



**HAL**  
open science

# The one dimensional semi-classical Bogoliubov-de Gennes Hamiltonian with PT symmetry: generalized Bohr-Sommerfeld quantization rules.

Abdelwaheb Ifa, Michel L. Rouleux

## ► To cite this version:

Abdelwaheb Ifa, Michel L. Rouleux. The one dimensional semi-classical Bogoliubov-de Gennes Hamiltonian with PT symmetry: generalized Bohr-Sommerfeld quantization rules.. Group 32 (32nd International Colloquium on Group Theoretical Methods in Physics), Jul 2018, Prague, Czech Republic. pp.012049, 10.1088/1742-6596/1194/1/012049 . hal-02049767

**HAL Id: hal-02049767**

**<https://hal.science/hal-02049767>**

Submitted on 26 Feb 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The one dimensional semi-classical Bogoliubov-de Gennes Hamiltonian with PT symmetry: generalized Bohr-Sommerfeld quantization rules

**A Ifa**

Université Tunis El-Manar, Tunis, Tunisia

**M Rouleux**

Aix-Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France

E-mail: [abdelwaheb.ifa@fsm.rnu.tn](mailto:abdelwaheb.ifa@fsm.rnu.tn); [rouleux@univ-tln.fr](mailto:rouleux@univ-tln.fr)

**Abstract.** We present a method for computing first order asymptotics of semiclassical spectra for 1-D Bogoliubov-de Gennes (BdG) Hamiltonian from Supraconductivity, which models the electron/hole scattering through two SNS junctions. This involves: 1) reducing the system to Weber equation near the branching point at the junctions; 2) constructing local sections of the fibre bundle of microlocal solutions; 3) normalizing these solutions for the “flux norm” associated to the microlocal Wronskians; 4) finding the relative monodromy matrices in the gauge group that leaves invariant the flux norm; 5) from this we deduce Bohr-Sommerfeld (BS) quantization rules that hold precisely when the fibre bundle of microlocal solutions (depending on the energy parameter  $E$ ) has trivial holonomy. Such a semi-classical treatment reveals interesting continuous symmetries related to monodromy. Details will appear elsewhere.

## 1. Bogoliubov-de Gennes Hamiltonian

BdG Hamiltonian describes the dynamics of a pair of quasi-particles electron/hole in the Theory of Supraconductivity [2]. We consider a narrow metallic 1-D wire (Normal Metal N) connected to Supraconducting bulks S through a SNS junction, and compute the excitation spectrum in the normal contact region as a function of gate voltage, when electronic levels transform into phase sensitive Andreev levels. The wire, or lead, is identified with a 1-D structure, the interval  $x \in [-L, L]$  (case of a perfect junction) or  $x \in [-L + \ell/2, L - \ell/2]$  (“dirty junction”), where  $\ell \ll L$ . The reference energy in the lead is Fermi level  $E_F$ . The pair electron/hole is acted upon by two kinds of potentials:

(1) the “order parameter”  $\Delta(x)$  times a phase function  $e^{i\phi(x)/2}$ , which is the potential due to Cooper pairs in the supraconducting bulk. This potential, subject to self-consistency relations, is priori unknown. Namely, inside S,  $\Delta(x)e^{i\phi(x)/2}$  is a solution of Ginzburg-Landau (or Pitaevskiy) equations, and shows typically a vortex profile (in 2-D). In BdG Hamiltonian it is assumed, however, that  $\Delta(x)e^{i\phi(x)/2}$  is an “effective” potential. Inside N, superconducting gap  $\Delta(x) \equiv 0$ : quasi-particles live in the “clean metal”. For  $|x| \geq L + \ell$ ,  $\Delta(x) = \Delta_0 > 0$ .

We assume that the phase function  $\phi(x)$  is constant near the junction, and gauge the interaction by  $\phi_- = -\phi_+ = -\phi$  in the superconducting banks, so that  $\phi(x) = \text{sgn}(x)\phi$ . We

assume further that this equality holds everywhere: since  $\Delta(x) = 0$  inside N, the discontinuity of  $x \mapsto \phi(x)$  is irrelevant.

(2) a smooth chemical potential  $\mu(x)$ : typically  $\mu(x)$  is flat in N and drops smoothly to the band bottom in the superconducting banks S. In our model we assume again  $\mu(x)$  to be constant in the superconducting bank, i.e.  $\mu(x) = \mu_0$  when  $|x| \geq L + \ell$ . Andreev currents at energy  $E$  occur only if  $\mu(x) \geq E$  in  $[-L, L]$ .

The case of a perfect junction ( $\Delta$  “hard-wall potential”) has been considered in [5], see also [4] for a SFS junction, and makes use scattering matrix techniques. In this work, justifying semi-classical techniques as in [8] (also in the multi-dimensional case) we rather consider an imperfect (or “dirty”) junction:  $\Delta(x)e^{i\phi(x)/2}$  is a smooth function. In a neighborhood of  $[-L, L]$ , say  $x \in [-L - \ell, L + \ell]$ , the system is described at the classical level by BdG Hamiltonian

$$\mathcal{P}(x, \xi) = \begin{pmatrix} \xi^2 - \mu(x) & \Delta(x)e^{i\phi(x)/2} \\ \Delta(x)e^{-i\phi(x)/2} & -\xi^2 + \mu(x) \end{pmatrix} \quad (1)$$

The energy surface:  $\Sigma_E = \{\det(\mathcal{P} - E) = -(\xi^2 - \mu(x))^2 - \Delta(x)^2 + E^2 = 0\} = \Lambda_E^< \cup \Lambda_E^>$  splits into 2 branches separated in momentum space, so consists of two microlocal wells. Interaction between these wells gives the imaginary parts of the resonances for the electron/hole scattering, and will be ignored in this paper. Because of smoothness of  $x \mapsto \Delta(x)$ , the reflections occur inside  $[-L, L]$ , we denote by  $(\pm x_E, \xi_E) \in \Lambda_E^>$ , the one-parameter family of “branching points” defined by  $\Delta(\pm x_E) = E$  with  $x_E$  near  $x_0 \in [L - \frac{\ell}{2}, L + \frac{\ell}{2}]$ ,  $\Delta(x_0) > 0$ . We do not consider the problem of “clustering” of eigenvalues as  $E \rightarrow 0 = E_F$  (Fermi level). In the “hard wall potential” limit for  $x$  near  $x_0$ , the potential  $\Delta(x)$  can be safely approximated by a linear function such that  $\Delta(x_0) = E_0$ , and  $\mu(x)$  by a constant  $\mu$ . So near  $x_0$  we assume that

$$\phi(x) = \phi, \quad \mu(x) = \mu > E, \quad \Delta(x) = E + \alpha(x - x_E)$$

for large  $\alpha > 0$ . Condition  $a_E = (x_E, \xi_E) \in \Sigma_E$  gives  $\xi_E^2 = \mu > E$ ,  $\Delta(x_E) = E$ .

