Hypocoercivity of linear kinetic equations via Harris's Theorem

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Abstract

We study convergence to equilibrium of the linear relaxation Boltzmann (also known as linear BGK) and the linear Boltzmann equations either on the torus $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ or on the whole space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with a confining potential. We present explicit convergence results in total variation or weighted total variation norms (alternatively L^1 or weighted L^1 norms). The convergence rates are exponential when the equations are posed on the torus, or with a confining potential growing at least quadratically at infinity. Moreover, we give algebraic convergence rates when subquadratic potentials considered. We use a method from the theory of Markov processes known as Harris's Theorem.

Contents

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1 Introduction

The goal of this paper is to give quantitative rates of convergence to equilibrium for some linear kinetic equations, using a method based on Harris's Theorem from the theory of Markov processes [28, 35, 27] that we believe is very well adapted to hypocoercive, nonlocal equations. We consider equations of the type

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}f,$$

where $f = f(t, x, v)$, with time $t \geq 0$, space $x \in \mathbb{T}^d$ (the d-dimensional unit torus), and velocity $v \in \mathbb{R}^d$. The operator $\mathcal{L}$ acts only on the v variable, and it must typically be the generator of a stochastic semigroup for our method to work. We give explicit results for $\mathcal{L}$ equal to the linear relaxation Boltzmann operator (sometimes known as linear BGK operator), and for $\mathcal{L}$ equal to the linear Boltzmann operator (see below for a full description). We also consider the equations posed on the whole space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with a confining potential Φ:

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}f.$$

We are able to give exponential convergence results on the d-dimensional torus, or with confining potentials growing at least quadratically at ∞, always in total variation or weighted total variation norms (alternatively, L^1 or weighted L^1 norms). For subquadratic potentials we give algebraic convergence rates, again in the same kind of weighted L^1 norms. Some results were already available for these equations [11, 21, 38, 18, 29, 22]. We will give a more detailed account of them after we describe them more precisely. Previous proofs of convergence to equilibrium used strongly weighted L^2 norms (typically with a weight which is the inverse of a Gaussian), so one advantage of our method is that it directly yields convergence for a much wider range of initial conditions. The result works, in particular, for initial conditions with slow decaying tails, and for measure initial conditions with very bad local regularity. The method gives also existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases they can be nontrivial. We also note that our results for subquadratic potentials are to our knowledge new. Apart from these new results, our aim is to present a new application of a probabilistic method, using mostly PDE arguments, and which is probably useful for a wide range of models.

The study of the speed of relaxation to equilibrium for kinetic equations is a well known problem, both for linear and nonlinear models. The central obstacle is that dissipation happens only on the v variable via the effect of the operator $\mathcal{L}$, while only transport takes place in x. The transport then "mixes" the dissipation into the x variable, and one has to find a way to estimate this effect. The theory of hypocoercivity was developed in [39, 29, 30] precisely to overcome these problems for linear operators. In a landmark result, [17] proved that the full nonlinear Boltzmann equation converges to equilibrium at least at an algebraic rate. Exponential convergence results for the (linear) Fokker-Planck equation were given in [16], and a theory for a range of linear kinetic equations has been given in [18]. All of these results give convergence in exponentially weigthed L^2 norms or H^1 norms; convergence to equilibrium in weigthed L^1 norms can then be proved for several kinetic models by using the techniques in [25]. There are also works which deal directly with the L^1 theory via abstract semigroup

methods. We mention [5, 36] which give results for linear scattering equations on the torus with spatially degenerate jump rates.

Let us describe our equations more precisely. The *linear relaxation Boltzmann equation* is given by

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f, \tag{1}$$

where

$$\mathcal{L}^+ f = \left(\int f(t, x, u) \mathrm{d}u \right) \mathcal{M}(v),$$

and $\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$. The function $f = f(t, x, v)$ depends on time $t \geq 0$, space $x \in \mathbb{R}^d$, and velocity $v \in \mathbb{R}^d$, and the potential $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{C}^2$ function of x. Alternatively, we consider this equation on the torus; that is, for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, assuming periodic boundary conditions. In that case we omit Φ (which corresponds to $\Phi = 0$ in the above equation):

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f. \tag{2}$$

This simple equation is well studied in kinetic theory and can be thought of as a toy model with similar properties to either the non-linear BGK equation or linear Boltzmann equation. It is also one of the simplest examples of a hypocoercive equation. Convergence to equilibrium in H^1 for this equation has been shown in [11], at a rate faster than any function of t. It was then shown to converge exponentially fast in both H^1 and L^2 using hypocoercivity techniques in [29, 38, 18].

The linear Boltzmann equation is of a similar type:

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}), \tag{3}$$

where Φ is a $\mathcal{C}^2$ potential and $\mathcal{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$ as before, and Q is the Boltzmann operator

$$Q(f, g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \sigma) \left(f(v') g(v_*') - f(v) g(v_*) \right) \, \mathrm{d}\sigma \, \mathrm{d}v_*,$$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

and B is the *collision kernel*. We always assume that B is a hard kernel and can be written as a product

$$B(|v - v_*|, \sigma) = |v - v_*|^\gamma \, b \left(\sigma \cdot \frac{v - v_*}{|v - v_*|} \right), \tag{4}$$

for some $\gamma \geq 0$ and b integrable and uniformly positive on $[-1, 1]$; that is, there exists $C_b > 0$ such that

$$b(z) \geq C_b \qquad \text{for all } z \in [-1, 1]. \tag{5}$$

This assumption includes for example the physical hard-spheres collision kernel, for which $b \equiv 1$. The so-called *non-cutoff* kernels, for which b is not integrable, are not considered in this work.

As before, alternatively we consider the same equation posed for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, without any potential Φ:

$$\partial_t f + v \cdot \nabla_x f = Q(f, \mathcal{M}). \tag{6}$$

This equation models gas particles interacting with a background medium which is already in equilibrium. Moreover, it has been used in describing many other systems like radiative transfer, neutron transportation, cometary flow and dust particles. The spatially homogeneous case has been studied in [33, 6, 10, 32]. The kinetic equations (3) or (6) fit into the general framework in [38, 18], so convergence to equilibrium in weighted L^2 norms may be proved by using the techniques described there.

We denote by $\mathcal{P}(\Omega)$ the set of probability measures on a set $\Omega \subseteq \mathbb{R}^k$ (that is, the probability measures defined on the Borel σ-algebra of Ω). We state our main results on the torus, and then on $\mathbb{R}^d$ with a confining potential:

Theorem 1.1 (Exponential convergence results on the torus). *Suppose that $t \mapsto f_t$ is the solution to (2) or (6) with initial data $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. In the case of equation (6) we also assume (4) with $\gamma \geq 0$ and (5). Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that*

$$\|f_t - \mu\|_* \leq C e^{-\lambda t} \|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (that is, $\mu(x, v) = \mathcal{M}(v)$). The norm $\|\cdot\|_*$ is just the total variation norm $\|\cdot\|_{\mathrm{TV}}$ for equation (2),

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} |f_0 - \mu| \, \mathrm{d}x \, \mathrm{d}v \qquad \text{for equation (2)},$$

and it is a weigthed total variation norm in the case of equation (6):

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} (1 + |v|^2) |f_0 - \mu| \, \mathrm{d}x \, \mathrm{d}v \qquad \text{for equation (6)}.$$

Theorem 1.2 (Exponential convergence results with a confining potential). *Suppose that $t \mapsto f_t$ is the solution to (1) or (3) with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below, and satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d,$$

for some positive constants γ_1, γ_2, A. Define $\langle x \rangle = \sqrt{1 + |x|^2}$. In the case of equation (6) we also assume (4) with $\gamma \geq 0$, (5) and

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \leq C e^{-\lambda t} \|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{d}\mu = \mathcal{M}(v) e^{-\Phi(x)} \mathrm{d}v \mathrm{d}x.$$

The norm $\|\cdot\|_*$ is a weighted total variation norm defined by

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu| \, \mathrm{d}v \, \mathrm{d}x.$$

In all results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

We also look at Harris type theorems with weaker controls on moments to give analogues of all our theorems when the confining potential is weaker and give algebraic rates of convergence with rates depending on the assumption we make on the confining potential. Subgeometric convergence for kinetic Fokker-Planck equations with weak confinement has been shown in [19, 1, 13]. To our knowledge this is the only work showing this type of convergence in a quantitative way for the equations we present.

Theorem 1.3 (Subgeometric convergence results with weak confining potentials)**.** *Suppose that $t \mapsto f_t$ is the solution to* (1) *in the whole space with a confining potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below. Define $\langle x \rangle = \sqrt{1 + |x|^2}$. Assume that for some β in $(0,1)$ the confining potential satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{ \|f_0 - \mu\|_{\mathrm{TV}}, \ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) (1+t)^{-\beta/(1-\beta)} \right\}.$$

Similarly if $t \mapsto f_t$ is the solution to (3) *in the whole space, satisfies* (4) (5) *and*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{1+\beta} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta},$$

for some positive constants $\gamma_1, \gamma_2, \gamma_3 A, \beta$, then there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{ \|f_0 - \mu\|_{\mathrm{TV}}, \ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right) (1+t)^{-\beta} \right\}.$$

We carry out all of our proofs using variations of Harris's Theorem from probability. Harris's Theorem originated in the paper [28] where Harris gave conditions for existence and uniqueness of a steady state for Markov processes. It was then pushed forward by Meyn and Tweedie [35] to show exponential convergence to equilibrium. The last paper [27] gives an efficient way of getting quantitative rates for convergence to equilibrium once you have quantitatively verified the assumptions, we use this version of the result. Harris's Theorem says, broadly speaking, that if you have a good confining property and some uniform mixing property in the centre of the state space then you have exponentially fast convergence to equilibrium in a weighted total variation norm. We give the precise statement in the next section.

Harris's Theorem is based on the classical Doeblin's Theorem. The use of such tools to analyse PDEs was first introduced in [4, 31] to study integro-differential equations for scatterers. These results are very broad and include both models with non-equilibrium steady states and spatially degenerate jump rates. Doeblin's Theorem has also been used in [3] to show ergodicity for a kinetic equation. In all these papers the authors do not seek to find explicit rates. Harris's Theorem has already been used to show convergence to equilibrium for some kinetic equations. In [34], the authors show convergence to equilibrium for the kinetic Fokker-Planck equation with non-quantitative rates. In

[14], the authors show quantitative exponential convergence to a non-equilibrium steady state for some non-linear kinetic equations on the torus using Doeblin's Theorem.

This method is also applicable to some integro-PDEs describing several biological and physical phenomena. In [24], Doeblin's argument is used to show exponential relaxation to equilibrium for the conservative renewal equation which is a common model in population dynamics, often referred as the McKendrick-von Foerster equation. In [12], the authors show existence of a spectral gap property in the linear (no-connectivity) setting for elapsed-time structured neuron networks by using Doeblin's Theorem. Relaxation to the stationary state for the original nonlinear equation is then proved by a perturbation argument where the non-linearity is weak. Moreover, in [20] the authors consider a nonlinear model which is derived from mean-field description of an excitatory network made up of leaky integrate-and-fire neurons. In the case of weak connectivity, the authors demonstrate the uniqueness of a stationary state and its global exponential stability by using Doeblin's type of contraction argument for the linear case. Also in [2], the authors extend similar ideas to obtain quantitative estimates in total variation distance for positive semigroups, that can be non-conservative and non-homogeneous. They provide a speed of convergence for periodic semigroups and new bounds in the homogeneous setting.

