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# Hardy-Sobolev inequalities with singularities on non smooth boundary. Part 2: Influence of the global geometry in small dimensions

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**HARDY-SOBOLEV INEQUALITIES WITH SINGULARITIES ON  
NON SMOOTH BOUNDARY. PART 2: INFLUENCE OF THE  
GLOBAL GEOMETRY IN SMALL DIMENSIONS.**

HUSSEIN CHEIKH ALI

ABSTRACT. We consider Hardy-Sobolev nonlinear equations on domains with singularities. We introduced this problem in Cheikh-Ali [4]. Under a local geometric hypothesis, namely that the generalized mean curvature is negative (see (6) below), we proved the existence of extremals for the relevant Hardy-Sobolev inequality for large dimensions. In the present paper, we tackle the question of small dimensions that was left open. We introduce a "mass", that is a global quantity, the positivity of which ensures the existence of extremals in small dimensions. As a byproduct, we prove the existence of solutions to a perturbation of the initial equation via the Mountain Pass Lemma.

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1. INTRODUCTION

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 3$ . We fix  $s \in [0, 2]$  and  $\gamma \in \mathbb{R}$ . It follows from the classical Caffarelli-Kohn-Nirenberg inequalities [3] that if  $\gamma < \frac{(n-2)^2}{4}$ , there exists  $K > 0$  such that

$$(1) \quad \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_{\Omega} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx,$$

for all  $u \in D^{1,2}(\Omega)$ , where  $2^*(s) := \frac{2(n-s)}{n-2}$  and  $D^{1,2}(\Omega)$  is the completion of  $C_c^\infty(\Omega)$  with respect to the norm  $u \mapsto \|\nabla u\|_2$ . We define the Hardy constant by

$$\gamma_H(\Omega) := \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}; u \in D^{1,2}(\Omega) \setminus \{0\} \right\} > 0.$$

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The classical Hardy inequality reads  $\gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$  and therefore, we have that  $\gamma_H(\Omega) \geq \frac{(n-2)^2}{4}$ . We refer to [4] for discussions and properties of the Hardy constant. As one checks, for any  $\gamma < \gamma_H(\Omega)$ , there exists  $K = K(\Omega, \gamma, s) > 0$  such that (1) holds for all  $u \in D^{1,2}(\Omega)$ . For  $a \in L^\infty(\Omega)$ , we define

$$\mu_{\gamma,s,a}(\Omega) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} J_{\gamma,s,a}^\Omega(u),$$

where

$$J_{\gamma,s,a}^\Omega(u) := \frac{\int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx}{\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}},$$

so that

$$(2) \quad \mu_{\gamma,s,a}(\Omega) \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx,$$

for all  $u \in D^{1,2}(\Omega)$ . As in [4], we address the question of the existence of extremals for (2), more precisely

**Q:** Is there  $u \in D^{1,2}(\Omega) \setminus \{0\}$  equality holds in (2)?

When  $0 \in \Omega$ , there are no extremals for  $\mu_{\gamma,s,0}(\Omega)$  (see [7]). From now on, we assume that  $0 \in \partial\Omega$ . When  $\Omega$  is a smooth domain, criteria for existence are in Ghoussoub-Robert [8]: in particular, there is a dichotomy between large dimension (where the criterion is local) and the small dimensions (where the criterion is global). In [4], we studied the case of domains that are modeled on cones:

**Definition 1.** We fix  $1 \leq k \leq n$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . We say that  $x_0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  if there exist  $U, V$  open subsets of  $\mathbb{R}^n$  such that  $0 \in U$ ,  $x_0 \in V$  and there exists  $\phi \in C^\infty(U, V)$  a diffeomorphism such that  $\phi(0) = x_0$  and

$$\phi(U \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial(\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0  $d\phi_0$  is an isometry.

In the sequel, we write  $\mathbb{R}^{k+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ . We have that (see [4])

$$\gamma_H(\mathbb{R}^{k+,n-k}) = \frac{(n-2+2k)^2}{4}.$$

We have proved the following:

**Theorem 1.1** (Cheikh-Ali [4]). Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$  and  $0 \leq \gamma < \gamma_H(\Omega)$ . Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We assume that

$$(3) \quad \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \text{ that is } n \geq n_{\gamma,k} := \sqrt{4\gamma+1} + 2 - 2k.$$

Then there are extremals for  $\mu_{\gamma,s,0}(\Omega)$  if

$$GH_{\gamma,s}(\Omega) < 0$$

where  $GH_{\gamma,s}(\Omega)$  is the **generalized mean curvature** defined below in (6).

This result is for large dimension  $n \geq n_{\gamma,k}$  (see (3)). In the present article, we tackle the case of the remaining small dimensions. The argument based on local geometry performed for the proof of Theorem 1.1 is not working here. Here, the global geometry has an impact: in order to obtain extremals, we must introduce a "mass" in the spirit of Schoen [14] and Schoen-Yau [15]. Concerning low dimension phenomena, we refer to the pioneer work of Brezis-Nirenberg [2], Jannelli [13] and the more recent reference Ghoussoub-Robert [7] for further discussions. Our main theorem is the following:

**Theorem 1.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k}.$$

We assume that the operator  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive and has a mass  $m_{\gamma,a}(\Omega)$  (see Definition 3), and that  $m_{\gamma,a}(\Omega) > 0$ . Then there are extremals for  $\mu_{\gamma,s,a}(\Omega)$ . In particular, there exists  $u \in C^{2,\theta}(\Omega) \cap D^{1,2}(\Omega)$  such that

$$(4) \quad \begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the second part of this paper, we consider the perturbative Hardy-Schrödinger equation. Given  $a, h \in C^{0,\theta}(\Omega)$  for some  $\theta \in (0, 1)$  and  $q \in (1, 2^* - 1)$ , we investigate the existence of solutions  $u \in C^2(\Omega) \cap D^{1,2}(\Omega)$  to

$$(5) \quad \begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^*(s)-1}}{|x|^s} + h(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We refer to Brezis-Nirenberg [2] ( $\gamma = 0$  and  $s = 0$  on a smooth domain  $\Omega$ ), Ghoussoub-Yuan [10] ( $\gamma = 0$ ,  $s > 0$  and  $0 \in \Omega$ ), Ghoussoub-Kang [9] and Jaber [12] ( $\gamma = 0$ ,  $s > 0$  and  $0 \in \partial\Omega$ ). In the Riemannian context with no boundary, still for  $\gamma = 0$ , we refer to Djadli [5] when  $s = 0$ , and to Jaber [11] for  $s > 0$  and  $h \equiv 0$ .

The case  $a, h \equiv 0$  was tackled in [4] for  $n \geq n_{\gamma,k}$  for nonsmooth domains. We prove the following:

**Theorem 1.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . Let  $a, h \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ) such that  $-\Delta - (\gamma|x|^{-2} + a)$  is coercive and  $h \geq 0$ . Consider  $s \in [0, 2)$  and  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We fix  $q \in (1, 2^* - 1)$ . Then, there exists a positive mountain pass solution  $u \in D^{1,2}(\Omega)$  to the perturbative Hardy-Schrödinger equation (5) under one of the following conditions:*

- $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and
 
$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q + 1 < \frac{2n-2}{n-2}, \\ c_1 GH_{\gamma,s}(\Omega) - c_2 h(0) < 0 & \text{if } q + 1 = \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q + 1 > \frac{2n-2}{n-2}, \end{cases}$$

- $0 \leq \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and
 
$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2}{n-2}, \end{cases}$$
- $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and
 
$$\begin{cases} m_{\gamma,a}(\Omega) > 0 & \text{if } q+1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ c_3 m_{\gamma,a}(\Omega) + c_2 h(0) > 0 & \text{if } q+1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

where  $\alpha_+ - \alpha_- = 2\sqrt{\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma}$  (see (7) below),  $c_1, c_2, c_3 > 0$  are defined in (69) and  $m_{\gamma,a}(\Omega)$  is the mass of  $\Omega$  at 0.

This result shows how the subcritical nonlinearity has an impact on the existence of solutions. When the subcritical nonlinearity is close to being linear, only the geometry of  $\Omega$  commands the existence. Conversely, when it is close to being critical, the subcritical nonlinearity commands the existence, whatever the geometry is.

**Notation:** In the sequel,  $C$  denotes a positive constant. The value of the might change from a page to another, and even from one line to another.

## 2. DEFINITION OF THE GENERALIZED CURVATURE AND THE MASS

### Generalized curvature.

#### Definition 2.

$$\Omega_i := \phi(U \cap \{x_i > 0\}) \text{ for all } i = 1, \dots, k,$$

where  $(\phi, U)$  is a chart as in Definition 1. We have that:

- For all  $i = 1, \dots, k$ ,  $\Omega_i$  is smooth around  $0 \in \partial\Omega_i$ .
- Up to permutation, the  $\Omega_i$ 's are locally independent of the chart  $\phi$ .
- The  $\Omega_i$ 's define locally  $\Omega$ : there exists  $\delta > 0$  such that

$$\Omega \cap B_\delta(0) = \bigcap_{i=1}^k \Omega_i \cap B_\delta(0).$$

We set  $\Sigma := \bigcap_{i=1}^k \partial\Omega_i$  where  $k \in \{1, \dots, n\}$ . The vector  $\vec{H}_0^\Sigma$  denotes the mean-curvature vector at 0 of the  $(n-k)$ -submanifold  $\Sigma$ . For any  $m = 1, \dots, k$ ,  $II_0^{\partial\Omega_m}$  denotes the second fundamental form at 0 of the oriented  $(n-1)$ -submanifold  $\partial\Omega_m$ . The **generalized mean curvature** of  $\Omega$  is defined by:

$$(6) \quad GH_{\gamma,s}(\Omega) := c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \\ + c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q)$$

where for any  $m = 1, \dots, k$ ,  $\vec{\nu}_m$  is the outward normal vector at 0 of  $\partial\Omega_m$  and  $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$  are positive explicit constants. We refer to [4] for details on this curvature.

**The mass.** Let  $\alpha \in \mathbb{R}$  be a real number and fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . Then

$$\left(-\Delta - \frac{\gamma}{|x|^2}\right) S_\alpha = 0 \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\},$$

where:

$$(7) \quad S_\alpha := |x|^{-\alpha-k} \prod_{i=1}^k x_i \text{ and } \alpha_\pm = \alpha_\pm(\gamma, n, k) := \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma}.$$

The functions  $S_{\alpha_-}, S_{\alpha_+}$  are prototypes of solution to (4) vanishing on  $\partial\mathbb{R}^{k+,n-k}$ .