The physical mechanism goes roughly as follows (see [5] for a detailed exposition): An electron  $e^-$  moving in the metallic lead, say, to the right, with energy  $0 < E \leq \Delta$  below the gap and kinetic energy  $K_+(x) = \mu(x) + \sqrt{E^2 - \Delta(x)^2}$  is reflected back as a hole  $e^+$  from the supraconductor, injecting a Cooper pair into the superconducting contact. The hole has kinetic energy  $K_-(x) = \mu(x) - \sqrt{E^2 - \Delta(x)^2}$ , and a momentum of the same sign as this of the electron. When  $\inf_{[-L, L]} K_-(x) > 0$  it bounces along the lead to the left and picks up a Cooper pair in the supraconductor, transforming again to the original electron state, a process known as Andreev reflection. This works also the other way in  $\Lambda_E^<$ , since Hamiltonian system conserves both charge and energy. Actually, the hole can propagate throughout the lead only if  $\inf_{[-L, L]} \mu(x) \geq E$ . Otherwise, it is reflected from the potential  $\mu(x)$  in the junction, and Andreev levels are quenched at higher energies, i.e. transform into localized electronic states.

For a rescaled “Planck constant”  $h$  so that  $h \ll \ell$ , we consider Weyl  $h$ -quantization of BdG Hamiltonian  $\mathcal{P}(x, hD_x)$  on  $L^2(I) \otimes \mathbf{C}^2$ ,  $I = [-(L + \ell), L + \ell]$ , which is self-adjoint when imposing Dirichlet boundary conditions at  $\partial I$ . Phase-sensitive Andreev states carry supercurrents that turn out to be proportional to the  $\phi$ -derivative of the eigen-energies of  $\mathcal{P}(x, hD_x)$ .

We have  $\sigma^y \mathcal{P}(\phi) \sigma^y = -\mathcal{P}(-\phi)$ , with  $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ , accounting for “negative energies”. We shall assume here  $E > 0$ . When potentials are even functions (typical for metals),  $\mathcal{P}(x, hD_x)$  verifies PT symmetry  ${}^V \mathcal{I} \mathcal{P}(x, hD_x) = \mathcal{P}(x, hD_x) \mathcal{I}^V$  which is essential for our approach to work.

At least formally, since BdG is only defined locally near N, removing boundary conditions leads to “resonances” (i.e. metastable states or quasi-particles with a finite life-time). Thus for simplicity we have assumed that (1), together with its semi-classical quantization, describes the system not only in  $I$ , but on the whole real line, provided  $h \ll \ell \ll L$ . Thus  $\mathcal{P}(x, hD_x)$  extends to  $L^2(\mathbf{R}) \otimes \mathbf{C}^2$ ,

Our general goal is to give a precise mathematical meaning to these “resonances”. Here we content to compute their real parts through Bohr-Sommerfeld quantization rules.

## 2. Monodromy operator, scattering matrix: an outlook

a) *Schrödinger operator on the real line.*

We first recall from [1] basic facts for a 1-D Schrödinger operator with a compactly supported potential  $V$ . The generalized wave-functions  $u$  with energy  $E = k^2 > 0$  satisfy

$$-h^2 u''(x) + V(x)u(x) = Eu(x) \quad (2)$$

and outside  $\text{supp} V$ ,

$$-h^2 u''(x) = k^2 u(x) \quad (3)$$

defines the state space  $\mathcal{Z} \approx \mathbf{C}^2$  of the “free particle”, spanned by  $f_1(x) = e^{ikx/h}$ ,  $f_2(x) = e^{-ikx/h}$ . The monodromy operator  $M(k) : f_1 + Bf_2 \mapsto Af_1$  is such that

$$M(k) = \begin{pmatrix} 1/\bar{A} & -\bar{B}/\bar{A} \\ -B/A & 1/A \end{pmatrix} \in \text{SU}(1, 1)$$

In particular,  $|A|^2 + |B|^2 = 1$ . We call  $|A|^2$  the *transmission coefficient* and  $|B|^2$  the *reflection coefficient*. Along with the passage from the left to the right of the support of  $V$ , consider the passage from the right to the left. The corresponding solution  $v$  of (2) is  $e^{-ikx/h} + B_2 e^{ikx/h}$  to the right of  $\text{supp} V$ , and  $A_2 e^{-ikx/h}$  to the left. The *scattering matrix* is defined as

$$S(k) = \begin{pmatrix} A & B \\ -\bar{B}A/\bar{A} & A \end{pmatrix} \in \text{U}(2)$$

$S(k)$  remains unitary and symmetric for complex values of  $k$ . *Resonances* of (2) are then defined as  $E = k^2 \in \mathbf{C}$ , where  $k$  is a pole of  $S$ , and *physical resonances* those with  $\text{Im}k > 0$ . Thus  $E$  is a resonance iff the solution of (3) is purely outgoing as  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ . The poles coincide with the poles of meromorphic extension of the resolvent  $(P - k^2)^{-1}$  from the physical half-plane  $\text{Im}E < 0$  to the second sheet  $\text{Im}E > 0$ .

b) *Monodromy matrix for BdG equation: heuristics.*

Now we discuss BdG equation  $(\mathcal{P}(x, hD_x) - E)U = 0$  for large  $|x|$ , i.e. (within our approximation above) when  $|x| \geq L + \ell$ , so  $\Delta(x) = \Delta_0$ ,  $\mu(x) = \mu_0 > E$ . Solutions are of the form

$$U(x; h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{ikx/h} \\ e^{i\ell x/h} \end{pmatrix}$$

$\mu_0 + E \pm i\Delta_0 \in \{k^2, \ell^2\}$ , so eigenfrequencies are  $(\pm k, \pm \bar{k})$ ,  $k = \sqrt{\mu_0 + E + i\Delta_0}$ , and the corresponding solutions as follows:

Let  $\phi(x) = \text{sgn}(x)\phi$ ,  $\mathcal{Z}$  be the 2-D complex line bundle spanned by  $F_1^\pm(x) = \begin{pmatrix} e^{i\phi(x)/2} \\ -i \end{pmatrix} e^{\pm ikx/h}$  (associated with the scattering process  $e^+ \rightarrow e^-$ ), and  $\bar{\mathcal{Z}}$  the 2-D complex line bundle spanned by  $F_2^\pm(x) = \begin{pmatrix} e^{i\phi(x)/2} \\ i \end{pmatrix} e^{\pm i\bar{k}x/h}$  (associated with the scattering process  $e^- \rightarrow e^+$ ).