Using Harris's Theorem gives an alternative and very different strategy for proving quantitative exponential decay to equilibrium. It allows us to look at hypocoercive effects on the level of stochastic processes and to look at specific trajectories which might allow one to produce quantitative theorems based on more trajectorial intuition. Another difference is that the confining behaviour is shown here by exploiting good behaviour of moments rather than a Poincaré inequality, this means looking at point wise bounds rather than integral controls on the operator. These are often equivalent for time reversible processes [1, 15] and have advantages and disadvantages. However, the condition on the moments used here might be much easier to verify in the case where the equilibrium state cannot be made explicit. This is the motivation behind [4, 14]. These works also allow us to look at a large class of initial data. We only need f_0 to be a probability measure where $\|f_0 - \mu\|$ is finite. Harris's Theorem has a restriction which is that we can only consider Markov processes. Many kinetic equations are linear Markov processes but this excludes the study of linearized non-linear equations which are not necessarily mass preserving.

The plan of the paper is as follows. We introduce Harris's Theorem in Section 2. Then we have a section for each of our equations where we prove our results.

2 Harris's Theorem

Now let us be more specific about Harris's Theorem. We give the theorems and assumption as in the setting of [27] where they make it clear how the rates depend on those in the assumptions. Markov operators can be defined by means of *transition probability functions*. We always assume that $(\Omega, \mathcal{S})$ is a measurable space. A function $S \colon \Omega \times \mathcal{S} \to \mathbb{R}$ is a transition probability function on a finite measure space if $S(x, \cdot)$ is a probability measure for every x and $x \mapsto S(x, A)$ is a measurable function for every $A \in \mathcal{S}$. We can then define $\mathcal{P}$, the associated stochastic operator on probability

measures by

$$\mathcal{P}\mu(\cdot) = \int_\Omega \mu(\mathrm{d}x)S(x,\cdot).$$

In a similar way we can define the action of S on functions (observables) by

$$(\mathcal{P}^*\psi)(x) := \int_\Omega \psi(y)S(x,\mathrm{d}y).$$

Since we are looking at a process we have Markov transition kernel S_t for each $t > 0$. We also define $\mathcal{P}_t$ from S_t as above. In our situation $\mathcal{P}_t\mu$ is the weak solution to the PDE with initial data μ. If we define $\mathcal{M}(\Omega)$ as the space of finite measures on $(\Omega, \mathcal{S})$ then we have that $\mathcal{P}_t$ is a *linear* map

$$\mathcal{P}_t : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega).$$

From the conditions on S_t we see that $\mathcal{P}_t$ will be *linear, mass preserving* and *positivity preserving*.

We can define the *forwards operator* $\mathcal{U}$, associated to S_t as the operator which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_t^*\psi\Big|_{t=0} = \mathcal{U}\psi, \tag{7}$$

for all $\psi \in C_c^\infty(\Omega)$.

We begin by looking at Doeblin's Theorem. Harris's Theorem is a natural successor to Doeblin's Theorem. Harris's and Doeblin's theorems are normally stated for a fixed time t_*. In our theorems we work to choose an appropriate t_*.

Hypothesis 1 (Doeblin's Condition). *We assume $(\mathcal{P}_t)_{t\geq 0}$ is a stochastic semigroup, coming from a Markov transition kernel, and that there exists $t_* > 0$, a probability distribution ν and $\alpha \in (0,1)$ such that for any z in the state space we have*

$$\mathcal{P}_{t_*}\delta_z \geq \alpha\nu.$$

Using this we prove

Theorem 2.1 (Doeblin's Theorem). *If we have a stochastic semigroup $(\mathcal{P}_t)_{t\geq 0}$ satisfying Doeblin's condition (Hypothesis 1) then for any two measures μ_1 and μ_2 and any integer $n \geq 0$ we have that*

$$\|\mathcal{P}_{t_*}^n\mu_1 - \mathcal{P}_{t_*}^n\mu_2\|_{\mathrm{TV}} \leq (1-\alpha)^n\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{8}$$

As a consequence, the semigroup has a unique equilibrium probability measure μ_*, and for all μ

$$\|\mathcal{P}_t(\mu - \mu_*)\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha}e^{-\lambda t}\|\mu - \mu_*\|_{\mathrm{TV}}, \qquad t \geq 0, \tag{9}$$

where

$$\lambda := \frac{\log(1-\alpha)}{t_*} > 0.$$

Proof. This proof is classical and can be found in various versions in [27] and many other places.

Firstly we show that if $\mathcal{P}_t\delta_z \geq \alpha\nu$ for every z, then we also have $\mathcal{P}_t\mu \geq \alpha\nu$ for every probability measure μ. Here since $\mathcal{P}_t$ comes from a Markov transition kernel we have

$$\mathcal{P}_t\delta_z(\cdot) = \int S_t(z', \cdot)\delta_z(\mathrm{d}z') = S_t(z, \cdot).$$

Therefore our condition says that

$$S_t(z, \cdot) \geq \alpha\nu(\cdot)$$

for every z. Therefore,

$$\mathcal{P}_t\mu(\cdot) = \int S_t(z, \cdot)\mu(\mathrm{d}z) \geq \alpha\int \nu(\cdot)\mu(\mathrm{d}z) = \alpha\nu(\cdot).$$

By the triangle inequality we have

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq \|\mathcal{P}_{t_*}\mu_1 - \alpha\nu\|_{\mathrm{TV}} + \|\mathcal{P}_{t_*}\mu_2 - \alpha\nu\|_{\mathrm{TV}}.$$

Now, since $\mathcal{P}_{t_*}\mu_1 \geq \alpha\nu$, we can write

$$\|\mathcal{P}_{t_*}\mu_1 - \alpha\nu\|_{\mathrm{TV}} = \int (\mathcal{P}_{t_*}\mu_1 - \alpha\nu) = \int \mu_1 - \alpha = 1 - \alpha,$$

due to mass conservation, and similarly for the term $\|\mathcal{P}_{t_*}\mu_2 - \alpha\nu\|_{\mathrm{TV}}$. This gives

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq 2(1 - \alpha) = (1 - \alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}$$

if μ_1, μ_2 have disjoint support. By homogeneity, this inequality is obviously also true for any nonnegative μ_1, μ_2 having disjoint support with $\int \mu_1 = \int \mu_2$. We obtain the inequality in general for any μ_1, μ_2 with the same integral by writing $\mu_1 - \mu_2 = (\mu_1 - \mu_2)_+ - (\mu_2 - \mu_1)_+$, which is a difference of nonnegative measures with the same integral. This proves

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq (1 - \alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{10}$$

We then iterate this to obtain (8). The contractivity (10) shows that the operator $\mathcal{P}_{t_*}$ has a unique fixed point, which we call μ_*. In fact, μ_* is a stationary state of the whole semigroup since for all $s \geq 0$ we have

$$\mathcal{P}_{t_*}\mathcal{P}_s\mu_* = \mathcal{P}_s\mathcal{P}_{t_*}\mu_* = \mathcal{P}_s\mu_*,$$

which shows that $\mathcal{P}_s\mu_*$ (which is again a probability measure) is also a stationary state of $\mathcal{P}_{t_*}$; due to uniqueness,

$$\mathcal{P}_s\mu_* = \mu_*.$$

Hence the only stationary state of $\mathcal{P}_t$ must be μ_*, since any stationary state of $\mathcal{P}_t$ is in particular a stationary state of $\mathcal{P}_{t_*}$.

In order to show (9), for any probability measure μ and any $t \geq 0$ we write

$$k := \left\lfloor \frac{t}{t_*} \right\rfloor,$$

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(where $\lfloor \cdot \rfloor$ denotes the integer part) so that

$$\frac{t}{t_*} - 1 < k \le \frac{t}{t_*}.$$

Then,

$$\|\mathcal{P}_t(\mu - \mu_*)\|_{\mathrm{TV}} = \|\mathcal{P}_{t-kt_*}\mathcal{P}_{kt_*}(\mu - \mu_*)\|_{\mathrm{TV}} \le \|\mathcal{P}_{kt_*}(\mu - \mu_*)\|_{\mathrm{TV}}$$

$$\le (1-\alpha)^k \|\mu - \mu_*\|_{\mathrm{TV}} \le \frac{1}{1-\alpha} \exp\left(t \log\left(\frac{1-\alpha}{t_*}\right)\right) \|\mu - \mu_*\|_{\mathrm{TV}}. \quad \square$$

Harris's Theorem extends this to the setting where we cannot prove minorisation uniformly on the whole of the state space. The idea is to use the argument given above on the centre of the state space then exploit the Lyapunov structure to show that any stochastic process will return to the centre infinitely often.

We make two assumptions on the behaviour of $\mathcal{P}_{t_*}$, for some fixed t_*:

Hypothesis 2 (Lyapunov condition). *There exists some function $V : \Omega \to [0, \infty)$ and constants $D \ge 0, \alpha \in (0,1)$ such that*

$$(\mathcal{P}_{t_*}^* V)(z) \le \alpha V(z) + D.$$

Remark. We use the name Lyapunov condition as it is the standard name used for this condition in probability literature. However, we should stress this condition is not related to the Lyapunov method for proving convergence to equilibrium. We do not prove monotonicity of a functional.

Remark. In our situation, we have an equation on the law $f(t) \equiv \mathcal{P}_t f_0$. This is equivalent to the statement

$$\int_\Omega f(t,z)V(z)\mathrm{d}z \le \alpha \int_\Omega f_0(z)V(z)\mathrm{d}z + D. \tag{11}$$

We normally verify this by showing that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(t,z)V(z)\mathrm{d}z \le -\lambda \int_\Omega f(t,z)V(z)\mathrm{d}z + K,$$

for some positive constants K and λ, which then implies (11) for $\alpha = e^{-\lambda t}$ and $D = \frac{K}{\lambda}(1 - e^{-\lambda t}) \le Kt$. Equivalently, one can show that

$$\mathcal{U}V \le -\lambda V + K,$$

where $\mathcal{U}$ is the forwards operator defined in (7).

The idea behind verifying the Lyapunov structure in our case comes from [34] where they use similar Lyapunov structures for the kinetic Fokker-Planck equation. When we work on the torus the Lyapunov structure is only needed in the v variable and the result is purely about how moments in v are affected by the collision operator.

The next assumption is a minorisation condition as in Doeblin's Theorem

Hypothesis 3. *There exists a probability measure ν and a constant $\beta \in (0,1)$ such that*

$$\inf_{z \in \mathcal{C}} \mathcal{P}_{t_*}\delta_z \ge \beta\nu,$$

where

$$\mathcal{C} = \{z : V(z) \le R\}$$

for some $R > 2D/(1-\alpha)$.

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Remark. Production of quantitative lower bounds as a way to quantify the *positivity* of a solution has been proved and used in kinetic theory before. For example it is an assumption required for the works of Desvillettes and Villani [16, 17]. Such lower bounds have been proved for the non-linear Boltzmann equation in [37, 7, 8].

This second assumption is more challenging to verify in our situations. Here we use a strategy based on our observation about how noise is transferred from the v to the x variable as described earlier. The actual calculations are based on the PDE governing the evolution and iteratively using Duhamel's formula.

We define a weighted total variation norm on measures for each a by:

$$\|\mu_1 - \mu_2\|_{V,a} = \int (1 + aV(z))|\mu_1 - \mu_2|(\mathrm{d}z).$$

Theorem 2.2 (Harris's Theorem as in [27]). *If Hypotheses 2 and 3 hold then there exist $\bar{\alpha} \in (0,1)$ and $a > 0$ such that*

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{V,a} \leq \bar{\alpha}\|\mu_1 - \mu_2\|_{V,a}. \tag{12}$$

Explicitly if we choose $\beta_0 \in (0, \beta)$ and $\alpha_0 \in (\alpha + 2D/R, 1)$ then we can set $\gamma = \beta_0/K$ and $\bar{\alpha} = \max\{1 - (\beta - \beta_0), (2 + R\gamma\alpha_0)/(2 + R\gamma)\}$.

