Hardy-Sobolev inequalities with singularities on non smooth boundary. Part 2: Influence of the global geometry in small dimensions

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HARDY-SOBOLEV INEQUALITIES WITH SINGULARITIES ON NON SMOOTH BOUNDARY. PART 2: INFLUENCE OF THE GLOBAL GEOMETRY IN SMALL DIMENSIONS.

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Abstract. We consider Hardy-Sobolev nonlinear equations on domains with singularities. We introduced this problem in Cheikh-Ali [4]. Under a local geometric hypothesis, namely that the generalized mean curvature is negative (see (7) below), we proved the existence of extremals for the relevant Hardy-Sobolev inequality for large dimensions. In the present paper, we tackle the question of small dimensions that was left open. We introduce a "mass", that is a global quantity, the positivity of which ensures the existence of extremals in small dimensions. As a byproduct, we prove the existence of solutions to a perturbation of the initial equation via the Mountain-Pass Lemma.

Contents

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 3$. We fix $s \in [0,2]$ and $\gamma \in \mathbb{R}$. It follows from the classical Caffarelli-Kohn-Nirenberg inequalities [3] that if $\gamma < \frac{(n-2)^2}{4}$, there exists $K > 0$ such that

$$\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^\star(s)}} \leq K \int_\Omega \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx, \tag{1}$$

for all $u \in D^{1,2}(\Omega)$, where $2^\star(s) := \frac{2(n-s)}{n-2}$ and $D^{1,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $u \mapsto \|\nabla u\|_2$. We define the Hardy constant by

$$\gamma_H(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx}; u \in D^{1,2}(\Omega) \backslash \{0\} \right\} > 0.$$

Date: May 8th, 2020.

The classical Hardy inequality reads $\gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$ and therefore, we have that $\gamma_H(\Omega) \geq \frac{(n-2)^2}{4}$. We refer to [4] for discussions and properties of the Hardy constant. As one checks, for any $\gamma < \gamma_H(\Omega)$, there exists $K = K(\Omega, \gamma, s) > 0$ such that (1) holds for all $u \in D^{1,2}(\Omega)$. For $a \in L^\infty(\Omega)$, we define

$$\mu_{\gamma,s,a}(\Omega) = \inf_{u \in D^{1,2}(\Omega)\setminus\{0\}} J^\Omega_{\gamma,s,a}(u),$$

where

$$J^\Omega_{\gamma,s,a}(u) := \frac{\int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + a(x)\right) u^2\right) dx}{\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s} dx\right)^{\frac{2}{2^\star(s)}}},$$

so that

$$\mu_{\gamma,s,a}(\Omega) \left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \leq \int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + a(x)\right) u^2\right) dx, \tag{2}$$

for all $u \in D^{1,2}(\Omega)$. As in [4], we address the question of the existence of extremals for (2), more precisely

Q: Does there exist $u \in D^{1,2}(\Omega)\setminus\{0\}$ for which equality holds in (2)?

When $0 \in \Omega$, there are no extremals for $\mu_{\gamma,s,0}(\Omega)$ (see [7]). From now on, we assume that $0 \in \partial\Omega$. When Ω is a smooth domain, criteria for existence are in Ghoussoub-Robert [8]: in particular, there is a dichotomy between large dimension (where the criterion is local) and the small dimensions (where the criterion is global). In [4], we studied the case of domains that are modeled on cones:

Definition 1. *We fix $1 \leq k \leq n$. Let Ω be a domain of $\mathbb{R}^n$. We say that $x_0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ if there exist U, V open subsets of $\mathbb{R}^n$ such that $0 \in U$, $x_0 \in V$ and there exists a diffeomorphism $\phi \in C^\infty(U, V)$ such that $\phi(0) = x_0$ and*

$$\phi(U \cap \left(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}\right)) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial\left(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}\right)) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0, namely $d\phi_0$, is an isometry.

In the sequel, we write $\mathbb{R}^{k_+,n-k} := \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$. We have that (see [4])

$$\gamma_H(\mathbb{R}^{k_+,n-k}) = \frac{(n-2+2k)^2}{4}.$$

We have proved the following:

Theorem 1.1 (Cheikh-Ali [4])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $0 \leq s < 2$ and $0 \leq \gamma < \gamma_H(\Omega)$. In addition, suppose that either $\{s > 0\}$ or $\{s = 0, n \geq 4$ and $\gamma > 0\}$. We assume that*

$$\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \quad \text{that is } n \geq n_{\gamma,k} := \sqrt{4\gamma + 1} + 2 - 2k. \tag{3}$$

Then there are extremals for $\mu_{\gamma,s,0}(\Omega)$ if

$$GH_{\gamma,s}(\Omega) < 0$$

where $GH_{\gamma,s}(\Omega)$ is the **generalized mean curvature** *defined below in* (7)*.*

The assumption

$$\{s > 0\} \text{ or } \{s = 0, n \geq 4 \text{ and } \gamma > 0\} \tag{4}$$

will be reminiscent in the statements below. Its sole utility is to ensure the existence of extremals for $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ (see Ghoussoub-Robert [8]).

This result is for large dimension $n \geq n_{\gamma,k}$ (see (3)). In the present article, we tackle the case of the remaining small dimensions. The argument based on local geometry performed for the proof of Theorem 1.1 is not working here. Here, the global geometry has an impact: in order to obtain extremals, we must introduce a "mass" in the spirit of Schoen [14] and Schoen-Yau [15]. Concerning low dimension phenomena, we refer to the pioneer work of Brezis-Nirenberg [2], Jannelli [13] and the more recent reference Ghoussoub-Robert [7] for further discussions. Our main theorem is the following:

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $0 \leq s < 2$, $\gamma < \gamma_H(\Omega)$ and $a \in C^{0,\theta}(\Omega)$ ($\theta \in (0,1)$). We assume that condition (4) holds and that*

$$\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k}.$$

We assume that the operator $-\Delta - (\gamma|x|^{-2} + a(x))$ is coercive and has a mass $m_{\gamma,a}(\Omega)$ (see Definition 3), and that $m_{\gamma,a}(\Omega) > 0$. Then there are extremals for $\mu_{\gamma,s,a}(\Omega)$. In particular, there exists $u \in C^{2,\theta}(\Omega) \cap D^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

In the second part of this paper, we consider the perturbative Hardy-Schrödinger equation. Given $a, h \in C^{0,\theta}(\Omega)$ for some $\theta \in (0,1)$ and $q \in (1, 2^\star - 1)$ where $2^\star = 2^\star(0)$, we investigate the existence of solutions $u \in C^2(\Omega) \cap D^{1,2}(\Omega)$ to

$$\begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^\star(s)-1}}{|x|^s} + h(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

We refer to Brezis-Nirenberg [2] ($\gamma = 0$ and $s = 0$ on a smooth domain Ω), Ghoussoub-Yuan [10] ($\gamma = 0$, $s > 0$ and $0 \in \Omega$), Ghoussoub-Kang [9] and Jaber [12] ($\gamma = 0$, $s > 0$ and $0 \in \partial\Omega$). In the Riemannian context with no boundary, still for $\gamma = 0$, we refer to Djadli [5] when $s = 0$, and to Jaber [11] for $s > 0$ and $h \equiv 0$.

The case $a, h \equiv 0$ was tackled in [4] for $n \geq n_{\gamma,k}$ for nonsmooth domains. We prove the following:

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. Let $a, h \in C^{0,\theta}(\Omega)$ ($\theta \in (0,1)$) be such that $-\Delta - (\gamma|x|^{-2} + a)$ is coercive and $h \geq 0$. Consider $s \in [0, 2)$ and $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$. We assume that condition (4) holds and we fix $q \in (1, 2^\star - 1)$, $2^\star = 2^\star(0)$. Then, there exists a positive Mountain-Pass solution $u \in D^{1,2}(\Omega)$ to the perturbative Hardy-Schrödinger equation (6) under one of the following conditions:*

- $0 \leq \gamma < \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, *and*

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q+1 < \frac{2n-2}{n-2}, \\ c_1 GH_{\gamma,s}(\Omega) - c_2 h(0) < 0 & \text{if } q+1 = \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

- $0 \leq \gamma = \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, *and*

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

- $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, *and*

$$\begin{cases} m_{\gamma,a}(\Omega) > 0 & \text{if } q+1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ c_3 m_{\gamma,a}(\Omega) + c_2 h(0) > 0 & \text{if } q+1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

where $GH_{\gamma,s}(\Omega)$ is the generalized curvature (see (7)), $m_{\gamma,a}(\Omega)$ is the mass of Ω at 0 (see Definition 3), $\alpha_+ - \alpha_- = 2\sqrt{\gamma_H(\mathbb{R}^{k_+,n-k}) - \gamma}$ (see (8) below) and $c_1, c_2, c_3 > 0$ are defined in (70).

This result shows how the subcritical nonlinearity has an impact on the existence of solutions. When the subcritical nonlinearity is close to being linear, only the geometry of Ω commands the existence. Conversely, when it is close to being critical, the subcritical nonlinearity commands the existence, whatever the geometry is.

Notation: In the sequel, C denotes a positive constant, the value of which may change from one page to another and even from one line to the next.

This paper is organized as follows. Section 2 is devoted to the definitions of the generalized curvature and the singular interior mass. In Section 3, we introduce some preliminary results that will be of use in the sequel. In Section 4, we prove Theorem 1.2, which is a existence of extremals for $\mu_{\gamma,s,a}(\Omega)$ is ensured for small dimensions when the mass $m_{\gamma,a}(\Omega)$ is positive. In Section 6, we prove Theorem 1.3, which is a general existence result for a Mountain-Pass solution for equation (6). In Section 7, we make test-function estimates in order to obtain a sufficient condition of existence for (6).

2. Definition of the generalized curvature and the mass

Generalized curvature.

Definition 2. *Let Ω be a domain in $\mathbb{R}^n$ with $n \geq 3$ such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$. We define*

$$\Omega_i := \phi(U \cap \{x_i > 0\}) \text{ for all } i = 1, ..., k,$$

where (ϕ, U) is a chart as in Definition 1. We have that:

- *For all $i = 1, ..., k$, Ω_i is smooth around $0 \in \partial\Omega_i$.*
- *Up to permutation, the Ω_i's are locally independent of the chart ϕ.*
- *The Ω_i's define locally Ω: there exists $\delta > 0$ such that*

$$\Omega \cap B_\delta(0) = \bigcap_{i=1}^{k} \Omega_i \cap B_\delta(0).$$

We set $\Sigma := \cap_{i=1}^k \partial\Omega_i$ where $k \in \{1, ..., n\}$. The vector $\vec{H}_0^\Sigma$ denotes the mean-curvature vector at 0 of the $(n-k)-$submanifold Σ. For any $m = 1, ..., k$, $II_0^{\partial\Omega_m}$ denotes the second fundamental form at 0 of the oriented $(n-1)-$submanifold $\partial\Omega_m$. The **generalized mean curvature** of Ω is defined by:

$$GH_{\gamma,s}(\Omega) := c^1_{\gamma,s} \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c^2_{\gamma,s} \sum_{i,m=1,\, i\neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) + c^3_{\gamma,s} \sum_{p,q,m=1,\, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\overrightarrow{\nu}_p, \overrightarrow{\nu}_q) \tag{7}$$

where for any $m = 1, ..., k$, $\vec{\nu}_m$ is the outward normal vector at 0 of $\partial\Omega_m$ and $c^1_{\gamma,s}, c^2_{\gamma,s}, c^3_{\gamma,s}$ are positive explicit constants. We refer to [4] for details on this curvature.