The space of solutions of exponential type for BdG is  $\mathcal{Z} \oplus \bar{\mathcal{Z}}$ , and  $\mathcal{Z}, \bar{\mathcal{Z}}$  are orthogonal for the usual pointwise Hermitian product in  $\mathbf{C}^2$ . Declare that  $E \in \mathbf{C}$  is a  $\mathcal{Z}$ -*resonance* iff the  $\mathcal{Z}$ -component of the wave function solving BdG equation is outgoing and evanescent (“physical solution”) at infinity, i.e.

$$\begin{aligned} U(x, h) &= A \begin{pmatrix} e^{i\phi/2} \\ -i \end{pmatrix} e^{ikx/h}, x \rightarrow +\infty \\ U(x, h) &= B \begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{-ikx/h}, x \rightarrow -\infty \end{aligned}$$

Similarly we say that  $E$  is a  $\overline{\mathcal{Z}}$ -resonance iff the  $\overline{\mathcal{Z}}$ -component of the wave function is outgoing (and evanescent) at infinity, i.e.

$$\begin{aligned} U(x, h) &= A \begin{pmatrix} e^{i\phi/2} \\ i \end{pmatrix} e^{-i\overline{k}x/h}, x \rightarrow +\infty \\ U(x, h) &= B \begin{pmatrix} e^{-i\phi/2} \\ i \end{pmatrix} e^{i\overline{k}x/h}, x \rightarrow -\infty \end{aligned}$$

So for both sets of resonances, the corresponding solution is simultaneously decaying, and outgoing at  $\pm\infty$ . These sets of resonances need not coincide (although they come up in pairs), but their real parts are given by Bohr-Sommerfeld quantization rules. Namely, define the monodromy operator  $M^{\mathcal{Z}}(k)$  acting on  $\mathcal{Z}$  according to the formula

$$\begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{ikx/h} + B \begin{pmatrix} e^{-i\phi/2} \\ -i \end{pmatrix} e^{-ikx/h} \mapsto A \begin{pmatrix} e^{i\phi/2} \\ -i \end{pmatrix} e^{ikx/h}$$

and similarly for  $M^{\overline{\mathcal{Z}}}(k)$ . It is plausible to expect that  $M^{\mathcal{Z}}(k), M^{\overline{\mathcal{Z}}}(k) \in \text{U}(1, 1)$ , and that the corresponding scattering matrices  $S^{\mathcal{Z}}(k), S^{\overline{\mathcal{Z}}}(k)$  have a meromorphic extension to the complex plane, their poles defining the resonances  $E^{\mathcal{Z}}$  and  $E^{\overline{\mathcal{Z}}}$ . Actually, we shall construct “relative monodromy operators” in the “classically allowed region”. In particular the relative monodromy operators are in  $\text{U}(1, 1)$  for some specific Lorenzian form which is constructed below.

### 3. Bohr-Sommerfeld quantization rules

In this work, we content to determine the real parts of the resonances, extending to this setting the method of positive commutators elaborated in [12], [9] and [10]. Imaginary parts may be determined as in [11]. We obtain Bohr-Sommerfeld quantization rules for the quasi-particle, alternating even and odd quantum numbers associated with the electron and the hole. In the sequel we will sketch a proof of the following result:

**Theorem 1:** Let  $\int_{-x_0}^{x_0} \eta^\rho(y; h) dy$  be the semi-classical actions (see Proposition 8 below)  $\rho = 1$  for the electron,  $\rho = -1$  for the hole. Bohr-Sommerfeld quantization conditions near  $E_0$  are given at first order by:

$$\oint_{\gamma_E} \eta^\rho(y; h) dy - h\phi + h\pi + \mathcal{O}(h^2) = 2\pi n h; \quad n \in \mathbf{Z}$$

Here  $\oint_{\gamma_E}$  denotes integral over the loop  $\gamma_E$  obtained by gluing together  $\Lambda_E^>$  and  $\Lambda_E^<$ , if we ignore tunneling in momentum space.

### 4. Microlocal solutions in Fourier representation near the branching points

a) *Reduction of the system.*

In  $h$ -Fourier representation,  $\mathcal{F}_h u(\xi) = (2\pi h)^{-1/2} \int e^{-ix\xi/h} u(x) dx$  the local Hamiltonian near  $a = a_E = (x_E, \xi_E)$ ,  $\mathcal{P}^a$  takes the form :

$$\mathcal{P}^a(-hD_\xi, \xi) = \begin{pmatrix} \xi^2 - \mu & e^{i\phi/2}(E - \alpha h D_\xi - \alpha x_E) \\ e^{-i\phi/2}(E - \alpha h D_\xi - \alpha x_E) & -\xi^2 + \mu \end{pmatrix} \quad (4)$$

By PT symmetry  $\mathcal{P}^{a'} = \mathcal{I}\mathcal{P}^a\mathcal{I}$  near  $a' = a'_E = (-x_E, \xi_E)$ . Solving the system  $\mathcal{P}^a(-hD_\xi, \xi)\widehat{U} = 0$ ,  $\widehat{U} = \begin{pmatrix} \widehat{\varphi}_1 \\ \widehat{\varphi}_2 \end{pmatrix}$  gives second order ODE for  $u(\xi) = \exp[-i \int^\xi g(s)ds/h]\widehat{\varphi}_2(\xi)$ ,

$$P^a(-hD_\xi, \xi, h)u(\xi) = \frac{E^2}{\alpha^2}u(\xi) \quad (5)$$

$$P^a(-hD_\xi, \xi, h) = (hD_\xi)^2 + \alpha^{-2}(\xi^2 - \mu)^2 + h^2(\xi^2 - \mu - E)^{-2}(2\xi^2 + \mu + E)$$