**Definition 3.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ . such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). We say that a coercive operator  $-\Delta - (\gamma|x|^{-2} + a)$  has a mass if there exists  $G \in C^2(\Omega) \cap D_{loc,0}^{1,2}(\Omega)$  such that

$$(8) \quad \begin{cases} -\Delta G - \left(\frac{\gamma}{|x|^2} + a(x)\right) G = 0 & \text{in } \Omega, \\ G > 0 & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

and there exists  $c \in \mathbb{R}$  such that

$$(9) \quad G(x) = \prod_{i=1}^k d(x, \partial\Omega_i) (|x|^{-\alpha_+ - k} + c|x|^{-\alpha_- - k} + o(|x|^{-\alpha_- - k})) \text{ as } x \rightarrow 0.$$

Then the function  $G$  is unique, and we define  $m_{\gamma,a}(\Omega) := c$  as the boundary mass of the operator  $-\Delta - (\gamma|x|^{-2} + a)$ .

Examples of domains with positive or negative mass are in Section 5 below.

### 3. SOME BACKGROUND RESULTS

We start with the following classical result:

**Theorem 3.1.** [see Cheikh-Ali [4]] Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . Assume that  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $0 \leq s \leq 2$ , et  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Then there are extremals for  $\mu_{\gamma,s,a}(\Omega)$ .

Indeed, Theorem 3.1 was proved in [4] when  $a \equiv 0$ . The proof extends to the general case with no effort. Recall now an optimal regularity theorem.

**Theorem 3.2.** [See Felli-Ferrero [6] and [4]] Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that

$$(10) \quad |f(x, v)| \leq C|v| \left(1 + \frac{|v|^{2^*(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega, v \in \mathbb{R}.$$

Let  $u \in D^{1,2}(\Omega)_{loc,0}$ , be a weak solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x, u) \text{ in } D^{1,2}(\Omega)_{loc,0}$$

for some  $\tau > 0$ . Then there exists  $K \in \mathbb{R}$  such that

$$\lambda^{\alpha_-} u(\lambda\phi(x)) \rightarrow K|x|^{-\alpha_-} \frac{\prod_{i=1}^k x_i}{|x|^k} \text{ in } B_1(0) \cap \mathbb{R}^{k+,n-k},$$

uniformly in  $C^1$  as  $\lambda \rightarrow 0$ , where  $\phi$  is a chart as in Definition 1.

In section 4, we will need the following lemma:

**Lemma 3.1.** [See [4]] Assume the  $u \in D^{1,2}(\mathbb{R}^{k_+,n-k})_{loc,0}$  is a weak solution of

$$\begin{cases} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\mathbb{R}^{k_+,n-k})_{loc,0}, \\ u = 0 & \text{on } B_{2\delta}(0) \cap \partial\mathbb{R}^{k_+,n-k}, \end{cases}$$

for some  $\tau > 0$  and  $\alpha \in \{\alpha_-, \alpha_+\}$ . Assume there exists  $c > 0$  such that

$$|u(x)| \leq c|x|^{-\alpha} \text{ for } x \rightarrow 0, x \in \mathbb{R}^{k_+,n-k}.$$

- Then, there exists  $c_1 > 0$  such that

$$|\nabla u(x)| \leq c_1|x|^{-\alpha-1} \text{ as } x \rightarrow 0, x \in \mathbb{R}^{k_+,n-k}.$$

- If  $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$ , then  $\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla u(x)| = 0$ .

#### 4. TEST-FUNCTIONS ESTIMATES FOR THE MASS: PROOF OF THEOREM 1.2

Let  $U \in D^{1,2}(\mathbb{R}^{k_+,n-k})$  be a positive extremal for  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ . Then

$$J_{\gamma,s,0}^{\mathbb{R}^{k_+,n-k}}(U) = \frac{\int_{\mathbb{R}^{k_+,n-k}} (|\nabla U|^2 - \gamma|x|^{-2}U^2) dx}{\left(\int_{\mathbb{R}^{k_+,n-k}} |x|^{-s}|U|^{2^*(s)} dx\right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}).$$

Therefore, there exists  $\xi > 0$  such that

$$(11) \quad \begin{cases} -\Delta U - \gamma|x|^{-2}U = \xi|x|^{-s}U^{2^*(s)-1} & \text{in } \mathbb{R}^{k_+,n-k}, \\ U > 0 & \text{in } \mathbb{R}^{k_+,n-k}, \\ U = 0 & \text{on } \partial\mathbb{R}^{k_+,n-k}. \end{cases}$$

For  $r > 0$ , we define

$$(12) \quad B_r := B_r(0) \text{ and } B_{r,+} := B_r(0) \cap \mathbb{R}^{k_+,n-k}.$$

Therefore, with  $\delta > 0$  small, the chart  $\phi$  of Definition 1 yields

$$\phi(B_{3\delta} \cap \mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \Omega \text{ and } \phi(B_{3\delta} \cap \partial\mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \partial\Omega.$$

We fix  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that

$$(13) \quad \eta(x) = \begin{cases} 1 & \text{for } x \in B_\delta, \\ 0 & \text{for } x \notin B_{2\delta}. \end{cases}$$

Define also for convenience,

$$(14) \quad p(x) := \prod_{i=1}^k d(x, \partial\Omega_i) \text{ for all } x \in \Omega \text{ and } v(x) := \prod_{i=1}^k x_i \text{ for all } x \in \mathbb{R}^{k_+,n-k}.$$

Equation (9) allows us to define  $\Theta \in \Omega \rightarrow \mathbb{R}$  such that

$$G(x) = (\eta v|x|^{-\alpha-k}) \circ \phi^{-1}(x) + \Theta(x) \text{ for any } x \in \Omega,$$

where  $\phi$  as in Definition 1. We then get that  $\Theta \in D^{1,2}(\Omega)$  and

$$(15) \quad \Theta(x) = m_{\gamma,\alpha}(\Omega)p(x)|x|^{-\alpha-k} + o(p(x)|x|^{-\alpha-k}) \text{ as } x \rightarrow 0.$$

Note that that

$$(16) \quad \left\{ \gamma > \gamma_H(\mathbb{R}_+^k \times \mathbb{R}^{n-k}) - \frac{1}{4} \right\} \Leftrightarrow \{\alpha_+ - \alpha_- < 1\} \Leftrightarrow \{n < n_{\gamma,k}\}.$$

Since  $U$  satisfies (11), Theorem 3.2 yields  $K_1 > 0$  such that

$$(17) \quad \lim_{\lambda \rightarrow 0^+} \lambda^{\alpha-k} U(\lambda x) = K_1 v(x)|x|^{-\alpha-k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+,n-k}.$$

The regularity applied to the Kelvin transform  $x \mapsto \bar{U}(x) := |x|^{2-n}U(\frac{x}{|x|^2})$  yields

$$(18) \quad \lim_{\lambda \rightarrow +\infty} \lambda^{\alpha+} U(\lambda x) = K_2 v(x) |x|^{-\alpha+-k} \text{ in } B_1(0) \cap \mathbb{R}^{k+,n-k},$$

for some  $K_2 > 0$ . Up to multiplying  $U$  by a positive constant, we assume that  $K_2 = 1$ . Equation (17), the Kelvin transform and Lemma 3.1 yield

$$(19) \quad |U(x)| \leq C|x|^{-\alpha+} \text{ and } |\nabla U(x)| \leq C|x|^{-1-\alpha+} \text{ for any } x \in \mathbb{R}^{k+,n-k}.$$

For  $\epsilon > 0$ , we define

$$(20) \quad U_\epsilon(x) := \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}x) \text{ for all } x \in \mathbb{R}^{k+,n-k}$$

and

$$(21) \quad u_\epsilon(x) := (\eta U_\epsilon) \circ \phi^{-1}(x) \text{ for } x \in \Omega \text{ and } \tilde{u}_\epsilon := u_\epsilon + \epsilon^{\frac{\alpha+-\alpha-}{2}} \Theta.$$

The main result of this paper is the following:

**Proposition 4.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). Assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k},$$

and that the operator  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive with a mass  $m_{\gamma,a}(\Omega)$ . We let  $(\tilde{u}_\epsilon)_\epsilon \in D^{1,2}(\Omega)$  as in (21). Then

$$J_{\gamma,s,a}^\Omega(\tilde{u}_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k}) (1 - \zeta_{\gamma,s}^0 m_{\gamma,a}(\Omega) \epsilon^{\alpha+-\alpha-} + o(\epsilon^{\alpha+-\alpha-})) \text{ as } \epsilon \rightarrow 0,$$

where

$$(22) \quad \zeta_{\gamma,s}^0 := (\alpha+ - \alpha-) C_{k,n} \left( \xi \int_{\mathbb{R}^{k+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1} > 0,$$

where  $C_{k,n}$  is defined in (24).

As one checks, Theorem 1.2 is a direct consequence of the combination of Proposition 4.1 and Theorem 3.1.

This section is devoted to the proof of Proposition 4.1.

*Proof of Proposition 4.1:* It follows from the uniform convergence in  $C^1$  of the equation (18), the definitions of  $\tilde{u}_\epsilon$  and  $G$ , we denote that

$$(23) \quad \lim_{\epsilon \rightarrow 0} \frac{\tilde{u}_\epsilon}{\epsilon^{\frac{\alpha+-\alpha-}{2}}} = G \text{ dans } C_{loc}^1(\Omega) \cap D_{loc,0}^{1,2}(\Omega).$$

Define the constant

$$(24) \quad C_{k,n} := \int_{S^{n-1} \cap \mathbb{R}^{k+,n-k}} \left( \prod_{i=1}^k x_i \right)^2 d\sigma.$$

In the sequel,  $\vartheta_\rho^\epsilon$  will denote a quantity such

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \vartheta_\rho^\epsilon = 0.$$

For convenience, we define

$$N_{\gamma,a}(w) := |\nabla w|^2 - (\gamma|x|^{-2} + a) w^2.$$



**Step 4.1.** For any  $\rho > 0$ , we claim that

$$\int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon) dx = \epsilon^{\alpha_+ - \alpha_-} (\alpha_+ C_{k,n} \rho^{n-2\alpha_+ - 2} + m_{\gamma,a}(\Omega)(n-2)C_{k,n} + \vartheta_\rho^\epsilon),$$

as  $\epsilon \rightarrow 0$  with the constant  $C_{k,n}$  is defined in (24).