The mass. Let $\alpha \in \mathbb{R}$ be a real number and fix $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$. Then

$$\left(-\Delta - \frac{\gamma}{|x|^2}\right) S_\alpha = 0 \ \Leftrightarrow \ \alpha \in \{\alpha_-, \alpha_+\},$$

where:

$$S_\alpha := |x|^{-\alpha-k} \prod_{i=1}^k x_i \text{ and } \alpha_\pm = \alpha_\pm(\gamma, n, k) := \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k_+,n-k}) - \gamma}. \tag{8}$$

The functions $S_{\alpha_-}, S_{\alpha_+}$ are prototypes of solution to (5) vanishing on $\partial\mathbb{R}^{k_+,n-k}$.

Definition 3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $\gamma < \gamma_H(\Omega)$ and $a \in C^{0,\theta}(\Omega)$ ($\theta \in (0,1)$). We say that a coercive operator $-\Delta - (\gamma|x|^{-2} + a)$ has a mass if there exists $G \in C^2(\Omega) \cap D^{1,2}_{loc,0}(\Omega)$ such that*

$$\begin{cases} -\Delta G - \left(\frac{\gamma}{|x|^2} + a(x)\right) G = 0 & \text{in } \Omega, \\ G > 0 & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega\backslash\{0\}, \end{cases} \tag{9}$$

and there exists $c \in \mathbb{R}$ such that

$$G(x) = \prod_{i=1}^k d(x, \partial\Omega_i) \left(|x|^{-\alpha_+-k} + c|x|^{-\alpha_--k} + o(|x|^{-\alpha_--k})\right) \text{ as } x \to 0, \tag{10}$$

where $\alpha_\pm$ is defined in (8). We can therefore define the quantity $m_{\gamma,a}(\Omega) := c$ as the boundary mass of the operator $-\Delta - (\gamma|x|^{-2} + a)$. The function G is unique, so that the definition of the mass makes sense.

Examples of domains with positive or negative mass are in Section 5 below.

3. Some background results

We start with the following result that is reminiscent for critical elliptic problems:

Theorem 3.1. *[see Cheikh-Ali [4]] Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. Assume that $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$, $0 \leq s \leq 2$, and $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$. Then there are extremals for $\mu_{\gamma,s,a}(\Omega)$.*

Indeed, Theorem 3.1 was proved in [4] when $a \equiv 0$. The proof extends to the general case with no effort. Recall now an optimal regularity theorem.

Theorem 3.2. *[See Felli-Ferrero [6] and [4]] Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$|f(x,v)| \leq C|v| \left(1 + \frac{|v|^{2^*(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega, v \in \mathbb{R}. \tag{11}$$

Let $u \in D^{1,2}(\Omega)_{loc,0}$, be a weak solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u) \text{ in } D^{1,2}(\Omega)_{loc,0}$$

for some $\tau > 0$. Then there exists $K \in \mathbb{R}$ such that

$$\lambda^{\alpha_-} u(\lambda\phi(x)) \to K|x|^{-\alpha_-} \frac{\prod_{i=1}^k x_i}{|x|^k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+,n-k},$$

uniformly in C^1 as $\lambda \to 0$, where ϕ is a chart as in Definition 1.

In section 4, we will need the following lemma:

Lemma 3.1. *[See [4]] Assume that $u \in D^{1,2}(\mathbb{R}^{k_+,n-k})_{loc,0}$ is a weak solution of*

$$\begin{cases} -\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\mathbb{R}^{k_+,n-k})_{loc,0}, \\ u = 0 & \text{on } B_{2\delta}(0) \cap \partial\mathbb{R}^{k_+,n-k}, \end{cases}$$

for some $\tau > 0$ and $\alpha \in \{\alpha_-, \alpha_+\}$. Assume there exists $c > 0$ such that

$$|u(x)| \leq c|x|^{-\alpha} \text{ for } x \to 0, x \in \mathbb{R}^{k_+,n-k}.$$

- *Then, there exists $c_1 > 0$ such that*

$$|\nabla u(x)| \leq c_1|x|^{-\alpha-1} \text{ as } x \to 0, x \in \mathbb{R}^{k_+,n-k}.$$

- *If $\lim_{x\to 0} |x|^\alpha u(x) = 0$, then $lim_{x\to 0}|x|^{\alpha+1}|\nabla u(x)| = 0$.*

4. Test-functions estimates for the mass: proof of Theorem 1.2

Let $U \in D^{1,2}(\mathbb{R}^{k_+,n-k})$ be a positive extremal for $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$. Then

$$J^{\mathbb{R}^{k_+,n-k}}_{\gamma,s,0}(U) = \frac{\int_{\mathbb{R}^{k_+,n-k}} \left(|\nabla U|^2 - \gamma|x|^{-2}U^2\right) dx}{\left(\int_{\mathbb{R}^{k_+,n-k}} |x|^{-s}|U|^{2^\star(s)}\, dx\right)^{\frac{2}{2^\star(s)}}} = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}).$$

Therefore, there exists $\xi > 0$ such that

$$\begin{cases} -\Delta U - \gamma|x|^{-2}U = \xi|x|^{-s}U^{2^\star(s)-1} & \text{in } \mathbb{R}^{k_+,n-k}, \\ U > 0 & \text{in } \mathbb{R}^{k_+,n-k}, \\ U = 0 & \text{on } \partial\mathbb{R}^{k_+,n-k}. \end{cases} \tag{12}$$

For $r > 0$, we define

$$B_r := B_r(0) \text{ and } B_{r,+} := B_r(0) \cap \mathbb{R}^{k_+,n-k}. \tag{13}$$

Therefore, with $\delta > 0$ small, the chart ϕ of Definition 1 yields

$$\phi(B_{3\delta} \cap \mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \Omega \text{ and } \phi(B_{3\delta} \cap \partial\mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \partial\Omega.$$

We fix $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B_\delta, \\ 0 & \text{for } x \notin B_{2\delta}. \end{cases} \tag{14}$$

Define also for convenience,

$$p(x) := \prod_{i=1}^k d(x, \partial\Omega_i) \text{ for all } x \in \Omega \text{ and } v(x) := \prod_{i=1}^k x_i \text{ for all } x \in \mathbb{R}^{k_+,n-k}. \tag{15}$$

Equation (10) allows us to define $\Theta : \Omega \to \mathbb{R}$ such that

$$G(x) = (\eta v|x|^{-\alpha_+-k}) \circ \phi^{-1}(x) + \Theta(x) \text{ for any } x \in \Omega,$$

where ϕ is as in Definition 1. We then get that $\Theta \in D^{1,2}(\Omega)$ and

$$\Theta(x) = m_{\gamma,a}(\Omega)p(x)|x|^{-\alpha_- -k} + o(p(x)|x|^{-\alpha_- -k}) \text{ as } x \to 0. \tag{16}$$

Note that

$$\left\{\gamma > \gamma_H(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) - \frac{1}{4}\right\} \Leftrightarrow \{\alpha_+ - \alpha_- < 1\} \Leftrightarrow \{n < n_{\gamma,k}\}. \tag{17}$$

Since U satisfies (12), Theorem 3.2 yields $K_1 > 0$ such that

$$\lim_{\lambda \to 0^+} \lambda^{\alpha_-} U(\lambda x) = K_1 v(x)|x|^{-\alpha_- -k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+,n-k}. \tag{18}$$

The regularity applied to the Kelvin transform $x \mapsto \overline{U}(x) := |x|^{2-n}U(\frac{x}{|x|^2})$ yields

$$\lim_{\lambda \to +\infty} \lambda^{\alpha_+} U(\lambda x) = K_2 v(x)|x|^{-\alpha_+-k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+,n-k}, \tag{19}$$

for some $K_2 > 0$. Up to multiplying U by a positive constant, we assume that $K_2 = 1$. Equation (18), the Kelvin transform and Lemma 3.1 yield

$$|U(x)| \le C|x|^{-\alpha_+} \text{ and } |\nabla U(x)| \le C|x|^{-1-\alpha_+} \text{ for any } x \in \mathbb{R}^{k_+,n-k}. \tag{20}$$

For $\epsilon > 0$, we define

$$U_\epsilon(x) := \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}x) \text{ for all } x \in \mathbb{R}^{k_+,n-k} \tag{21}$$

and

$$u_\epsilon(x) := (\eta U_\epsilon) \circ \phi^{-1}(x) \text{ for } x \in \Omega \text{ and } \tilde{u}_\epsilon := u_\epsilon + \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\Theta. \tag{22}$$

The main result of this paper is the following:

Proposition 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 3$ such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $0 \le s < 2$, $\gamma < \gamma_H(\Omega)$ and $a \in C^{0,\theta}(\Omega)$ ($\theta \in (0,1)$). Assume that there exists a positive extremal U for $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$. We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k},$$

and that the operator $-\Delta - (\gamma|x|^{-2} + a(x))$ is coercive with the mass $m_{\gamma,a}(\Omega)$. We let $(\tilde{u}_\epsilon)_\epsilon \in D^{1,2}(\Omega)$ be as in (22). Then

$$J^\Omega_{\gamma,s,a}(\tilde{u}_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})\left(1 - \zeta^0_{\gamma,s} m_{\gamma,a}(\Omega)\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-})\right) \text{ as } \epsilon \to 0,$$

where

$$\zeta^0_{\gamma,s} := (\alpha_+ - \alpha_-)C_{k,n}\left(\xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^\star(s)}}{|x|^s}\, dx\right)^{-1} > 0, \tag{23}$$

where $C_{k,n}$ is defined in (25).

Remark: as noted in the introduction, the existence of extremals for $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ is a consequence of (4).

As one checks, Theorem 1.2 is a direct consequence of the combination of Proposition 4.1 and Theorem 3.1.

This section is devoted to the proof of Proposition 4.1.
Proof of Proposition 4.1: From the definitions of $\tilde{u}_\epsilon$ and G, and the uniform C^1-convergence in (19), it follows that

$$\lim_{\epsilon\to 0} \frac{\tilde{u}_\epsilon}{\epsilon^{\frac{\alpha_+-\alpha_-}{2}}} = G \text{ in } C^1_{loc}(\Omega)\cap D^{1,2}_{loc,0}(\Omega). \tag{24}$$

Define the constant

$$C_{k,n} := \int_{S^{n-1}\cap\mathbb{R}^{k_+,n-k}} \left(\prod_{i=1}^k x_i\right)^2 d\sigma. \tag{25}$$

In the sequel, ϑ^ϵ_ρ will denote a quantity such

$$\lim_{\rho\to 0}\lim_{\epsilon\to 0} \vartheta^\epsilon_\rho = 0.$$

For convenience, we define

$$N_{\gamma,a}(w) := |\nabla w|^2 - \left(\gamma|x|^{-2}+a\right)w^2.$$

Step 4.1. *For any $\rho>0$, we claim that*

$$\int_{\Omega\setminus\phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon)\,dx = \epsilon^{\alpha_+-\alpha_-}\left(\alpha_+ C_{k,n}\rho^{n-2\alpha_+-2} + m_{\gamma,a}(\Omega)(n-2)C_{k,n} + \vartheta^\epsilon_\rho\right),$$

as $\epsilon\to 0$ where the constant $C_{k,n}$ is defined in (25).