Remark. We have that

$$\min\{1, a\}\|\mu_1 - \mu_2\|_{V,1} \leq \|\mu_1 - \mu_2\|_{V,a} \leq \max\{1, a\}\|\mu_1 - \mu_2\|_{V,1}.$$

We can also iterate Theorem 2.2 to get

$$\|\mathcal{P}_{nt_*}\mu_1 - \mathcal{P}_{nt_*}\mu_2\|_{V,a} \leq \bar{\alpha}^n\|\mu_1 - \mu_2\|_{V,a}.$$

Therefore we have that

$$\|\mathcal{P}_{nt_*}\mu_1 - \mathcal{P}_{nt_*}\mu_2\|_{V,1} \leq \bar{\alpha}^n \frac{\max\{1, a\}}{\min\{1, a\}}\|\mu_1 - \mu_2\|_{V,1}.$$

Remark. In this paper we always consider functions V where $V(z) \to \infty$ as $|z| \to \infty$. In this case, we can replace $\mathcal{C}$ in Hypothesis 3 with some ball of radius R' which will contain $\mathcal{C}$.

Doeblin's Theorem corresponds to the irreducibility property for Markov processes in the bounded state space. But when the state space is unbounded it is expected that the process may drift arbitrarily far away and we cannot prove a uniform minorisation condition on the whole state space. Harris's Theorem is one way to extend the ideas of Doeblin to the unbounded state space by finding a Lyapunov functional with small level sets to show that transition probabilities of the process converge towards a unique invariant measure. Therefore Harris's Theorem is based on providing a combination of a minorisation and a geometric drift conditions. The minorisation condition can be thought as finding a bound on the probability of transitioning in one step from any initial state to some specified region which corresponds to the small level sets of the Lyapunov functional.

Another way to prove convergence towards a unique invariant measure for a Markov process is to show the Markov semigroup has the *strong Feller property*, meaning that

the semigroup maps bounded measurable functions to bounded continuous functions. We refer the reader to [26] for further comments on this.

There are versions of Harris's Theorem adapted to weaker Lyapunov conditions which give subgeometric convergence [19]. We use the following theorem which can be found in Section 4 of [26].

Theorem 2.3 (Subgeometric Harris's Theorem)**.** *Given the forwards operator, $\mathcal{U}$, of our Markov semigroup $\mathcal{P}$, suppose that there exists a continuous function V valued in $[1, \infty)$ with pre compact level sets such that*

$$\mathcal{U}V \leq K - \phi(V),$$

for some constant K and some strictly concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$ and increasing to infinity. Assume that for every $C > 0$ we have the minorisation condition like Hypothesis 3. i.e. for some t_* a time and ν a probability distribution and $\alpha \in (0, 1)$, then for all z with $V(z) \leq C$:

$$\mathcal{P}_{t_*}\delta_z \geq \alpha\nu.$$

With these conditions we have that

- *There exists a unique invariant measure μ_* for the Markov process and it satisfies*

$$\int \phi(V(z))\mathrm{d}\mu_* \leq K.$$

- *Let H_ϕ be the function defined by*

$$H_\phi = \int_1^u \frac{\mathrm{d}s}{\phi(s)}.$$

Then there exists a constant C such that

$$\|\mathcal{P}_t\mu - \mu_*\|_{\mathrm{TV}} \leq \frac{C\mu(V)}{H_\phi^{-1}(t)} + \frac{C}{(\phi \circ H_\phi^{-1})(t)}$$

holds for every μ where $\mu(V) = \int V(z)\mu(\mathrm{d}z)$.

We will apply the subgeometric Harris's Theorem to the PDEs we study to show convergence when only a weaker confinement condition is available.

3 The linear relaxation Boltzmann equation

3.1 On the torus

This is the simplest operator on the torus, so we do not in fact need to use Harris's Theorem. We can instead use Doeblin's Theorem where we have a uniform minorisation condition.

We consider

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}f, \tag{13}$$

posed for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T}^d$ is the d-dimensional torus of side 1 and

$$\mathcal{L}f(x, v) := \mathcal{L}^+ f(x, v) - f(x, v) := \left(\int_{\mathbb{R}^d} f(x, u) \, \mathrm{d}u \right) \mathcal{M}(v) - f(x, v), \qquad (14)$$

which is a well defined operator from $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ to $L^1(\mathbb{T}^d \times \mathbb{R}^d)$, and can also be defined as an operator from $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ to $\mathcal{M}(\mathbb{T}^d \times \mathbb{R}^d)$ with the same expression (where $\int_{\mathbb{R}^d} f(x, u) \, \mathrm{d}u$ now denotes the marginal of the measure f with respect to u). We define $(T_t)_{t \geq 0}$ as the transport semigroup associated to the operator $-v \cdot \nabla_x f$ in the space of measures with the bounded Lipschitz topology (see for example [9]); that is, $t \mapsto T_t f_0$ solves the equation $\partial_t f + v \cdot \nabla_x f = 0$ with initial condition f_0. In this case one can write T_t explicitly as

$$T_t f_0(x, v) = f_0(x - tv, v). \qquad (15)$$

Using Duhamel's formula repeatedly one can obtain that, if f is a solution of (13) with initial data f_0, then

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s. \qquad (16)$$

We will now check two properties, which we list as lemmas. The first one says that the operator $\mathcal{L}$ always allows jumps to any small velocity. We always use the notation $\mathbb{1}_A$ to denote the characteristic function of a set A (if A is a set), or the function which is 1 where the condition A is met, and 0 otherwise (if A is a condition).

Lemma 3.1. *For all $\delta_L > 0$ there exists $\alpha_L > 0$ such that for all $g \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ we have*

$$\mathcal{L}^+ g(x, v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x, u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq \delta_L\}} \qquad (17)$$

for almost all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Proof. Given any δ_L it is enough to choose $\alpha_L := \mathcal{M}(v)$ for any v with $|v| = \delta_L$. $\qquad \square$

The second one is regarding to the behaviour of the transport part alone. It says that if we start at any point inside a ball of radius R, and we are allowed to start with any small velocity, then we can reach any point in the ball of radius R with a predetermined bound on the final velocity. We use $B(\delta)$ to denote the open ball $\{x \in \mathbb{R}^d \mid |x| < \delta\}$, and in general we will use the notation $B(z, \delta)$ to denote the open ball of radius δ centered at $z \in \mathbb{R}^d$. The lemma is:

Lemma 3.2. *Given any time $t_0 > 0$ and radius $R > 0$ there exist $\delta_L, R' > 0$ such that for all $t \geq t_0$ it holds that*

$$\int_{B(R')} T_t \left(\delta_{x_0}(x) \mathbb{1}_{\{|v| \leq \delta_L\}} \right) \mathrm{d}v \geq \frac{1}{t^d} \mathbb{1}_{\{|x| \leq R\}} \qquad \text{for all } x_0 \text{ with } |x_0| < R. \qquad (18)$$

In particular, if we take $R > \sqrt{d}$, there exist $\delta_L, R' > 0$ such that

$$\int_{B(R')} T_t \left(\delta_{x_0}(x) \mathbb{1}_{\{|v| \leq \delta_L\}} \right) \mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d. \qquad (19)$$

Proof. Take $t, R > 0$. We have

$$T_t \left(\delta_{x_0}(x) \mathbb{1}_{B(\delta_L)}(v) \right) = \delta_{x_0}(x - vt) \mathbb{1}_{B(\delta_L)}(v).$$

Integrating this and changing variables gives that

$$\int_{B(R')} T_t \left(\delta_{x_0}(x) \mathbb{1}_{B(\delta_L)}(v) \right) \, \mathrm{d}v = \frac{1}{t^d} \int_{B(x,tR')} \delta_{x_0}(y) \mathbb{1}_{B(\delta_L)} \left(\frac{x-y}{t} \right) \mathrm{d}y.$$

Since $|x - y| \le |x| + |y|$ we have that

$$\mathbb{1}_{B(\delta_L)} \left(\frac{x-y}{t} \right) \ge \mathbb{1}_{B(\delta_L/2)} \left(\frac{x}{t} \right) \, \mathbb{1}_{B(\delta_L/2)} \left(\frac{y}{t} \right).$$

Therefore if we take $\delta_L > 2R/t$ we have

$$\mathbb{1}_{B(\delta_L)} \left(\frac{x-y}{t} \right) \ge \mathbb{1}_{B(R)}(x) \mathbb{1}_{B(R)}(y).$$

On the other hand, if we take $|x| < R$ and $R' > 2R/t$ then

$$B(x, tR') \supseteq B(x, 2R) \supseteq B(R).$$

Hence if $\delta_L > 2R/t$ and $R' > 2R/t$,

$$\int_{B(R')} T_t \left(\delta_{x_0}(x) \mathbb{1}_{B(\delta_L)}(v) \right) \, \mathrm{d}v \ge \frac{1}{t^d} \mathbb{1}_{B(R)}(x),$$

which proves the result. $\qquad\square$

Lemma 3.3 (Doeblin condition for the linear relaxation Botzmann equation on the torus). *For any $t_* > 0$ there exist constants $\alpha, \delta_L > 0$ (depending on t_*) such that any solution f to equation (13) with initial condition $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfies*

$$f(t_*, x, v) \ge \alpha \mathbb{1}_{\{|v| \le \delta_L\}}, \tag{20}$$

where the inequality is understood in the sense of measures.

Proof. It is enough to prove it for $f_0 := \delta_{(x_0, v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point. From Lemma 3.2 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exists $\delta_L > 0$ such that

$$\int_{\mathbb{R}^d} T_t \left(\delta_{x_0}(x) \mathbb{1}_{\{|v| \le \delta_L\}} \right) \mathrm{d}v \ge \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d, \, t > t_0.$$

Also, Lemma 3.1 gives an $\alpha_L > 0$ such that

$$\mathcal{L}^+ g \ge \alpha_L \left(\int_{\mathbb{R}^d} g(x, u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \le \delta_L\}}.$$

Take any $r > 0$. Since $T_r f_0 = \delta_{(x_0 - v_0 r, v_0)}$, using this shows

$$\mathcal{L}^+ T_r f_0 \ge \alpha_L \, \delta_{x_0 - v_0 r}(x) \, \mathbb{1}_{\{|v| \le \delta_L\}}.$$

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Hence, whenever $s - r > t_0$ we have

$$\mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \geq \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r} \mathcal{L}^+ T_r f_0 \, du \right) \mathbb{1}_{\{|v| \leq \delta_L\}}$$

$$\geq \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r} \left(\delta_{x_0 - v_0 r}(x) \, \mathbb{1}_{\{|u| \leq \delta_L\}} \right) du \right) \mathbb{1}_{\{|v| \leq \delta_L\}}$$

$$\geq \frac{1}{(s-r)^d} \alpha_L^2 \, \mathbb{1}_{\{|v| \leq \delta_L\}}.$$

Finally, for the movement along the flow T_{t-s}, notice that

$$T_t \left(\mathbb{1}_{\mathbb{T}^d}(x) \mathbb{1}_{\{|v| < \delta_L\}}(v) \right) = \mathbb{1}_{\mathbb{T}^d}(x) \mathbb{1}_{\{|v| < \delta_L\}}(v) \qquad \text{for all } t \geq 0.$$

This means that for all $t > s > r > 0$ such that $s - r > t_0$ we have

$$T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \geq \frac{1}{(s-r)^d} \alpha_L^2 \, \mathbb{1}_{\{|v| \leq \delta_L\}}.$$