*Proof of Step 4.1:* From the equation (23), we observe that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-(\alpha_+ - \alpha_-)} \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon) dx = \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(G) dx.$$

Since  $G$  satisfies (8) and vanishes on  $\partial\Omega \setminus \{0\}$ , integrations by parts yield

$$\begin{aligned} \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(G) dx &= \int_{\Omega \setminus \phi(B_{\rho,+})} (-\Delta G - (\gamma|x|^2 + a(x))G) dx \\ &\quad - \int_{\phi(\partial(B_{\rho,+}))} G \partial_\nu G d\sigma \\ (25) \qquad \qquad \qquad &= - \int_{(\partial B_\rho(0)) \cap \mathbb{R}^{k_+, n-k}} (G \circ \phi) \partial_{\phi_* \nu} (G \circ \phi) d(\phi^* \sigma), \end{aligned}$$

where  $\nu(x)$  is the outer normal vector of  $B_\rho(0)$  at  $x \in \partial B_\rho(0)$ . We will now find the value of  $(G \circ \phi) \partial_{\phi_* \nu} (G \circ \phi)$ . The definitions of  $v$  and  $G$  yields,

$$(26) \quad (G \circ \phi)(x) = v(x)|x|^{-\alpha_+ - k} + m_{\gamma,a}(\Omega)v(x)|x|^{-\alpha_- - k} + o(v(x)|x|^{-\alpha_- - k}) \text{ as } x \rightarrow 0.$$

From  $\Theta$  and the uniform convergence in  $C^1$  of  $G$ , we have for all  $l = 1, \dots, n$  that

$$(27) \quad \partial_l(\Theta \circ \phi) = \partial_l(m_{\gamma,a}(\Omega)v|x|^{-\alpha_- - k}) + o(|x|^{-\alpha_- - 1}) \text{ as } x \rightarrow 0.$$

Moreover, the definition of  $G$  yields,

$$\begin{aligned} \partial_l(G \circ \phi) &= \partial_l v (|x|^{-\alpha_+ - k} + m_{\gamma,a}(\Omega)|x|^{-\alpha_- - k}) \\ &\quad - x_l v ((\alpha_+ + k)|x|^{-\alpha_+ - k - 2} + (\alpha_- + k)m_{\gamma,a}(\Omega)|x|^{-\alpha_- - k - 2}) + o(|x|^{-\alpha_- - 1}). \end{aligned}$$

In view of,

$$\phi_* \nu(x) = \frac{x}{|x|} + O(|x|) \text{ as } x \rightarrow 0 \text{ and } \alpha_+ < \alpha_- + 1,$$

we obtain as  $x \rightarrow 0$  that,

$$(28) \quad \partial_{\phi_* \nu}(G \circ \phi) = -v(\alpha_+ |x|^{-\alpha_+ - k - 1} + m_{\gamma,a}(\Omega)\alpha_- |x|^{-\alpha_- - k - 1}) + o(|x|^{-\alpha_- - 1}).$$

We combine the equations (26), (28) and since  $\alpha_+ + \alpha_- = n - 2$ ,  $-2\alpha_- - 1 > 1 - n$ ,  $\alpha_+ - \alpha_- < 1$ , we get

$$-(G \circ \phi) \partial_{\phi_* \nu}(G \circ \phi) = v^2 (\alpha_+ |x|^{-2\alpha_+ - 2k - 1} + m_{\gamma,a}(\Omega)(n-2)|x|^{-n+1-2k}) + o(|x|^{1-n}).$$

Moreover, using again the definition of  $v$ ,

$$- \int_{\partial B_{\rho,+}} (G \circ \phi) \partial_{\phi_* \nu}(G \circ \phi) d(\phi^* \sigma) = \alpha_+ C_{k,n} \rho^{n-2\alpha_+ - 2} + m_{\gamma,a}(\Omega)(n-2)C_{k,n} + \vartheta_\rho,$$

where  $\lim_{\rho \rightarrow 0} \vartheta_\rho = 0$  and  $C_{k,n}$  is defined in (24). Plugging the last equation in (25) yields Step 4.1.  $\square$

**Step 4.2.** We claim that, as  $\epsilon \rightarrow 0$ ,

$$\int_{\Omega} N_{\gamma,a}(\tilde{u}_\epsilon) dx = \xi \int_{\mathbb{R}^{k_+, n-k}} |x|^{-s} U^{2^*(s)} dx + m_{\gamma,a}(\Omega)(n-2)C_{k,n} \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}).$$

*Proof of Step 4.2:* With the definition (21) of  $\tilde{u}_\epsilon$ , for any  $x \in \mathbb{R}^{k_+ + n - k}$ , we get

$$(29) \quad \tilde{u}_\epsilon \circ \phi(x) = U_\epsilon(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta \circ \phi(x) \text{ for all } x \in B_{\delta_+}.$$

Fix  $\rho \in ]0, \delta[$  that we will eventually let go to 0. We define

$$I_{\epsilon, \rho} := \int_{\phi(B_{\rho, +})} (|\nabla \tilde{u}_\epsilon|^2 - (\gamma|x|^{-2} + a) \tilde{u}_\epsilon^2) dx.$$

Let  $\phi^* Eucl$  be the pullback of the Euclidean metric. With (29), we get

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} \left( |\nabla(\tilde{u}_\epsilon \circ \phi)|_{\phi^* Eucl}^2 - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\tilde{u}_\epsilon \circ \phi)^2 \right) |Jac(\phi)| dx \\ &= \int_{B_{\rho, +}} \left( |\nabla U_\epsilon|_{\phi^* Eucl}^2 - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) U_\epsilon^2 \right) |Jac(\phi)| dx \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{\phi^* Eucl} - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\Theta \circ \phi) U_\epsilon \right) |Jac(\phi)| dx \\ &+ \epsilon^{\alpha_+ - \alpha_-} \int_{B_{\rho, +}} \left( |\nabla(\Theta \circ \phi)|_{\phi^* Eucl}^2 - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\Theta \circ \phi)^2 \right) |Jac(\phi)| dx. \end{aligned}$$

Since  $d\varphi_0 = Id_{\mathbb{R}^n}$ ,  $\phi^* Eucl = Eucl + O(|x|)$ . Since  $\Theta \in D^{1,2}(\Omega)$ , we get that

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} \left( |\nabla U_\epsilon|_{Eucl}^2 - \left( \frac{\gamma}{|x|^2} + a \circ \phi \right) U_\epsilon^2 \right) dx \\ &+ O \left( \int_{B_{\rho, +}} |x| (|\nabla U_\epsilon|_{Eucl}^2 + |x|^{-2} U_\epsilon^2) dx \right) \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \left( \frac{\gamma}{|x|^2} + a \circ \phi \right) (\Theta \circ \phi) U_\epsilon \right) dx \\ &+ O \left( \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} |x| (|\nabla U_\epsilon| \cdot |\nabla(\Theta \circ \phi)| + |x|^{-2} (\Theta \circ \phi) U_\epsilon) dx \right) + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \end{aligned}$$

as  $\epsilon \rightarrow 0$ . The explicit expression (20) of  $U_\epsilon$ , (19) and  $n > 2\alpha_+$  yield

$$(30) \quad \begin{aligned} \int_{B_{\rho, +}} U_\epsilon^2 dx &= O \left( \epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r^{n-2\alpha_+ - 1} dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

The definition of  $\Theta$  and  $\alpha_+ + \alpha_- = n - 2$  give

$$(31) \quad \begin{aligned} \int_{B_{\rho, +}} a \circ \phi(\Theta \circ \phi) U_\epsilon dx &= O \left( \epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

We combine the equations (17), (19), (27), (30) and (31),

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} \left( |\nabla U_\epsilon|_{Eucl}^2 - \frac{\gamma}{|x|^2} U_\epsilon^2 \right) dx \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \frac{\gamma}{|x|^2} (\Theta \circ \phi) U_\epsilon \right) dx + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Using again the integrations by parts and since  $U_\epsilon$  and  $\Theta \circ \phi$  vanish on  $\partial \mathbb{R}^{k_+, n-k} \setminus \{0\}$ , we have as  $\epsilon \rightarrow 0$  that

$$(32) \quad \begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} U_\epsilon (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx + \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon d\sigma \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( \int_{B_{\rho, +}} (\Theta \circ \phi) (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx \right. \\ &\left. + \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon d\sigma \right) + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

We claim as  $\epsilon \rightarrow 0$  that

$$(33) \quad \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon d\sigma = -\alpha_+ \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} m_{\gamma, a}(\Omega) C_{k, n} + o(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}}),$$

and

$$(34) \quad \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon d\sigma = -\alpha_+ C_{k, n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} + o(\epsilon^{\alpha_+ - \alpha_-} \rho^{n-2-2\alpha_+}).$$

We prove the claim. It follows from the uniform convergence in  $C^1$  of the equation (18), we have for all  $l = 1, \dots, n$

$$(35) \quad \lim_{\lambda \rightarrow +\infty} \lambda^{\alpha_+} \partial_l U(\lambda x) = |x|^{-\alpha_+ - k} \left( \delta_{l \leq k} \prod_{j=1; j \neq l}^k x_j - (\alpha_+ + k) \frac{v(x) x_l}{|x|^2} \right),$$

where  $v$  is defined in (14). The definition of  $U_\epsilon$  and (19) yield

$$\partial_l U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( |x|^{-\alpha_+ - k} \left( \delta_{l \leq k} \prod_{j=1; j \neq l}^k x_j - (\alpha_+ + k) \frac{x_l}{|x|^2} v \right) + o(|x|^{-\alpha_+ - 1}) \right).$$

Since  $\nu(x) = |x|^{-1} x$  is the outer normal vector of  $B_\rho(0)$ , we then get

$$(36) \quad \partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} (-\alpha_+ v |x|^{-\alpha_+ - k - 1} + o(|x|^{-\alpha_+ - 1})),$$

as  $\epsilon \rightarrow 0$  uniformly on compact subsets of  $\subset \overline{\mathbb{R}^{k_+, n-k}} \setminus \{0\}$ . From  $\Theta$  and  $\alpha_+ + \alpha_- = n - 2$ , and (16), we obtain as  $\epsilon \rightarrow 0$  that

$$(\Theta \circ \phi) \partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( -\alpha_+ m_{\gamma, a}(\Omega) \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} v^2 |x|^{-n+1-2k} + o(|x|^{1-n}) \right).$$

Therefore, we get (33). The definition of  $U_\epsilon$  and the equations (17) and (36) yield

$$U_\epsilon \partial_\nu U_\epsilon = -\alpha_+ \epsilon^{\alpha_+ - \alpha_-} v^2 |x|^{-2\alpha_+ - 2k - 1} + o(\epsilon^{\alpha_+ - \alpha_-} |x|^{-2\alpha_+ - 1}),$$

as  $\epsilon \rightarrow 0$  uniformly locally in  $\subset \overline{\mathbb{R}^{k_+, n-k}} \setminus \{0\}$ . This yields (34) and proves the claim.

