associated with Hamilton equations.

Moreover we can reduce this generating function by eliminating t by stationary phase. Again, of course, this holds only for small $t > 0$, before unfolding of Lagrangian singularities.

Example 2: We take H as in (8) with $n(x) = n(|x|)$, $n = 2$ for simplicity and

$$\Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi),\ \varphi \in \mathbf{R}, \omega \in \mathbf{S}^{n-1}\}$$

ψ being the usual angles parametrizing $\omega \in \mathbf{S}^{n-1}$. When $n = 2$ this is the wave-front set of Bessel function $f_h(x; h) = J_0(\frac{|x|}{h})$; such functions arise in the wave beam theory (see [Ki], [DoMaNa] and references therein), so we call f_h a "Bessel beam".

Computing the action we find $p\,dx|_\Lambda = d\varphi$ so coordinate φ will play the role of x in he previous Example. We have $n(|x|) = n(\varphi)$, and $\varphi = \text{Const.}$ on $L = \Lambda \cap \{H = 1\}$. The argument above extends readily to this setting, in particular as in (9)

$$S(t,\varphi, \psi) = S(0, \varphi, \psi) + \int_{(0,\varphi,\psi)}^{(t,\varphi,\psi)} p\,dx|_{\Lambda_+} = S(0, \varphi, \psi) + \int_{(0,\varphi,\psi)}^{(t,\varphi,\psi)} P(s, \varphi, \psi)\,dX(s, \varphi, \psi) =$$
$$S_0 + \int_0^t \langle P(s, \varphi, \psi), \dot{X}(s, \varphi, \psi)\rangle\,ds$$

and

$$\langle P(t, \varphi, \psi), \partial_\psi X(t, \varphi, \psi)\rangle = \langle P, \partial_\varphi X\rangle = 0$$

Now we can apply the results of [DNS], Sect.2.2 on eikonal coordinates for a Lagrangian manifold in a general position, with $X_{\widetilde{\psi}}(\widetilde{\varphi}, \widetilde{\psi})$ of rank k. Here $\widetilde{\varphi}, \widetilde{\psi}$ are coordinates in a local chart of Λ_+ in the extended phase space. More specifically, we take $\widetilde{\psi} = (t, \psi)$, $\widetilde{\varphi} = \varphi$, try to make a change of variables $\varphi = \varphi(x, t, \psi)$, and seek for a generating function of Λ_+ in the form $\Phi(x, (t, \psi, r))$ as in the previous Example. Details will be given elsewhere.

5. More examples and perspectives.

The methods above apply in a number of situations as:

(1) The water-wave Hamiltonian $H(x, hD_x; h)$ with $H_0(x, p) = |p| \tanh\big(|p|D(x)\big) - E$, together with $\Lambda = \{x = x_0\}$ has been discussed in [DoNa], [AnDoNaRo2], in relationship with (11) or Helmhotz operator $H(x, hD_x) = -h^2\Delta - \big(n(x)\big)^2$. The localized function f can be a Gaussian (even in x) or a Gaussian times a linear function (odd in x), or can be of antenna type, i.e. its Fourier transform localized in a cone in p.

(2) The kinetic part of Hamiltonian is of Lorenzian type (as $-p_0^2 + p_1^2$), and f a localized function (Gaussian) supported on $\Lambda = \{x = x_0\}$, so that the semi-classical Green function u_h is the linear response to f localized on Kelvin angle (or Mach cone).

It should also be possible to construct semi-classical Green functions in case the pair (Λ, Λ_+) is no longer intersecting cleanly, but glancing. We then need second-microlocalization, and introducing so called 2-phases, see e.g. [LaWi] in the standard (polyhomogeneous) calculus.

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