For any t_* we have then, recalling that $t_0 = t_*/3$,

$$\int_0^{t_*} \int_0^s T_{t_* - s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, dr \, ds \geq \alpha_L^2 \mathbb{1}_{\{|v| \leq \delta_L\}} \int_{2t_0}^{t_*} \int_0^{t_0} \frac{1}{(s-r)^d} \, dr \, ds$$

$$\geq \frac{t_0^2}{t_*^d} \alpha_L^2 \mathbb{1}_{\{|v| \leq \delta_L\}} = \frac{1}{9} t_*^{2-d} \alpha_L^2 \mathbb{1}_{\{|v| \leq \delta_L\}}.$$

Finally, from Duhamel's formula (16) we obtain

$$f(t_*, x, v) \geq \frac{1}{9} e^{-t_*} t_*^{2-d} \alpha_L^2 \mathbb{1}_{\{|v| \leq \delta_L\}},$$

which gives the result. □

Proof of Theorem 1.1 in the case of the linear relaxation Boltzmann equation. Lemma 3.3 allows us to apply directly Doeblin's Theorem 2.1 to obtain fast exponential convergence to equilibrium in the total variation distance. This rate is also explicitly calculable. Therefore, the proof follows. □

3.2 On the whole space with a confining potential

Now we consider the equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f = \mathcal{L}f, \tag{21}$$

where $\mathcal{L}$ is defined as in the previous section and $x, v \in \mathbb{R}^d$. We want to use a slightly different strategy to show the minorisation condition based on the fact that we instantaneously produce large velocities. We first need a result on the trajectories of particles under the action of the potential Φ. Always assuming that Φ is a $\mathcal{C}^2$ function, we consider the characteristic ordinary differential equations associated to the transport part of (21):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\nabla\Phi(x), \end{aligned} \tag{22}$$

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and we denote by $(X_t(x_0, v_0), V_t(x_0, v_0))$ the solution at time t to (22) with initial data $x(0) = x_0$, $v(0) = v_0$. Performing time integration twice, it clearly satisfies

$$X_t(x_0, v_0) = x_0 + v_0 t + \int_0^t \int_0^s \nabla\Phi(X_u(x_0, v_0)) \, \mathrm{d}u \, \mathrm{d}s \tag{23}$$

for any $x_0, v_0 \in \mathbb{R}^d$ and any t for which it is defined. Intuitively the idea is that for small times we can approximate (X_t, V_t) by $(X_t^{(0)}, V_t^{(0)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 0, \end{aligned} \tag{24}$$

whose explicit solution is $(X_t^{(0)}, V_t^{(0)}) = (x_0 + v_0 t, v_0)$. If we want to hit a point x_1 in time t then if we travel with the trajectory $X^{(0)}$ we just need to choose $v_0 = (x_1 - x_0)/t$. Now we choose an interpolation between $(X^{(0)}, V^{(0)})$ and (X, V). We denote it by $(X^{(\epsilon)}, V^{(\epsilon)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\epsilon^2 \nabla\Phi(x), \end{aligned} \tag{25}$$

still with initial data (x_0, v_0). We calculate that

$$X_t^{(\epsilon)}(x_0, v_0) = X_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right), \qquad V_t^{(\epsilon)}(x_0, v_0) = \epsilon V_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right).$$

Now we can see from the ODE representation (and we will make this more precise later) that (X, V) is a C^1 map of (t, ϵ, x, v). Therefore if we fix t and x_0 we can define a C^1 map

$$F \colon [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d,$$

by

$$F(\epsilon, v) = X_t^{(\epsilon)}(x_0, v).$$

Then for $\epsilon = 0$ we can find v_* such that $F(0, v_*) = x_1$ as given above. Furthermore $\nabla F(0, v_*) \neq 0$ so by the implicit function theorem for all ϵ less than some ϵ_* we have a C^1 function $v(\epsilon)$ such that $F(\epsilon, v(\epsilon)) = x_1$. This means that

$$X_{\epsilon t}\left(x_0, \frac{v(\epsilon)}{\epsilon}\right) = x_1.$$

So if we take $s < \epsilon_* t$ then we can choose v such that $X_s(x_0, v) = x_1$. We now need to get quantitative estimates on ϵ_*, and we do this by tracking the constants in the proof of the contraction mapping theorem.

In order to make these ideas quantitative and to check that the solution is in fact C^1 we need to get bounds on (X_t, V_t) and $\nabla\Phi(X_t)$ for t is some fixed intervals. For the potentials of interest we will have that the solutions to these ODEs will exist for infinite time. We prove bounds on the solutions and $\nabla\Phi(X_t)$ for any potential:

Lemma 3.4. *Assume that the potential Φ is C^2 in $\mathbb{R}^d$. Take $\lambda > 1$, $R > 0$ and $x_0, v_0 \in \mathbb{R}^d$ with $|x_0| \leq R$. The solution $t \mapsto X_t(x_0, v_0)$ to (22) is defined (at least) for $|t| \leq T$, with*

$$T := \min\left\{\frac{(\lambda - 1)R}{2|v_0|}, \frac{\sqrt{(\lambda - 1)R}}{\sqrt{2C_{\lambda R}}}\right\}, \qquad C_{\lambda R} := \max_{|x| \leq \lambda R} |\nabla\Phi(x)|.$$

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(It is understood that any term in the above minimum is $+\infty$ if the denominator is 0.)
Also, it holds that
$$|X_t(x_0, v_0)| \le \lambda R \qquad \text{for } |t| \le T.$$

from this we can deduce

$$|V_t(x_0, v_0)| \le |v_0| + C_{\lambda R} t \qquad \text{for } |t| \le T.$$

Proof. By standard ODE theory, the solution is defined in some maximal (open) time interval I containing 0; if this maximal interval has any finite endpoint t_*, then $X_t(x_0, v_0)$ has to blow up as t approaches t_*. Hence if the statement is not satisfied, there must exist $t \in I$ with $|t| \le T$ such that $|X_t(x_0, v_0)| \ge \lambda R$. By continuity, one may take $t_0 \in I$ to be the "smallest" time when this happens: that is, $|t_0| \le T$ and

$$X_{t_0}(x_0, v_0) = \lambda R,$$
$$|X_{t_0}(x_0, v_0)| \le \lambda R \qquad \text{for } |t| \le |t_0|.$$

By (23) and using that $|t_0| \le T$ we have

$$\lambda R = |X_{t_0}(x_0, v_0)| \le |x_0| + |v_0 t_0| + \frac{t_0^2}{2} \max\{|\nabla\Phi(X_t(x_0, v_0))| : t \le t_0\}$$
$$\le R + \frac{(\lambda - 1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2 = \frac{(\lambda + 1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2,$$

which implies that

$$(\lambda - 1)R \le C_{\lambda R} t_0^2.$$

If $C_{\lambda R} = 0$ this is false; if $C_{\lambda R} > 0$, then this contradicts with that $|t_0| \le T$. $\qquad\square$

We now follow the intuition given at the beginning of this section. However we collapse the variables ϵ and t together and consequently look at $X_t\left(x, \frac{v}{t}\right)$ which is intuitively less clear but algebraically simpler.

Lemma 3.5. *Assume that $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$, and take $x_0, x_1 \in \mathbb{R}^d$. Let $R := \max\{|x_0|, |x_1|\}$. There exists $0 < t_1 = t_1(R)$ such that for any $t \le t_1$ we can find a $|v_0| \le 4R$ such that*

$$X_t\left(x_0, \frac{v_0}{t}\right) = x_1.$$

In fact, it is enough to take $t_1 > 0$ such that

$$C t_1^2 e^{C t_1^2} \le \frac{1}{4}, \qquad t_1 \le \frac{\sqrt{R}}{\sqrt{2 C_{2R}}}, \qquad t_1 \le \frac{2\sqrt{R}}{\sqrt{C_{9R}}},$$

where

$$C := \sup_{|x| \le 9R} |D^2\Phi(x)|$$

, $C_{\lambda R}$ is defined in Lemma 3.4 and $D^2\Phi$ denotes the Hessian matrix of Φ.

Proof. We define

$$f(t, v) = X_t\left(x_0, \frac{v}{t}\right) - x_1, \qquad t \ne 0, \ v \in \mathbb{R}^d,$$
$$f(0, v) := x_0 + v - x_1, \qquad v \in \mathbb{R}^d.$$

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Notice that due to Lemma 3.4 with $\lambda = 9$, this is well-defined whenever

$$|t| \leq \frac{2\sqrt{R}}{\sqrt{C_{9R}}} =: t_2, \quad |v| \leq 4R.$$

Our goal is to find a neighbourhood of $t = 0$ on which there exists $v = v(t)$ with $f(t, v(t)) = 0$, for which we will use the implicit function theorem.

Now, notice that we have

$$f(0, x_1 - x_0) = 0$$

and

$$\frac{\partial f}{\partial v_i}(0, x_1 - x_0) = 1, \quad i = 1, \ldots, d.$$

We can apply the implicit function theorem to find a neighbourhood I of $t = 0$ and a function $v = v(t)$ such that $f(t, v(t)) = 0$ for $t \in I$. However, since we need to estimate the size of I and of $v(t)$, we carry out a constructive proof.

Take $v_0, v_1 \in \mathbb{R}^d$ with $|v_0|, |v_1| \leq 4R$, and denote $\tilde{v}_0 := v_0/t$, $\tilde{v}_1 := v_1/t$. By (23), for all $0 < t \leq t_2$ we have

$$X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) = (\tilde{v}_1 - \tilde{v}_0)t + \int_0^t \int_0^s \nabla\Phi(X_u(x_0, \tilde{v}_1)) - \nabla\Phi(X_u(x_0, \tilde{v}_0)) \, du \, ds. \tag{26}$$

Take any $t_1 \leq t_2$, to be fixed later. Then Lemma 3.4 implies, for all $0 \leq t \leq t_1$,

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0)| \leq |\tilde{v}_1 - \tilde{v}_0|t + Ct_1 \int_0^t |X_u(x_0, \tilde{v}_1) - X_u(x_0, \tilde{v}_0)| \, du.$$

by Gronwall's Lemma we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0)| \leq |\tilde{v}_1 - \tilde{v}_0|te^{Ct_1 t} \qquad \text{for } 0 < t \leq t_1.$$

Using this again in (26) we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| \leq |\tilde{v}_1 - \tilde{v}_0|Ct_1 \int_0^t ue^{Ct_1 u} \, du$$
$$\leq |\tilde{v}_1 - \tilde{v}_0|t\, Ct_1^2 e^{Ct_1^2}.$$

Taking t_1 such that

$$Ct_1^2 e^{Ct_1^2} \leq \frac{1}{4} \tag{27}$$

we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| \leq \frac{1}{4}|\tilde{v}_1 - \tilde{v}_0|t$$

which is the same as

$$\left| X_t\left(x_0, \frac{v_1}{t}\right) - X_t\left(x_0, \frac{v_0}{t}\right) - (v_1 - v_0) \right| \leq \frac{1}{4}|v_1 - v_0|, \tag{28}$$

for any $0 < t \leq t_1$ and any v_0, v_1 with $|v_0|, |v_1| \leq 4R$. Now, for any $0 \leq t \leq t_1$ and $|v| \leq 4R$ we define

$$A_t(v) = v - f(t, v).$$

A fixed point of $A_t(v)$ satisfies $f(t, v) = 0$, and by (28) $A_t(v)$ is contractive:

$$|A_t(v_1) - A_t(v_0)| \leq \frac{1}{4}|v_1 - v_0| \qquad \text{for } 0 \leq t \leq t_1, \ |v| \leq 4R.$$