We combine equations (32), (33) and (34) to get

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} U_\epsilon (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx - \alpha_+ C_{k, n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} (\Theta \circ \phi) (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx \\ &- 2\alpha_+ \epsilon^{\alpha_+ - \alpha_-} m_{\gamma, a}(\Omega) C_{k, n} + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

Since  $U$  satisfies the equation (11) and by the definition (20) of  $U_\epsilon$ , we have

$$-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon = \xi|x|^{-s}U_\epsilon^{2^*(s)-1}.$$

Therefore, we get as  $\epsilon \rightarrow 0$  that

$$\begin{aligned} I_{\epsilon,\rho} &= \xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx - \alpha_+ C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} \\ (37) \quad &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \xi \int_{B_{\rho,+}} (\Theta \circ \phi) \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} dx \\ &- 2\alpha_+ \epsilon^{\alpha_+ - \alpha_-} m_{\gamma,a}(\Omega) C_{k,n} + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

The definition (20) of  $U_\epsilon$  and (19) yield

$$\left| \xi \int_{\mathbb{R}^{k_+,n-k} \setminus (B_{\rho,+})} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right| \leq C \epsilon^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-)}.$$

Therefore, with  $2^*(s) > 2$ , we get

$$(38) \quad \xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx = \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx + o(\epsilon^{\alpha_+ - \alpha_-}) \text{ as } \epsilon \rightarrow 0.$$

The definition (14), (20) and the control (19) yield

$$\begin{aligned} \int_{\mathbb{R}^{k_+,n-k} \setminus (B_{\rho,+})} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx &= O\left(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{\epsilon^{-1}\rho}^{+\infty} r^{(1-\frac{2^*(s)}{2})(\alpha_+ - \alpha_-) - 1} dr\right) \\ &= \epsilon^{\frac{2^*(s)-1}{2}(\alpha_+ - \alpha_-)} \vartheta_\rho^\epsilon. \end{aligned}$$

Therefore, with the definition of  $\Theta$  we get as  $\epsilon \rightarrow 0$  that

$$\begin{aligned} \xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} \Theta \circ \phi dx &= \xi m_{\gamma,a}(\Omega) \int_{B_{\rho,+}} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx \\ &+ o\left(\int_{B_{\rho,+}} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx\right) \\ (39) \quad &= \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( m_{\gamma,a}(\Omega) \xi \int_{\mathbb{R}^{k_+,n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx + \vartheta_\epsilon^\rho \right). \end{aligned}$$

Since  $(-\Delta - \gamma|x|^{-2})(v|x|^{-\alpha_- - k}) = 0$ , using integrations by parts and since  $U$  vanishes on  $\partial\Omega \setminus \{0\}$ , we obtain that

$$\begin{aligned} \xi \int_{\mathbb{R}^{k_+,n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx &= \lim_{R \rightarrow +\infty} \int_{B_{R,+}} v|x|^{-\alpha_- - k} (-\Delta U - \gamma|x|^{-2}U) dx \\ &= \lim_{R \rightarrow +\infty} \left[ \int_{B_{R,+}} U (-\Delta - \gamma|x|^{-2})(v|x|^{-\alpha_- - k}) dx \right. \\ (40) \quad &\left. - \int_{\mathbb{R}^{k_+,n-k} \cap \partial B_R} \partial_\nu U v|x|^{-\alpha_- - k} d\sigma \right]. \end{aligned}$$

Arguing as for (36), it follows from (35), that, as  $R \rightarrow +\infty$

$$\partial_\nu U = -\alpha_+ v|x|^{-\alpha_+ - k - 1} + o(|x|^{-\alpha_+ - 1}) \text{ uniformly for } x \in \partial B_R(0) \cap \mathbb{R}^{k_+,n-k}.$$

Moreover, since  $\alpha_+ + \alpha_- = n - 2$  we get

$$\partial_\nu U v |x|^{-\alpha_- - k} = -\alpha_+ v^2 |x|^{-(n+2k-1)} + o(|x|^{1-n}).$$

The last equation yields,

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^{k_+, n-k} \cap \partial B_R(0)} \partial_\nu U v |x|^{-\alpha_- - k} d\sigma = -\alpha_+ C_{k,n}.$$

Then, by (40)

$$(41) \quad \xi \int_{\mathbb{R}^{k_+, n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx = \alpha_+ C_{k,n}.$$

Combining (39) and (41), we get

$$(42) \quad \xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} \Theta \circ \phi dx = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} (\alpha_+ m_{\gamma,a}(\Omega) C_{k,n} + \vartheta_\epsilon^\rho) \text{ as } \epsilon \rightarrow 0.$$

Next, the equations (37), (38) and (42) yields,

$$I_{\epsilon,\rho} = \xi \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx - \alpha_+ C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} + o(\epsilon^{\alpha_+ - \alpha_-}).$$

In the other hand, using Step 4.1 the definition of  $I_{\epsilon,\rho}$  and the last equation, we get Step 4.2  $\square$

**Step 4.3.** We claim as  $\epsilon \rightarrow 0$  that,

$$\int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx + 2^*(s) \alpha_+ m_{\gamma,a}(\Omega) \xi^{-1} C_{k,n} \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}).$$

*Proof of Step 4.3:* We fix  $\rho > 0$ . The definitions of  $\tilde{u}_\epsilon$  and  $\Theta$ , and  $2^*(s) > 2$  yield

$$(43) \quad \int_{\phi(B_{2\delta,+} \setminus B_{\delta,+})} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = o(\epsilon^{\alpha_+ - \alpha_-}),$$

with the definition (12). Equations (15), (17), (29) and (43) yield

$$\int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{B_{\delta,+}} \frac{|U_\epsilon + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} (\Theta \circ \phi)|^{2^*(s)}}{|x|^s} |(1 + O(|x|))| dx + o(\epsilon^{\alpha_+ - \alpha_-}),$$

as  $\epsilon \rightarrow 0$ .

$$\begin{aligned} \int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx &= \int_{B_{\delta,+}} \left( \frac{U_\epsilon^{2^*(s)}}{|x|^s} + 2^*(s) \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} (\Theta \circ \phi) \right) dx \\ &+ \int_{B_{\delta,+}} O \left( \epsilon^{\alpha_+ - \alpha_-} \frac{U_\epsilon^{2^*(s)-2}}{|x|^s} \Theta \circ \phi^2 + \epsilon^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-)} |\Theta \circ \phi|^{2^*(s)} \right) dx + o(\epsilon^{\alpha_+ - \alpha_-}). \end{aligned}$$

It follows he definitions of  $\Theta$  and  $U_\epsilon$ ,

$$(44) \quad \begin{aligned} \int_{B_{\delta,+}} \frac{U_\epsilon^{2^*(s)-2}}{|x|^s} (\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} (\Theta \circ \phi))^2 dx &= O \left( \epsilon^{2(\alpha_+ - \alpha_-)} \int_0^{\epsilon^{-1}\rho} r^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-) - 1} dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned}$$

And,

$$\begin{aligned} \int_{B_{\delta,+}} (\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta \circ \phi)^{2^*(s)} |x|^{-s} dx &= O \left( \epsilon^{(\alpha_+ - \alpha_-) \frac{2^*(s)}{2}} \int_0^\rho r^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-) - 1} dr \right) \\ (45) \qquad \qquad \qquad &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned}$$

The equations (44) et (45) yield as  $\epsilon \rightarrow 0$  that,

$$(46) \quad \int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{B_{\delta,+}} \left( \frac{U_\epsilon^{2^*(s)}}{|x|^s} + 2^*(s) \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} (\Theta \circ \phi) \right) dx + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho.$$

Therefore, for all  $\xi > 0$  the equations (38), (42) and (46) yield the result.  $\square$

**Step 4.4.** *We are now in position to prove Proposition 4.1.*

*Proof of Step 4.4:* By Step 4.3, we have that

$$\begin{aligned} \left( \int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &= \left( \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ (47) \qquad \qquad \qquad &+ 2\alpha_+ m_{\gamma,a}(\Omega) \xi^{-1} C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \left( \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)} - 1} \\ &+ o(\epsilon^{\alpha_+ - \alpha_-}). \end{aligned}$$

We go back to the definition of  $J_{\gamma,a,s}^\Omega$ , Step 4.3, equation (47) and since  $U$  satisfies (11), we get as  $\epsilon \rightarrow 0$  that

$$J_{\gamma,s,a}^\Omega(\tilde{u}_\epsilon) = J_{\gamma,s,0}^{\mathbb{R}^{k_+, n-k}}(U) (1 - m_{\gamma,a}(\Omega) \zeta_{\gamma,s}^0 \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})),$$

where  $\zeta_{\gamma,s}^0$  is defined in (22). This ends the proof of Proposition 4.1.  $\square$

Combining Proposition 4.1 and Theorem 3.1 yields Theorem 1.2 .