After  $E$ -dependent scalings  $\beta = \sqrt{\alpha}(2\xi_E)^{-3/2} > 0$ ,  $E_1 = (2\xi_E)^{-2}E$ ,  $\xi = 2\xi_E\beta\omega\xi' + \xi_E$ ,  $\omega = \pm 1$  ( $\xi'$  is ‘‘local momentum’’) we obtain  $P_\omega^a(-hD_{\xi'}, \xi', h)u_\omega(\xi') = \left(\frac{E_1}{\beta}\right)^2 u_\omega(\xi')$ , where

$$P_\omega^a(-hD_{\xi'}, \xi'; h) = (-hD_{\xi'})^2 + (\xi' + \beta\omega\xi'^2)^2 + h^2\beta^2 f(\omega\beta\xi')$$

is an anharmonic Schrödinger operator. The lower order term  $f(z) = (2z^2 + 2z + \frac{3}{4} + E_1)(z^2 + z - E_1)^{-2}$  has a pole on  $\Lambda_E^>$  where the linear approximation of  $\Delta(x)$  breaks down. The linear approximation only holds for small  $\xi'$ . Consider the map

$$\iota^a : \sum_{\omega=\pm 1} \text{Ker}_h(P_\omega^a - \left(\frac{E_1\omega}{\beta}\right)^2) \rightarrow \text{Ker}_h(\mathcal{P}^a - E) \quad (6)$$

where  $\text{Ker}_h$  denotes the microlocal kernel. The index  $\omega$  is to be chosen carefully with the complex germ of solutions having the right decay beyond the branching points  $\pm x_E$ . We shall endow the RHS of (6) with a Lorenzian structure and ‘‘diagonalize’’  $\iota^a$  in some orthogonal subspaces.

*b) The normal form of Helffer-Sjöstrand*

When  $E_1 < \frac{1}{4}$ , we take  $P_\omega^a$  microlocally to its normal form, namely:

**Proposition 2** [9]: There exists an analytic diffeomorphism  $t \mapsto F_0(t)$  defined in a neighborhood of 0,  $F_0(0) = 0$ , with inverse  $G_0$ , and a real analytic phase function  $\phi_\beta(\xi', \theta)$ , defined in a neighborhood of (0,0), of the form  $\phi_\beta(\xi', \theta) = \xi'\theta + g_\beta(\xi', \theta)$ ,  $g_\beta(\xi', \theta) = \mathcal{O}(|\xi', \theta|^3)$ , parametrizing the canonical transformation  $\kappa_\beta : (\partial_\theta\phi_\beta, \theta) \mapsto (\xi', \partial_{\xi'}\phi_\beta)$ , such that  $F_0 \circ p_\beta \circ \kappa_\beta = p_0$ . At the semi-classical level, there is a (formally) unitary FIO operator  $A$  defined microlocally near (0,0)

$$Av(\xi', h) = (2\pi h)^{-1} \int \int e^{i\varphi(\xi', \eta, \theta)/h} c(\xi', \eta, \theta, h) e^{ib(\xi', \eta, \theta, h)} v(\eta, h) d\eta d\theta$$

and a real valued analytic symbol

$$F(t, \beta, h) = F_0(t, \beta) + hF_1(t, \beta) + h^2F_2(t, \beta) + \dots$$

with  $F_1(t, \beta) = -\frac{1}{2}$  such that

$$A^*F(P_\omega, \beta, h)A = P_0(\eta, hD_\eta) = \frac{1}{2}((hD_\eta)^2 + \eta^2 - h), \quad A^*A \equiv \text{Id}$$

The function  $F_0$ , taking the period  $T(E)$  of Hamilton vector flow for  $P_\omega^a$  at energy  $(E_1/\beta)^2$  to  $2\pi$ , involves an elliptic integral, which requires sometimes the use of formal calculus.

*c) Weber equation and parabolic cylinder functions*

Weber equation  $P_0v = \nu hv$ , through change of variables  $\eta = (h/2)^{1/2}\zeta$ ,  $\tilde{v}(\zeta) = v(\eta)$  scales to

$$-\tilde{v}'' + \frac{1}{4}\zeta^2\tilde{v} = \left(\nu + \frac{1}{2}\right)\tilde{v}$$

Fundamental solutions express as *parabolic cylinder* functions  $D_\nu$ , entire in  $\mathbf{C}$ . The systems  $(D_\nu(\pm\zeta), D_{-\nu-1}(\pm i\zeta))$  are fundamental solutions for any choice of  $\pm$ . Integral representations give asymptotic solutions of  $(P_0 - \nu h)u(\eta) = 0$  by stationary phase for real  $\nu$ ,  $E_1^2 = 2\beta^2 F(\beta^{-2}E_1^2, \beta, h) = 2\beta^2(\nu + 1)h$ .

$$\begin{aligned} D_\nu(\varepsilon(h/2)^{-1/2}\eta) &= \frac{\Gamma(\nu+1)}{-2i\pi\sqrt{h}} h^{E^2/4h} \int_\infty^{(0^+)} \exp[i\Phi_\varepsilon^\nu(s; \eta)/h] ds \\ D_{-\nu-1}(i\varepsilon(h/2)^{-1/2}\eta) &= \frac{\Gamma(-\nu)}{2i\pi} h^{-E^2/4h} \int_\infty^{(0^+)} \exp[i\Phi_\varepsilon^{-\nu-1}(s; \eta)/h] \frac{ds}{s} \end{aligned}$$

with  $\varepsilon = \pm 1$ ,  $E = \sqrt{2(\nu + 1)h}$ , see [13]. This normalization is called Whittaker normalization. Classically forbidden regions  $|\eta| > E$  lie on Stokes lines, classically allowed region  $|\eta| < E$  in between, and 3 Stokes lines stem from each ‘‘turning point’’  $\eta = \pm E$ .

d) *Microlocal solutions.*

We apply asymptotic stationary phase to  $AD_j$ ,  $j \in \{\nu, -\nu-1\}$ . With  $h' = \beta^2 h$  as a ‘‘rescaled’’ Planck constant, we get:

**Proposition 3:** In Fourier representation, the image  $K_h^a(E) = \text{Ker}_h(\mathcal{P}^a(-hD_\xi, \xi) - E)$  of  $\iota^a$  is a 2-D vector space spanned by the spinors  $\widehat{U}_{\varepsilon, \omega}^j = (\widehat{\varphi}_2^j)_{\varepsilon, \omega}^j$ ,  $(j, \varepsilon, \omega) \in \{\nu, -\nu-1\} \times \{-1, 1\}^2$ , of the form:

$$\begin{aligned} \widehat{U}_{\varepsilon, \omega}^\nu &= C_{h'}^\nu \sum_{\theta_\omega = \pm \widehat{\theta}_\omega(\xi_1)} \begin{pmatrix} e^{i\phi/2}(\xi^2 - \mu - E)^{-1/2} X_{\varepsilon, \omega}^\nu \\ (\xi^2 - \mu - E)^{1/2} \end{pmatrix} |\widetilde{a}_{\varepsilon, \omega}^\nu| \exp[i(\Phi_{\varepsilon, \omega}^\nu + h'R_\omega^\nu)/h'] + \mathcal{O}(h') \\ \widehat{U}_{\varepsilon, \omega}^{-\nu-1} &= C_{h'}^{-\nu-1} \sum_{\theta_\omega = \pm \widehat{\theta}_\omega(\xi_1)} \varepsilon \text{sgn}(\theta_\omega) \begin{pmatrix} e^{i\phi/2}(\xi^2 - \mu - E)^{-1/2} X_{\varepsilon, \omega}^{-\nu-1} \\ (\xi^2 - \mu - E)^{1/2} \end{pmatrix} \\ &\quad |\widetilde{a}_{\varepsilon, \omega}^{-\nu-1}| \exp[i(\Phi_{\varepsilon, \omega}^{-\nu-1} + h'R_\omega^{-\nu-1})/h'] + \mathcal{O}(h') \end{aligned}$$

Here  $\widehat{\theta}_\omega(\xi_1)$  is a critical point (from stationary phase),  $\Phi_{\varepsilon, \omega}^j + h'R_\omega^j$  the  $h'$ -dependent phase functions, and  $X_{\varepsilon, \omega}^j$ ,  $|\widetilde{a}_{\varepsilon, \omega}^j|$  some positive amplitudes. Spinors  $U_{\varepsilon, \omega}^j$  verify the symmetry  $\dagger \widehat{U}_{-\varepsilon, -\omega}^j = \widehat{U}_{\varepsilon, \omega}^j$  for the ‘‘local time’’ reversal operator  $\dagger u(\xi_1) = u(-\xi_1)$ , and the constants  $C_{h'}^j$  (from Whittaker normalization of  $D_\nu, D_{-\nu-1}$ ) are related by  $C_{h'}^\nu C_{h'}^{-\nu-1} = ((2\sqrt{h'})^3 \pi^2 \sin \pi\nu)^{-1}$ .

## 5. Normalization

a) *The microlocal Wronskian.*

We extend to BdG Hamiltonian the classical ‘‘positive commutator method’’ using conservation of some quantity called a ‘‘quantum flux’’ ([12], [9], [11], [10]).

**Definition 4:** Let  $\mathcal{P}$  be (formally) self-adjoint, and  $U^a, V^a \in K_h(E)$  be supported on  $\Lambda_E^>$ . We call the sesquilinear form  $\mathcal{W}_\rho^a(U^a, V^a) = (\frac{i}{h}[\mathcal{P}, \chi^a]_\rho U^a | V^a) = (\frac{i}{h}[\mathcal{P}, \chi^a]_\rho \widehat{U}^a | \widehat{V}^a)$  the *microlocal Wronskian* of  $(U^a, \overline{V^a})$  in  $\omega_\rho^a$ . Here  $\frac{i}{h}[\mathcal{P}, \chi^a]_\rho$  denotes the part of the commutator supported microlocally on  $\omega_\rho^a$  (a small neighborhood of  $\text{supp}[\mathcal{P}, \chi^a] \cap \Lambda_E$  near  $\rho$ ).

A crucial property of the microlocal Wronskian is to be invariant by Fourier transformation:  $\mathcal{W}_\rho^a(U^a, V^a) = \mathcal{W}_\rho^a(\widehat{U}^a, \widehat{V}^a)$ . The relation  $\mathcal{W}_+^a(U^a, V^a) + \mathcal{W}_-^a(U^a, V^a) = 0$  doesn’t readily follow as in the scalar case [10], the microlocal solutions being neither smooth in spatial of Fourier representation near the branching point, but from a careful inspection, involving also formal calculus. This is used essentially in Propositions 5 and 8 below. Choosing  $\varepsilon, \omega$  such that  $\varepsilon\omega = 1$

we define a Lorenzian metric  $\mathcal{W}_\rho$  on the space of microlocal solutions near  $a$ . In the basis  $\widehat{U}_{\varepsilon,\omega}^j, j \in \{\nu, -\nu - 1\}$  we have, up to a constant factor:

$$\rho\mathcal{W}_\rho = \begin{pmatrix} |C_{h'}^\nu|^2 \mathcal{O}(h') & C_{h'}^\nu \overline{C_{h'}^{-\nu-1}} \exp[-i\pi E_1^2/4h'] (1 + \mathcal{O}(h')) \\ \overline{C_{h'}^\nu} C_{h'}^{-\nu-1} \exp[i\pi E_1^2/4h'] (1 + \mathcal{O}(h')) & |C_{h'}^{-\nu-1}|^2 \mathcal{O}(h') \end{pmatrix}$$

Changing Whittaker normalization for the  $D_\nu, D_{-\nu-1}$  functions, and the microlocal solutions by some constant phase factors, we can reduce to  $\rho\mathcal{W}_\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(h')$ , and prove:

**Proposition 5:** Under PT symmetry above the microlocal Wronskians  $\mathcal{W}_\rho^a$  endow  $K_h^a(E)$  (mod  $h'$ ) with a Lorenzian form  $\mathcal{W}^a = \frac{1}{2}(\mathcal{W}_+^a - \mathcal{W}_-^a)$ . The same holds at  $a'$ , and the corresponding structures on  $K_h^a \times K_h^{a*}$  and  $K_h^{a'} \times K_h^{a'*}$  are anti-isomorphic. The group of automorphisms preserving  $\mathcal{W}^a$  and  $\mathcal{W}^{a'}$  mod  $\mathcal{O}(h')$  is therefore  $U(1,1)$ .