(Equation (28) proves this for $0 < t \leq t_1$, and for $t = 0$ it is obvious.) In order to use the Banach fixed-point theorem we still need to show that the image of A_t is inside the set with $|v| \leq 4R$. Using (28) for $v_1 = 0$, $v_0 = v$ we also see that

$$\left| X_t(x_0, 0) - X_t\left(x_0, \frac{v}{t}\right) + v \right| \leq \frac{1}{4}|v|,$$

which gives

$$|A_t(v) + x_1 - X_t(x_0, 0)| \leq \frac{1}{4}|v|,$$

so

$$|A_t(v)| \leq \frac{1}{4}|v| + |x_1| + |X_t(x_0, 0)| \leq 2R + |X_t(x_0, 0)|. \tag{29}$$

If we take

$$t_1 \leq \frac{\sqrt{R}}{\sqrt{2C_{2R}}} \tag{30}$$

then Lemma 3.4 (used for $\lambda = 2$) shows that

$$|X_t(x_0, 0)| \leq 2R \qquad \text{for } 0 \leq t \leq t_1,$$

and from (29) we have

$$|A_t(v)| \leq 4R \quad \text{for} \quad 0 < t \leq t_1.$$

Hence, as long as t_1 satisfies (27) and (30), A_t has a fixed point $|v|$ for any $0 < t \leq t_1$, and this fixed point satisfies $|v| \leq 4R$. $\qquad\square$

Lemma 3.6. *Assume the potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ is bounded below, and let T_s denote the transport semigroup associated to the operator $f \mapsto -v \cdot \nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$. Given any $R > 0$ there exists a time $t_1 > 0$ such that for any $0 < s < t_1$ one can find constants $\alpha, R', R_2 > 0$ (depending on s and R) such that*

$$\int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \leq R_2\}}) \, dv \geq \alpha \mathbb{1}_{\{|x| \leq R\}}, \tag{31}$$

for any x_0 with $|x_0| \leq R$. The constants α, R', R_2 are uniformly bounded in bounded intervals of time; that is, for any closed interval $J \subseteq (0, t_1)$ one can find α, R', R_2 for which the inequality holds for all $s \in J$.

Proof. Since the statement is invariant if Φ changes by an additive constant, we may assume that $\Phi \geq 0$ for simplicity. Using Lemma 3.5 we find t_1 such that for any $s < t_1$ and every $x_1 \in B(R)$ there exists $v \in B(4R)$ (depending on x_0, x_1 and s) such that

$$X_s\left(x_0, \frac{v}{s}\right) = x_1.$$

Since $v/s \in B(4R/s)$, call $R_2 := 4R/s$. We see that for every $x_1 \in B(0, R)$ there is at least one $u \in \mathbb{R}^d$ such that

$$(x_1, u) \in T_s\left(\{x_0\} \times \{|v| \leq R_2\}\right).$$

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In other words,
$$X_s(x_0, \{|v| \le R_2\}) \supseteq B(0, R). \tag{32}$$

This essentially contains our result, and we just need to carry out a technical argument to complete it and estimate the constants α and R'. For any compactly supported, continuous and positive $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{|V_s(x,v)| < R'\}} \, \varphi(X_s(x, v)) \delta_{x_0}(x) \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x$$

$$= \int_{|v| \le R_2} \mathbb{1}_{\{|V_s(x_0,v)| < R'\}} \, \varphi(X_s(x_0, v)) \, \mathrm{d}v, \quad (33)$$

since the characteristics map $(x, v) \mapsto (X_s(x, v), V_s(x, v))$ is measure-preserving. If we write the energy as $H(x, v) = |v|^2/2 + \Phi(x)$ and call

$$E_0 := \sup \{ H(x, v) \ : \ |x| < R, |v| < R_2 \}.$$

Then for all $s \ge 0$

$$H(X_s(x_0, v), V_s(x_0, v)) \le E_0,$$

and in particular

$$|V_s(x_0, v)| \le \sqrt{2E_0}.$$

If we take $R' > \sqrt{2E_0}$ then the term $\mathbb{1}_{\{|V_s(x_0,v)| < R'\}}$ is always 1 in (33) and we get

$$\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x = \int_{|v| \le R_2} \varphi(X_s(x_0, v)) \, \mathrm{d}v.$$

Now, take an $M > 0$ such that $|\operatorname{Jac}_v X_s(x, v)| \le M$ for all (x, v) with $|x| \le R$ and $|v| \le R_2$. (Notice this M depends only on Φ, R and R_2.) Then

$$\int_{|v| \le R_2} \varphi(X_s(x_0, v)) \, \mathrm{d}v \ge \frac{1}{M} \int_{|v| \le R_2} \varphi(X_s(x_0, v)) |\operatorname{Jac}_v X_s(x_0, v)| \, \mathrm{d}v$$

$$= \frac{1}{M} \int_{X_s(x_0, \{|v| \le R_2\})} \varphi(x) \, \mathrm{d}x \ge \frac{1}{M} \int_{B(0,4R)} \varphi(x) \, \mathrm{d}x,$$

where we have used (32) in the last step. In sum we find that

$$\int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x \ge \frac{1}{M} \int_{B(0,R)} \varphi(x) \, \mathrm{d}x$$

for all compactly supported, continuous and positive functions φ. This directly implies the result. $\qquad \square$

Lemma 3.7 (Doeblin condition for linear relaxation Boltzmann equation with a confining potential). *Let the potential $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with compact level sets. Given $t > 0$ and $K > 0$ there exist constants $\alpha, \delta_X, \delta_V > 0$ such that any solution f to equation (21) with initial condition $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ supported on $B(0, K) \times B(0, K)$ satisfies*

$$f(t, x, v) \ge \alpha \mathbb{1}_{\{|x| < \delta_X\}} \, \mathbb{1}_{\{|v| < \delta_V\}}$$

in the sense of measures.

Proof. Fix any $t, K > 0$. Set

$$H_{\max}(K) = \max \left\{ H(x,v) = |v|^2/2 + \Phi(x) \, : \, x \in B(0,K), v \in B(0,K) \right\}$$

and then define

$$R := \max \left\{ |x| \, : \, \Phi(x) \leq H_{\max}(K) \right\}.$$

Since our conditions on Φ imply that its level sets are compact we know that R is finite. We use Lemma 3.6 to find constants $\alpha, R_2 > 0$ and an interval $[a,b] \subseteq (0,t)$ such that

$$\int_{\mathbb{R}^d} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \leq R_2\}}) \, \mathrm{d}v \geq \alpha \mathbb{1}_{\{|x| \leq R\}},$$

for any x_0 with $|x_0| \leq R$ and any $s \in [a,b]$. From Lemma 3.1 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x,v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x,u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq R_2\}} \tag{34}$$

for all nonnegative measures g. We first notice that we can do the same estimate as in formula (16), where now $(T_t)_{t \geq 0}$ represents the semigroup generated by the operator $-v \cdot \nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$:

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s. \tag{35}$$

Take $x_0, v_0 \in B(0,K)$, and call $f_0 := \delta_{(x_0,v_0)}$. For all r we have by the definition of R that

$$|X_r(x_0, v_0)| \leq R \qquad \text{for all } 0 \leq r. \tag{36}$$

For any $r > 0$, since $T_r f_0 = \delta_{(X_r(x_0,v_0), V_r(x_0,v_0))}$, using (17) gives

$$\mathcal{L}^+ T_r f_0 \geq \alpha_L \delta_{X_r(x_0,v_0)}(x) \mathbb{1}_{\{|v| \leq R_2\}}.$$

Then, using (36) and our two lemmas, whenever $s - r \in [a,b]$ we have

$$\mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \geq \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq R_2\}}$$

$$\geq \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r} \left(\delta_{X_r(x_0,v_0)}(x) \mathbb{1}_{\{|u| \leq R_2\}} \right) \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq R_2\}}$$

$$\geq \alpha_L^2 \alpha \, \mathbb{1}_{\{|x| \leq R\}} \mathbb{1}_{\{|v| \leq R_2\}}.$$

We now need to allow for a final bit of movement along the flow T_{t-s}. The time gradient of the flow is bounded by $|\nabla_x(\Phi(X_t(x,v)))| + |V_t(x,v)|$ and this quantity is bounded on sublevel sets of the Hamiltonian which are preserved by the flow so there exists a sufficiently small, quantifiable $\epsilon > 0$ so that for all $0 \leq \tau \leq \epsilon$ we have

$$T_\tau \left(\mathbb{1}_{B(R)}(x) \mathbb{1}_{B(R_2)}(v) \right) \geq \mathbb{1}_{B(R/2)}(x) \mathbb{1}_{B(R_2/2)}(v). \tag{37}$$

(We point out a way to quantify ϵ: since T_τ is measure-preserving, we have $T_\tau(h)(x,v) = h(X_\tau(x,v), V_t(x,v))$ for any function $h = h(x,v)$. Define the inverse flow of T_τ by G_τ, if we denote $G_\tau(x_0, v_0) = (Y_\tau(x_0, v_0), Z_\tau(x_0, v_0))$, then Y_τ, Z_τ satisfies

$$\dot{y} = -z$$
$$\dot{z} = W(y),$$

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with initial condition $\{x_0, y_0\}$. Hence (37) holds if $|Y_\tau(x, v)| \leq R$ and $|Z_\tau(x, v)| \leq R_2$ for all $|x| \leq R/2$, $|v| \leq R_2/2$. It's easily seen that the result of Lemma 3.4 will still hold for G_τ, so we can take

$$\epsilon = \min\left\{\frac{R}{2R_2}, \frac{\sqrt{R}}{2\sqrt{C_R}}, \frac{R_2}{2R}\right\}, \qquad C_R := \max_{|x| \leq R} |\nabla\Phi(x)|.$$

by Lemma 3.4.)

From (37), for all t, s, r such that $t - s \leq \epsilon$ and $s - r \in (a, b)$ we have

$$T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \geq \alpha_L^2 \alpha \, \mathbb{1}_{\{|x| \leq R/2\}} \, \mathbb{1}_{\{|v| \leq R_2/2\}}.$$

We have then

$$\int_0^t \int_0^s T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s \geq \alpha_L^2 \alpha \int_{t-\epsilon}^t \int_{s-b}^{s-a} \mathbb{1}_{\{|x| \leq R/2\}} \mathbb{1}_{\{|v| \leq R_2/2\}} \, \mathrm{d}r \, \mathrm{d}s$$
$$= \alpha_L^2 \alpha \epsilon (b - a) \mathbb{1}_{\{|x| \leq R/2\}} \mathbb{1}_{\{|v| \leq R_2/2\}}.$$

Finally, from Duhamel's formula (35) we obtain

$$f(t, x, v) \geq e^{-t} \alpha_L^2 \alpha \epsilon (b - a) \, \mathbb{1}_{\{|x| \leq R/2\}} \mathbb{1}_{\{|v| \leq R_2/2\}},$$

which gives the result. $\qquad\square$

Lemma 3.8 (Lyapunov condition). *Suppose that $\Phi(x)$ is a $\mathcal{C}^2$ function satisfying*

$$x \cdot \nabla\Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A$$

for positive constants A, γ_1 γ_2. Then we have that

$$V(x, v) = 1 + \Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2$$

is a function for which the semigroup satisfies Hypothesis 2.

Remark. If Φ is superquadratic at infinity (which is implied by earlier assumptions) then V is equivalent to $1 + H(x, v)$ where the energy is defined as $H(x, v) = |v|^2/2 + \Phi(x)$. So the total variation distance weighted by V is equivalent to the total variation distance weighted by $1 + H(x, v)$.