## 5. EXAMPLES OF MASS

In this section, we discuss the existence and the sign of the mass. An example of existence of mass is as follows:

**Proposition 5.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We assume that  $\gamma > \gamma_H(\mathbb{R}^{k_+, n-k}) - 1/4$  and that*

$$(48) \quad \Omega \cap B_\delta(0) = \mathbb{R}^{k_+, n-k} \cap B_\delta(0) \text{ for some } \delta > 0.$$

*We assume that  $\gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4} < \gamma < \gamma_H(\Omega)$ , that  $a \in C^{0,\theta}(\Omega)$  vanishes around 0 and that  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive. Then the mass is defined.*

*Proof of Proposition 5.1.* We fix  $\eta$  as in (13). For  $a \in C^{0,\theta}(\Omega)$  that vanishes around 0, define on  $\Omega$  the function

$$g := \left( -\Delta - \frac{\gamma}{|x|^2} - a(x) \right) (\eta S_{\alpha_+}),$$

where  $S_{\alpha_+}$  is defined in (7) such that  $-\Delta S_{\alpha_+} - \gamma|x|^{-2} S_{\alpha_+} = 0$  on  $\mathbb{R}^{k_+, n-k}$ . Note that this definition makes sense when the support of  $\eta$  is small enough due to (48) and  $a$  vanishes around 0. In particular  $g(x) = 0$  around 0. Therefore, we have

$g \in L^{\frac{2n}{n+2}}(\Omega) = (L^{2^*}(\Omega))' \subset (D^{1,2}(\Omega))'$ . Since the operator  $-\Delta - (\gamma|x|^{-2} + a)$  is coercive, there exists  $w \in D^{1,2}(\Omega)$  such that

$$\begin{cases} \left(-\Delta - \frac{\gamma}{|x|^2} - a(x)\right) w = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $g$  vanishes around 0, Theorem 3.2 and the change of variable  $y = \lambda x$  that there exists  $K \in \mathbb{R}$  such that

$$w(x) = K \frac{v(x)}{|x|^{\alpha_- + k}} + o\left(\frac{v(x)}{|x|^{\alpha_- + k}}\right) \text{ as } x \rightarrow 0,$$

where  $v$  is as in (14). For all  $x \in \bar{\Omega} \setminus \{0\}$ , we define the function  $G_0 := \eta S_{\alpha_+} - w$ . The definition of  $w$  yields

$$\begin{cases} \left(-\Delta - \frac{\gamma}{|x|^2} - a(x)\right) G_0 = 0 & \text{in } \Omega, \\ G_0 = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

For  $\delta_0 > 0$  small enough, the definitions of  $S_{\alpha_+}$ ,  $w$  and  $\alpha_- < \alpha_+$  yield

$$G_0(x) = v(x)|x|^{-\alpha_+ - k} (1 + o(1)) \text{ in } \mathbb{R}^{k_+, n-k} \cap B_{\delta_0},$$

with  $o(1) \rightarrow 0$  as  $x \rightarrow 0$ . Therefore,  $G_0 > 0$  in  $\mathbb{R}^{k_+, n-k} \cap B_{\delta_0}$ . Then coercivity and the comparison principle yield  $G_0 > 0$  in  $\Omega$ . Moreover, we have that

$$G_0(x) = v(x) (|x|^{-\alpha_+ - k} - K|x|^{-\alpha_- - k} + o(|x|^{-\alpha_- - k})),$$

as  $x \rightarrow 0$ . Then the mass at 0 of  $-\Delta - (\gamma|x|^{-2} + a(x))$  is defined and  $m_{\gamma, a}(\Omega) = -K$ . This proves Proposition 5.1.

We now discuss briefly examples of negative and positive mass. Here, the reference is Section 9 of Ghoussoub-Robert [8]. We still assume (48) and that  $\gamma > \gamma_H(\mathbb{R}^{k_+, n-k}) - 1/4$ , so that the mass  $m_{\gamma, 0}(\Omega)$  is defined. When  $\Omega \subset \mathbb{R}^{k_+, n-k}$ , due to the comparison principle, we get that  $G_0 < S_{\alpha_+}$ , and  $m_{\gamma, 0}(\Omega) < 0$ . Arguing as in [8], we are able to define the mass of a domain  $\tilde{\Omega} \supset \mathbb{R}^{k_+, n-k}$ , for which  $m_{\gamma, 0}(\tilde{\Omega}) > 0$ : then, defining  $\tilde{\Omega}_R := \tilde{\Omega} \cap B_R(0)$ , we get that  $\lim_{R \rightarrow +\infty} m_{\gamma, 0}(\tilde{\Omega}_R) = m_{\gamma, 0}(\tilde{\Omega}) > 0$ . So for  $R > 0$  large, we get examples of bounded domains with a singularity of type  $(k, n-k)$  at 0 and with positive mass.

## 6. PROOF OF THEOREM 1.3: FUNCTIONAL BACKGROUND FOR THE PERTURBED EQUATION

In this section, we proceed as in Jaber [11]. A Palais-Smale sequence for  $G : E \rightarrow \mathbb{R}$  is a sequence  $(u_m)_{m \in \mathbb{N}} \in E$  such that there exists  $\beta \in \mathbb{R}$  such that

$$G(u_m) \rightarrow \beta \text{ for all } m \in \mathbb{N} \text{ and } G'(u_m) \rightarrow 0 \text{ in } E',$$

as  $m \rightarrow +\infty$ . Here, we say that the Palais-Smale sequence is at level  $\beta$ . The main tool is the Mountain Pass Lemma of Ambrosetti-Rabinowitz [1]:

**Theorem 6.1** (Mountain-Pass Lemma [1]). *Let  $G \in C^1(E, \mathbb{R})$  where  $(E, \|\cdot\|)$  is a Banach space. We assume that  $G(0) = 0$  and that*

- *There exists  $\lambda, r > 0$  such that  $G(u) \geq \lambda$  for all  $u \in E$  such that  $\|u\| = r$ ,*
- *There exists  $u_0$  in  $E$  such that  $\limsup_{t \rightarrow +\infty} G(tu_0) < 0$ .*

We consider  $t_0 > 0$  large such that  $\|t_0 u_0\| > r$  and  $G(t_0 u_0) < 0$ , and

$$\beta = \inf_{c \in \Gamma} \sup G(c(t)),$$

where

$$\Gamma = \{c : [0, 1] \rightarrow E \text{ s.t. } c(0) = 0, c(1) = t_0 u_0\}.$$

Then, there exists a Palais-Smale sequence at level  $\beta$  for  $G$ . Moreover, we have that  $\beta \leq \sup_{t \geq 0} G(tu_0)$ .

Weak solutions of (5) are to the nonzero critical points of the functional

$$E_q(u) := \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u^2 \right) dx - \int_{\Omega} \frac{u_+^{2^*(s)}}{2^*(s)|x|^s} dx - \int_{\Omega} \frac{hu_+^{q+1}}{q+1} dx,$$

for any  $u \in D^{1,2}(\Omega)$  and where  $u_+ = \max\{u, 0\}$ . In the sequel, we assume that the operator  $-\Delta - \left( \frac{\gamma}{|x|^2} + a(x) \right)$  is coercive that there exists  $c_0 > 0$  such that

$$(49) \quad \int_{\Omega} \left( |\nabla w|^2 - \left( \frac{\gamma}{|x|^2} + a \right) w^2 \right) dx \geq c_0 \int_{\Omega} |\nabla w|^2 dx \text{ for all } w \in D^{1,2}(\Omega).$$

**Proposition 6.1.** *Fix  $u_0 \in D^{1,2}(\Omega)$  such that  $u_0 \geq 0$ ,  $u_0 \not\equiv 0$ . Then there exists a sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  that is a Palais-Smale sequence for  $E_q$  at level  $\beta$  such that  $0 < \beta \leq \sup_{t \geq 0} E_q(tu_0)$ .*

*Proof of Proposition 6.1:* Clearly  $E_q \in C^1(D^{1,2}(\Omega))$ . Note that  $E_q(0) = 0$ . It follows from (49) and the Sobolev and Hardy-Sobolev embeddings that there exist  $c_0, c_1, c_2 > 0$  such that

$$(50) \quad E_q(u) \geq c_0 \|u\|^2 - c_1 \|u\|^{2^*(s)} - c_2 \|u\|^{q+1} \text{ for all } u \in D^{1,2}(\Omega).$$

Define  $f(r) = r^2 [c_0 - c_1 r^{2^*(s)-2} - c_2 r^{q-1}] := r^2 g(r)$  and since  $2^*(s), q+1 > 2$  we have  $g(r) \rightarrow c_0$  as  $r \rightarrow 0$ . Then there exists  $r_0 > 0$  such that  $r < r_0$ , we have  $g(r) > \frac{c_0}{2}$ . Therefore, for all  $u \in D^{1,2}(\Omega)$  such that  $\|u\| = \frac{r_0}{2}$  and by (50), we have  $E_q(u) \geq \frac{c_0 r_0^2}{8} := \lambda$ . We fix  $u_0 \in D^{1,2}(\Omega)$ ,  $u_0 \not\equiv 0$ . We have that

$$\begin{aligned} E_q(tu_0) &= \frac{t^2}{2} \int_{\Omega} \left( |\nabla u_0|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u_0^2 \right) dx \\ &\quad - \frac{t^{2^*(s)}}{2^*(s)} \int_{\Omega} \frac{|u_0|^{2^*(s)}}{|x|^s} dx - \frac{t^{q+1}}{q+1} \int_{\Omega} h |u_0|^{q+1} dx \\ &:= \frac{t^2}{2} R_1 - \frac{t^{2^*(s)}}{2^*(s)} R_2 - \frac{t^{q+1}}{q+1} R_3 \leq t^{2^*(s)} \left( \frac{t^{2-2^*(s)}}{2} R_1 - R_2 \right), \end{aligned}$$

where  $R_1, R_2 > 0$  and  $R_3 \geq 0$ . Since  $2^*(s) > 2$ , we have  $E_q(tu_0) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Then  $\limsup_{t \rightarrow +\infty} E_q(tu_0) < 0$ . We consider  $t_0 > 0$  large such that  $\|t_0 u_0\| > r$  and  $E_q(t_0 u_0) < 0$ . For  $t \in [0, 1]$ , we have  $E_q(c(t)) \geq \lambda$  and then there exists

$$\beta := \inf_{c \in \Gamma} \sup E_q(c(t)) \geq \lambda > 0.$$

Proposition 6.1 then follows from Theorem 6.1.  $\square$



**Proposition 6.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $a, h \in C^{0,\theta}(\Omega)$ ,  $\theta \in (0, 1)$ . We assume that  $h \geq 0$  and that (49) holds. We fix  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$  and  $\beta \in \mathbb{R}$  such that*

$$(51) \quad \beta < \frac{2-s}{2(n-s)} \mu_{\gamma, s, 0}(\mathbb{R}^{k+, n-k})^{\frac{n-s}{2-s}}.$$

