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# SEMI-CLASSICAL GREEN FUNCTIONS

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*Abstract:* Let  $H(x, p) \sim H_0(x, p) + hH_1(x, p) + \dots$  be a semi-classical Hamiltonian on  $T^*\mathbf{R}^n$ , and  $\Sigma_E = \{H_0(x, p) = E\}$  a non critical energy surface. Consider  $f_h$  a semi-classical distribution (the “source”) microlocalized on a Lagrangian manifold  $\Lambda$  which intersects cleanly the flow-out  $\Lambda_+$  of the Hamilton vector field  $X_{H_0}$  in  $\Sigma_E$ . Using Maslov canonical operator, we look for a semi-classical distribution  $u_h$  satisfying the limiting absorption principle and  $H^w(x, hD_x)u_h = f_h$  (semi-classical Green function). In this report, we elaborate (still at an early stage) on some results announced in [AnDoNaRo1] and provide some examples, in particular from the theory of wave beams.

## 1. Introduction

Our general motivation is to solve equations like  $(H - E)u_h = f_h$ , with  $f_h(x) = h^{-n}f(\frac{x}{h})$ ,  $f \in L^2(\mathbf{R}^n)$ , a “localized function” at  $x_0 = 0$ . In case  $H = -h^2\Delta$ , we apply  $h$ -Fourier transform

$$\mathcal{F}_h u_h(\xi) = \int e^{-ix\xi/h} u_h(x) dx$$

to get  $\mathcal{F}_h u_h(\xi) = \frac{\mathcal{F}_1 f(\xi)}{\xi^2 - E}$ . If  $E < 0$ , this is an elliptic equation, and  $u_h$  has the same form as  $f_h$ . If  $E > 0$  however,

$$(1) \quad u_+(x; E, h) = (2\pi h)^{-n} \int e^{ix\xi/h} \frac{\mathcal{F}_1 f(\xi)}{\xi^2 - E - i0} d\xi$$

is defined through regularisation, so to satisfy the limiting absorption principle. Actually,

$$u_+(x; E, h) = E_+ f_h, \quad E_+ = \frac{i}{h} \int_0^\infty e^{-it(H-E)/h} dt$$

is the forward parametrix. When  $f_h(x)$  is replaced by  $\delta(x - x_0)$ ,  $u_+$  is the Green function. Such a distribution is very singular since there are infinitely many classical trajectories linking the “source”  $x_0$  to  $x$  in time  $t$ ; so it is suitable to consider instead  $\mathcal{F}_1 f$  with compact support.

This problem has of course received considerable attention [Ke], [Ba], [Ku]; our special purpose is to reformulate it in terms of Maslov canonical operator, and describe the solution globally, including unfolding of Lagrangian singularities; this is of special importance in the context of wave propagation.

More generally, let  $f_h = f(\cdot; h)$  be a semi-classical Lagrangian distribution (or oscillatory integral) defined on the manifold  $M$  (for all practical purposes, we shall assume here  $M = \mathbf{R}^n$ ), i.e. locally

$$f(x; h) = (2\pi h)^{-n/2} \int e^{i\varphi(x, \theta)/h} a(x, \theta; h) d\theta$$

where  $\varphi(x, \theta)$  is a non-degenerate phase function in the sense of [Hö], and  $a(x, \theta; h) = a_0(x, \theta) + ha_1(x, \theta) + \dots$  an amplitude. With  $f$  we associate the critical set  $C_\varphi = \{(x, \theta) \in M \times \mathbf{R}^N : \partial_\theta \varphi = 0\}$  and  $\iota_\varphi : C_\varphi \rightarrow T^*M$  with image the Lagrangian submanifold  $\Lambda_\varphi = \{(x, \partial_x \varphi) : (x, \theta) \in C_\varphi\}$ . Choose local coordinates  $\xi \in \mathbf{R}^d$  on  $\Lambda_\varphi$  and define the half-density in the local chart  $(C_\varphi, \iota_\varphi)$  by  $\sqrt{d\mu_\varphi} = |\det \varphi''|^{-1/2} |d\xi|^{1/2}$ . The (oscillating) principal symbol of  $f$  in  $\Lambda_\varphi$  is then defined (up to the factor  $\sqrt{d\mu_\varphi}$ ) as

$$e^{i\phi(\xi)/h} A_0(\xi) = e^{i\phi(\xi)/h} e^{i\pi \operatorname{sgn} \varphi''/4} a_0(x(\xi), \theta(\xi))$$

where  $\phi$  is a “reduced phase function”. Conversely, assume  $\iota : \Lambda \rightarrow T^*M$  is a smooth Lagrangian immersion, with a smooth positive half-density  $\sqrt{d\mu}$ , it can be parametrized locally by phase functions  $\varphi$  in canonical charts  $\iota_\varphi : U_\varphi \rightarrow T^*M$ . These phases can be chosen coherently, and define a class of “reduced phase functions”  $\phi$ , parametrizing  $\iota$  locally. This gives the fibre bundle of phases  $\mathcal{L}_h$ , including Maslov indices, equipped with transition functions. We are also given local smooth half-densities  $|d\mu_\varphi|^{1/2}$  on  $\Lambda$ , defining the fibre bundle of half-densities  $\Omega^{1/2}$ , equipped with transition functions. The collection of these objects make a fibre bundle  $\Omega^{1/2} \otimes \mathcal{L}_h$  over  $\Lambda$ . A section of  $\Omega^{1/2} \otimes \mathcal{L}_h$  will be written as