Proof. We look at the forwards operator acting on an observable ϕ,

$$\mathcal{U}\phi = v \cdot \nabla_x \phi - \nabla_x \Phi(x) \cdot \nabla_v \phi + \mathcal{L}^* \phi =: \mathcal{T}^* \phi + \mathcal{L}^* \phi,$$

where $\mathcal{L}^*$ is the adjoint of the linear relaxation Boltzmann operator $\mathcal{L}$, given by

$$\mathcal{L}^* \phi(x, v) = \int \phi(x, u)\mathcal{M}(u) \, \mathrm{d}u - \phi(x, v).$$

We want a function $V(x, v)$ such that

$$\mathcal{U}V \leq -\lambda V + K$$

for some constants $\lambda > 0, K \geq 0$. We need to make the assumption that

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A. \qquad (38)$$

for some positive constant A, γ_1, γ_2. We then try the function

$$V(x,v) = H(x,v) + ax \cdot v + b|x|^2 = \Phi(x) + \frac{1}{2}|v|^2 + ax \cdot v + b|x|^2,$$

with $a, b > 0$ to be fixed later. We want this to be positive so we impose $a^2 < 2b$. Using that

$$\mathcal{L}^*(|v|^2) = d - |v|^2, \qquad \mathcal{L}^*(x \cdot v) = -x \cdot v, \qquad \mathcal{L}^*(\Phi(x)) = \mathcal{L}^*(|x|^2) = 0$$

and that

$$\mathcal{T}^*(H(x,v)) = 0, \qquad \mathcal{T}^*(x \cdot v) = |v|^2 - x \cdot \nabla_x \Phi(x), \qquad \mathcal{T}^*(|x|^2) = 2x \cdot v,$$

we see that

$$
\begin{aligned}
\mathcal{U}(V) =& \frac{d}{2} - \frac{1}{2}|v|^2 - ax \cdot v + a|v|^2 - ax \cdot \nabla_x \Phi(x) + 2bx \cdot v \\
\leq& C' - \left(\frac{1}{2} - a\right)|v|^2 + (2b - a)x \cdot v - a\gamma_1 |x|^2 - a\gamma_2 \Phi(x),
\end{aligned}
$$

where we have used (38), and $C' := \frac{d}{2} + aA$. Now, taking $a = 1/4, b = 1/8$,

$$
\begin{aligned}
\mathcal{U}(V) =& C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}|x|^2 - \frac{\gamma_2}{4}\Phi(x) \\
\leq& C' - \frac{\min(\gamma_1, 1)}{4}(|x|^2 + |v|^2) - \frac{\gamma_2}{4}\Phi(x) \\
\leq& C' - \frac{\min(\gamma_1, 1)}{4}\left(\frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2\right) - \frac{\gamma_2}{4}\Phi(x).
\end{aligned}
$$

So $V(x,v)$ works with

$$\lambda = \frac{\min(\gamma_1, \gamma_2, 1)}{4}. \qquad \square$$

Proof of Theorem 1.2 in the case of the linear relaxation Boltzmann equation. The proof follows by applying Harris's Theorem since Lemmas 3.7 and 3.8 show that the equation satisfies the hypotheses of the theorem. $\qquad \square$

3.3 Subgeometric convergence

When we do not have the superquadratic behaviour of the confining potential at infinity we can still use a Harris type theorem to show convergence to equilibrium. This time we must pay the price of having subgeometric rates of convergence. We use the subgeometric Harris's Theorem given in Section 2 which can be found in Section 4 of [26]. Now instead of our earlier assumption on the confining potential Φ, we instead make a weaker assumption that Φ is a C^2 function satisfying

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constant A, γ_1, γ_2, where

$$\langle x \rangle = \sqrt{1 + |x|^2},$$

and $\beta \in (0, 1)$.

Proof of Theorem 1.3 in the case of the linear relaxation Boltzmann equation. We have already proved the minorisation condition. We can also replicate the calculations for the Lyapunov function to get that in this new situation, take the V in Lemma 3.8, we have for $a = 1/4, b = 1/8$ that

$$\mathcal{U}V \leq C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}\langle x \rangle^{2\beta} - \frac{\gamma_2}{4}\Phi(x).$$

For $x, y \geq 1$

$$(x + y)^\beta \leq x^\beta + y^\beta.$$

So we have

$$
\begin{aligned}
\mathcal{U}V \leq & C' - \frac{\min(\gamma_1, 1)}{4}\left(\langle v \rangle^2 + \langle x \rangle^{2\beta}\right) - \frac{\gamma_2}{4}\Phi(x) \\
\leq & C'' - \frac{\min(\gamma_1, 1)}{4}\left(1 + |x|^2 + |v|^2\right)^\beta - \frac{\gamma_2}{4}\Phi(x)^\beta \\
\leq & C'' - \lambda\left(1 + \frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2\right)^\beta - \lambda\Phi(x)^\beta \\
\leq & C'' - \lambda\left(\Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2\right)^\beta,
\end{aligned}
$$

for some constant $\lambda, C'' > 0$ that can be explicitly computed, so we have that

$$\mathcal{U}V \leq -\lambda V^\beta + C''.$$

This means we can take $\phi(s) = 1 + s^\beta$. Therefore, for u large

$$H_\phi(u) = \int_1^u \frac{1}{1 + t^\beta}dt \sim 1 + u^{1-\beta},$$

and for t large

$$H_\phi^{-1}(t) \sim 1 + t^{1/(1-\beta)}$$

and

$$\phi \circ H_\phi^{-1}(t) \sim (1 + t)^{\beta/(1-\beta)}.$$

$\square$

4 The linear Boltzmann Equation

We now look at the linear Boltzmann equation. This has been studied in the spatially homogeneous case in [6, 10]. Here the interest is partly that this is a more complex and physically relevant operator. Also, it presents less globally uniform behaviour in v which means that we have to use a Lyapunov function even on the torus. Apart from this, the strategy is very similar to that from the linear relaxation Boltzmann equation. The full Boltzmann equation has been studied as a Markov process in [23], the linear case is similar and more simple. It is well known that this equation preserves positivity and mass, which follows from standard techniques both in the spatially homogeneous case and the case with transport. The Lyapunov condition on the torus and the bound below on the jump operator have to be verified in this situation.

We consider for $x \in \mathbb{T}^d$

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma, |v - v_*|\right) (f(v')\mathcal{M}(v_*') - f(v)\mathcal{M}(v_*)) \, d\sigma dv_*. \tag{39}$$

We assume that B splits as

$$B\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma, |v - v_*|\right) = b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) |v - v_*|^\gamma. \tag{40}$$

We make a cutoff assumption that b is integrable in σ. In fact, we make a much stronger assumption that b is bounded below by a constant. We also work in the hard spheres/Maxwell molecules regime that is to suppose $\gamma \geq 0$. When working with the Boltzmann collision kernel we have a choice of parametrizations for the incoming velocities. Choosing this 'σ-parametrization' allows for simpler calculations, but is not otherwise essential. We notice that for the physical hard spheres kernel the angular kernel b is bounded below in the σ parametrization. We have

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - \kappa(v)f,$$

where $\kappa(v) \geq 0$ and $\kappa(v)$ behaves like $|v|^\gamma$ for large v; that is,

$$0 \leq \kappa(v) \leq (1 + |v|^2)^{\gamma/2}, \qquad v \in \mathbb{R}^d. \tag{41}$$

See [10] Lemma 2.1 for example.

We also look at the situation where the spatial variable is in $\mathbb{R}^d$ and we have a confining potential. With hard sphere, the operator $\mathcal{L}^+$ acting on $x \cdot v$ produces error terms which are difficult to deal with. We show that when we have hard spheres with $\gamma > 0$ we can still show exponential convergence when $\Phi(x)$ is growing at least as fast as $|x|^{\gamma+2}$. In the subgeometric case we suppose $\Phi(x)$ grows at least as fast as $|x|^{\epsilon+1}, \epsilon > 0$. The equation is

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi(x) \cdot \nabla_v f) = Q(f, \mathcal{M}). \tag{42}$$

We begin by proving lemmas which are useful for proving the Doeblin condition in both situations. We want to reduce to a similar situation to the linear relaxation Boltzmann equation.

Lemma 4.1. *Let f be a solution to* (39) *or* (42), *and define $H(x, v) := |v|^2/2$ on the torus for* (39) *or $H(x, v) := \Phi(x) + |v|^2/2$ in the whole space for* (42), *where Φ is*

a C^2 potential bounded below. Take $E_0 > 0$ and assume that f has initial condition $f_0 = \delta_{(x_0,v_0)}$ with

$$H(x_0, v_0) \leq E_0.$$

Then there exists a constant $C_1 > 0$ such that

$$f(t, x, v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r (\mathbb{1}_E f_0(x, v)) \, \mathrm{d}r \, \mathrm{d}s,$$

where

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E \mathcal{L}^+ g, \qquad E := \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \; : \; H(x, v) \leq E_0\}.$$

Proof. Call $(X_t(x, v), V_t(x, v))$ the solution to the backward characteristic equations obtained from the transport part of either (39) or (42). Let us call

$$\Sigma(s, t, x, v) = e^{\int_s^t \kappa(V_r(x,v)) \mathrm{d}r}.$$

Looking at Duhamel's formula again we get

$$f(t, x, v) = \Sigma(0, t, x, v) T_t f_0 + \int_0^t \Sigma(0, t - s, x, v)(T_{t-s} \mathcal{L}^+ f_s)(x, v) \, \mathrm{d}s$$

If a function $g = g(x, v)$ has support on the set

$$E := \{(x, v) \; : \; H(x, v) \leq E_0\},$$

then the same is true of $T_t g$ (since the transport part preserves energy). On the set E we have, using (41),

$$\int_s^t \kappa(V_r(x, v)) \, \mathrm{d}r \leq (t - s)C \left(1 + 2E_0\right)^{\gamma/2} =: (t - s)C_1, \qquad (x, v) \in E.$$

Hence

$$f(t, x, v) \geq \Sigma(0, t, x, v) T_t (\mathbb{1}_E f_0) + \int_0^t \Sigma(0, t - s, x, v)(T_{t-s}(\mathbb{1}_E \mathcal{L}^+ f_s))(x, v) \, \mathrm{d}s$$

$$\geq e^{-tC_1} T_t (\mathbb{1}_E f_0) + \int_0^t e^{-(t-s)C_1}(T_{t-s}(\mathbb{1}_E \mathcal{L}^+ f_s))(x, v) \, \mathrm{d}s$$

$$= e^{-tC_1} T_t f_0 + \int_0^t e^{-(t-s)C_1}(T_{t-s}(\widetilde{\mathcal{L}}^+ f_s))(x, v) \, \mathrm{d}s,$$

where we define

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E \mathcal{L}^+ g.$$

Iterating this formula we obtain the result. $\qquad\square$

The next lemma we want to prove is a local version of Lemma 3.1, which states that the operator $\mathcal{L}^+$ allows jumps between any two velocities with a probability which is bounded below, provided the size of the two velocities is bounded by a fixed number. In order to do this, let us first rewrite the operator $\mathcal{L}^+$. We have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) |v - v_*|^\gamma f(v') \mathcal{M}(v'_*) \mathrm{d}\sigma \mathrm{d}v_*.$$

Using the Carleman representation we rewrite this as

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \frac{f(v')}{|v-v'|^{d-1}} \int_{E_{(v,v')}} B(|u|,\xi)\mathcal{M}(v'_*)\mathrm{d}v'_* \, \mathrm{d}v',$$

where $E_{(v,v')}$ denotes the hyperplane $\{v'_* \in \mathbb{R}^d \mid (v-v')\cdot(v-v'_*) = 0\}$, and the integral in v'_* is understood to be with respect to $(n-1)$-dimensional measure on this hyperplane. We want to bound this in the manner of Lemma 3.1 from the first part. We look at hard spheres and no angular dependence, which means

$$B(|u|,\xi) = C|u|^\gamma \xi^{d-2}$$

with $\gamma \geq 0$. We also have that

$$\xi = \frac{|v-v'|}{|2v-v'-v'_*|}, \quad |u| = |2v-v'-v'_*|.$$

So we have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d} \frac{f(v')}{|v-v'|} \int_{E_{(v,v')}} |2v-v'-v'_*|^{\gamma-d-2}\mathcal{M}(v'_*)\mathrm{d}v'_* \, \mathrm{d}v'.$$