*Then, for any Palais-Smale sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  for  $E_q$  at level  $\beta$ , there exists  $u \in D^{1,2}(\Omega)$  such that  $E_q(u) = \beta$  and we have  $u_m$  converges strongly in  $D^{1,2}(\Omega)$  up to a subsequence. Moreover, we have  $E'_q(u) = 0$ .*

*Proof of Proposition 6.2:* Take  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  a Palais-Smale sequence for  $E_q$  such that

$$E_q(u_m) \rightarrow \beta \text{ and } E'_q(u_m) \rightarrow 0 \text{ in } D^{1,2}(\Omega)'.$$

**Step 6.1.** *We claim that  $u_m$  is bounded in  $D^{1,2}(\Omega)$ .*

*Proof of Step 6.1:* The coercivity (21) and the definition of  $E_q$  yield

$$(52) \quad \|u_m\|^2 \leq 2c_0^{-1} \left( E_q(u_m) + \frac{1}{2^*(s)} \int_{\Omega} (u_m)_+^{2^*(s)} dx + \frac{1}{q+1} \int_{\Omega} h(u_m)_+^{q+1} dx \right).$$

Since  $E'_q(u_m) \rightarrow 0$  in  $D^{1,2}(\Omega)'$ , we observe that

$$\int_{\Omega} \left( |\nabla u_m|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u_m^2 \right) dx = \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx + \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|).$$

The definition of the energy  $E_q$  and the last equation yield

$$(53) \quad 2E_q(u_m) = \left( 1 - \frac{2}{2^*(s)} \right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx + \left( 1 - \frac{2}{q+1} \right) \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|).$$

Moreover, since  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ ,  $h \geq 0$  and  $q+1 > 2$ , we obtain that

$$\begin{aligned} \left( 1 - \frac{2}{2^*(s)} \right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx &= 2E_q(u_m) - \left( 1 - \frac{2}{q+1} \right) \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|) \\ &\leq 2\beta + o(\|u_m\|), \end{aligned}$$

therefore,

$$(54) \quad \left( 1 - \frac{2}{2^*(s)} \right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx = O(1) + o(\|u_m\|).$$

Similar and but  $2^*(s) > 2$ , we have

$$(55) \quad \left( 1 - \frac{2}{q+1} \right) \int_{\Omega} h(u_m)_+^{q+1} dx = O(1) + o(\|u_m\|).$$

(52) and (53) give

$$(56) \quad \|u_m\|^2 \leq c_0^{-1} \left( \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx + \int_{\Omega} h(u_m)_+^{q+1} dx \right) + o(\|u_m\|).$$

The equations (54), (55) and (56) yields,

$$\|u_m\|^2 = O(1) + o(\|u_m\|),$$

as  $m \rightarrow +\infty$ . This proves Step 6.1.  $\square$

Therefore, up to a subsequence, there exists  $u \in D^{1,2}(\Omega)$  such that

$$(57) \quad \begin{cases} u_m \rightharpoonup u & \text{weakly in } D^{1,2}(\Omega), \\ u_m \rightarrow u & \text{strongly in } L^p(\Omega) \text{ for all } 1 < p < 2^*. \end{cases}$$

Moreover, we have  $E'_q(u) = 0$ .

**Step 6.2.** We claim that, as  $m \rightarrow +\infty$

$$(58) \quad \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx = \int_{\Omega} \frac{(u_m - u)_+^{2^*(s)}}{|x|^s} dx + o(1),$$

and,

$$(59) \quad \frac{2-s}{2(n-s)} \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx \leq \beta + o(1).$$

*Proof of Step 6.2:* We denote that

$$\begin{aligned} \langle E'_q(u_m), \varphi \rangle &= \int_{\Omega} \left( (\nabla u_m, \nabla \varphi) - \left( \frac{\gamma}{|x|^2} + a \right) u_m \varphi \right) dx \\ &\quad - \int_{\Omega} \frac{(u_m)_+^{2^*(s)-1}}{|x|^s} \varphi dx - \int_{\Omega} h(u_m)_+^q \varphi dx, \end{aligned}$$

for all  $\varphi \in D^{1,2}(\Omega)$ . We observe that

$$(60) \quad \begin{aligned} o(1) &= \langle E'_q(u_m) - E'_q(u), u_m - u \rangle \\ &= \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \left( \frac{\gamma}{|x|^2} + a \right) (u_m - u)^2 \right) dx \\ &\quad - \int_{\Omega} \left( (u_m)_+^{2^*(s)-1} - u_+^{2^*(s)-1} \right) \frac{(u_m - u)}{|x|^s} dx \\ &\quad - \int_{\Omega} h \left( (u_m)_+^q - u_+^q \right) (u_m - u) dx. \end{aligned}$$

Since  $u_m \rightharpoonup u$  weakly in  $D^{1,2}(\Omega)$ , integration theory yields

$$(61) \quad \lim_{m \rightarrow +\infty} \int_{\Omega} \frac{(u_m)_+^{2^*(s)-1}}{|x|^s} u dx = \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx = \lim_{m \rightarrow +\infty} \int_{\Omega} \frac{u_+^{2^*(s)-1}}{|x|^s} u_m dx.$$

The equation (57) yields,

$$(62) \quad \int_{\Omega} h(u_m - u) \left( (u_m)_+^q - u_+^q \right) dx = \int_{\Omega} h(u_m - u)^{q+1} dx + o(1) = o(1),$$

as  $m \rightarrow +\infty$ . Combining (60), (61) and (62), we get as  $m \rightarrow +\infty$  that

$$(63) \quad \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \left( \frac{\gamma}{|x|^2} + a \right) (u_m - u)^2 \right) dx = \int_{\Omega} \left( (u_m)_+^{2^*(s)} - u_+^{2^*(s)} \right) \frac{dx}{|x|^s} + o(1).$$

Since  $2^*(s) > 1$ , we get that

$$\left| (u_m)_+^{2^*(s)} - u_+^{2^*(s)} - (u_m - u)_+^{2^*(s)} \right| \leq C \left( |u_m - u|^{2^*(s)-1} |u| + |u|^{2^*(s)-1} |u_m - u| \right),$$

for some  $C > 0$  independent of  $m$ . Therefore, but (57) we have

$$(64) \quad \int_{\Omega} \left( (u_m)_+^{2^*(s)} - (u_m - u)_+^{2^*(s)} \right) \frac{dx}{|x|^s} = \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx + o(1).$$

Since  $u_m \rightarrow u$  strongly in  $L^2(\Omega)$  as  $m \rightarrow +\infty$  and by (63), (64), we obtain (58). With (57) we have that

$$\begin{aligned} E_q(u_m) - E_q(u) &= \frac{1}{2} \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx \\ &\quad - \frac{1}{2^*(s)} \int_{\Omega} \left( (u_m)_+^{2^*(s)} - u_+^{2^*(s)} \right) \frac{dx}{|x|^s} + o(1). \end{aligned}$$

With (58), we get

$$E_q(u_m) - E_q(u) = \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx + o(1).$$

Since  $u$  is a solution of (5) then  $E_q(u) \geq 0$ , and  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ . We then get (59). This proves Step 6.2.  $\square$

**Step 6.3.** *We claim that*

$$(65) \quad \lim_{m \rightarrow +\infty} u_m = u \text{ in } D^{1,2}(\Omega).$$

*Proof of Step 6.3:* Let  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$  for all  $k \in \{1, \dots, n\}$  and by the Proposition 2.1 in Cheikh-Ali [4], then for all  $\epsilon > 0$  there exists  $c_\epsilon > 0$  such that for all  $v \in D^{1,2}(\Omega)$ ,

$$\left( \int_{\Omega} \frac{|v|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{-1} + \epsilon) \int_{\Omega} \left( |\nabla v|^2 - \frac{\gamma}{|x|^2} v^2 \right) dx + c_\epsilon \int_{\Omega} v^2 dx.$$

Take  $\theta_m = u_m - u$ . Since  $u_m$  converges to  $u$  in  $L^2(\Omega)$  taking  $v = \theta_m$  yields

$$(66) \quad \left( \int_{\Omega} \frac{(\theta_m)_+^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{-1} + \epsilon) \int_{\Omega} \left( |\nabla \theta_m|^2 - \frac{\gamma}{|x|^2} \theta_m^2 \right) dx + o(1).$$

We write  $N(\theta_m) := \int_{\Omega} \left( |\nabla \theta_m|^2 - \frac{\gamma}{|x|^2} \theta_m^2 \right) dx$ . By (58) and (66), we get that

$$N(\theta_m)^{\frac{2}{2^*(s)}} \left( 1 - (\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{-1} + \epsilon) N(\theta_m)^{1 - \frac{2}{2^*(s)}} \right) \leq o(1).$$

With (59) and the last inequation, we get that, as  $\epsilon \rightarrow 0$ ,

$$(67) \quad N(\theta_m)^{\frac{2}{2^*(s)}} \left( 1 - (\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{-1} + \epsilon) \left( \frac{2(n-s)\beta}{2-s} \right)^{\frac{2^*(s)-2}{2^*(s)}} + o(1) \right) \leq o(1).$$

With the assumption (51) and (67), taking  $\epsilon > 0$  small enough, we get that  $N(\theta_m) \rightarrow 0$  as  $m \rightarrow +\infty$  and by coercivity, we obtain (65).  $\square$

With Step 6.3 and since  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ , we get that  $E_q(u) = \beta$ . This ends the proof of Proposition 6.2.  $\square$

**Theorem 6.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . We fix  $a \in C^{0,\theta}(\Omega)$  such that  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive, and  $h \in C^{0,\theta}(\Omega)$  such that  $h \geq 0$ . We fix  $0 \leq s < 2$  and  $1 < q < 2^* - 1$ . We assume that there exists  $u_0 \in D^{1,2}(\Omega)$ ,  $u_0 \not\equiv 0$ , such that*

$$(68) \quad \sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}}.$$

Then equation (5) has a non-vanishing solution in  $D^{1,2}(\Omega)$ .