$$f(x; h) = [K_{(\Lambda, \mu)}^h A](x; h)$$

where  $K_{(\Lambda, \mu)}^h$  is called Maslov canonical operator. The “reduced phase function” and the “principal symbol” of  $f$  are defined invariantly. See [M], [Du], [Iv], [DoZh], [DNS] for details.

So our general aim is to construct, in term of Maslov canonical operators, a representation of  $E_+ f$ . Here are some examples of  $f$  (expressed in a single chart):

(1)  $\Lambda = \{(x, \partial_x \varphi) : x \in M\}$ . WKB functions  $f(x, h) = e^{i\varphi(x)/h} a(x, h)$  or in Fourier representation

$$(2) \quad f(x; h) = \frac{e^{i\pi n/4}}{(2\pi h)^{n/2}} \int e^{i(xp+S(p))/h} A(p; h) dp$$

(such an integral conveniently normalized including a phase factor, will be written  $\int^*(\dots)$ ; here  $\operatorname{sgn}(x \cdot p)'' = -n$ ).

(2) Gaussian functions  $f(x; h) = \frac{1}{h^n} \exp(-\frac{\omega^2 \cdot x^2}{2h})$  or more general (superposition of) coherent states.

(3) A conormal distribution with  $\Lambda = T_N^* \mathbf{R}^n$ ,  $N = \{x_n = 0\}$

$$(3) \quad f(x; h) = \int^* e^{ix_n p_n/h} A(x', p_n) dp_n$$

(4)  $\Lambda = \{x = X(\varphi, \psi) = \varphi \omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}\}$ ,  $\omega \in \mathbf{S}^{n-1}$ ,  $f$  identifies with a “Bessel beam”, see Sect. 4.

## 2. Hypotheses and main result

Let  $\Lambda_0 \rightarrow T^*M$ ,  $\Lambda_1 \rightarrow T^*M$  be smooth embedded Lagrangian manifolds,  $\Lambda_1$  with a smooth boundary  $\partial\Lambda_1$  (isotropic manifold). Following [MelUh], we say that  $(\Lambda_0, \Lambda_1)$  is an *intersecting pair* iff  $\Lambda_0 \cap \Lambda_1 = \partial\Lambda_1$  and the intersection is *clean*, i.e.

$$\forall \rho \in \partial\Lambda_1 \quad T_\rho \Lambda_0 \cap T_\rho \Lambda_1 = T_\rho \partial\Lambda_1$$

In particular, near each  $\rho \in T^*M$ , there is a canonical transform mapping  $\Lambda_0$  to  $T_0^*\mathbf{R}^n = \{x = 0\}$  and  $\Lambda_1$  to

$$\Lambda_+^0 = \{(x, \xi) : x = (0, x_n), \xi = (\xi', 0), x_n \geq 0\}$$

i.e. the flow-out of  $T_0^*\mathbf{R}^n$  by the Hamilton vector field  $X_{\xi_n} = ((0, \dots, 0, 1), 0)$  of  $\xi_n$ , passing through  $x = 0, \xi_0 = (\xi', 0)$ .

Let now  $H(x, p; h) = H_0(x, p) + hH_1(x, p) + \dots$  be a symbol of class  $S^0(m)$  and  $H^w(x, hD_x; h)$  its Weyl quantization, as a bounded operator on  $L^2(M)$ . Assume

(A.1) The energy surface  $\Sigma_E = H_0^{-1}(E)$  is non critical.

(A.2) The set  $L = \Sigma_E \cap \Lambda$  is compact.

(A.3) The Hamiltonian vector field  $X_{H_0}$  is transverse to  $\Lambda$  along  $L$ .

For  $t \geq 0$ , let  $g_{H_0}^t = \exp tX_{H_0}$ ,  $\Lambda_t = g_{H_0}^t(\Lambda)$ , and  $\Lambda_+ = \bigcup_{t \geq 0} g_{H_0}^t(L)$ . The pair  $(\Lambda, \Lambda_+)$  define a Lagrangian intersection, and  $L = \partial\Lambda_+$ . We assume also

(A.4) The Hamilton flow  $g_{H_0}^t|_L$  extends for all  $t > 0$ .

(A.5)  $|\pi_x(g_{H_0}^t|_L)| \rightarrow \infty$  as  $t \rightarrow \infty$  (non trapping condition).