Proof of Step 4.1: Indeed, it follows from (24) that

$$\lim_{\epsilon\to 0}\epsilon^{-(\alpha_+-\alpha_-)}\int_{\Omega\setminus\phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon)\,dx = \int_{\Omega\setminus\phi(B_{\rho,+})} N_{\gamma,a}(G)\,dx.$$

Since G satisfies (9) and vanishes on $\partial\Omega\setminus\{0\}$, integrating by parts yields

$$\begin{aligned}\int_{\Omega\setminus\phi(B_{\rho,+})} N_{\gamma,a}(G)\,dx &= \int_{\Omega\setminus\phi(B_{\rho,+})} G\left(-\Delta G - (\gamma|x|^{-2}+a(x))G\right)\,dx\\ &- \int_{\phi(\partial(B_{\rho,+}))} G\partial_\nu G\,d\sigma\\ &= -\int_{(\partial B_\rho(0))\cap\mathbb{R}^{k_+,n-k}} (G\circ\phi)\partial_{\phi_*\nu}(G\circ\phi)\,d(\phi^*\sigma), \end{aligned}\tag{26}$$

where $\nu(x)$ is the outer normal vector of $B_\rho(0)$ at $x\in\partial B_\rho(0)$. We will now find the value of $(G\circ\phi)\partial_{\phi_*\nu}(G\circ\phi)$. Using (15) and the definition of G, we have that

$$(G\circ\phi)(x) = v(x)|x|^{-\alpha_+-k} + m_{\gamma,a}(\Omega)v(x)|x|^{-\alpha_--k} + o(v(x)|x|^{-\alpha_--k}) \text{ as } x\to 0. \tag{27}$$

From Θ and the uniform convergence in C^1 of G, we have for all $l=1,...,n$ that

$$\partial_l(\Theta\circ\phi) = \partial_l\left(m_{\gamma,a}(\Omega)v|x|^{-\alpha_--k}\right) + o(|x|^{-\alpha_--1}) \text{ as } x\to 0. \tag{28}$$

Moreover, it follows from the definition of G that

$$\begin{aligned}\partial_l(G\circ\phi) &= \partial_l v\left(|x|^{-\alpha_+-k}+m_{\gamma,a}(\Omega)|x|^{-\alpha_--k}\right)\\ &\quad -x_l v\left((\alpha_++k)|x|^{-\alpha_+-k-2}+(\alpha_-+k)m_{\gamma,a}(\Omega)|x|^{-\alpha_--k-2}\right)+o(|x|^{-\alpha_--1}).\end{aligned}$$

Since

$$\phi_*\nu(x)=\frac{x}{|x|}+O(|x|) \text{ as } x\to 0 \text{ and } \alpha_+<\alpha_-+1,$$

we obtain as $x\to 0$ that,

$$\partial_{\phi_*\nu}(G\circ\phi)=-v\left(\alpha_+|x|^{-\alpha_+-k-1}+m_{\gamma,a}(\Omega)\alpha_-|x|^{-\alpha_--k-1}\right)+o(|x|^{-\alpha_--1}). \tag{29}$$

We combine the equations (27), (29) and since $\alpha_++\alpha_-=n-2$, $-2\alpha_--1>1-n$, $\alpha_+-\alpha_-<1$, we get

$$-(G\circ\phi)\partial_{\phi_*\nu}(G\circ\phi)=v^2\left(\alpha_+|x|^{-2\alpha_+-2k-1}+m_{\gamma,a}(\Omega)(n-2)|x|^{-n+1-2k}\right)+o(|x|^{1-n}).$$

Moreover, using again the definition of v,

$$-\int_{\partial B_{\rho,+}}(G\circ\phi)\partial_{\phi_*\nu}(G\circ\phi)d(\phi^*\sigma) = \alpha_+C_{k,n}\rho^{n-2\alpha_+-2}+m_{\gamma,a}(\Omega)(n-2)C_{k,n}+\vartheta_\rho,$$

where $\lim_{\rho\to 0}\vartheta_\rho=0$ and $C_{k,n}$ is defined in (25). Plugging the last equation in (26) yields Step 4.1. □

Step 4.2. *We claim that, as $\epsilon\to 0$,*

$$\int_\Omega N_{\gamma,a}(\tilde u_\epsilon)\,dx=\xi\int_{\mathbb{R}^{k_+,n-k}}|x|^{-s}U^{2^\star(s)}\,dx+m_{\gamma,a}(\Omega)(n-2)C_{k,n}\epsilon^{\alpha_+-\alpha_-}+o(\epsilon^{\alpha_+-\alpha_-}).$$

Proof of Step 4.2: The definition (22) of $\tilde u_\epsilon$ rewrites

$$\tilde u_\epsilon\circ\phi(x)=U_\epsilon(x)+\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\Theta\circ\phi(x) \text{ for all } x\in B_{\delta,+}. \tag{30}$$

Fix $\rho\in]0,\delta[$ that we will eventually let go to 0. We define

$$I_{\epsilon,\rho}:=\int_{\phi(B_{\rho,+})}\left(|\nabla\tilde u_\epsilon|^2-\left(\gamma|x|^{-2}+a\right)\tilde u_\epsilon^2\right)\,dx.$$

Let ϕ^*Eucl be the pullback of the Euclidean metric. With (30), we get

$$\begin{aligned}I_{\epsilon,\rho} &= \int_{B_{\rho,+}}\left(|\nabla(\tilde u_\epsilon\circ\phi)|^2_{\phi^*Eucl}-\left(\frac{\gamma}{|\phi(x)|^2}+a\circ\phi\right)(\tilde u_\epsilon\circ\phi)^2\right)|Jac(\phi)|\,dx\\ &= \int_{B_{\rho,+}}\left(|\nabla U_\epsilon|^2_{\phi^*Eucl}-\left(\frac{\gamma}{|\phi(x)|^2}+a\circ\phi\right)U_\epsilon^2\right)|Jac(\phi)|\,dx\\ &+ 2\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_{\rho,+}}\left(\langle\nabla U_\epsilon,\nabla(\Theta\circ\phi)\rangle_{\phi^*Eucl}-\left(\frac{\gamma}{|\phi(x)|^2}+a\circ\phi\right)(\Theta\circ\phi)U_\epsilon\right)|Jac(\phi)|\,dx\\ &+ \epsilon^{\alpha_+-\alpha_-}\int_{B_{\rho,+}}\left(|\nabla(\Theta\circ\phi)|^2_{\phi^*Eucl}-\left(\frac{\gamma}{|\phi(x)|^2}+a\circ\phi\right)(\Theta\circ\phi)^2\right)|Jac(\phi)|\,dx.\end{aligned}$$

Since $d\phi_0 = Id_{\mathbb{R}^n}$, $\phi^* Eucl = Eucl + O(|x|)$ and $\Theta \in D^{1,2}(\Omega)$, we get that

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} \left(|\nabla U_\epsilon|^2_{Eucl} - \left(\frac{\gamma}{|x|^2} + a \circ \phi \right) U_\epsilon^2 \right) dx \\ &+ O\left(\int_{B_{\rho,+}} |x| \left(|\nabla U_\epsilon|^2_{Eucl} + |x|^{-2} U_\epsilon^2 \right) dx \right) \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho,+}} \left(\langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \left(\frac{\gamma}{|x|^2} + a \circ \phi \right) (\Theta \circ \phi) U_\epsilon \right) dx \\ &+ O\left(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho,+}} |x| \left(|\nabla U_\epsilon| \cdot |\nabla(\Theta \circ \phi)| + |x|^{-2} (\Theta \circ \phi) U_\epsilon \right) dx \right) + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \end{aligned}$$

as $\epsilon \to 0$. The explicit expression (21) of U_ϵ, (20) and $n > 2\alpha_+$ yield

$$\begin{aligned} \int_{B_{\rho,+}} U_\epsilon^2 \, dx &= O\left(\epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r^{n-2\alpha_+ - 1} \, dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned} \tag{31}$$

The definition of Θ and $\alpha_+ + \alpha_- = n - 2$ give

$$\begin{aligned} \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho,+}} a \circ \phi (\Theta \circ \phi) U_\epsilon \, dx &= O\left(\epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r \, dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned} \tag{32}$$

We combine the equations (18), (20), (28), (31) and (32),

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} \left(|\nabla U_\epsilon|^2_{Eucl} - \frac{\gamma}{|x|^2} U_\epsilon^2 \right) dx \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho,+}} \left(\langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \frac{\gamma}{|x|^2} (\Theta \circ \phi) U_\epsilon \right) dx + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \text{ as } \epsilon \to 0. \end{aligned}$$

Integrating by parts and since both U_ϵ and $\Theta \circ \phi$ vanish on $\partial \mathbb{R}^{k_+, n-k} \backslash \{0\}$, we get as $\epsilon \to 0$ that

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} U_\epsilon \left(-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon \right) dx + \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon \, d\sigma \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left(\int_{B_{\rho,+}} (\Theta \circ \phi) \left(-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon \right) dx \right. \\ &+ \left. \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon \, d\sigma \right) + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned} \tag{33}$$

We claim as $\epsilon \to 0$ that

$$\int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon \, d\sigma = -\alpha_+ \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} m_{\gamma,a}(\Omega) C_{k,n} + o(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}}), \tag{34}$$

and

$$\int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon \, d\sigma = -\alpha_+ C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} + o(\epsilon^{\alpha_+ - \alpha_-} \rho^{n-2-2\alpha_+}). \tag{35}$$

We prove the claim. It follows from the uniform C^1-convergence in (19) that we have for all $l = 1, ..., n$

$$\lim_{\lambda\to+\infty} \lambda^{\alpha_+}\partial_l U(\lambda x) = |x|^{-\alpha_+-k}\left(\delta_{l\leq k}\prod_{j=1;j\neq l}^{k} x_j - (\alpha_+ + k)\frac{v(x)x_l}{|x|^2}\right), \tag{36}$$

where $\delta_{l\leq k}$ is such that $\delta_{l\leq k} = 1$ if $l \leq k$, and $\delta_{l\leq k} = 0$ otherwise, and v is defined in (15). The definition of U_ϵ and (20) yield

$$\partial_l U_\epsilon = \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(|x|^{-\alpha_+-k}\left(\delta_{l\leq k}\prod_{j=1;j\neq l}^{k} x_j - (\alpha_+ + k)\frac{x_l}{|x|^2}v\right) + o(|x|^{-\alpha_+-1})\right).$$

Since $\nu(x) = |x|^{-1}x$ is the outer normal vector of $B_\rho(0)$, we then get

$$\partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(-\alpha_+ v|x|^{-\alpha_+-k-1} + o(|x|^{-\alpha_+-1})\right), \tag{37}$$

as $\epsilon \to 0$ uniformly on compact subsets of $\overline{\mathbb{R}^{k_+,n-k}}\backslash\{0\}$. From the definition of Θ and $\alpha_+ + \alpha_- = n-2$, and (17), we obtain as $\epsilon \to 0$ that

$$(\Theta\circ\phi)\partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(-\alpha_+ m_{\gamma,a}(\Omega)v^2|x|^{-n+1-2k} + o(|x|^{1-n})\right).$$

Therefore, we get (34). The definition of U_ϵ and the equations (19) and (37) yield

$$U_\epsilon\partial_\nu U_\epsilon = -\alpha_+\epsilon^{\alpha_+-\alpha_-}v^2|x|^{-2\alpha_+-2k-1} + o(\epsilon^{\alpha_+-\alpha_-}|x|^{-2\alpha_+-1}),$$

as $\epsilon \to 0$ uniformly locally in $\overline{\mathbb{R}^{k_+,n-k}}\backslash\{0\}$. This yields (35) and proves the claim.