With this we can give the following lower bound of $\mathcal{L}^+$, which the reader can compare to Lemma 3.1:

Lemma 4.2. *Consider the positive part $\mathcal{L}^+$ of the linear Boltzmann operator for hard spheres, assuming (4) with $\gamma \geq 0$, and (5). For all $R_L, r_L > 0$, there exists $\alpha > 0$ such that for all $g \in \mathcal{P}$*

$$\mathcal{L}^+ g(v) \geq \alpha \int_{B(R_L)} g(u)\,\mathrm{d}u \qquad \text{for all } v \in \mathbb{R}^d \text{ with } |v| \leq r_L.$$

Proof. First we note that on $E_{(v,v')}$ we have

$$|2v-v'-v'_*|^{-d-2} \geq C_d \exp\left(-\frac{1}{2}|v-v'_*|^2 - \frac{1}{2}|v-v'|^2\right).$$

Then since $\gamma \geq 0$ we have

$$|2v-v'-v'_*|^\gamma = \left(|v-v'|^2 + |v-v'_*|^2\right)^{\gamma/2} \geq |v-v'_*|^\gamma.$$

So this means that

$$\int_{E_{(v,v')}} |2v-v'-v'_*|^{\gamma-d-2}\mathcal{M}(v'_*)\mathrm{d}v'_*$$

$$\geq Ce^{-|v-v'|^2/2} \int_{E_{(v,v')}} |v-v'_*|^\gamma \exp\left(-\frac{1}{2}|v-v'_*|^2 - \frac{1}{2}|v'_*|^2\right)\mathrm{d}v'_*$$

$$\geq Ce^{-|v-v'|^2/2-|v|^2/2} \int_{E_{(v,v')}} |v-v'_*|^\gamma e^{-|v-v'_*|^2}\mathrm{d}v'_*$$

$$= C'e^{-|v-v'|^2/2-|v|^2/2}.$$

So we have that

$$\mathcal{L}^+ f(v) \geq C \int_{\mathbb{R}^d} f(v')|v - v'|^{-1} e^{-|v-v'|^2/2 - |v|^2/2} \mathrm{d}v'$$

$$\geq C \int_{\mathbb{R}^d} f(v') e^{-2|v'|^2 - 3|v|^2} \, \mathrm{d}v'$$

$$\geq C e^{-2R_L^2} e^{-3|v|^2} \int_{B(0,R_L)} f(v') \mathrm{d}v',$$

which is a similar bound to the one we found in Lemma 3.1. This gives the result by choosing $\alpha := C \exp(-2R_L^2 - 3|r_L|^2)$. $\qquad\square$

4.1 On the torus

Now we work specifically on the torus. For the minorisation we can argue almost exactly as for the linear relaxation Boltzmann equation.

Lemma 4.3 (Doeblin condition). *Assume (4) with $\gamma \geq 0$, and (5). Given $t_* > 0$ and $R > 0$ there exist constants $0 < \alpha < 1$, $\delta_L > 0$ such that any solution $f = f(t, x, v)$ to the linear Boltzmann equation (39) on the torus with initial condition $f_0 = \delta_{(x_0,v_0)}$ with $|v_0| \leq R$ satisfies*

$$f(t_*, x, v) \geq \alpha \mathbb{1}_{\{|v| \leq \delta_L\}}$$

in the sense of measures.

Proof. Take $f_0 := \delta_{(x_0,v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point with $|v_0| \leq R$. From Lemma 3.2 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exist $\delta_L, R' > 0$ such that

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x) \mathbb{1}_{\{|v| \leq \delta_L\}}\Big) \mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d, \, t > t_0. \tag{43}$$

Also, Lemma 4.2 gives an $\alpha > 0$ such that

$$\mathcal{L}^+ g \geq \alpha \left(\int_{B(R_L)} g(x, u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq \delta_L\}}, \tag{44}$$

where $R_L := \max\{R', R\}$. Finally, from Lemma 4.1 we can find $C_1 > 0$ (depending on R) such that

$$f(t, x, v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r(\mathbb{1}_E \delta_{(x_0,v_0)}) \, \mathrm{d}r \, \mathrm{d}s,$$

where E is the set of points with energy less than E_0, with

$$E_0 := \max\{R^2/2, \delta_L^2/2\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. Due to our choice of E_0, we see that equation (43) also holds with $\widetilde{\mathcal{L}}^+$ in the place of $\mathcal{L}^+$. One can then carry out the same proof as in Lemma 3.3, using estimates (43) and (44) instead of the corresponding ones there. $\qquad\square$

Since our Doeblin condition holds only on sets which are bounded in $|v|$, we do need a Lyapunov functional in this case (as opposed to the linear relaxation Boltzmann equation, where Lemma 3.3 gives a lower bound for all starting conditions (x, v)). Testing with $V = v^2$ involves proving a result similar to the moment control result from [6]. Instead of the σ representation we use the n-representation for the collisions:

$$v' = v - n(u \cdot n), \qquad v'_* = v_* + n(u \cdot n).$$

By our earlier assumption, the collision kernel can be written as

$$\tilde{B}(|v - v_*|, |\xi|) = |v - v_*|^\gamma \tilde{b}(|\xi|),$$

where

$$\xi := \frac{u \cdot n}{|u|}, \qquad u := v - v_*.$$

Here the $\tilde{B}, \tilde{b}$ are different from those in the σ representation because of the change of variables. We also have by assumption that $\tilde{b}$ is normalised, that is,

$$\int_{\mathbb{S}^d} \tilde{b}(|w \cdot n|) \, \mathrm{d}n = 1$$

for all unit vectors $w \in \mathbb{S}^{d-1}$.

Lemma 4.4. *The function $V(x, v) = |v|^2$ is a Lyapunov function for the linear Boltzmann equation on the torus in the sense that it is a function for which the associated semigroup satisfies Hypothesis 2.*

Proof. We define $\mathcal{L}$ to be the linear Boltzmann operator. Using the weak formulation of the operator,

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \, \mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)|v - v_*|^\gamma \tilde{b}(|\xi|)(|v'|^2 - |v|^2) \, \mathrm{d}n \, \mathrm{d}v \, \mathrm{d}v_*.$$

In other words,

$$\mathcal{L}^*(|v|^2) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathcal{M}(v_*)|v - v_*|^\gamma \tilde{b}(|\xi|)(|v'|^2 - |v|^2) \, \mathrm{d}n \, \mathrm{d}v_*.$$

We are going to prove the Lyapunov condition by showing that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mathcal{L}(f) + \mathcal{T}(f))|v|^2 \, \mathrm{d}x \, \mathrm{d}v \le -\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f|v|^2 \, \mathrm{d}x \, \mathrm{d}v + K \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, \mathrm{d}x \, \mathrm{d}v,$$

where $\mathcal{T}f = -v\nabla_x f$ is the transport operator. The transport part plays no role, since

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{T}(f)|v|^2 \, \mathrm{d}x \, \mathrm{d}v = 0.$$

For the collisional part, we notice that

$$
\begin{aligned}
|v'|^2 - |v|^2 = |v_*|^2 - |v'_*|^2 &= -(u \cdot n)^2 - 2(v_* \cdot n)(u \cdot n) \\
&= -|u|^2\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2 \\
&= -|v|^2\xi^2 - |v_*|^2\xi^2 + 2v \cdot v_*\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2.
\end{aligned}
$$

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Note that the first term is negative and quadratic in v, and the rest of the terms are of lower order in v. Hence, calling $\gamma_b := \int_{\mathbb{S}^{d-1}} \xi^2 \tilde{b}(|\xi|) \, d\xi$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \, dv =& - \gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \\
& - \gamma_b \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \\
& + 2\gamma_b \int_{\mathbb{R}^d} v f(v) \int_{\mathbb{R}^d} v_* \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \\
& - 2 \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} (v \cdot n) f(v) \int_{\mathbb{R}^d} (v_* \cdot n) \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \, dn \\
& + \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} (v_* \cdot n)^2 \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \, dn \\
\leq& - \gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \\
& + (2 + \gamma_b) \int_{\mathbb{R}^d} |v| f(v) \int_{\mathbb{R}^d} |v_*| \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv \\
& + \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \, dv.
\end{aligned}
$$

We can now use the following bound, which holds for all $k \geq 0$ and some constants $0 < A_k \leq C_k$ depending on k:

$$
A_k(1 + |v|^\gamma) \leq \int_{\mathbb{R}^d} |v_*|^k \mathcal{M}(v_*)|v - v_*|^\gamma \, dv_* \leq C_k(1 + |v|^\gamma), \qquad v \in \mathbb{R}^d.
$$

Choosing $\epsilon > 0$ we get

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \, dv \leq& - A_0 \gamma_b \int_{\mathbb{R}^d} |v|^2 (1 + |v|^\gamma) f(v) \, dv + C_1(2 + \gamma_b) \int_{\mathbb{R}^d} |v|(1 + |v|^\gamma) f(v) \, dv \\
& + C_2 \int_{\mathbb{R}^d} f(v)(1 + |v|^\gamma) \, dv \\
\leq& \int_{\mathbb{R}^d} f(v)(C_2 + C_1(1 + \gamma_b/2)/\epsilon)(1 + |v|^\gamma) \, dv \\
& - (A_0 \gamma_b - \epsilon C_1(1 + \gamma_b/2)) \int_{\mathbb{R}^d} |v|^2 (1 + |v|^\gamma) f(v) dv \\
\leq& \int_{\mathbb{R}^d} f(v) \left(C_2 + C_1(1 + \gamma_b/2)/\epsilon + (\epsilon C_1(1 + \gamma_b/2) - A_0 \gamma_b)|v|^2 \right)(1 + |v|^\gamma) f(v) dv \\
& - (A_0 \gamma_b - \epsilon C_1(1 + \gamma_b/2)) \int_{\mathbb{R}^d} |v|^2 f(v) dv \\
\leq& \alpha_1 \int_{\mathbb{R}^d} f(v) dv - \alpha_2 \int_{\mathbb{R}^d} |v|^2 f(v) dv.
\end{aligned}
$$

Here we choose ϵ sufficiently small to make the constant in front of the second moment negative. This also means that

$$
(C_2 + C_1(1 + \gamma_b/2)/\epsilon + (\epsilon C_1(1 + \gamma_b/2) - A_0 \gamma_b)|v|^2(1 + |v|^\gamma)
$$

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is bounded above. These things together give that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mathcal{L}(f) + \mathcal{T}(f)) |v|^2 \, \mathrm{d}x \, \mathrm{d}v \leq -\alpha_2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f|v|^2 \, \mathrm{d}x \, \mathrm{d}v + \alpha_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, \mathrm{d}x \, \mathrm{d}v,$$

which finishes the proof. $\qquad\square$

Proof of Theorem 1.1 in the case of the linear Boltzmann equation. We have the Doeblin condition from Lemma 4.3 and the Lyapunov structure from Lemma 4.4. Harris's Theorem gives the result. $\qquad\square$

4.2 On the whole space with a confining potential

We now work on the whole space with a confining potential. As we stated earlier, we cannot verify the Lyapunov condition in the hard spheres case. However, the proof for the Doeblin's condition is the same in the hard sphere or Maxwell molecule case. We need to combine the Lemmas 3.6, 4.1 and 4.2.