## 6. Spinors in the spatial representation

We compute  $U_{\varepsilon,\omega}^{a,j}, U_{\varepsilon,\omega}^{a',j}$  in spatial representation, then extend along the branches  $\rho = \pm 1$  of  $\Lambda_E^>$  with WKB solutions.

a) *Spinors near the branching points.*

Near  $a, a'$  we apply inverse  $h$ -Fourier transform and get:

**Proposition 6:** Up to a constant phase factor

$$\begin{aligned} U_{\varepsilon,\omega}^\nu(x, h) &= 2\omega\beta\xi_E e^{ix\xi_E/h} \sum_{\rho=\pm} \left( \frac{e^{i\phi/2}(\xi^2 - \mu - E)^{-1/2} X_{\varepsilon,\omega}^\nu}{(\xi^2 - \mu - E)^{1/2}} \right) |a_{\varepsilon,\omega}^\nu| \Big|_{\theta_1=\theta_\omega(\xi_1), \xi_1=\xi_\omega^\rho(x)} \\ &\quad \times \left( \frac{L_\omega^\rho(x)}{i} \right)^{-1/2} \exp[i(\Psi_{\varepsilon,\omega}^{\nu,\rho}(x) + h'R_{\varepsilon,\omega}^{\nu,\rho}(x))/h'] (1 + \mathcal{O}(h')) \\ U_{\varepsilon,\omega}^{-\nu-1}(x, h) &= 2\omega\beta\xi_E e^{ix\xi_E/h} \sum_{\rho=\pm} \varepsilon \operatorname{sgn}(\theta_1) \left( \frac{e^{i\phi/2}(\xi^2 - \mu - E)^{-1/2} X_{\varepsilon,\omega}^{-\nu-1}}{(\xi^2 - \mu - E)^{1/2}} \right) |\widetilde{a}_{\varepsilon,\omega}^{-\nu-1}| \Big|_{\theta_1=\theta_\omega(\xi_1), \xi_1=\xi_\omega^\rho(x)} \\ &\quad \times \left( \frac{L_\omega^\rho(x)}{i} \right)^{-1/2} \exp[i(\Psi_{\varepsilon,\omega}^{\nu,\rho}(x) + h'R_{\varepsilon,\omega}^{\nu,\rho}(x))/h'] (1 + \mathcal{O}(h')) \end{aligned}$$

Here  $(L_\omega^\rho(x))^{-1/2}$  is a real density (singular at  $x = x_E$ ), and  $\rho$  labels the branch of the Lagrangian manifold. The phases  $\Psi_{\varepsilon,\omega}^{j,\rho}(x) + h'R_{\varepsilon,\omega}^{j,\rho}(x)$ ,  $j \in \{\nu, -\nu - 1\}$  differ only by a constant.

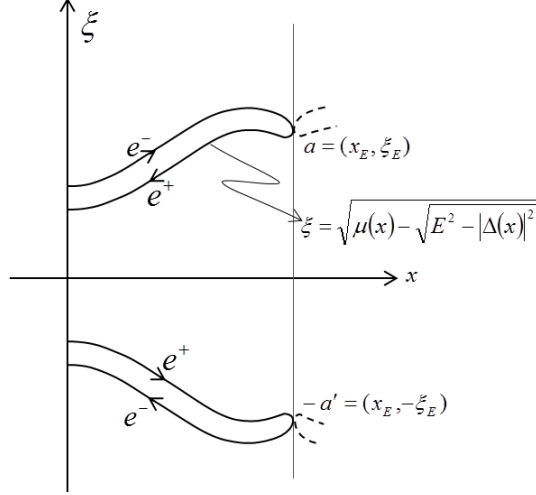
b) *WKB spinors away from the branching points*

The Lagrangian manifold  $\Lambda_E^>$  consists of 2 branches  $\Lambda_E^{>\rho}$  (or simply  $\rho$ )  $\rho = \pm 1$  so that  $\rho = +1$  belongs to the electronic state ( $\xi_1 > 0$  in the local coordinates near  $a$  above), resp.  $\rho = -1$  to the hole state ( $\xi_1 < 0$ ). These states mix up when  $\Delta(x) \neq 0$ , but we can sort them out semiclassically, outside  $a, a'$ . Call the vector space of  $\mathbf{C}^2$  generated by  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  the space of (pure) *electronic states*, or *electronic spinors*, and this by  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  the space of (pure) *hole states*, or *hole spinors*.

The principal symbol  $\mathcal{P}(x, \xi)$  has eigenvalues  $\lambda_\rho = \rho\lambda(x, \xi) = \rho\sqrt{\Delta(x)^2 + (\xi^2 - \mu(x))^2}$ . By diagonalizing, we obtain a line bundle  $\Lambda_E^\rho$  with fiber

$$Y_\rho(x, \xi) = (\Delta^2 + (-\xi^2 + \mu + \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2})^2)^{-1/2} \begin{pmatrix} \Delta e^{i\phi/2} \\ -\xi^2 + \mu + \rho\sqrt{\Delta^2 + (\xi^2 - \mu)^2} \end{pmatrix}$$





**Figure 1.** Phase-space picture

Looking at the electronic state, we choose  $\rho = +1$  so that  $\lambda_\rho(x_\rho, \xi_\rho) - E = 0$ , while  $\lambda_{-\rho}(x_\rho, \xi_\rho) - E$  is elliptic. and similarly when looking at the hole state.

**Proposition 7** The microlocal kernel  $\text{Ker}_h(\mathcal{P} - E)$  on  $\Lambda_E^{>,\rho}$  is one-dimensional space spanned by

$$W^\rho(x, h) = e^{iS_\rho(x, h)/h} (w_0^\rho(x, h) Y_\rho(x, \partial_x S_\rho) + \mathcal{O}(h)) = e^{iS_\rho(x, h)/h} \widetilde{W}^\rho(x, h)$$

where  $w_0^\rho(x) |dx|^{1/2}$  is a smooth half-density. By the uniqueness property of WKB solutions along simple bicharacteristics, the  $h$  (or  $h'$ )-dependent phase function  $S_\rho(x, h)$  should coincide, up to a constant (in a punctured neighborhood of  $a$ ) with either one of  $\Psi_{\varepsilon, \omega}^{j, \rho}(x) + h' R_{\varepsilon, \omega}^{j, \rho}(x)$  above,  $j \in \{\nu, -\nu - 1\}$ , and similarly for the half-densities.