We combine equations (33), (34) and (35) to get

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} U_\epsilon\left(-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon\right)dx - \alpha_+ C_{k,n}\epsilon^{\alpha_+-\alpha_-}\rho^{n-2\alpha_+-2} \\ &+ 2\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_{\rho,+}}(\Theta\circ\phi)\left(-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon\right)dx \\ &- 2\alpha_+\epsilon^{\alpha_+-\alpha_-}m_{\gamma,a}(\Omega)C_{k,n} + \epsilon^{\alpha_+-\alpha_-}\vartheta^\epsilon_\rho. \end{aligned}$$

Since U satisfies equation (12), with the definition (21) of U_ϵ, we get

$$-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon = \xi|x|^{-s}U_\epsilon^{2^\star(s)-1}.$$

Therefore, we get as $\epsilon \to 0$ that

$$\begin{aligned} I_{\epsilon,\rho} &= \xi\int_{B_{\rho,+}}\frac{U_\epsilon^{2^\star(s)}}{|x|^s}dx - \alpha_+ C_{k,n}\epsilon^{\alpha_+-\alpha_-}\rho^{n-2\alpha_+-2} \\ &+2\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\xi\int_{B_{\rho,+}}(\Theta\circ\phi)\frac{U_\epsilon^{2^\star(s)-1}}{|x|^s}dx \\ &-2\alpha_+\epsilon^{\alpha_+-\alpha_-}m_{\gamma,a}(\Omega)C_{k,n} + \epsilon^{\alpha_+-\alpha_-}\vartheta^\epsilon_\rho. \end{aligned} \tag{38}$$

It follow from the definition (21) of U_ϵ and the first estimate in (20) that

$$\left|\xi\int_{\mathbb{R}^{k_+,n-k}\backslash(B_{\rho,+})}\frac{U_\epsilon^{2^\star(s)}}{|x|^s}dx\right| \leq C\epsilon^{\frac{2^\star(s)}{2}(\alpha_+-\alpha_-)}.$$

Therefore, with $2^\star(s) > 2$, we get

$$\xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^\star(s)}}{|x|^s}\,dx = \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^\star(s)}}{|x|^s}\,dx + o(\epsilon^{\alpha_+-\alpha_-}) \text{ as } \epsilon \to 0. \tag{39}$$

The definition (15), (21) and the first control in (20) yield

$$\begin{aligned}\int_{\mathbb{R}^{k_+,n-k}\setminus(B_{\rho,+})} v\frac{U_\epsilon^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx &= O\left(\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{\epsilon^{-1}\rho}^{+\infty} r^{(1-\frac{2^\star(s)}{2})(\alpha_+-\alpha_-)-1}\,dr\right)\\ &= \epsilon^{\frac{2^\star(s)-1}{2}(\alpha_+-\alpha_-)}\vartheta_\rho^\epsilon.\end{aligned}$$

Therefore, with the definition of Θ, we get as $\epsilon \to 0$ that

$$\begin{aligned}\xi\int_{B_{\rho,+}} \frac{U_\epsilon^{2^\star(s)-1}}{|x|^s}\Theta\circ\phi\,dx &= \xi m_{\gamma,a}(\Omega)\int_{B_{\rho,+}} v\frac{U_\epsilon^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx\\ &+ o\left(\int_{B_{\rho,+}} v\frac{U_\epsilon^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx\right)\\ &= \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(m_{\gamma,a}(\Omega)\xi\int_{\mathbb{R}^{k_+,n-k}} v\frac{U^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx + \vartheta_\epsilon^\rho\right).\end{aligned} \tag{40}$$

Since $(-\Delta-\gamma|x|^{-2})\left(v|x|^{-\alpha_--k}\right) = 0$ and U vanishes on $\partial\mathbb{R}^{k_+,n-k}\setminus\{0\}$, integrating by parts, we get that

$$\begin{aligned}\xi\int_{\mathbb{R}^{k_+,n-k}} v\frac{U^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx &= \lim_{R\to+\infty}\int_{B_{R,+}} v|x|^{-\alpha_--k}\left(-\Delta U-\gamma|x|^{-2}U\right)\,dx\\ &= \lim_{R\to+\infty}\left[\int_{B_{R,+}} U\left(-\Delta-\gamma|x|^{-2}\right)\left(v|x|^{-\alpha_--k}\right)\,dx\right.\\ &- \left.\int_{\mathbb{R}^{k_+,n-k}\cap\partial B_R}\partial_\nu U v|x|^{-\alpha_--k}\,d\sigma\right].\end{aligned} \tag{41}$$

Arguing as for (37), it follows from (36) that, as $R \to +\infty$

$$\partial_\nu U = -\alpha_+ v|x|^{-\alpha_+-k-1} + o(|x|^{-\alpha_+-1}) \text{ uniformly for } x \in \partial B_R(0)\cap\mathbb{R}^{k_+,n-k}.$$

Moreover, since $\alpha_+ + \alpha_- = n-2$ we get

$$\partial_\nu U v|x|^{-\alpha_--k} = -\alpha_+ v^2|x|^{-(n+2k-1)} + o(|x|^{1-n}).$$

This latest equation yields

$$\lim_{R\to+\infty}\int_{\mathbb{R}^{k_+,n-k}\cap\partial B_R(0)}\partial_\nu U v|x|^{-\alpha_--k}\,d\sigma = -\alpha_+ C_{k,n}.$$

Then, by (41)

$$\xi\int_{\mathbb{R}^{k_+,n-k}} v\frac{U^{2^\star(s)-1}}{|x|^{s+\alpha_-+k}}\,dx = \alpha_+ C_{k,n}. \tag{42}$$

Combining (40) and (42), we get

$$\xi\int_{B_{\rho,+}} \frac{U_\epsilon^{2^\star(s)-1}}{|x|^s}\Theta\circ\phi\,dx = \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(\alpha_+ m_{\gamma,a}(\Omega)C_{k,n} + \vartheta_\epsilon^\rho\right) \text{ as } \epsilon\to 0. \tag{43}$$

Next, the equations (38), (39) and (43) yield

$$I_{\epsilon,\rho} = \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U_\epsilon^{2^\star(s)}}{|x|^s}\,dx - \alpha_+ C_{k,n}\epsilon^{\alpha_+-\alpha_-}\rho^{n-2\alpha_+-2} + o(\epsilon^{\alpha_+-\alpha_-}).$$

On the other hand, using Step 4.1 the definition of $I_{\epsilon,\rho}$ and the last equation, we get Step 4.2 . □

Step 4.3. *We claim as $\epsilon \to 0$ that,*

$$\int_\Omega \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx = \int_{\mathbb{R}^{k_+,n-k}} \frac{U_\epsilon^{2^\star(s)}}{|x|^s}\,dx + 2^\star(s)\alpha_+ m_{\gamma,a}(\Omega)\xi^{-1}C_{k,n}\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-}).$$

Proof of Step 4.3: We fix $\rho > 0$. The definitions of $\tilde{u}_\epsilon$ and Θ, and $2^\star(s) > 2$ yield

$$\int_{\phi(B_{2\delta,+}\setminus B_{\rho,+})} \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx = o(\epsilon^{\alpha_+-\alpha_-}), \tag{44}$$

as $\epsilon \to 0$. Equations (16), (18), (30) and (44) yield

$$\int_\Omega \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx = \int_{B_{\rho,+}} \frac{\left|U_\epsilon + \epsilon^{\frac{\alpha_+-\alpha_-}{2}}(\Theta\circ\phi)\right|^{2^\star(s)}}{|x|^s}|(1+O(|x|)|\,dx + o(\epsilon^{\alpha_+-\alpha_-}),$$

as $\epsilon \to 0$, and

$$\int_\Omega \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx = \int_{B_{\rho,+}} \left(\frac{U_\epsilon^{2^\star(s)}}{|x|^s} + 2^\star(s)\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\frac{U_\epsilon^{2^\star(s)-1}}{|x|^s}(\Theta\circ\phi)\right)dx$$
$$+ \int_{B_{\rho,+}} O\left(\epsilon^{\alpha_+-\alpha_-}\frac{U_\epsilon^{2^\star(s)-2}}{|x|^s}(\Theta\circ\phi)^2 + \epsilon^{\frac{2^\star(s)}{2}(\alpha_+-\alpha_-)}|\Theta\circ\phi|^{2^\star(s)}\right)dx + o(\epsilon^{\alpha_+-\alpha_-}).$$

Using the asymptotics (16) and (18) of Θ and U, we get that

$$\begin{aligned}\int_{B_{\rho,+}} \frac{U_\epsilon^{2^\star(s)-2}}{|x|^s}(\epsilon^{\frac{\alpha_+-\alpha_-}{2}}(\Theta\circ\phi))^2\,dx &= O\left(\epsilon^{2(\alpha_+-\alpha_-)}\int_0^{\epsilon^{-1}\rho} r^{\frac{2^\star(s)}{2}(\alpha_+-\alpha_-)-1}\,dr\right)\\ &= \epsilon^{\alpha_+-\alpha_-}\vartheta_\epsilon^\rho,\end{aligned} \tag{45}$$

and, from the definition of Θ and the control (16), we get that

$$\begin{aligned}\int_{B_{\rho,+}} (\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\Theta\circ\phi)^{2^\star(s)}|x|^{-s}\,dx &= O\left(\epsilon^{(\alpha_+-\alpha_-)\frac{2^\star(s)}{2}}\int_0^\rho r^{\frac{2^\star(s)}{2}(\alpha_+-\alpha_-)-1}\,dr\right)\\ &= \epsilon^{\alpha_+-\alpha_-}\vartheta_\epsilon^\rho.\end{aligned} \tag{46}$$

The equations (45) and (46) yield as $\epsilon \to 0$ that

$$\int_\Omega \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx = \int_{B_{\rho,+}} \left(\frac{U_\epsilon^{2^\star(s)}}{|x|^s} + 2^\star(s)\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\frac{U_\epsilon^{2^\star(s)-1}}{|x|^s}(\Theta\circ\phi)\right)dx + \epsilon^{\alpha_+-\alpha_-}\vartheta_\epsilon^\rho. \tag{47}$$

Therefore, for all $\xi > 0$ the equations (39), (43) and (47) yield the result. □

Step 4.4. *We are now in a position to prove Proposition 4.1.*

Proof of Step 4.4: By Step 4.3, we have that

$$\left(\int_\Omega \frac{\tilde{u}_\epsilon^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} = \left(\int_{\mathbb{R}^{k_+,n-k}} \frac{U_\epsilon^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}$$

$$+ 2\alpha_+ m_{\gamma,a}(\Omega)\xi^{-1}C_{k,n}\epsilon^{\alpha_+-\alpha_-}\left(\int_{\mathbb{R}^{k_+,n-k}} \frac{U_\epsilon^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}-1} \tag{48}$$

$$+ o(\epsilon^{\alpha_+-\alpha_-}).$$

We go back to the definition of $J^\Omega_{\gamma,a,s}$. Step 4.3, Equation (48) and (12) yield

$$J^\Omega_{\gamma,s,a}(\tilde{u}_\epsilon) = J^{\mathbb{R}^{k_+,n-k}}_{\gamma,s,0}(U)\left(1 - m_{\gamma,a}(\Omega)\zeta^0_{\gamma,s}\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-})\right),$$

as $\epsilon \to 0$, where $\zeta^0_{\gamma,s}$ is defined in (23). This ends the proof of Proposition 4.1. □

Combining Proposition 4.1 and Theorem 3.1 yields Theorem 1.2 .