Lemma 4.5. *Let the potential $\Phi \colon \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with compact level sets. Given $t > 0$ and $K > 0$ there exist constants $\alpha, \delta_X, \delta_V > 0$ such that for any (x_0, v_0) with $|x_0|, |v_0| < K$ the solution f to (42) with initial data $\delta_{(x_0, v_0)}$ satisfies*

$$f_t \geq \alpha \mathbb{1}_{\{|x| \leq \delta_X\}} \mathbb{1}_{\{|v| \leq \delta_V\}}.$$

Proof. We fix $R > 0$ as in Lemma 3.7. We use Lemma 3.6 to find constants $\alpha, R_2, R' > 0$ and an interval $[a, b] \subseteq (0, t)$ such that

$$\int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \leq R_2\}}) \, \mathrm{d}v \geq \alpha \mathbb{1}_{\{|x| \leq R\}},$$

for any x_0 with $|x_0| \leq R$ and any $s \in [a, b]$. From Lemma 4.2 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x, v) \geq \alpha_L \left(\int_{R_L} g(x, u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq R_2\}} \tag{45}$$

for all nonnegative measures g, where $R_L := \max\{R, R'\}$. From Lemma 4.1 we can find $C_1 > 0$ (depending on R) such that

$$f(t, x, v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r (\mathbb{1}_E \delta_{(x_0, v_0)}) \, \mathrm{d}r \, \mathrm{d}s,$$

where E is the set of points with energy less than E_0, with

$$E_0 := \max\{H(x, v) \,:\, |x| \leq R, |v| \leq \max\{R_L, R_2\}\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. These three estimates allow us to carry out a proof which is completely analogous to that of Lemma 3.7; notice that the only difference is the appearance of R' here, and the need to use $\widetilde{\mathcal{L}}^+$ (which still satisfies a bound of the same type). $\qquad\square$

Now we need to find a Lyapunov functional. As before we will look at V of the form

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2.$$

for some $\alpha, \beta > 0$. In this case we need a stronger bound for Φ, as stated in the following:

Lemma 4.6. *Assume that for some $\gamma_1, \gamma_2 > 0$ and some $A \in \mathbb{R}$ we have,*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

where γ is the exponent in (40). We can find α, β such that

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2$$

is a function for which the semigroup associated to equation (42) satisfies Hypothesis 2.

Proof. We are going to show that, for an appropriate choice of α, β it holds that

$$(\mathcal{T}^* + \mathcal{L}^*)(V) \leq -\lambda V + K,$$

for some $\lambda, K > 0$, where $\mathcal{L}^*$ is the dual of the Boltzmann collision operator, and $\mathcal{T}^* f = v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f$ (the dual of the transport operator, which has the same expression as $-\mathcal{T}$). This will show Hypotheses 2 (see Remark 2).

Let us look at how the collision operator acts on the different terms. First,

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v - v_*|^\gamma \left(|v'|^2 - |v|^2\right) \mathrm{d}n\mathrm{d}v\mathrm{d}v_*.$$

Repeating the same calculation as in the proof of Lemma 4.4 in this case, we see that

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \mathrm{d}v \leq -\alpha_1 \int_{\mathbb{R}^d} \langle v \rangle^{\gamma+2} f(v)\mathrm{d}v + \alpha_2 \int_{\mathbb{R}^d} f(v)\mathrm{d}v.$$

That is,

$$\mathcal{L}^*(|v|^2) \leq -\alpha_1 \langle v \rangle^{\gamma+2} + \alpha_2. \tag{46}$$

Similarly we have

$$\int_{\mathbb{R}^d} \mathcal{L}(f)x \cdot v\mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v - v_*|^\gamma \left(v' \cdot x - v \cdot x\right) \mathrm{d}n\mathrm{d}v\mathrm{d}v_*.$$

We can see that

$$v' \cdot x - v \cdot x = (v \cdot n)(x \cdot n) - (v_* \cdot n)(x \cdot n).$$

Integrating this gives that

$$\int_{\mathbb{R}^d} \mathcal{L}(f)x \cdot v\mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v - v_*|^\gamma(v \cdot n)(x \cdot n)\mathrm{d}v_*\mathrm{d}v\mathrm{d}n$$

$$\leq \int_{\mathbb{R}^d} f(v)\langle v \rangle^{\gamma+1}|x|\mathrm{d}v,$$

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so

$$\mathcal{L}^*(x \cdot v) \le \langle v \rangle^{\gamma+1} |x|. \tag{47}$$

For the effect of $\mathcal{T}^*$, notice that

$$V(x,v) = \Phi(x) + |v|^2/2 + \alpha x \cdot v + \beta |x|^2 = H(x,v) + \alpha x \cdot v + \beta |x|^2,$$

where $H(x,v)$ denotes the energy. We have

$$\mathcal{T}^*(H(x,v)) = 0, \qquad \mathcal{T}^*(|x|^2) = 2x \cdot v, \qquad \mathcal{T}^*(x \cdot v) = |v|^2 - x \cdot \nabla_x \Phi(x)$$

Using this together with (46) and (47) we have

$$(\mathcal{L}^* + \mathcal{T}^*)(V) \le -\frac{\alpha_1}{2} \langle v \rangle^{\gamma+2} + \frac{\alpha_2}{2} + \alpha \langle v \rangle^{\gamma+1} |x| + \alpha |v|^2 - \alpha x \cdot \nabla_x \Phi(x) + 2\beta x \cdot v$$

$$\le \left(\alpha - \frac{\alpha_1}{2} \right) \langle v \rangle^{\gamma+2} + (\alpha + 2\beta)|x|\langle v \rangle^{\gamma+1} - \alpha\gamma_1 \langle x \rangle^{\gamma+2} - \alpha\gamma_2 \Phi(x) + \frac{\alpha_2}{2} + \alpha A.$$

Setting $\beta = \alpha$, $\alpha \le \alpha_1/4$, and using Young's inequality $AB \le \frac{1}{p}A^p + \frac{1}{q}B^q$ on $(|x|/\epsilon)(\langle v \rangle^{\gamma+1}\epsilon)$ with $p = (\gamma+2)/(\gamma+1)$ and $q = \gamma+2$, we get

$$(\mathcal{L}^* + \mathcal{T}^*)(V) \le \left(-\frac{\alpha_1}{4} + 3\epsilon^{\frac{\gamma+2}{\gamma+1}} \frac{\gamma+1}{\gamma+2} \right) \langle v \rangle^{\gamma+2} + \left(\frac{3\alpha^{\gamma+2}}{(\gamma+2)} \frac{1}{\epsilon^{\gamma+2}} - \alpha\gamma_1 \right) \langle x \rangle^{\gamma+2}$$

$$- \alpha\gamma_2 \Phi(x) + \frac{\alpha_2}{2} + \alpha A.$$

Now we can choose ϵ small enough so that the $\langle v \rangle^{\gamma+2}$ term is negative and then for this ϵ choose α small enough so that the $\langle x \rangle^{\gamma+2}$ term is negative (since $\gamma + 2 \ge 1$). Then, since $\langle z \rangle^{\gamma+2}$ grows faster than $|z|^2$ at infinity this gives

$$(\mathcal{L}^* + \mathcal{T}^*)(V) \le -\lambda_1(|x|^2 + |v|^2) - \lambda_2 \Phi(x) + K.$$

Then using equivalence between the quadratic forms

$$|x|^2 + |v|^2 \quad \text{and} \quad \frac{1}{2}|v|^2 + \alpha x \cdot v + \alpha |x|^2,$$

when $\alpha < 1/2$ we have the result in the Lemma. $\qquad \square$

Proof of Theorem 1.2 in the case of the linear Boltzmann equation. We have the minorisation condition in Lemma 4.5 and the Lyapunov condition from Lemma 4.6. Therefore we can apply Harris's Theorem. $\qquad \square$

4.3 Subgeometric convergence

As with the linear relaxation Boltzmann equation, the minorisation results in Lemma 4.5 holds for Φ which are not sufficiently confining to prove the Lyapunov structure. However in this situation we can still prove subgeometric rates of convergence. Here in order to find a Lyapunov functional we need to be more precise about how $\mathcal{L}$ acts on the $x \cdot v$ moment.

We need $\Phi(x)$ to provide a stronger confinement if we are in the hard potential case. We want

$$x \cdot \nabla_x \Phi(x) \ge \gamma_1 \langle x \rangle^{1+\delta} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \le \gamma_3 \langle x \rangle^{1+\delta} \tag{48}$$

for some $\gamma_1, \gamma_2, \gamma_3, \delta > 0$. Then we have

Lemma 4.7. *Assume that Φ is a C^2 potential satisfying* (48). *Then there exist some $\alpha, \beta > 0$ satisfying $4\alpha^2 < \beta$ such that the function*

$$V(x, v) = \Phi(x) + \frac{1}{2}|v|^2 + \frac{\alpha x \cdot v}{\langle x \rangle} + \beta \langle x \rangle$$

satisfies

$$\mathcal{U}(V) \leq -\lambda V^{\delta/(1+\delta)} + K,$$

for some positive constants λ, K.

Remark 4.8. Notice that

$$V(x, v) \geq \Phi(x) + \frac{1}{4}|v|^2 + (\beta - 4\alpha^2)\langle x \rangle,$$

so that the sub level sets of V are bounded.

Proof. Using (46), (47) and that

$$\mathcal{T}^*(H(x, v)) = 0, \qquad \mathcal{T}^*(\langle x \rangle) = \frac{x \cdot v}{\langle x \rangle}, \qquad \mathcal{T}^*\left(\frac{x \cdot v}{\langle x \rangle}\right) = \frac{|v|^2}{\langle x \rangle} - \frac{(x \cdot v)^2}{\langle x \rangle^2} - \nabla_x \Phi \cdot \frac{x}{\langle x \rangle},$$

we have the following

$$(\mathcal{L}^* + \mathcal{T}^*)(V(x, v)) \leq -\alpha_1 \langle v \rangle^{\gamma+2} + \alpha_2 + \alpha \langle v \rangle^{\gamma+1} + \alpha|v|^2 - \frac{\alpha x \cdot \nabla_x \Phi(x)}{\langle x \rangle} + \frac{\beta x \cdot v}{\langle x \rangle}.$$

Using (48), $\dfrac{x \cdot v}{\langle x \rangle} \leq \langle v \rangle \leq \langle v \rangle^{2+\gamma}$ and $\Phi(x)^{\delta/(1+\delta)} \leq \gamma_3^{\delta/(1+\delta)} \langle x \rangle^\delta$ we obtain

$$(\mathcal{L}^* + \mathcal{T}^*)(V(x, v))$$

$$\leq (2\alpha - \alpha_1 + \beta)\langle v \rangle^{\gamma+2} - \alpha\gamma_1 \langle x \rangle^\delta - \alpha\gamma_2 \frac{\Phi(x)}{\langle x \rangle} + \alpha_2 + \alpha A$$

$$\leq \lambda_1 \left(-|v|^2 - \langle x \rangle^\delta - \Phi(x)^{\delta/(1+\delta)} + C\right) \leq \lambda V(x, v)^{\delta/(1+\delta)} + K,$$

for some $\lambda_1, K > 0$. To make the last two inequalities valid we choose α and β so that $\alpha_1 > 2\alpha + \beta$ and $4\alpha^2 < \beta$ so that

$$V(x, v) \geq \Phi(x) + \frac{1}{4}|v|^2 + (\beta - 4\alpha^2)\langle x \rangle.$$

$\square$

Proof of Theorem 1.3 *in the linear Boltzmann case.* We have the minorisation condition in Lemma 4.5 and the Lyapunov condition from Lemma 4.7. Therefore we can apply Harris's Theorem. $\square$

Acknowledgements

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