Let  $\psi$  be a system of coordinates on  $L$ , completed (locally near  $L$ ) to coordinates  $(\psi, \tau)$  on  $\Lambda$  and to coordinates  $(\psi, t)$  on  $\Lambda_+$ , so that the positive measure on  $\Lambda$  (square of half-density) takes the form  $d\mu = |d\tau \wedge d\psi|$ . This implies the measures  $d\mu_t = (g_{H_0}^{t*})^{-1}(d\mu)$  on  $\Lambda_t$  and  $d\mu_+ = |dt \wedge d\psi|$  on  $\Lambda_+$ .

We shall consider a partition of unity subordinated to a covering of the pair  $(\Lambda, \Lambda_+)$ , with cut-off functions (of small enough support)  $\tilde{\chi}_0 \in C_0^\infty(\Lambda)$ , that depends only on  $\tau$  (the ‘‘radial variable’’ on  $\Lambda$ ), with  $\tilde{\chi}_0(\tau) \equiv 1$  near  $L = \{\tau = 0\}$ , and  $\chi_0 \in C_0^\infty(\Lambda_+)$ , that depends only on  $t$ ,  $\chi_0(t) \equiv 1$  near  $t = 0$ . Let also  $\chi_T \in C^\infty(\mathbf{R}^+)$ ,  $\chi_T \equiv 1$  for  $t \leq T/2$ ,  $\chi_T \equiv 0$  for  $t \geq T/2$  where  $T$  is so large that  $x \notin \pi_x(g_{H_0}^t|_L)$  if  $t \geq T$  (using the non-trapping condition (A.5)).

For simplicity, we state our main result [AnDoNaRo1] in the case  $\Lambda = T_0^*M$ , so that all solutions of Hamilton equations start from some  $(0, p)$ , with  $p \in \text{supp } A$ . Representing  $E_+$  as a sum of Maslov canonical operators associated with the pair  $(\Lambda, \Lambda_+)$  in particular we shall retrieve the theorem on propagation of singularities for wave-front sets  $\text{WF}_h u \subset \text{WF}_h(Hu) \cup \text{Char } H$ .

**Theorem 1:** *Let  $\Omega \subset M$  be a bounded domain. Under hypotheses (A.1)-(A.5) above, there is  $u = u_h$ , solution of  $H^w(x, hD_x; h)u_h = f_h$ , satisfying the limiting absorption principle, which can generically be cast in the following form*

$$(4) \quad \begin{aligned} u(x, h) &= \int^* \frac{(1 - \tilde{\chi}_0(\tau(p)))A(p)}{H(0, p) - E} e^{ipx/h} dp \\ &+ \int_0^{\varepsilon_0} dt \int^* e^{i\Phi(t, x, \eta)/h} e^{-i\Theta(t, x, \eta)} |\det J(t, x, \eta)|^{-1/2} \chi_T(t) \chi_0(t) \tilde{\chi}_0(\tau(\eta)) A(\eta) d\eta dt \\ &+ K_{(\Lambda_+, d\mu_+)}^h [\chi_T(\cdot)(1 - \chi_0(\cdot))e^{-i\Theta} A_+] \end{aligned}$$

up to  $\mathcal{O}(h)$  terms; the summands on the right hand side are boundary, transient and wave part, respectively. In Eq.(4),  $\Phi(t, x; \eta)$  solves Hamilton-Jacobi equation (see Sect.3), and  $\Theta(t, x, \eta) = \int_0^t H_1 \circ$

$\exp(sX_{H_0}(0, \eta)) ds$  is the integral of the sub-principal 1-form, where we recall  $H_1$  is the sub-principal symbol of  $H$ . Moreover  $J(t, x, \eta)$  is a Jacobian computed from Hamilton-Jacobi equation.

The case when  $\Lambda$  is slightly “tilted” with respect to the vertical plane, i.e. has generating function  $\phi(x, \xi) = x\xi + S(\xi)$  can be formulated in a very similar way (see Sect.3). For more general  $f$ 's, the boundary part should be written as  $[K_{(\Lambda, \mu)}^h \frac{(1-\tilde{\chi}_0)A}{H}](x; h)$ , and the transient part as the integral over  $t$  of a phase factor times  $[K_{(\Lambda_t, \mu_t)}^h \chi_T(\cdot) \tilde{\chi}_0(\cdot) \chi_0(\cdot) A_t](x; h)$ ,  $A_t$  being the solution of the transport equation along the projection of  $\exp tX_{H_0}$  with initial data  $A(\eta)$ .

The boundary part is microlocalized on  $\Lambda$ , the wave part on  $\Lambda_+$ . Let us make first a few comments:

1) (4) can be interpreted as “integrated” Van Vleck Formula [CdV], which expresses the semi-classical propagator  $e^{-itH/h}$  acting on a localized function.

2) The wave part contributes generally at  $x \in \Omega$  only if  $t \mapsto \Phi(t, x, \eta)$  has a non-degenerate critical point for all  $\eta \in \text{supp } f$ : let  $hD_{x_n}$  be the “model” operator, and  $f(x; h) = \int^* e^{ixp/h} A(p) dp$ . Then [MelUh]

$$u_h(x) = \frac{i}{h} \int_0^\infty \chi_T(t) dt \int^* e^{i(x'\xi' + (x_n - t)\xi_n)/h} A(\xi) d\xi$$

verifies the limiting absorption principle and  $hD_{x_n} u(x, h) = f(x, h) + \mathcal{O}(h^\infty)$  for  $x_n \leq T/2$ ; still  $u_h$  has no “wave part”.