5. Examples of mass

In this section, we discuss the existence and the sign of the mass. An example of existence of mass is as follows:

Proposition 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We assume that $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$ and that*

$$\Omega \cap B_\delta(0) = \mathbb{R}^{k_+,n-k} \cap B_\delta(0) \text{ for some } \delta > 0. \tag{49}$$

We assume that $\gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} < \gamma < \gamma_H(\Omega)$, that $a \in C^{0,\theta}(\Omega)$ vanishes around 0 and that $-\Delta - (\gamma|x|^{-2} + a(x))$ is coercive. Then the mass is defined.

Proof of Proposition 5.1. We fix η as in (14). For $a \in C^{0,\theta}(\Omega)$ that vanishes around 0, define on Ω the function

$$g := \left(-\Delta - \frac{\gamma}{|x|^2} - a(x)\right)\left(\eta S_{\alpha_+}\right),$$

where S_{α_+} is defined in (8) such that $-\Delta S_{\alpha_+} - \gamma|x|^{-2}S_{\alpha_+} = 0$ on $\mathbb{R}^{k_+,n-k}$. Note that this definition makes sense when the support of η is small enough due to (49) and a vanishes around 0. In particular $g(x) = 0$ around 0. Therefore, we have that $g \in L^{\frac{2n}{n+2}}(\Omega) = \left(L^{2^\star}(\Omega)\right)' \subset \left(D^{1,2}(\Omega)\right)'$. Since the operator $-\Delta - (\gamma|x|^{-2} + a)$ is coercive, there exists $w \in D^{1,2}(\Omega)$ such that

$$\begin{cases} \left(-\Delta - \frac{\gamma}{|x|^2} - a(x)\right)w = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since g vanishes around 0, Theorem 3.2 yields the existence of $K \in \mathbb{R}$ such that

$$w(x) = K\frac{v(x)}{|x|^{\alpha_-+k}} + o\left(\frac{v(x)}{|x|^{\alpha_-+k}}\right) \text{ as } x \to 0,$$

where v is as in (15). For all $x \in \overline{\Omega}\backslash\{0\}$, we define the function $G_0 := \eta S_{\alpha_+} - w$. The definition of w yields

$$\begin{cases} \left(-\Delta - \frac{\gamma}{|x|^2} - a(x)\right)G_0 = 0 & \text{in } \Omega, \\ G_0 = 0 & \text{on } \partial\Omega\backslash\{0\}. \end{cases}$$

For $\delta_0 > 0$ small enough, the definitions of S_{α_+}, w and $\alpha_- < \alpha_+$ yield

$$G_0(x) = v(x)|x|^{-\alpha_+ - k}\,(1 + o(1)) \text{ in } \mathbb{R}^{k_+,n-k} \cap B_{\delta_0},$$

with $o(1) \to 0$ as $x \to 0$. Therefore, $G_0 > 0$ in $\mathbb{R}^{k_+,n-k} \cap B_{\delta_0}$. Then coercivity and the comparison principle yield $G_0 > 0$ in Ω. Moreover, we have that

$$G_0(x) = v(x)\left(|x|^{-\alpha_+ - k} - K|x|^{-\alpha_- - k} + o(|x|^{-\alpha_- - k})\right),$$

as $x \to 0$. Then the mass at 0 of $-\Delta - (\gamma|x|^{-2} + a(x))$ is defined and $m_{\gamma,a}(\Omega) = -K$. This proves Proposition 5.1.

We now discuss briefly examples of negative and positive mass. Here, the reference is Section 9 of Ghoussoub-Robert [8]. We still assume (49) and that $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$, so that the mass $m_{\gamma,0}(\Omega)$ is defined. When $\Omega \subset \mathbb{R}^{k_+,n-k}$, due to the comparison principle, we get that $G_0 < S_{\alpha_+}$, and $m_{\gamma,0}(\Omega) < 0$. Arguing as in [8], we are able to define the mass of a domain $\tilde{\Omega} \supset \mathbb{R}^{k_+,n-k}$, for which $m_{\gamma,0}(\tilde{\Omega}) > 0$: then, defining $\tilde{\Omega}_R := \tilde{\Omega} \cap B_R(0)$, we get that $\lim_{R\to+\infty} m_{\gamma,0}(\tilde{\Omega}_R) = m_{\gamma,0}(\tilde{\Omega}) > 0$. So for $R > 0$ large, we get examples of bounded domains with a singularity of type $(k, n-k)$ at 0 and with positive mass.

6. Proof of Theorem 1.3: functional background for the perturbed equation

In this section, we proceed as in Jaber [11]. For any function $G \in C^1(E, \mathbb{R})$ where $(E, \|.\|)$ is a Banach space, we say that $(u_m)_{m\in\mathbb{N}} \in E$ is a Palais-Smale sequence of G if there exists $\beta \in \mathbb{R}$ such that

$$G(u_m) \to \beta \text{ and } G'(u_m) \to 0 \text{ in } E' \text{ as } m \to +\infty.$$

Here, we say that the Palais-Smale sequence is at level β. The main tool is the Mountain-Pass Lemma of Ambrosetti-Rabinowitz [1]:

Theorem 6.1 (Mountain-Pass Lemma [1])**.** *Consider $G \in C^1(E, \mathbb{R})$ where $(E, \|.\|)$ is a Banach space. We assume that $G(0) = 0$ and that*

- *There exists $\lambda, r > 0$ such that $G(u) \geq \lambda$ for all $u \in E$ such that $\|u\| = r$,*
- *There exists u_0 in E such that $\limsup_{t\to+\infty} G(tu_0) < 0$.*

We consider $t_0 > 0$ sufficiently large such that $\|t_0u_0\| > r$ and $G(t_0u_0) < 0$, and

$$\beta = \inf_{c\in\Gamma} \sup_{t\in[0,1]} G(c(t)),$$

where

$$\Gamma = \{c : [0,1] \to E \text{ s.t. } c(0) = 0,\, c(1) = t_0u_0\}.$$

Then, there exists a Palais-Smale sequence at level β for G. Moreover, we have that $\beta \leq \sup_{t\geq 0} G(tu_0)$.

Weak solutions to (6) are to the nonzero critical points of the functional

$$E_q(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \left(\frac{\gamma}{|x|^2} + a\right)u^2\right)dx - \int_\Omega \frac{u_+^{2^\star(s)}}{2^\star(s)|x|^s}\,dx - \int_\Omega \frac{hu_+^{q+1}}{q+1}\,dx,$$

for any $u \in D^{1,2}(\Omega)$ and where $u_+ = \max\{u, 0\}$. In the sequel, we assume that the operator $-\Delta - \left(\frac{\gamma}{|x|^2} + a(x)\right)$ is coercive, so that there exists $c_0 > 0$ such that

$$\int_\Omega \left(|\nabla w|^2 - \left(\frac{\gamma}{|x|^2} + a\right)w^2\right)dx \geq c_0 \int_\Omega |\nabla w|^2\,dx \text{ for all } w \in D^{1,2}(\Omega). \tag{50}$$

Proposition 6.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $a, h \in C^{0,\theta}(\Omega)$, $\theta \in (0,1)$. We assume that $h \geq 0$ and that (50) holds. Fix $u_0 \in D^{1,2}(\Omega)$ such that $u_0 \geq 0$, $u_0 \not\equiv 0$, and $q \in (1, 2^\star - 1)$. Then there exists a sequence $(u_m)_{m\in\mathbb{N}} \in D^{1,2}(\Omega)$ that is a Palais-Smale sequence for E_q at level β such that $0 < \beta \leq \sup_{t\geq 0} E_q(tu_0)$.*

Proof of Proposition 6.1: Clearly $E_q \in C^1(D^{1,2}(\Omega))$. Note that $E_q(0) = 0$. It follows from (50) and the Sobolev and Hardy-Sobolev embeddings that there exist $c_0, c_1, c_2 > 0$ such that

$$E_q(u) \geq c_0\|u\|^2 - c_1\|u\|^{2^\star(s)} - c_2\|u\|^{q+1} \text{ for all } u \in D^{1,2}(\Omega). \tag{51}$$

Define $f(r) = r^2\left[c_0 - c_1 r^{2^\star(s)-2} - c_2 r^{q-1}\right] := r^2 g(r)$ and since $2^\star(s), q+1 > 2$ we have $g(r) \to c_0$ as $r \to 0$. Then there exists $r_0 > 0$ such that $r < r_0$, we have $g(r) > \frac{c_0}{2}$. Therefore, for all $u \in D^{1,2}(\Omega)$ such that $\|u\| = \frac{r_0}{2}$ and by (51), we have $E_q(u) \geq \frac{c_0 r_0^2}{8} := \lambda$. We fix $u_0 \in D^{1,2}(\Omega)$, $u_0 \not\equiv 0$. We have that

$$\begin{aligned} E_q(tu_0) &= \frac{t^2}{2}\int_\Omega \left(|\nabla u_0|^2 - (\frac{\gamma}{|x|^2} + a)u_0^2\right) dx \\ &- \frac{t^{2^\star(s)}}{2^\star(s)}\int_\Omega \frac{|u_0|^{2^\star(s)}}{|x|^s}dx - \frac{t^{q+1}}{q+1}\int_\Omega h|u_0|^{q+1}dx \\ &:= \frac{t^2}{2}R_1 - \frac{t^{2^\star(s)}}{2^\star(s)}R_2 - \frac{t^{q+1}}{q+1}R_3 \leq t^{2^\star(s)}\left(\frac{t^{2-2^\star(s)}}{2}R_1 - R_2\right), \end{aligned}$$

where $R_1, R_2 > 0$ and $R_3 \geq 0$. Since $2^\star(s) > 2$, we have $E_q(tu_0) \to -\infty$ as $t \to +\infty$. Then $\limsup_{t\to+\infty} E_q(tu_0) < 0$. We consider $t_0 > 0$ large such that $\|t_0 u_0\| > r$ and $E_q(t_0 u_0) < 0$. For $t \in [0,1]$, we have $E_q(c(t)) \geq \lambda$ and then there exists

$$\beta := \inf_{c\in\Gamma}\sup E_q(c(t)) \geq \lambda > 0.$$

Proposition 6.1 then follows from Theorem 6.1. □

Proposition 6.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1, ..., n\}$. We fix $a, h \in C^{0,\theta}(\Omega)$, $\theta \in (0,1)$. We assume that $h \geq 0$ and that (50) holds. We fix $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ and $\beta \in \mathbb{R}$ such that*

$$\beta < \frac{2-s}{2(n-s)}\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{n-s}{2-s}}. \tag{52}$$

Then, for any Palais-Smale sequence $(u_m)_{m\in\mathbb{N}} \in D^{1,2}(\Omega)$ for E_q at level β, there exists $u \in D^{1,2}(\Omega)$ such that $E_q(u) = \beta$ and we have that (u_m) converges strongly in $D^{1,2}(\Omega)$ as $m \to +\infty$ up to a subsequence. Moreover, we have that $E_q'(u) = 0$.