## 7. Relative monodromy matrices

Now we look for connexion formulas. For each  $\varepsilon, \omega, \rho = \pm 1, j \in \{\nu, -\nu - 1\}$ , the normalized microlocal solutions  $U_{\varepsilon, \omega}^{a', j, \rho}$  are related to the extension  $U_{-\varepsilon, -\omega, \text{ext}}^{a, k, \rho}$  of the normalized microlocal solutions  $U_{\varepsilon, \omega}^{a, k, \rho}$  along the bicharacteristics by a monodromy matrix

$$\mathcal{M}^{a, a', \rho} = \begin{pmatrix} d_{11}^\rho & d_{12}^\rho \\ d_{21}^\rho & d_{22}^\rho \end{pmatrix} \in U(1, 1)$$

(defined at least mod  $\mathcal{O}(h')$ ) which we call a *relative monodromy matrix*. Since there is a pair of particles, the symmetry between the  $\mathcal{M}^{a, a', \rho}$  and  $\mathcal{M}^{a', a, \rho}$  is order 4;  $\mathcal{M}^{a', a, \rho} \in U(1, 1)$  is obtained by extending from the left to the right, and applying symmetry

$$\rho \mathcal{M}^{a', a, \rho} = \mathcal{I}(\mathcal{M}^{a, a', \rho})^{-1} \mathcal{I}, \quad \rho = \pm 1 \quad (7)$$

where  $\mathcal{I}$  denotes complex conjugation. We compute the coefficients  $d_{ij} = d_{ij}^\rho$ . Considering behavior of  $U_{\varepsilon, \omega}^{a', j, \rho}$  in the classically forbidden region (according to scattering process  $e^+ \rightarrow e^-$  or  $e^- \rightarrow e^+$ ) we obtain

$$\mathcal{M}^{a, a', \rho} = \begin{pmatrix} 0 & d_{12}^\rho \\ d_{21}^\rho & 0 \end{pmatrix}, \quad \overline{d_{12}^\rho} d_{21}^\rho = 1$$

Note that if we do not look too closely at the relevant complex branches, as is the case when computing BS, it makes no difference to choose instead  $\mathcal{M}^{a,a',\rho} = \begin{pmatrix} d_{11}^\rho & 0 \\ 0 & d_{22}^\rho \end{pmatrix}$ , with  $\overline{d_{11}^\rho} d_{22}^\rho = 1$ .

As in [12], [9], [11], [10], the argument consists now in extending microlocal solutions obtained above from  $a$  to  $a'$ , and computing the resulting semi-classical action. So take first  $U_1$  equal to  $U_1^a = U_{\varepsilon,\omega}^{\nu,a}$  near  $a$ , extend it along to  $a'$  along the bicharacteristics  $\rho = \pm 1$  by WKB. Evaluating on  $\rho$  near  $a'$  we find  $U_1^{a',\rho} = U_{\varepsilon,\omega,\text{ext}}^{\nu,a,\rho} = d_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a',\rho}$ . Similarly, take  $U_2$  starting at  $a'$  and with  $-\nu - 1$  instead of  $\nu$ , we get  $U_1^{a,\rho} = U_{\varepsilon,\omega,\text{ext}}^{-\nu-1,a',\rho} = e_{12}^\rho U_{\varepsilon,\omega}^{\nu,a,\rho}$ , where  $e_{12}^\rho = \rho(d_{21}^\rho)^{-1}$  is the matrix element of  $\mathcal{M}^{a',a,\rho}$  given in (7). We compute  $d_{21}^\rho$  in two different ways and compare the result.

(1) Using time-reversal and PT symmetries in the microlocal Wronskians, we get

$$\begin{aligned} \left(\frac{i}{\hbar} [\mathcal{P}^{a'}, \chi^{a'}]_\rho U_1 | U_{\varepsilon,\omega}^\nu \right) &= d_{21}^\rho \left(\frac{i}{\hbar} [\mathcal{P}^{a'}, \chi^{a'}]_\rho U_{\varepsilon,\omega}^{-\nu-1} | U_{\varepsilon,\omega}^\nu \right) = \\ &= d_{21}^\rho \mathcal{W}_\rho^{a'}(U_{\varepsilon,\omega}^{-\nu-1}, U_{\varepsilon,\omega}^\nu) = d_{21}^\rho \mathcal{W}_\rho^{a'}(\widehat{U}_{\varepsilon,\omega}^{-\nu-1}, \widehat{U}_{\varepsilon,\omega}^\nu) = \\ &= -d_{21}^\rho \overline{\mathcal{W}_\rho^a(\widehat{U}_{-\varepsilon,-\omega}^{-\nu-1}, \widehat{U}_{-\varepsilon,-\omega}^\nu)} = -d_{21}^\rho \mathcal{W}_\rho^a(\widehat{U}_{-\varepsilon,-\omega}^\nu, \widehat{U}_{-\varepsilon,-\omega}^{-\nu-1}) = -d_{21}^\rho \end{aligned}$$

(2) Using the extensions described in Proposition 7. Near  $a'$  we have  $U_{1,\text{ext}}^\rho = e^{i\phi/2} W^\rho(x, h) = d_{21}^\rho U_{\varepsilon,\omega}^{-\nu-1,a',\rho}$  (by solving transport equation along  $\rho$  the amplitude picks up the phase factor  $e^{i\phi/2}$ ), so we need to compute  $\left(\frac{i}{\hbar} [\mathcal{P}^{a'}, \chi^{a'}]_\rho W^\rho(x, h) | U_{\varepsilon,\omega}^\nu \right)$ . The amplitude  $W^\rho(x, h)$  is actually defined up to a real, constant factor  $\widetilde{C}^\rho$ .

**Proposition 8:** Let  $\widetilde{\Psi}_{\varepsilon,\omega}^{\nu,a',\rho}(x) = x \xi_E + \frac{(2\xi_E)^3}{\alpha} \Psi_{\varepsilon,\omega}^{\nu,a',\rho}(x)$ . We have

$$\left(\frac{i}{\hbar} [\mathcal{P}^{a'}, \chi^{a'}]_\rho W^\rho | U_{\varepsilon,\omega}^{\nu,a',\rho} \right) = 2 \widetilde{C}^\rho e^{i\pi/4} \int \exp[i(\widetilde{S}_\rho(x; h)/h)] \beta(x, h) (\chi_1^{a'})'(x) dx \quad (8)$$

where the amplitude  $\beta(x, h)$ , real mod  $\mathcal{O}(h)$ , is computed from the WKB solutions in Proposition 7, and

$$\begin{aligned} \widetilde{S}_\rho(x, h) &= S_\rho(x; h) - (x \xi_E + \widetilde{\Psi}_{\varepsilon,\omega}^{\nu,a',\rho}(x) - h R_{-\omega}^\nu(\theta_{-\omega}(\xi_{-\omega}^\rho(-x))) = \\ &= \frac{(2\xi_E)^3}{\alpha} \Psi_{-\varepsilon,-\omega}^{\nu,a,\rho}(x_0) - \int_{-x_0}^{x_0} \eta^\rho(y; h) dy + h R_{-\omega}^\nu(\theta_{-\omega}(0)) \end{aligned}$$