3) Formula (4) is also valid near a focal point  $x \in \Omega$ , or more generally when  $x$  is linked to  $x_0$  through a trajectory containing several focal points. The wave-part simplifies outside the focal points to a WKB form, involving non trivial Maslov indices passing the first focal point.

4) The phase function  $\Phi(t, x, \eta)$  can be also replaced by a Lagrangian action, in the spirit of [DNS]. This will be discussed in detail in a future work.

We illustrate Theorem 1 by computing  $u_h$  explicitly in the 2-D case for Helmholtz operator with constant coefficient as in (1), but  $f$  with compact support. Let  $f$  also be radially symmetric; its Fourier transform  $g = \mathcal{F}_1 f$  is again of the form  $g(p) = g(|p|) = g(r)$  and extends holomorphically to  $\mathbf{C}^2$ . For  $E = k^2$ ,  $k > 0$ , we rewrite (1) as  $u_h(x) = u(x) = u_0(x) + u_1(x)$  with

$$u_0(x) = \frac{k + i\varepsilon}{(2\pi h)^2} \int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r \cos \theta/h] \frac{g(r)}{r^2 - (k + i\varepsilon)^2} dr$$

$$u_1(x) = \frac{1}{(2\pi h)^2} \int_0^{2\pi} d\theta \int_0^\infty \exp[i|x|r \cos \theta/h] \frac{g(r)}{r + k + i\varepsilon} dr$$

To compute  $u_0$  we use contour integrals. When  $\theta \in ] -\frac{\pi}{2}, \frac{\pi}{2}[$ , we shift the contour of integration to the positive imaginary axis and get by the residues formula

$$(5) \quad \int_0^\infty \exp[i|x|r \cos \theta/h] \frac{g(r)}{r^2 - (k + i\varepsilon)^2} dr + \int_0^\infty \exp[-|x|r \cos \theta/h] \frac{g(ir)}{r^2 + (k + i\varepsilon)^2} idr =$$

$$2i\pi \frac{g(k + i\varepsilon)}{2(k + i\varepsilon)} \exp[i|x|(k + i\varepsilon) \cos \theta/h]$$

while for  $\theta \in ]\frac{\pi}{2}, \frac{3\pi}{2}[$ ,

$$(6) \quad \int_0^\infty \exp[i|x|r \cos \theta/h] \frac{g(r)}{r^2 - (k + i\varepsilon)^2} dr - \int_0^\infty \exp[|x|r \cos \theta/h] \frac{g(-ir)}{r^2 + (k + i\varepsilon)^2} idr = 0$$

Summing up (5) and (6), integrating over  $\theta \in ]0, 2\pi[$  and letting  $\varepsilon \rightarrow 0$ , we obtain

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k \cos \theta/h] d\theta + \int_0^\infty \frac{dr}{r^2 + k^2} \left[ \int_{-\pi/2}^{\pi/2} g(ir) - \int_{\pi/2}^{3\pi/2} g(-ir) \right] \exp[-|x| |\cos \theta|/h] d\theta$$

Since  $g(ir) = g(-ir)$ , the latter integral vanishes, so we end up with

$$u_0(x) = \frac{i\pi g(k)}{(2\pi h)^2} \int_{-\pi/2}^{\pi/2} \exp[i|x|k \cos \theta/h] d\theta$$

It is readily seen that

$$\text{WF}_h u_0 \subset \{x = 0\} \cup \left\{ \left( x, k \frac{x}{|x|} \right), x \neq 0 \right\} = \Lambda \cup \Lambda_+$$

Consider now  $u_1$ . We let  $\varepsilon \rightarrow 0$  and set  $\tilde{g}(r) = \frac{g(r)}{r(r+k)}$ . Since  $\tilde{g}(r)\sqrt{r} \in L^1(\mathbf{R}_+)$ , we have  $u_1(x) = H_0(\tilde{g})\left(\frac{|x|}{h}\right)$ , where  $H_0$  denotes Hankel transform of order 0.

Let  $\chi \in C_0^\infty(\mathbf{R}^2)$  be radially symmetric, and equal to 1 near 0, since  $\text{WF}_h f_h = \{x = 0\}$ , we have

$$g = \mathcal{F}_h(\chi f_h) + \mathcal{O}(h^\infty) = (2\pi h)^{-2} \mathcal{F}_h(\chi) * g + \mathcal{O}(h^\infty)$$