Proof of Proposition 6.2: Let $(u_m)_{m\in\mathbb{N}} \in D^{1,2}(\Omega)$ be a Palais-Smale sequence for E_q such that

$$E_q(u_m) \to \beta \text{ and } E_q'(u_m) \to 0 \text{ in } D^{1,2}(\Omega)'.$$

Step 6.1. *We claim that $(u_m)_m$ is bounded in $D^{1,2}(\Omega)$.*

Proof of Step 6.1: The coercivity (50) and the definition of E_q yield

$$\|u_m\|^2 \leq 2c_0^{-1}\left(E_q(u_m) + \frac{1}{2^\star(s)}\int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\,dx + \frac{1}{q+1}\int_\Omega h(u_m)_+^{q+1}\,dx\right). \tag{53}$$

Since $E_q'(u_m) \to 0$ in $D^{1,2}(\Omega)'$, we observe that

$$\int_\Omega \left(|\nabla u_m|^2 - \left(\frac{\gamma}{|x|^2} + a \right) u_m^2 \right) dx = \int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\, dx + \int_\Omega h (u_m)_+^{q+1}\, dx + o(\|u_m\|).$$

The definition of the energy E_q and the last equation yield

(54)
$$2E_q(u_m) = \left(1 - \frac{2}{2^\star(s)}\right) \int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\, dx + \left(1 - \frac{2}{q+1}\right) \int_\Omega h (u_m)_+^{q+1}\, dx + o(\|u_m\|).$$

Moreover, since $E_q(u_m) \to \beta$ as $m \to +\infty$, $h \geq 0$ and $q+1 > 2$, we obtain that

$$\begin{aligned} \left(1 - \frac{2}{2^\star(s)}\right) \int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\, dx &= 2E_q(u_m) - \left(1 - \frac{2}{q+1}\right) \int_\Omega h (u_m)_+^{q+1}\, dx + o(\|u_m\|) \\ &\leq 2\beta + o(\|u_m\|), \end{aligned}$$

therefore,

$$\left(1 - \frac{2}{2^\star(s)}\right) \int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\, dx = O(1) + o(\|u_m\|). \tag{55}$$

Similarly, we have that

$$\left(1 - \frac{2}{q+1}\right) \int_\Omega h (u_m)_+^{q+1}\, dx = O(1) + o(\|u_m\|). \tag{56}$$

Relations (53) and (54) give

$$\|u_m\|^2 \leq c_0^{-1} \left(\int_\Omega \frac{(u_m)_+^{2^\star(s)}}{|x|^s}\, dx + \int_\Omega h (u_m)_+^{q+1}\, dx \right) + o(\|u_m\|). \tag{57}$$

The equations (55), (56) and (57) yield,

$$\|u_m\|^2 = O(1) + o(\|u_m\|),$$

as $m \to +\infty$. This proves Step 6.1. □

Therefore, up to a subsequence, there exists $u \in D^{1,2}(\Omega)$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } D^{1,2}(\Omega), \\ u_m \to u & \text{strongly in } L^p(\Omega) \text{ for all } 1 < p < 2^\star. \end{cases} \tag{58}$$

Moreover, we have $E_q'(u) = 0$.

Step 6.2. *We claim that, as $m \to +\infty$*

$$\int_\Omega \left(|\nabla (u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx = \int_\Omega \frac{(u_m - u)_+^{2^\star(s)}}{|x|^s}\, dx + o(1), \tag{59}$$

and,

$$\frac{2-s}{2(n-s)} \int_\Omega \left(|\nabla (u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx \leq \beta + o(1). \tag{60}$$

Proof of Step 6.2: We have that

$$\begin{aligned} \langle E_q'(u_m), \varphi \rangle = &\int_\Omega \left((\nabla u_m, \nabla \varphi) - \left(\frac{\gamma}{|x|^2} + a \right) u_m \varphi \right) dx \\ &- \int_\Omega \frac{(u_m)_+^{2^\star(s)-1}}{|x|^s} \varphi\, dx - \int_\Omega h (u_m)_+^q \varphi\, dx, \end{aligned}$$

for all $\varphi \in D^{1,2}(\Omega)$. We observe that

$$\begin{aligned} o(1) &= \langle E_q'(u_m) - E_q'(u), u_m - u\rangle \\ &= \int_\Omega \left(|\nabla(u_m-u)|^2 - (\frac{\gamma}{|x|^2}+a)(u_m-u)^2 \right) dx \\ &- \int_\Omega \left((u_m)_+^{2^\star(s)-1} - u_+^{2^\star(s)-1} \right) \frac{(u_m-u)}{|x|^s}\, dx \\ &- \int_\Omega h\left((u_m)_+^q - u_+^q \right)(u_m-u)\, dx. \end{aligned} \tag{61}$$

Since $u_m \rightharpoonup u$ weakly in $D^{1,2}(\Omega)$ as $m \to \infty$, integration theory yields

$$\lim_{m\to+\infty} \int_\Omega \frac{(u_m)_+^{2^\star(s)-1}}{|x|^s} u\, dx = \int_\Omega \frac{u_+^{2^\star(s)}}{|x|^s}\, dx = \lim_{m\to+\infty} \int_\Omega \frac{u_+^{2^\star(s)-1}}{|x|^s} u_m\, dx. \tag{62}$$

Equation (58) yields

$$\int_\Omega h(u_m-u)\left((u_m)_+^q - u_+^q \right) dx = \int_\Omega h(u_m-u)^{q+1}\, dx + o(1) = o(1), \tag{63}$$

as $m \to +\infty$. Combining (61), (62) and (63), we get as $m \to +\infty$ that

$$\int_\Omega \left(|\nabla(u_m-u)|^2 - \left(\frac{\gamma}{|x|^2}+a\right)(u_m-u)^2 \right) dx = \int_\Omega \left((u_m)_+^{2^\star(s)} - u_+^{2^\star(s)} \right) \frac{dx}{|x|^s} + o(1). \tag{64}$$

Since $2^\star(s) > 1$, we get that

$$\left| (u_m)_+^{2^\star(s)} - u_+^{2^\star(s)} - (u_m-u)_+^{2^\star(s)} \right| \le C\left(|u_m-u|^{2^\star(s)-1}|u| + |u|^{2^\star(s)-1}|u_m-m| \right),$$

for some $C > 0$ independent of m. Therefore, with (58), we have that

$$\int_\Omega \left((u_m)_+^{2^\star(s)} - (u_m-u)_+^{2^\star(s)} \right) \frac{dx}{|x|^s} = \int_\Omega \frac{u_+^{2^\star(s)}}{|x|^s} dx + o(1). \tag{65}$$

Since $u_m \to u$ strongly in $L^2(\Omega)$ as $m \to +\infty$ and by (64), (65), we obtain (59). With (58) we have that

$$\begin{aligned} E_q(u_m) - E_q(u) &= \frac{1}{2}\int_\Omega \left(|\nabla(u_m-u)|^2 - \gamma\frac{(u_m-u)^2}{|x|^2} \right) dx \\ &\quad -\frac{1}{2^\star(s)} \int_\Omega ((u_m)_+^{2^\star(s)} - u_+^{2^\star(s)}) \frac{dx}{|x|^s} + o(1). \end{aligned}$$

With (59), we get

$$E_q(u_m) - E_q(u) = \left(\frac{1}{2} - \frac{1}{2^\star(s)} \right) \int_\Omega \left(|\nabla(u_m-u)|^2 - \gamma\frac{(u_m-u)^2}{|x|^2} \right) dx + o(1).$$

Since u is a solution to (6) then $E_q(u) \ge 0$. Moreover $E_q(u_m) \to \beta$ as $m \to +\infty$. Then we then get (60). This proves Step 6.2. □

Step 6.3. *We claim that*

$$\lim_{m\to+\infty} u_m = u \text{ in } D^{1,2}(\Omega). \tag{66}$$

Proof of Step 6.3: Since $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ for all $k \in \{1,...,n\}$ and by the Proposition 2.1 in Cheikh-Ali [4], then for all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that for all $v \in D^{1,2}(\Omega)$,

$$\left(\int_\Omega \frac{|v|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \leq \left(\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{-1}+\epsilon\right)\int_\Omega \left(|\nabla v|^2 - \frac{\gamma}{|x|^2}v^2\right)dx + c_\epsilon\int_\Omega v^2\,dx.$$

Take $\theta_m = u_m - u$. Since (u_m) converges strongly to u in $L^2(\Omega)$, taking $v = \theta_m$ yields

(67)
$$\left(\int_\Omega \frac{(\theta_m)_+^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \leq \left(\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{-1}+\epsilon\right)\int_\Omega \left(|\nabla \theta_m|^2 - \frac{\gamma}{|x|^2}\theta_m^2\right)dx + o(1).$$

We write $N(\theta_m) := \int_\Omega \left(|\nabla\theta_m|^2 - \frac{\gamma}{|x|^2}\theta_m^2\right)dx$. By (59) and (67), we get that

$$N(\theta_m)^{\frac{2}{2^\star(s)}}\left(1 - \left(\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{-1}+\epsilon\right)N(\theta_m)^{1-\frac{2}{2^\star(s)}}\right) \quad \leq \quad o(1).$$

With (60) and the last inequation, we get that, as $m \to \infty$,

(68)
$$N(\theta_m)^{\frac{2}{2^\star(s)}}\left(1 - \left(\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{-1}+\epsilon\right)\left(\frac{2(n-s)\beta}{2-s}\right)^{\frac{2^\star(s)-2}{2^\star(s)}} + o(1)\right) \leq o(1).$$

With the assumption (52) and (68), taking $\epsilon > 0$ small enough, we get that $N(\theta_m) \to 0$ as $m \to +\infty$ and by coercivity, we obtain (66). □

With Step 6.3 and since $E_q(u_m) \to \beta$ as $m \to +\infty$, we get that $E_q(u) = \beta$. This ends the proof of Proposition 6.2. □

Theorem 6.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$ is a singularity of type $(k, n-k)$ for some $k \in \{1,...,n\}$. We fix $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$, $a \in C^{0,\theta}(\Omega)$ such that $-\Delta - (\gamma|x|^{-2} + a(x))$ is coercive, and $h \in C^{0,\theta}(\Omega)$ such that $h \geq 0$ and let $0 \leq s < 2$ and $1 < q < 2^\star - 1$. Assume that there exists $u_0 \in D^{1,2}(\Omega)$, $u_0 \not\equiv 0$, such that*

$$\sup_{t\geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)}\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{n-s}{2-s}}, \tag{69}$$

then equation (6) has a non-vanishing solution in $D^{1,2}(\Omega)$ of Mountain-Pass type.