Moreover,  $\beta(x, h)$  is also independent of  $x$ , so that, comparing the former expression (1) and (8) for a suitable choice of  $\widetilde{C}^\rho$ , we get

$$d_{21}^\rho = -e^{i\tau^\rho(h)/h} \int (\chi_1^{a'})'(x) dx = e^{i\tau^\rho(h)/h} \quad (9)$$

Here  $\tau^\rho(h) = h \frac{\phi}{2} + h \frac{\pi}{4} - \int_{-x_0}^{x_0} \eta^\rho(y; h) dy + \text{Const.}$ , where Const. is evaluated at the boundaries  $x = \pm x_E$ , and depends only on  $E_1'$ . It will eventually disappear from the final formula, by adding to BS the contribution of the lower branch  $\Lambda_E^{\leq, \rho}$ . Note that  $\int_{-x_0}^{x_0} \eta^\rho(y; h) dy$ ,  $\eta^\rho(y; h)$  being the derivative of the  $h'$ -depending phase function, is the semi-classical action.

## 8. Bohr-Sommerfeld quantization rules

We set  $F_{\varepsilon,\omega}^{j,a,\rho} = \frac{i}{\hbar} [\mathcal{P}^a, \chi^a]_\rho U_{\varepsilon,\omega}^{j,a,\rho}$ , and similarly with  $a'$ . The set  $\{G_{\varepsilon,\omega}^{j,b} = F_{\varepsilon,\omega}^{j,b,+} - F_{\varepsilon,\omega}^{j,b,-} : j \in \{\nu, -\nu - 1\}, b \in \{a, a'\}\}$  (or their  $h'$ -Fourier transform) can be interpreted as a basis of the

microlocal co-kernel of  $\mathcal{P}$  near  $a, a'$ . Following [10], we introduce Gram matrix  $\mathcal{G}^\rho$  of vectors  $\widehat{U}_1^\rho$  and  $\widehat{U}_2^\rho$  in this basis, namely  $\mathcal{G} = \begin{pmatrix} (\widehat{U}_1 | \widehat{G}_{\varepsilon, \omega}^{-\nu-1, a}) & (\widehat{U}_2 | \widehat{G}_{\varepsilon, \omega}^{-\nu-1, a}) \\ (\widehat{U}_1 | \widehat{G}_{\varepsilon, \omega}^{\nu, a'}) & (\widehat{U}_2 | \widehat{G}_{\varepsilon, \omega}^{\nu, a'}) \end{pmatrix}$ . Using symmetries we get

$$\mathcal{G} = \mathcal{G}^\rho = 2 \begin{pmatrix} 1 & e_{12}^\rho \\ -d_{21}^\rho & -1 \end{pmatrix}$$

The condition  $\det(\mathcal{G}^{(\rho)}) = 0$  means that  $U_1$  is colinear to  $U_2$ , i.e. there is a global section of  $\text{Ker}_h(\mathcal{P} - E)$ . Recall  $e_{12}^\rho = \rho(\overline{d_{21}^\rho})^{-1}$ ; for  $\rho = +1$  (electronic state) we get  $\text{Im}d_{21}^+ = 0$ , that is  $\sin\left(\frac{\tau^{(+)}(h)}{h}\right) = 0$ . We eventually obtain BS by “surgery”: namely (ignoring tunneling) we cut and paste the half-bicharacteristic  $\Lambda_E^{>,+}$  in the upper-half plane  $\xi > 0$  with its symmetric part  $\Lambda_E^{<,-}$  in  $\xi < 0$  and add together the contributions. By symmetry, the constant term  $\text{Const.}$  in  $\tau^+(h)$  drops out, while the other terms  $h\frac{\phi}{2} + h\frac{\pi}{4} - \int_{-x_0}^{x_0} \eta^\rho(y; h) dy$  add up, which yields BS for the electronic state. We argue similarly for the hole state. This eventually gives Theorem 1.

*Acknowledgements:* We thank Timur Tudorovskiy for having introduced us to the problem. This work has been partially supported by the grant PRC CNRS/RFBR 2017-2019 No.1556 “Multi-dimensional semi-classical problems of Condensed Matter Physics and Quantum Dynamics”.

- [1] Arnold V 1983 *Geometrical methods in the theory of ordinary differential equations* (Springer, Berlin)
- [2] Bardeen J, Cooper L and Schrieffer J 1959 *Phys. Rev.* **108(5)** 1175
- [3] Bensouissi A, M’hadbi N and Rouleux M 2011 *Proc. “Days of Diffraction 2011” (Saint-Petersburg)* (IEEE 101109/DD.2011.6094362) 39
- [4] Cayssol J and Montambaux G 2004 *Phys.Rev.B* **70** 224520
- [5] Chtchelkatchev N, Lesovik G and Blatter G 2000 *Phys.Rev.B* **62(5)** 3559
- [6] de Gennes P G 1966 *Superconductivity of Metals and Alloys* (Benjamin, New York)
- [7] Gérard C and Sigal I M 1992 *Comm. Math. Phys.* **145** 281
- [8] Duncan K P and Györfy B L 2002 *Annals of. Phys.* **298** 273
- [9] Helffer B and Sjöstrand J 1989 *Soc. Math. de France, Mémoire 39* **117(4)**
- [10] Ifa A, Louati H and Rouleux M 2018 *J. Math. Sci. Univ. Tokyo* **25** 1
- [11] Rouleux M 1999 *Tunneling effects for h-Pseudodifferential Operators, Feshbach resonances and the Born-Oppenheimer approximation* (Adv. Part. Diff. Eq. **16**) ed M Demuth, E Schrohe *et al.* (Wiley VCH, Berlin)
- [12] Sjöstrand J 1990 Density of states oscillations for magnetic Schrödinger operators *Proc. Diff. Eq. Math. Phys. (Univ. Alabama, Birmingham)* ed Bennowitz
- [13] Whittaker E T and Watson G N 1980 *A Course of Modern Analysis* (Cambridge Univ.Press)