so in the expression for  $u_1$  we may replace mod  $\mathcal{O}(h^\infty)$ ,  $\tilde{g}(r)$  by a constant times  $\hat{g}(r) = \frac{(\mathcal{F}_h(\chi) * g)(r)}{r(r+m)}$  (see [Bad] for 2-D convolution and Fourier transform in polar coordinates). To estimate  $\text{WF}_h u_1$ , we compute again the Fourier transform of  $(1 - \tilde{\chi})\hat{g}$  where  $\tilde{\chi}$  is a cut-off equal to 1 near 0, and we find it is again  $\mathcal{O}(h^\infty)$  if  $\chi \equiv 1$  on  $\text{supp } \tilde{\chi}$ . This shows that  $\text{WF}_h u_1 \subset \{x = 0\}$ . See also [MelUh], Prop. 2.3. Note that the decomposition  $u_0 + u_1$  is directly related with the corresponding one in Theorem 1 as the sum of the boundary ( $u_1$ ), and the wave part ( $u_0$ ). We conjecture that the transient part in Theorem 1 can be removed (taking a limit  $\text{supp } \chi_0 \rightarrow \emptyset$ ) when the Hamiltonian flow  $X_{H_0}$  enjoys some non-degeneracy properties (see Sect.3), as in the case of a geodesic flow.

### 3. Maslov canonical operators associated with $(\Lambda, \Lambda_+)$

Proof of Theorem 1 consists first in looking at the propagator  $e^{-itH/h}$  acting on Lagrangian distribution  $f_h$  as above, i.e. the solution of the Cauchy problem  $(hD_t + H^w(x, hD_x))v = 0$ ,  $v|_{t=0} = f_h$ . Next step is to integrate with respect to  $t$  after introducing the partition of unity above.

In this report, we shall content to construct the phase functions by solving Hamilton-Jacobi equation. First we recall from [Hö], Thm 6.4.5 the following:

**Theorem 2:** *Denote the variable in  $T^*\mathbf{R}^d$  by  $(y, \eta) = (y', y_d; \eta', \eta_d)$ . Let  $\mathcal{H}$  be a real valued, smooth Hamiltonian near  $(0, \eta)$  such that  $\mathcal{H}(0, \eta) = 0$ ,  $\partial_{\eta_d} \mathcal{H}(0, \eta) \neq 0$ , and let  $\phi$  be a real valued, smooth function on  $\mathbf{R}^{d-1}$  such that  $\partial_{y'} \phi = \eta'$ . Then there exists in a neighborhood of  $0 \in \mathbf{R}^d$  a unique real valued solution  $\Phi(y; \eta)$  of Hamilton-Jacobi equation  $\mathcal{H}(y, \partial_y \Phi) = 0$  satisfying the boundary condition*

$$\Phi(y', 0; \eta) = \phi(y'), \quad \frac{\partial \Phi}{\partial y}(0; \eta) = \eta$$

We consider here the case of a “maximally singular” chart  $U$  for  $\Lambda$  where  $\Lambda = \Lambda_\phi$  has generating function  $\phi(x, \xi) = S(\xi) + x\xi$ . Even if  $\Lambda$  is a plane, the outgoing manifold  $\Lambda_+$  may be very complicated far away from  $x_0$ , but changing the canonical charts, we can proceed step by step.

Without loss of generality we can also assume here  $E = 0$ ,  $S(0) = 0$ ,  $\partial_\xi S(0) = 0$ . Let  $\eta, \tau$  such that  $\tau + H(0, \eta) = 0$ , by Theorem 2 (after slightly changing the notations), there exists  $\Phi(x, t; \eta, \tau)$  such that

$$(7) \quad \begin{aligned} \frac{\partial \Phi}{\partial t} + H(x, \frac{\partial \Phi}{\partial x}) &= 0 \\ \Phi(x, 0; \eta, \tau) &= S(\eta) + x\eta \\ \frac{\partial \Phi}{\partial x}(0, 0; \eta, \tau) &= \eta, \quad \frac{\partial \Phi}{\partial t}(0, 0; \eta, \tau) = \tau \end{aligned}$$

Moreover, when  $\tau$  is small enough, the intersecting pair  $(\Lambda, \Lambda_+)$  for energy level  $H = 0$  extends to a smooth family of intersecting pairs  $(\Lambda, \Lambda_+(\tau))$  for energy levels  $H = -\tau$

$$\Lambda_+(\tau) = \{(x, \xi) \in T^*M, \exists t \geq 0, \exists (y, \eta) \in \Lambda, (x, \xi) = \exp tX_H(y, \eta), \tau + H(y, \eta) = 0\}$$

intersecting along  $L(\tau)$  given by  $\eta_n = \tilde{\eta}_n(\eta', \tau)$ , with  $\tilde{\eta}_n(\xi'_0, 0) = 0$ .

So near  $x = x_0$  we can assume, possibly after permuting the  $\xi$ -coordinates, that  $H(0; \xi'_0, 0) = 0$ ,  $\frac{\partial H}{\partial \xi_n}(0; \xi'_0, 0) \neq 0$ , and for  $\tau$  small enough, the equation  $\tau + H(x, \xi) = 0$  is equivalent to  $\xi_n = \tilde{\xi}_n(x, \xi', \tau)$ , with  $\tilde{\xi}_n(0, \xi'_0, 0) = 0$ .