Proof of Theorem 6.2: By Proposition 6.1, there exists a Palais-Smale sequence $(u_m)_{m\in\mathbb{N}} \in D^{1,2}(\Omega)$ for E_q at level $\beta > 0$ such that $\beta \leq \sup_{t\geq 0} E_q(tu_0)$. It then follows from Proposition 6.2 that, up to a subsequence, (u_m) converges strongly to u in $D^{1,2}(\Omega)$. Then $E_q(u) = \beta > 0$, so $u \not\equiv 0$, and $E_q'(u) = 0$. Coercivity and $E_q'(u)[u_-] = 0$ yield $u \geq 0$. Regularity theory and Hopf's principle yield $u \in C^{2,\theta}(\Omega)$ and $u > 0$. Then u is a solution of (6). This proves Theorem 6.2. □

7. Proof of Theorem 1.3: Test-Functions estimates

The main result of this section is the following:

Proposition 7.1. *We fix $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ and $0 \leq s < 2$. We assume that there are extremals for $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ and we let U as in (12) be such an extremal. We let $(u_\epsilon)_\epsilon$ and $(\tilde{u}_\epsilon)_\epsilon$ be as in (22). Then,*

(a) For $0 \leq \gamma < \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we have that

$$\sup_{t\geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon) & \text{if } q+1 < \frac{2n-2}{n-2}, \\ \left(c_1 GH_{\gamma,s}(\Omega) - c_2 h(0)\right)\epsilon + o(\epsilon) & \text{if } q+1 = \frac{2n-2}{n-2}, \\ -c_2 h(0)\epsilon^{n-\frac{(q+1)(n-2)}{2}} + o(\epsilon^{n-\frac{(q+1)(n-2)}{2}}) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(b) For $0 \leq \gamma = \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we have that

$$\sup_{t\geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega)\epsilon \ln\left(\frac{1}{\epsilon}\right) + o(\epsilon \ln\left(\frac{1}{\epsilon}\right)) & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ -c_2 h(0)\epsilon^{n-\frac{(q+1)(n-2)}{2}} + o(\epsilon^{n-\frac{(q+1)(n-2)}{2}}) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(c) For $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we have that

$$\sup_{t\geq 0} E_q(t\tilde{u}_\epsilon) := \beta_0 + \begin{cases} -c_3 m_\gamma(\Omega)\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-}) & \text{if } q+1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ -\left(c_3 m_\gamma(\Omega) + c_2 h(0)\right)\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-}) & \text{if } q+1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ -c_2 h(0)\epsilon^{n-\frac{(q+1)(n-2)}{2}} + o(\epsilon^{n-\frac{(q+1)(n-2)}{2}}) & \text{if } q+1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

where $\beta_0 = \frac{2-s}{2(n-s)}\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{n-s}{2-s}}$,

$$\text{(70)} \qquad \begin{cases} c_1 &= \frac{\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{2^\star(s)}{2^\star(s)-2}}}{2}\left(\xi\int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^\star(s)}}{|x|^s}\,dx\right)^{-1}, \\ c_2 &= \frac{\xi^{\frac{q+1}{2^\star(s)-2}}}{q+1}\int_{\mathbb{R}^{k_+,n-k}} U^{q+1}\,dx, \\ c_3 &= \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{2^\star(s)}{2^\star(s)-2}}\frac{\alpha_+-\alpha_-}{2}\frac{\int_{S^{n-1}\cap\mathbb{R}^{k_+,n-k}}\left(\prod_{i=1}^k x_i\right)^2 d\sigma}{\xi\int_{\mathbb{R}^{k_+,n-k}}\frac{U^{2^\star(s)}}{|x|^s}\,dx}. \end{cases}$$

Theorem 6.2 and Proposition 7.1 yield Theorem 1.3.

Proof of Proposition 7.1: We define the test-function sequence $(Z_\epsilon)_{\epsilon>0}$ by

$$Z_\epsilon(x) := \begin{cases} u_\epsilon & \text{if } \gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}, \\ \tilde{u}_\epsilon & \text{if } \gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}, \end{cases}$$

where u_ϵ and $\tilde{u}_\epsilon$ are as in the definition (22). We have:

$$E_q(tZ_\epsilon) = \frac{t^2}{2}R_\epsilon - \frac{t^{2^\star(s)}}{2^\star(s)}B_\epsilon - \frac{t^{q+1}}{q+1}C_{h,\epsilon},$$

when $\epsilon \to 0$ where:

$$R_\epsilon := \int_\Omega \left(|\nabla Z_\epsilon|^2 - \left(\frac{\gamma}{|x|^2} + a(x)\right)Z_\epsilon^2\right)dx$$

$$B_\epsilon := \int_\Omega \frac{Z_\epsilon^{2^\star(s)}}{|x|^s}\,dx \text{ and } C_{h,\epsilon} := \int_\Omega h Z_\epsilon^{q+1}\,dx.$$

Step 7.1. *We fix $f \in C^{0,\theta}(\Omega)$, $\theta \in (0,1)$, and $p \in [1, 2^\star)$. We claim that*

$$\int_\Omega f|Z_\epsilon|^{p+1}\,dx = \begin{cases} f(0)\epsilon^{n-\frac{n-2}{2}(p+1)}\int_{\mathbb{R}^{k_+,n-k}} U^{p+1}\,dx + o\left(\epsilon^{n-\frac{n-2}{2}(p+1)}\right) & \text{if } n < (p+1)\alpha_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } n = (p+1)\alpha_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\right) & \text{if } n > (p+1)\alpha_+. \end{cases}$$

Moreover, we have that

$$\text{(71)} \qquad \int_\Omega f|Z_\epsilon|^{p+1}\,dx \to 0 \text{ as } \epsilon \to 0.$$

Proof of Step 7.1: Note that it follows from (20) that

$$0 < U_\epsilon(x) \le C\epsilon^{\frac{\alpha_+-\alpha_-}{2}}|x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^{k_+,n-k} \text{ and } \epsilon > 0. \tag{72}$$

We first prove Step 7.1 for u_ϵ, postponing the case of $\tilde{u}_\epsilon$, and then Z_ϵ, to the end of the proof. We distinguish three cases:

Case 1: We assume that $n > (p+1)\alpha_+$. It follows from (72) that

$$\left|\int_\Omega f|u_\epsilon|^{p+1}\,dx\right| \le C\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\int_\Omega |x|^{-(p+1)\alpha_+}\,dx \le C\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}$$

as $\epsilon \to 0$. This proves Step 7.1 for u_ϵ when $n > (p+1)\alpha_+$.

Case 2: We assume that $n = (p+1)\alpha_+$. With (72), we get that

$$\begin{aligned}\left|\int_\Omega f|u_\epsilon|^{p+1}\,dx\right| &\le C\epsilon^{n-\frac{n-2}{2}(p+1)} + C\int_{B_{\delta,+}}|u_\epsilon|^{p+1}\,dx\\ &\le C\epsilon^{n-\frac{n-2}{2}(p+1)} + C\epsilon^{n-\frac{n-2}{2}(p+1)}\int_{B_{\delta\epsilon^{-1},+}}U^{p+1}\,dx\\ &\le C\epsilon^{n-\frac{n-2}{2}(p+1)} + C\epsilon^{n-\frac{n-2}{2}(p+1)}\int_1^{\epsilon^{-1}\delta} r^{-1}\,dr\\ &\le C\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\ln\left(\frac{1}{\epsilon}\right)\end{aligned}$$

Case 3: We assume that $n < (p+1)\alpha_+$. For $\rho > 0$ small enough, it follows from (72) that

$$\int_{\Omega\setminus\phi(B_{\rho,+})} f|u_\epsilon|^{p+1}\,dx = O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\right) \text{ as } \epsilon \to 0.$$

Independently, since $f \in C^{0,\theta}(\Omega)$, we have that

$$\begin{aligned}\int_{\phi(B_{\rho,+})} f|u_\epsilon|^{p+1}\,dx &= \int_{B_{\rho,+}} f\circ\phi\cdot U_\epsilon^{p+1}|\text{Jac }\phi|\,dx\\ &= \epsilon^{n-\frac{n-2}{2}(p+1)}f(0)\int_{B_{\rho\epsilon^{-1},+}}U^{p+1}\,dx + O\left(\int_{B_{\rho,+}}|x|^\theta|U_\epsilon|^{p+1}\,dx\right)\end{aligned} \tag{73}$$

Since $n < (p+1)\alpha_+$, it follows from (20) that $U \in L^{p+1}(\mathbb{R}^{k_+,n-k})$ and that

$$\begin{aligned}\int_{B_{\rho\epsilon^{-1},+}}U^{p+1}\,dx &= \int_{\mathbb{R}^{k_+,n-k}}U^{p+1}\,dx + O\left(\int_{\mathbb{R}^{k_+,n-k}\setminus B_{\rho\epsilon^{-1},+}}U^{p+1}\,dx\right)\\ &= \int_{\mathbb{R}^{k_+,n-k}}U^{p+1}\,dx + O\left(\int_{\epsilon^{-1}\rho}^\infty r^{n-(p+1)\alpha_+-1}\,dr\right)\\ &= \int_{\mathbb{R}^{k_+,n-k}}U^{p+1}\,dx + O\left(\epsilon^{(p+1)\alpha_+-n}\right)\end{aligned} \tag{74}$$

We claim that

$$\int_{B_{\rho,+}}|x|^\theta|U_\epsilon|^{p+1}\,dx = o\left(\epsilon^{n-\frac{n-2}{2}(p+1)}\right) \text{ as } \epsilon \to 0. \tag{75}$$

Indeed, when $\theta + n > (p+1)\alpha_+$, we argue as in Case 1. When $\theta + n = (p+1)\alpha_+$, we argue as in Case 2. When $\theta + n < (p+1)\alpha_+$, we make a change of variable

$y = \epsilon^{-1}x$ and we argue as in (74). This yields (75). Putting (74) and (75) in (73) yields Step 7.1 for u_ϵ in Case 2.