*Proof of Theorem 6.2:* By Proposition 6.1, there exists a Palais-Smale sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  for  $E_q$  at level  $\beta > 0$  such that  $\beta \leq \sup_{t \geq 0} E_q(tu_0)$ . It then follows from Proposition 6.2 that, up to a subsequence,  $(u_m)$  converges strongly to  $u$  in  $D^{1,2}(\Omega)$ . Then  $E_q(u) = \beta > 0$ , so  $u \not\equiv 0$ , and  $E'_q(u) = 0$ . Coercivity and  $E'_q(u)[u_-] = 0$  yield  $u \geq 0$ . Regularity theory and Hopf's principle yield  $u \in C^{2,\theta}(\Omega)$  and  $u > 0$ . Then  $u$  is a solution of (5). This proves Theorem 6.2.  $\square$

## 7. PROOF OF THEOREM 1.3: TEST-FUNCTIONS ESTIMATES

The main result of this section is the following:

**Proposition 7.1.** *For  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$  and fix  $0 \leq s < 2$ . We assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ , we let  $U$  as in (11) be such an extremal. We let  $(u_\epsilon)_\epsilon$  and  $(\tilde{u}_\epsilon)_\epsilon$  as in (21). Then,*

(a) *For  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega) \epsilon + o(\epsilon) & \text{if } q+1 < \frac{2n-2}{n-2}, \\ (c_1 GH_{\gamma,s}(\Omega) - c_2 h(0)) \epsilon + o(\epsilon) & \text{if } q+1 = \frac{2n-2}{n-2}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o(\epsilon^{n - \frac{(q+1)(n-2)}{2}}) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(b) *For  $0 \leq \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega) \epsilon \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o\left(\epsilon^{n - \frac{(q+1)(n-2)}{2}}\right) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(c) *For  $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(t\tilde{u}_\epsilon) := \beta_0 + \begin{cases} -c_3 m_\gamma(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) & \text{if } q+1 < \frac{2n-2(\alpha_+ - \alpha_-)}{n-2}, \\ -(c_3 m_\gamma(\Omega) + c_2 h(0)) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) & \text{if } q+1 = \frac{2n-2(\alpha_+ - \alpha_-)}{n-2}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o\left(\epsilon^{n - \frac{(q+1)(n-2)}{2}}\right) & \text{if } q+1 > \frac{2n-2(\alpha_+ - \alpha_-)}{n-2}, \end{cases}$$

where  $\beta_0 = \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}}$ ,

$$(69) \quad \begin{cases} c_1 = \frac{\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{2^*(s)}{2^*(s)-2}}}{2} \left( \xi \int_{\mathbb{R}^{k+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1}, \\ c_2 = \frac{\xi^{\frac{q+1}{2^*(s)-2}}}{q+1} \int_{\mathbb{R}^{k+,n-k}} U^{q+1} dx, \\ c_3 = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{2^*(s)}{2^*(s)-2}} \frac{\alpha_+ - \alpha_-}{2} \frac{\int_{S^{n-1} \cap \mathbb{R}^{k+,n-k}} \left( \prod_{i=1}^k x_i \right)^2 d\sigma}{\xi \int_{\mathbb{R}^{k+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx}. \end{cases}$$

Theorem 6.2 and Proposition 7.1 yield Theorem 1.3.

*Proof of Proposition 7.1:* We define the test-function sequence  $(Z_\epsilon)_{\epsilon > 0}$  by

$$Z_\epsilon(x) := \begin{cases} u_\epsilon & \text{if } \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}, \\ \tilde{u}_\epsilon & \text{if } \gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}, \end{cases}$$

where  $u_\epsilon$  and  $\tilde{u}_\epsilon$  are as in the definition (21). We have:

$$E_q(tZ_\epsilon) = \frac{t^2}{2} R_\epsilon - \frac{t^{2^*(s)}}{2^*(s)} B_\epsilon - \frac{t^{q+1}}{q+1} C_{h,\epsilon},$$

when  $\epsilon \rightarrow 0$  where:

$$R_\epsilon := \int_{\Omega} \left( |\nabla Z_\epsilon|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) Z_\epsilon^2 \right) dx$$

$$B_\epsilon := \int_{\Omega} \frac{Z_\epsilon^{2^*(s)}}{|x|^s} dx \text{ and } C_{h,\epsilon} := \int_{\Omega} h Z_\epsilon^{q+1} dx.$$

**Step 7.1.** We fix  $f \in C^{0,\theta}(\Omega)$ ,  $\theta \in (0, 1)$ , and  $p \in [1, 2^*)$ . We claim that

$$\int_{\Omega} f |Z_\epsilon|^{p+1} dx = \begin{cases} f(0) \epsilon^{n - \frac{n-2}{2}(p+1)} \int_{\mathbb{R}^{k_+, n-k}} U^{p+1} dx + o\left(\epsilon^{n - \frac{n-2}{2}(p+1)}\right) & \text{if } n < (p+1)\alpha_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } n = (p+1)\alpha_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)}\right) & \text{if } n > (p+1)\alpha_+. \end{cases}$$

Moreover, we have

$$(70) \quad \int_{\Omega} f |Z_\epsilon|^{p+1} dx \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

*Proof of Step 7.1:* Note that it follows from (19) that

$$(71) \quad 0 < U_\epsilon(x) \leq C \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} |x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^{k_+, n-k} \text{ and } \epsilon > 0.$$

We first prove Step 7.1 for  $u_\epsilon$ , postponing the case of  $\tilde{u}_\epsilon$ , and then  $Z_\epsilon$ , to the end of the proof. We distinguish three cases:

*Case 1:* We assume that  $n > (p+1)\alpha_+$ . It follows from (71) that

$$\left| \int_{\Omega} f |u_\epsilon|^{p+1} dx \right| \leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \int_{\Omega} |x|^{-(p+1)\alpha_+} dx \leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)}$$

as  $\epsilon \rightarrow 0$ . This proves Step 7.1 for  $u_\epsilon$  when  $n > (p+1)\alpha_+$ .

*Case 2:* We assume that  $n = (p+1)\alpha_+$ . With (71), we get that

$$\begin{aligned} \left| \int_{\Omega} f |u_\epsilon|^{p+1} dx \right| &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \int_{B_{\delta,+}} |u_\epsilon|^{p+1} dx \\ &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \epsilon^{n - \frac{n-2}{2}(p+1)} \int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx \\ &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \epsilon^{n - \frac{n-2}{2}(p+1)} \int_1^{\epsilon^{-1}\delta} r^{-1} dr \\ &\leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \ln\left(\frac{1}{\epsilon}\right) \end{aligned}$$

*Case 3:* We assume that  $n < (p+1)\alpha_+$ . It follows from (71) that

$$\int_{\Omega \setminus \phi(B_{\delta,+})} f |u_\epsilon|^{p+1} dx = O\left(\epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)}\right) \text{ as } \epsilon \rightarrow 0.$$

Independently, since  $f \in C^{0,\theta}(\Omega)$ , we have that

$$(72) \quad \begin{aligned} \int_{\phi(B_{\delta,+})} f |u_\epsilon|^{p+1} dx &= \int_{B_{\delta,+}} f \circ \phi \cdot U_\epsilon^{p+1} |\text{Jac } \phi| dx \\ &= \epsilon^{n - \frac{n-2}{2}(p+1)} f(0) \int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx + O\left(\int_{B_{\delta,+}} |x|^\theta |U_\epsilon|^{p+1} dx\right) \end{aligned}$$

Since  $n < (p+1)\alpha_+$ , it follows from (19) that  $U \in L^{p+1}(\mathbb{R}^{k_+,n-k})$  and that

$$\begin{aligned}
\int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx &= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O\left(\int_{\mathbb{R}^{k_+,n-k} \setminus B_{\delta\epsilon^{-1},+}} U^{p+1} dx\right) \\
&= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O\left(\int_{\epsilon^{-1}\delta}^{\infty} r^{n-(p+1)\alpha_+-1} dr\right) \\
(73) \qquad &= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O\left(\epsilon^{(p+1)\alpha_+-n}\right)
\end{aligned}$$

We claim that

$$(74) \qquad \int_{B_{\delta,+}} |x|^\theta |U_\epsilon|^{p+1} dx = o\left(\epsilon^{n-\frac{n-2}{2}(p+1)}\right) \text{ as } \epsilon \rightarrow 0.$$

Indeed, when  $\theta + n > (p+1)\alpha_+$ , we argue as in Case 1. When  $\theta + n = (p+1)\alpha_+$ , we argue as in Case 2. When  $\theta + n < (p+1)\alpha_+$ , we make a change of variable  $y = \epsilon^{-1}x$  and we argue as in (73). This yields (74). Putting (73) and (74) in (72) yields Step 7.1 for  $u_\epsilon$  in Case 2.

We now prove Step 7.1. When  $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ ,  $Z_\epsilon = u_\epsilon$ , and we are done. When  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ ,  $Z_\epsilon = \tilde{u}_\epsilon$ . With the definition (21), we get that

$$\begin{aligned}
(75) \quad \int_{\Omega} f|\tilde{u}_\epsilon|^{p+1} dx &= \int_{\Omega} f \left| u_\epsilon + \epsilon^{\frac{\alpha_+-\alpha_-}{2}} \Theta \right|^{p+1} dx \\
&= \int_{\Omega} f|u_\epsilon|^{p+1} dx + O\left(\epsilon^{\frac{\alpha_+-\alpha_-}{2}} \int_{\Omega} |u_\epsilon|^p |\Theta| dx\right) + O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)} \int_{\Omega} |\Theta|^{p+1} dx\right)
\end{aligned}$$

Since  $\Theta \in D^{1,2}(\Omega)$  and  $p+1 < 2^*$ , we get that  $\Theta \in L^{p+1}(\Omega)$ . It follows from (15) that  $|\Theta(x)| \leq C|x|^{-\alpha_-}$  for all  $x \in \Omega$ . Arguing as in Cases 1, 2, 3 above, we get that the second term in the right-hand-side of (75) is dominated by  $\int_{\Omega} |u_\epsilon|^{p+1} dx$ . Then Step 7.1 for  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$  follows.  $\square$

By Cheikh-Ali [4] and Step 7.1 for the case  $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$  and Steps 4.2 and 4.3 for the case  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$ , we get that, as  $\epsilon \rightarrow 0$ ,

$$(76) \quad R_\epsilon \rightarrow R_0 := \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \text{ and } B_\epsilon \rightarrow B_0 := \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx.$$

**Step 7.2.** We claim that for all  $\epsilon > 0$ , then there exists a unique  $t_\epsilon$  such that

$$(77) \qquad \sup_{t \geq 0} E_q(tZ_\epsilon) = E_q(t_\epsilon Z_\epsilon).$$

Moreover,  $t_\epsilon$  verifies

$$(78) \qquad t_\epsilon = S_\epsilon [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})],$$

where  $S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}$ ,  $C_0 > 0$  and  $t_\epsilon \rightarrow t_0$  as  $\epsilon \rightarrow 0$ .