Now we want to set  $\tau = 0$  and therefore, given  $(x, \eta)$ , solve the equation  $\frac{\partial \Phi}{\partial t} = 0$  for  $t \geq 0$ . As we have seen, uniqueness of solutions doesn't always hold, as shows the “model case”  $hD_{x_n}$ . Namely, the phase function given by Hamilton-Jacobi theory is  $\Phi(x, t; \eta) = \phi(x', x_n - t, \eta)$ , so  $\partial_t \Phi = 0$  iff  $\eta_n = 0$ , for all  $t$ . However the phase parametrizing  $\Lambda_+$  is given for small  $t$  by Taylor expansion

$$\Phi(x, t; \eta) = x\eta + S(\eta) - tH(x, \eta) + \frac{t^2}{2} \partial_\xi H(x, \eta) \partial_x H(x, \eta) + \dots$$

The following assumption in turn ensures existence of a finite number of solutions  $t = t(x; \eta) > 0$  of  $\frac{\partial \Phi}{\partial t} = 0$ . Denote by  $\text{Exp}_{x_0}^t \eta = \pi_x(\exp tX_{H_0}(x_0, \eta))$  the projection of the bicharacteristic of  $H$  starting from  $(x_0, \eta)$  near  $(x_0, \xi_0)$  at time 0. Assume:

(A'\_0)  $\exists \Omega_\xi \subset \mathbf{R}^n$  open (small) neighborhood of  $\xi_0$ , such that if  $x = \text{Exp}_{x_0}^t \eta$  for some  $t > 0$ , and  $\eta \in \Omega_\xi$ , then the map  $\xi \mapsto \text{Exp}_{x_0}^t \xi$  is a local diffeomorphism near  $\eta$ ; in other terms,  $x_0$  and  $x$  are not conjugated along any trajectory that links them together within time  $t$ , with initial momentum  $\xi$ .

The set of such  $x$  is an open set  $\Omega_x$ .

**Proposition 1:** Under hypothesis (A'\_0), for all  $(x, \eta) \in \Omega_x \times \Omega_\xi$  there is a finite number of  $t_j = t_j(x, \eta) > 0$  ( $1 \leq j \leq N$ ) solutions of  $\partial_t \Phi(x, t; \eta) = 0$  and  $t_j$  are non degenerate critical points.

Moreover (A'\_0) is generically fulfilled before occurrence of the first focal point, as show the following examples.

Let  $H_0$  be a geodesic flow,  $E > 0$ ,  $f$  conormal to  $N = \{x_0\}$ ;  $\Omega_x$  is a small enough ball centered at  $x_0$ , (geodesic ball), or  $\Omega_x$  is a neighborhood of a minimal geodesic  $\{y = \text{Exp}_{x_0}^s \eta, 0 \leq s \leq t_0\}$  for  $\eta$

in some small neighborhood of  $\eta_0$ . This applies when  $f$  is as in (2) and  $\text{supp } A$  is localized near  $\eta_0$ . The same holds with  $f$  as in (3) conormal to the hypersurface  $N = \{x_n = 0\}$  (or more generally to a surface  $N$  of positive codimension in  $M$ ). For instance when  $N = \{x_n = 0\}$ ,  $(A'_0)$  holds with  $\Omega_\xi$  an open (small) neighborhood of  $\eta = (0, \dots, 1)$ , and  $\Omega_x$  a neighborhood of a minimal geodesic from  $N$  to some  $x_1$ . This follows from a well-known property of minimal geodesics (see e.g. [HeSj], Proposition 6.3). In case of a Schrödinger operator with principal symbol  $H_0(x, \xi) = \xi^2 + V(x) - E$  (where  $E$  is a scattering energy) we use Maupertuis-Jacobi principle to reduce again to a Riemannian metric  $d_E$  given by  $ds = (E - V(x))^{1/2}|dx|$ . So we arrive at the same conclusions, the metric  $d_E$  being conformal to the standard metric. We proved the following:

**Proposition 2:** *Under the minimality assumptions above,  $(A'_0)$  holds. For  $\eta \in \Omega_\xi$  and  $x = \text{Exp}_{x_0}^t(\eta) \in \Omega_x$ ,  $0 < t < t_0$ , the Lagrangian manifold  $\Lambda_t = \text{exp } tX_H(\Lambda_\phi)$  has same rank as  $\Lambda$ , and is of the form  $\Lambda_{\Phi(t, \cdot)}$ . In particular*

$$\text{exp } tX_{H_0}(\Lambda_\phi) = \{(x, \partial_x \Phi(t, x; \eta)) : \partial_\eta \Phi(t, x; \eta) = 0\} \subset \{\text{exp } tX_{H_0}(x, \eta), (x, \eta) \in T^*N^\perp\}$$

Moreover for all  $(x, \eta) \in \Omega_x \times \Omega_\xi$  there is a unique  $t(x, \eta) > 0$  solution of  $\partial_t \Phi(x, t; \eta) = 0$  and  $t(x, \eta)$  is a non degenerate critical point.