We now prove Step 7.1. When $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, $Z_\epsilon = u_\epsilon$, and we are done. When $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, $Z_\epsilon = \tilde{u}_\epsilon$. With the definition (22), we get that

$$\begin{aligned}(76)\ \int_\Omega f|\tilde{u}_\epsilon|^{p+1}\,dx &= \int_\Omega f\left|u_\epsilon + \epsilon^{\frac{\alpha_+-\alpha_-}{2}}\Theta\right|^{p+1}\,dx \\ &= \int_\Omega f|u_\epsilon|^{p+1}\,dx + O\left(\epsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_\Omega |u_\epsilon|^p|\Theta|\,dx\right) + O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\int_\Omega |\Theta|^{p+1}\,dx\right)\end{aligned}$$

Since $\Theta \in D^{1,2}(\Omega)$ and $p+1 < 2^\star$, we get that $\Theta \in L^{p+1}(\Omega)$. It follows from (16) that $|\Theta(x)| \leq C|x|^{-\alpha_-}$ for all $x \in \Omega$. Arguing as in Cases 1, 2, 3 above, we get that the second term in the right-hand-side of (76) is dominated by $\int_\Omega |u_\epsilon|^{p+1}\,dx$. Then Step 7.1 for $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$ follows. □

By Cheikh-Ali [4] and Step 7.1 for the case $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$ and Steps 4.2 and 4.3 for the case $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$, we get that, as $\epsilon \to 0$,

$$(77)\qquad R_\epsilon \to R_0 := \xi\int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^\star(s)}}{|x|^s}\,dx \text{ and } B_\epsilon \to B_0 := \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^\star(s)}}{|x|^s}\,dx.$$

Step 7.2. *We claim that for all $\epsilon > 0$, then there exists a unique t_ϵ such that*

$$(78)\qquad \sup_{t\geq 0} E_q(tZ_\epsilon) = E_q(t_\epsilon Z_\epsilon).$$

Moreover, t_ϵ verifies

$$(79)\qquad t_\epsilon = S_\epsilon\left[1 - C_0C_{h,\epsilon} + o(C_{h,\epsilon})\right],$$

where $S_\epsilon := \left(R_\epsilon B_\epsilon^{-1}\right)^{\frac{1}{2^\star(s)-2}}$, $C_0 > 0$ and $t_\epsilon \to t_0$ as $\epsilon \to 0$.

Proof of Step 7.2: We have that $\partial_t E_q(tZ_\epsilon) = 0$ iff $t = 0$ or $g_\epsilon(t) = R_\epsilon$ where $g_\epsilon(t) := B_\epsilon t^{2^\star(s)-2} + C_{h,\epsilon}t^{q-1}$. Since $B_\epsilon, C_{h,\epsilon} \geq 0$ and g_ϵ is a strictly increasing map i.e $g_\epsilon(t) - R_\epsilon$ also, and since $R_\epsilon > 0$ we have $g_\epsilon(0) - R_\epsilon < 0$ then, there exists $t_\epsilon > 0$ unique verifying $g_\epsilon(t_\epsilon) = R_\epsilon$ such that (78) holds. Since $g_\epsilon(t_\epsilon) = R_\epsilon$, we get

$$t_\epsilon \leq S_\epsilon := \left(R_\epsilon B_\epsilon^{-1}\right)^{\frac{1}{2^\star(s)-2}}.$$

We are using (77), (71) and (12) to get that $S_\epsilon \to \left(R_0B_0^{-1}\right)^{\frac{1}{2^\star(s)-2}} = \xi^{\frac{1}{2^\star(s)-2}}$ as $\epsilon \to 0$. Therefore, t_ϵ is bounded and there exists t_0 such that $t_\epsilon \to t_0$ up to extraction. Since $g_\epsilon(t_\epsilon) = R_\epsilon$ and $C_{h,\epsilon} \to 0$ as $\epsilon \to 0$, we obtain that

$$\begin{aligned} t_\epsilon &= \left[R_\epsilon B_\epsilon^{-1} - C_{h,\epsilon}B_\epsilon^{-1}t_\epsilon^{q-1}\right]^{\frac{1}{2^\star(s)-2}} \\ &= S_\epsilon\left[1 - C_{h,\epsilon}R_\epsilon^{-1}t_\epsilon^{q-1}\right]^{\frac{1}{2^\star(s)-2}} = S_\epsilon\left[1 - C_0C_{h,\epsilon} + o(C_{h,\epsilon})\right],\end{aligned}$$

where $C_0 := \frac{R_0^{-1}t_0^{q-1}}{2^\star(s)-2}$ and $t_0 = \xi^{\frac{1}{2^\star(s)-2}}$. This yields (79) and Step 7.2. □

Step 7.3. *We claim that, as $\epsilon \to 0$,*

$$E_q(t_\epsilon Z_\epsilon) = \frac{2-s}{2(n-s)}\left(J^\Omega_{\gamma,s,a}(Z_\epsilon)\right)^{\frac{2^\star(s)}{2^\star(s)-2}} - \frac{\xi^{\frac{q+1}{2^\star(s)-2}}}{q+1}C_{h,\epsilon} + o(C_{h,\epsilon}).$$

Proof of Step 7.3: The expression (79) of Step 7.2 and (71) yield

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{t_\epsilon^2}{2}R_\epsilon - \frac{t_\epsilon^{2^\star(s)}}{2^\star(s)}B_\epsilon - \frac{t_\epsilon^{q+1}}{q+1}C_{h,\epsilon} \\ &= \frac{S_\epsilon^2\left[1-C_0C_{h,\epsilon}+o(C_{h,\epsilon})\right]^2}{2}R_\epsilon - \frac{S_\epsilon^{2^\star(s)}\left[1-C_0C_{h,\epsilon}+o(C_{h,\epsilon})\right]^{2^\star(s)}}{2^\star(s)}B_\epsilon \\ &\quad -\frac{S_\epsilon^{q+1}\left[1-C_0C_{h,\epsilon}+o(C_{h,\epsilon})\right]^{q+1}}{q+1}C_{h,\epsilon} \\ &= \frac{S_\epsilon^2\left[1-2C_0C_{h,\epsilon}+o(C_{h,\epsilon})\right]}{2}R_\epsilon - \frac{S_\epsilon^{2^\star(s)}\left[1-C_02^\star(s)C_{h,\epsilon}+o(C_{h,\epsilon})\right]}{2^\star(s)}B_\epsilon \\ &\quad -\frac{S_\epsilon^{q+1}\left[1-(q+1)C_0C_{h,\epsilon}+o(C_{h,\epsilon})\right]}{q+1}C_{h,\epsilon}, \end{aligned}$$

then,

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{S_\epsilon^2}{2}R_\epsilon - \frac{S_\epsilon^{2^\star(s)}}{2^\star(s)}B_\epsilon - \frac{S_\epsilon^{q+1}}{q+1}C_{h,\epsilon} \\ &\quad -C_0C_{h,\epsilon}\left[S_\epsilon^2R_\epsilon - S_\epsilon^{2^\star(s)}B_\epsilon - S_\epsilon^{q+1}C_{h,\epsilon}\right] + o(C_{h,\epsilon}). \end{aligned}$$

Since $S_\epsilon := \left(R_\epsilon B_\epsilon^{-1}\right)^{\frac{1}{2^\star(s)-2}}$ and $C_{h,\epsilon} \to 0$ as $\epsilon \to 0$, this yields Step 7.3. □

Proof of Proposition 7.1 when $0 \le \gamma \le \gamma_{\boldsymbol{H}}(\mathbb{R}^{\boldsymbol{k}_+,\boldsymbol{n-k}}) - \frac{\boldsymbol{1}}{\boldsymbol{4}}$. In this case, we recall that $Z_\epsilon(x) = u_\epsilon(x)$. Note that

$$\left\{\gamma < (=)\gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}\right\} \Leftrightarrow \left\{\alpha_+ - \alpha_- > (=)1\right\}.$$

It was proved in Proposition 5.1 in Cheikh-Ali [4] that

- For $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we have that

$$J^\Omega_{\gamma,s,0}(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})\left(1+\kappa GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)\right). \tag{80}$$

- For $\gamma = \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we have that

$$J^\Omega_{\gamma,s,0}(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})\left(1+\kappa GH_{\gamma,s}(\Omega)\epsilon\ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon\ln\left(\frac{1}{\epsilon}\right)\right)\right), \tag{81}$$

where $\kappa := \left(\xi\int_{\mathbb{R}^{k_+,n-k}}\frac{U^{2^\star(s)}}{|x|^s}\,dx\right)^{-1}$ and $GH_{\gamma,s}(\Omega)$ is defined in (7). It follows from Step 7.1 that $\int_\Omega u_\epsilon^2\,dx = o(\epsilon)$ if $\alpha_+ - \alpha_- > 1$, and $O(\epsilon)$ if $\alpha_+ - \alpha_- = 1$. Therefore (80) and (81) hold unchanged with the potential a.

Case 1: We assume that $n < (q+1)\alpha_+$. It follows from Step 7.1 that

$$C_{h,\epsilon} = \int_\Omega h|u_\epsilon|^{q+1}\,dx = h(0)\epsilon^{n-\frac{n-2}{2}(q+1)}\int_{\mathbb{R}^{k_+,n-k}}U^{q+1}\,dx + o\left(\epsilon^{n-\frac{n-2}{2}(q+1)}\right)$$

as $\epsilon \to 0$. Then, when $n < (q+1)\alpha_+$, Case (a) of Proposition 7.1 follows by combining Step 7.3, (80), (81), the estimate of $C_{h,\epsilon}$ and studying the relative positions of $n - \frac{n-2}{2}(q+1)$ and 1.

Case 2: We assume that $n \geq (q+1)\alpha_+$. Since $\alpha_+ - \alpha_- \geq 1$ and $q > 1$, we then get that

$$n - \frac{n-2}{2}(q+1) - 1 = (n - (q+1)\alpha_+) + \frac{q+1}{2}(\alpha_+ - \alpha_-) - 1 > 0.$$

Then, for $n \geq (q+1)\alpha_+$, Cases (a) and (b) of Proposition 7.1 follows by the same arguments as in Case 1.

This proves Cases (a) and (b) of Proposition 7.1. □

Proof of Proposition 7.1 when $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$**.** Proposition 4.1 yields

$$J^{\Omega}_{\gamma,s,a}(\tilde{u}_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) \left(1 - \zeta^0_{\gamma,s} m_{\gamma,a}(\Omega)\epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-})\right), \quad (82)$$

as $\epsilon \to 0$. Here, we compare $n - \frac{n-2}{2}(q+1)$ and $\alpha_+ - \alpha_-$. Note that

$$n - \frac{n-2}{2}(q+1) - (\alpha_+ - \alpha_-) = n - (q+1)\alpha_+ + \frac{q-1}{2}(\alpha_+ - \alpha_-).$$

Therefore, since $q > 1$, when $n \geq (q+1)\alpha_+$, we have that $n - \frac{n-2}{2}(q+1) > \alpha_+ - \alpha_-$. As for the case $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$, we get Case (b) of Proposition 7.1 by studying the relative positions of $n - \frac{n-2}{2}(q+1)$ and $\alpha_+ - \alpha_-$ and using Step 7.1 and (82).

This proves Case (c) of Proposition 7.1. □

All these cases prove Proposition 7.1. As already mentioned, Theorem 6.2 and Proposition 7.1 yield Theorem 1.3.

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