*Proof of Step 7.2:* We have that  $\partial_t E_q(tZ_\epsilon) = 0$  iff  $t = 0$  or  $g_\epsilon(t) = R_\epsilon$  where  $g_\epsilon(t) := B_\epsilon t^{2^*(s)-2} + C_{h,\epsilon} t^{q-1}$ . Since  $B_\epsilon, C_{h,\epsilon} \geq 0$  and  $g_\epsilon$  is a strictly increasing map i.e  $g_\epsilon(t) - R_\epsilon$  also, and since  $R_\epsilon > 0$  we have  $g_\epsilon(0) - R_\epsilon < 0$  then, there exists  $t_\epsilon > 0$  unique verifying  $g_\epsilon(t_\epsilon) = R_\epsilon$  such that (77) holds. Since  $g_\epsilon(t_\epsilon) = R_\epsilon$ , we get

$$t_\epsilon \leq S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}.$$

We are using (76), (70) and (11) to get that  $S_\epsilon \rightarrow (R_0 B_0^{-1})^{\frac{1}{2^*(s)-2}} = \xi^{\frac{1}{2^*(s)-2}}$  as  $\epsilon \rightarrow 0$ . Therefore,  $t_\epsilon$  is bounded and there exists  $t_0$  such that  $t_\epsilon \rightarrow t_0$  up to extraction. Since  $g_\epsilon(t_\epsilon) = R_\epsilon$  and  $C_{h,\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , we obtain that

$$\begin{aligned} t_\epsilon &= [R_\epsilon B_\epsilon^{-1} - C_{h,\epsilon} B_\epsilon^{-1} t_\epsilon^{q-1}]^{\frac{1}{2^*(s)-2}} \\ &= S_\epsilon [1 - C_{h,\epsilon} R_\epsilon^{-1} t_\epsilon^{q-1}]^{\frac{1}{2^*(s)-2}} = S_\epsilon [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})], \end{aligned}$$

where  $C_0 := \frac{R_0^{-1} t_0^{q-1}}{2^*(s)-2}$  and  $t_0 = \xi^{\frac{1}{2^*(s)-2}}$ . This yields (78) and Step 7.2.  $\square$

**Step 7.3.** *We claim that*

$$E_q(t_\epsilon Z_\epsilon) = \frac{2-s}{2(n-s)} (J_{\gamma,s,a}^\Omega(Z_\epsilon))^{\frac{2^*(s)}{2^*(s)-2}} - \frac{\xi^{\frac{q+1}{2^*(s)-2}}}{q+1} C_{h,\epsilon} + o(C_{h,\epsilon}).$$

*Proof of Step 7.3:* The expression (78) of Step 7.2 and (70) yield

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{t_\epsilon^2}{2} R_\epsilon - \frac{t_\epsilon^{2^*(s)}}{2^*(s)} B_\epsilon - \frac{t_\epsilon^{q+1}}{q+1} C_{h,\epsilon} \\ &= \frac{S_\epsilon^2 [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^2}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)} [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^{2^*(s)}}{2^*(s)} B_\epsilon \\ &\quad - \frac{S_\epsilon^{q+1} [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^{q+1}}{q+1} C_{h,\epsilon} \\ &= \frac{S_\epsilon^2 [1 - 2C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)} [1 - C_0 2^*(s) C_{h,\epsilon} + o(C_{h,\epsilon})]}{2^*(s)} B_\epsilon \\ &\quad - \frac{S_\epsilon^{q+1} [1 - (q+1)C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]}{q+1} C_{h,\epsilon}, \end{aligned}$$

then,

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{S_\epsilon^2}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)}}{2^*(s)} B_\epsilon - \frac{S_\epsilon^{q+1}}{q+1} C_{h,\epsilon} \\ &\quad - C_0 C_{h,\epsilon} \left[ S_\epsilon^2 R_\epsilon - S_\epsilon^{2^*(s)} B_\epsilon - S_\epsilon^{q+1} C_{h,\epsilon} \right] + o(C_{h,\epsilon}). \end{aligned}$$

Since  $S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}$  and  $C_{h,\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , this yields Step 7.3.  $\square$

**Proof of Proposition 7.1 when  $0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ .** In this case, we recall that  $Z_\epsilon(x) = u_\epsilon(x)$ . Note that

$$\left\{ \gamma < (=) \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \right\} \Leftrightarrow \{ \alpha_+ - \alpha_- > (=) 1 \}.$$

It was proved in Proposition 5.1 in Cheikh-Ali [4] that

- For  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have that

$$(79) \quad J_{\gamma,s,0}^\Omega(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k}) (1 + \kappa GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)).$$

- For  $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have that

$$(80) \quad J_{\gamma,s,0}^\Omega(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k}) \left( 1 + \kappa GH_{\gamma,s}(\Omega)\epsilon \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon \ln\left(\frac{1}{\epsilon}\right)\right) \right),$$

where  $\kappa := \left( \xi \int_{\mathbb{R}^{k_+, n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1}$  and  $GH_{\gamma, s}(\Omega)$  is defined in (6). It follows from Step 7.1 that  $\int_{\Omega} u_{\epsilon}^2 dx = o(\epsilon)$  if  $\alpha_+ - \alpha_- > 1$ , and  $O(\epsilon)$  if  $\alpha_+ - \alpha_- = 1$ . Therefore (79) and (80) hold unchanged with the potential  $a$ .

*Case 1:* We assume that  $n < (q+1)\alpha_+$ . It follows from Step 7.1 that

$$C_{h, \epsilon} = \int_{\Omega} h |u_{\epsilon}|^{q+1} dx = h(0) \epsilon^{n - \frac{n-2}{2}(q+1)} \int_{\mathbb{R}^{k_+, n-k}} U^{q+1} dx + o\left(\epsilon^{n - \frac{n-2}{2}(q+1)}\right)$$

as  $\epsilon \rightarrow 0$ . Then, when  $n < (q+1)\alpha_+$ , we get Case (a) of Proposition 7.1 follows by combining Step 7.3, (79), (80), the estimate of  $C_{h, \epsilon}$  and studying the relative positions of  $n - \frac{n-2}{2}(q+1)$  and 1.

*Case 2:* We assume that  $n \geq (q+1)\alpha_+$ . Since  $\alpha_+ - \alpha_- \geq 1$  and  $q > 1$ , we then get that

$$n - \frac{n-2}{2}(q+1) - 1 = (n - (q+1)\alpha_+) + \frac{q+1}{2}(\alpha_+ - \alpha_-) - 1 > 0.$$

Then, for  $n \geq (q+1)\alpha_+$ , Cases (a) and (b) of Proposition 7.1 follows by the same arguments as in Case 1.

This proves Cases (a) and (b) of Proposition 7.1.  $\square$

**Proof of Proposition 7.1 when  $\gamma > \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$ .** Proposition 4.1 yields

$$(81) \quad J_{\gamma, s, a}^{\Omega}(\tilde{u}_{\epsilon}) = \mu_{\gamma, s, 0}(\mathbb{R}^{k_+, n-k}) \left( 1 - \zeta_{\gamma, s}^0 m_{\gamma, a}(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) \right),$$

as  $\epsilon \rightarrow 0$ . Here, we compare  $n - \frac{n-2}{2}(q+1)$  and  $\alpha_+ - \alpha_-$ . Note that

$$n - \frac{n-2}{2}(q+1) - (\alpha_+ - \alpha_-) = n - (q+1)\alpha_+ + \frac{q-1}{2}(\alpha_+ - \alpha_-).$$

Therefore, since  $q > 1$ , when  $n \geq (q+1)\alpha_+$ , we have that  $n - \frac{n-2}{2}(q+1) > \alpha_+ - \alpha_-$ . As for the case  $\gamma \leq \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$ , we get Case (b) of Proposition 7.1 by studying the relative positions of  $n - \frac{n-2}{2}(q+1)$  and  $\alpha_+ - \alpha_-$  and using Step 7.1 and (81).

This proves Case (c) of Proposition 7.1.  $\square$

All these cases prove Proposition 7.1. As already mentioned, Theorem 6.2 and Proposition 7.1 yield Theorem 1.3.

## REFERENCES

- [1] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.
- [2] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.
- [3] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275.
- [4] H. Cheikh Ali, *Hardy–Sobolev inequalities with singularities on non smooth boundary: Hardy constant and extremals. Part I: Influence of local geometry*, Nonlinear Anal. **182** (2019), 316–349.
- [5] Z. Djadli, *Nonlinear elliptic equations with critical Sobolev exponent on compact Riemannian manifolds*, Calc. Var. Partial Differential Equations **8** (1999), no. 4, 293–326.
- [6] V. Felli and A. Ferrero, *Almgreen-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 5, 957–1019.
- [7] N. Ghoussoub and F. Robert, *Sobolev inequalities for the Hardy–Schrödinger operator: extremals and critical dimensions*, Bull. Math. Sci. **6** (2016), no. 1, 89–144.



- [8] ———, *Hardy-singular boundary mass and Sobolev-critical variational problems*, Anal. PDE **10** (2017), no. 5, 1017–1079.
- [9] N. Ghoussoub and X. S. Kang, *Hardy-Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 767–793.
- [10] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703–5743.
- [11] H. Jaber, *Mountain pass solutions for perturbed Hardy-Sobolev equations on compact manifolds*, Analysis (Berlin) **36** (2016), no. 4, 287–296.
- [12] ———, *Influence of Mean Curvature on Mountain-Pass Solutions for Hardy-Sobolev Equations*, Rocky Mountain J. Math (to appear). <https://projecteuclid.org/euclid.rmjm/1542942040>.
- [13] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations **156** (1999), no. 2, 407–426.
- [14] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.
- [15] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), no. 1, 47–71.

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