Extending the geodesic  $\gamma$  beyond the first focal point occuring in some  $\Lambda_t$  for some  $t = t_0$ . we only need another representation of the phase function. Using the mixed representation for Lagrangian manifolds, we know that for any  $k = 0, \dots, n$ , ( $k = 0$  corresponds to a maximally singular chart,  $k = n$  to a regular chart, they are mapped onto each other by Fourier transform), there exists a partition of variables  $x = (x', x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$  and  $\xi = (\xi', \xi'')$ , such that if  $\tilde{\pi} : \mathbf{R}^{2n} \rightarrow \mathbf{R}^n$ ,  $(x, \xi) \mapsto (x', \xi'')$ , then  $\text{rank } d\tilde{\pi} = n$ . In such a chart, the generating function for  $\Lambda_t$  takes the form  $\phi(x, \eta) = x''\eta'' + S(x', \eta'')$ . We can reformulate Hamilton-Jacobi equations as in (7) in these coordinates, which has again a unique solution for small  $t - t_0$ . Generically (i.e. under an assumption  $(A'_k)$  modeled after  $(A'_0)$  in the new  $(x, \xi)$  coordinates), one still obtains a non degenerate phase function, which contributes to the fibre bundle  $\mathcal{L}_h$  over  $\Lambda_+$  together with Maslov index of  $\gamma$ . More explicit formulae will be given elsewhere. Of course,  $\gamma$  may contribute a finite number of times in the expression for the Green function at  $x$ , and a finite number of  $\gamma$  contribute to the wave part of the phase function.

#### 4. Using eikonal coordinates.

The computations above can be simplified using special coordinates adapted to  $\Lambda$ , called *eikonal coordinates*.

Let  $\iota : \Lambda \rightarrow T^*M$  be a smooth embedded Lagrangian manifold. The 1-form  $p dx$  is closed on  $\Lambda$ , so locally exact, and  $p dx = dS$  on any simply connected domain  $U$ . Such a  $S$  is called an *eikonal* (or action) and is defined up to a constant. Assume  $dS \neq 0$  on  $\Lambda$ , then  $S$  can be chosen as a coordinate on  $U$ , which we complete by smooth functions  $\psi \in \mathbf{R}^{n-1}$ .

We use here eikonal coordinates to construct a phase function solving Hamilton-Jacobi equation in case of a positively homogeneous Hamiltonian of degree  $m$  with respect to  $p$ . To fix ideas, we take



$n = 2$  (for simplicity)  $m = 1$  and

$$(8) \quad H(x, p) = \frac{|p|}{n(x)}$$

**Example 1:**  $\Lambda = \{x = 0\}$  intersects the energy surface  $H = 1$  along  $L$ . Let us compute the eikonal  $S$  on  $\Lambda_+$ . Integrating Hamilton equations we have  $x = X(t, \psi), p = P(t, \psi)$  where  $\psi \in \mathbf{R}^2$ , hence  $dS = p dx|_{\Lambda_+} = \langle P(t, \psi), dX(t, \psi) \rangle$ . Since  $dx = 0$  on  $\Lambda$ , we have  $S(0, \psi) = \text{Const.} = S_0$ , and

$$(9) \quad S(t, \psi) = S(0, \psi) + \int_{(0, \psi)}^{(t, \psi)} p dx|_{\Lambda_+} = S_0 + \int_0^t \langle P(s, \psi), \dot{X}(s, \psi) \rangle ds$$

By Hamilton equations and Euler identity, we have on  $H = 1$ .

$$\langle P(s, \psi), \dot{X}(s, \psi) \rangle = \langle P(s, \psi), \partial_p H(X, P) \rangle = mH(X, P) = m$$

and  $S(t, \psi) = S_0 + mt$  is the action on  $\Lambda_+$ . So

$$m dt = \langle P(t, \psi), dX(t, \psi) \rangle = \langle P(t, \psi), \dot{X}(t, \psi) \rangle dt + \langle P(t, \psi), X_\psi(t, \psi) \rangle d\psi$$

It follows that  $\langle P(t, \psi), \dot{X}(t, \psi) \rangle = m$  and  $\langle P(t, \psi), \partial_\psi X(t, \psi) \rangle = 0$ . Now we complete the coordinate system  $\psi$  on  $\Lambda$  by a smooth function  $r$  such that  $L$  is given by  $r = 1$ , and set

$$\Phi(x, (t, \psi, r)) = mt + r \langle P(t, \psi), x - X(t, \psi) \rangle$$

where  $r$  can be interpreted as a Lagrange multiplier. Let us check that  $\Phi$  satisfies Hamilton-Jacobi equation. We have

$$(10) \quad \begin{aligned} \partial_t \Phi &= \dot{\Phi} = m + r \langle \dot{P}, x - X(t, \psi) \rangle - r \langle P, \dot{X} \rangle = \\ &= m(1 - r) + r \langle \dot{P}(t, \psi), x - X(t, \psi) \rangle \\ \partial_r \Phi &= \langle P(t, \psi), x - X(t, \psi) \rangle \\ \partial_\psi \Phi &= r \langle \partial_\psi P(t, \psi), x - X(t, \psi) \rangle \end{aligned}$$

Last 2 equations in  $(\dot{\Phi}, \partial_r \Phi, \partial_\psi \Phi) = 0$  give an homogeneous linear system with determinant  $\det(P, P_\psi)$ . On  $x = 0$  we get  $|p| = n(0) > 0$ , and in dimension  $n = 2$ ,  $p = n(0)^t (\cos \psi, \sin \psi)$ . It follows that  $\det(P, P_\psi) = |n(0)|^2$  so for small  $t$ , we get  $x - X(t, \psi) = 0$ , so the phase is critical with respect to  $(\psi, r)$  for  $x = X(t, \psi)$ . Substituting into the last equation (10) we get  $\dot{\Phi} = 0$  when  $r = 1$ . We complete the proof that  $\Phi$  is a generating function for  $\Lambda_+$  by checking  $d\partial_t \Phi, d\partial_\psi \Phi, d\partial_r \Phi$  are linearly independent on the set  $x = X(t, \psi)$ . This can be done by examining the variational system associated with Hamilton equations.

Moreover we can reduce this generating function by eliminating  $t$  by stationary phase. Again, of course, this holds only for small  $t > 0$ , before unfolding of Lagrangian singularities.

**Example 2:** We take  $H$  as in (8) with  $n(x) = n(|x|)$ ,  $n = 2$  for simplicity and

$$\Lambda = \{x = X(\varphi, \psi) = \varphi\omega(\psi), p = P(\varphi, \psi) = \omega(\psi), \varphi \in \mathbf{R}, \omega \in \mathbf{S}^{n-1}\}$$

$\psi$  being the usual angles parametrizing  $\omega \in \mathbf{S}^{n-1}$ . When  $n = 2$  this is the wave-front set of Bessel function  $f_h(x; h) = J_0(\frac{|x|}{h})$ ; such functions arise in the wave beam theory (see [Ki], [DoMaNa] and references therein), so we call  $f_h$  a ‘‘Bessel beam’’.

Computing the action we find  $p dx|_\Lambda = d\varphi$  so coordinate  $\varphi$  will play the role of  $x$  in the previous Example. We have  $n(|x|) = n(\varphi)$ , and  $\varphi = \text{Const.}$  on  $L = \Lambda \cap \{H = 1\}$ . The argument above extends readily to this setting, in particular as in (9)

$$S(t, \varphi, \psi) = S(0, \varphi, \psi) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} p dx|_{\Lambda_+} = S(0, \varphi, \psi) + \int_{(0, \varphi, \psi)}^{(t, \varphi, \psi)} P(s, \varphi, \psi) dX(s, \varphi, \psi) = S_0 + \int_0^t \langle P(s, \varphi, \psi), \dot{X}(s, \varphi, \psi) \rangle ds$$

and

$$\langle P(t, \varphi, \psi), \partial_\psi X(t, \varphi, \psi) \rangle = \langle P, \partial_\varphi X \rangle = 0$$

Now we can apply the results of [DNS], Sect.2.2 on eikonal coordinates for a Lagrangian manifold in a general position, with  $X_{\tilde{\varphi}}(\tilde{\varphi}, \tilde{\psi})$  of rank  $k$ . Here  $\tilde{\varphi}, \tilde{\psi}$  are coordinates in a local chart of  $\Lambda_+$  in the extended phase space. More specifically, we take  $\tilde{\psi} = (t, \psi)$ ,  $\tilde{\varphi} = \varphi$ , try to make a change of variables  $\varphi = \varphi(x, t, \psi)$ , and seek for a generating function of  $\Lambda_+$  in the form  $\Phi(x, (t, \psi, r))$  as in the previous Example. Details will be given elsewhere.

## 5. More examples and perspectives.

The methods above apply in a number of situations as:

(1) The water-wave Hamiltonian  $H(x, hD_x; h)$  with  $H_0(x, p) = |p| \tanh(|p|D(x)) - E$ , together with  $\Lambda = \{x = x_0\}$  has been discussed in [DoNa], [AnDoNaRo2], in relationship with (11) or Helmholtz operator  $H(x, hD_x) = -h^2\Delta - (n(x))^2$ . The localized function  $f$  can be a Gaussian (even in  $x$ ) or a Gaussian times a linear function (odd in  $x$ ), or can be of antenna type, i.e. its Fourier transform localized in a cone in  $p$ .

(2) The kinetic part of Hamiltonian is of Lorenzian type (as  $-p_0^2 + p_1^2$ ), and  $f$  a localized function (Gaussian) supported on  $\Lambda = \{x = x_0\}$ , so that the semi-classical Green function  $u_h$  is the linear response to  $f$  localized on Kelvin angle (or Mach cone).

It should also be possible to construct semi-classical Green functions in case the pair  $(\Lambda, \Lambda_+)$  is no longer intersecting cleanly, but glancing. We then need second-microlocalization, and introducing so called 2-phases, see e.g. [LaWi] in the standard (polyhomogeneous) calculus.

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