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Abstract

In this work we prove the nonlinear instability of inhomogeneous steady states solutions to the Hamiltonian Mean Field (HMF) model. We first study the linear instability of this model under a simple criterion by adapting the techniques developed in [19]. In a second part, we extend to the inhomogeneous case some techniques developed in [14, 17, 18] and prove a nonlinear instability result under the same criterion.

1 Introduction

In this paper, we are interested in the nonlinear instability of inhomogeneous steady states of the Hamiltonian Mean Field (HMF) system. The HMF system is a kinetic model describing particles moving on a unit circle interacting via an infinite range attractive cosine potential. This 1D model holds many qualitative properties of more realistic long-range interacting systems as the Vlasov-Poisson model. The HMF model has been the subject of many works in the physical community, for the study of non equilibrium phase transitions [11, 26, 2, 24], of travelling clusters [6, 27] or of relaxation processes [28, 3, 12]. The long-time validity of the N-particle approximation for the HMF model has been investigated in [8, 9] and the Landau-damping phenomenon near a spatially homogeneous state has been studied recently in [13]. The formal linear stability of inhomogeneous steady states has been studied in [10, 23, 7]. In particular, a simple criterion of linear stability has been derived in [23]. In [22], the authors of the present paper have proved that, under the same criterion $\kappa_0 < 1$ (see below for a precise formulation), the inhomogeneous steady states of HMF that are nonincreasing functions of the microscopic energy are nonlinearly stable. The aim of the present paper is to show, in a certain sense, that this criterion is sharp: we show that if $\kappa_0 > 1$, the HMF model can develop instabilities, from both the linear and the nonlinear points of view.

In [15], Guo and Lin have derived a sufficient criterion for linear instability to 3D Vlasov-Poisson by extending an approach developped in [19] for BGK waves. Let us also mention that both works have adapted some techniques presented in [16] to prove the nonlinear instability of the 3D Vlasov-Poisson system. In the first part of this article we adapt these techniques and prove the linear instability of nonhomogeneous steady states to the HMF system. In [17], a nonlinear instability result for 1D Vlasov-Poisson equation was obtained for an initial data close to stationary homogeneous profiles that satisfy a Penrose instability criterion by using an approach developed in [14]. In [18], starting with the N- particles version of the HMF model, a nonlinear instability result is obtained for the corresponding Vlasov approximation by also considering a Penrose instability condition for stationary homogeneous profiles. In the second part of this article our aim is to prove the nonlinear instability of non-homogeneous steady states of HMF by adapting the techniques developed in [14, 17, 18].

In the HMF model, the distribution function of particles $f(t,\theta,v)$ solves the initial-valued problem

$$\begin{aligned} &\partial_t f + v\partial_\theta f - \partial_\theta \phi_f \partial_v f = 0, \qquad (t,\theta,v) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \\ &f(0,\theta,v) = f_{\text{init}}(\theta,v) \geqslant 0, \end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is the flat torus $[0,2\pi]$ and where the self-consistent potential ϕ_f associated with a distribution function f is defined by

$$\phi_f(\theta) = -\int_0^{2\pi} \rho_f(\theta')\cos(\theta-\theta')d\theta', \qquad \rho_f(\theta) = \int_{\mathbb{R}} f(\theta,v)dv. \tag{1.2}$$

Introducing the so-called magnetization vector defined by

$$M_f = \int_0^{2\pi} \rho_f(\theta)u(\theta)d\theta, \qquad \text{with} \quad u(\theta) = (\cos\theta, \sin\theta)^T \tag{1.3}$$

we have

$$\phi_f(\theta) = -M_f \cdot u(\theta). \tag{1.4}$$

In this work will consider steady states of (1.1) of the form

$$f_0(\theta,v) = F(e_0(\theta,v)), \tag{1.5}$$

where F is a given nonnegative function and where the microscopic energy $e_0(\theta,v)$ is given by

$$e_0(\theta,v) = \frac{v^2}{2} + \phi_0(\theta) \quad \text{with } \phi_0 = \phi_{f_0}. \tag{1.6}$$

Without loss of generality, we assume that $\phi_0(\theta) = -m_0\cos\theta$ with $m_0 > 0$. Here m_0 is the magnetization of the stationary state f_0 defined by $m_0 = \int \rho_{f_0}\cos\theta d\theta$.

It is shown in [22] that (essentially) if F is decreasing then f_0 is nonlinearly stable by the HMF flow (1.1) provided that the criterion $\kappa_0 < 1$ is satisfied, where κ_0 is given

by

$$\kappa_0 = -\int_0^{2\pi}\int_{-\infty}^{+\infty} F'(e_0(\theta,v)) \left(\frac{\displaystyle\int_{\mathcal{D}_{e_0(\theta,v)}} (\cos\theta - \cos\theta')(e_0(\theta,v) + m_0\cos\theta')^{-1/2} d\theta'}{\displaystyle\int_{\mathcal{D}_{e_0(\theta,v)}} (e_0(\theta,v) + m_0\cos\theta')^{-1/2} d\theta'} \right)^2 d\theta dv, \tag{1.7}$$

with

$$\mathcal{D}_e = \left\{ \theta' \in \mathbb{T} \,:\, m_0\cos\theta' > -e \right\}.$$

In this paper, we explore situations where this criterion is not satisfied, i.e when $\kappa_0 > 1$. Let us now state our two main results. The first one concerns the linearized HMF equation given by

$$\partial_t f = Lf, \tag{1.8}$$

where

$$Lf := -v\partial_\theta f + \partial_\theta\phi_0\partial_v f + \partial_\theta\phi_f\partial_v f_0. \tag{1.9}$$

Theorem 1.1 (Linear instability)**.** *Let $f_0 \in L^1(\mathbb{T},\mathbb{R})$ be a stationary solution of* (1.1) *of the form* (1.5)*, where F is a nonnegative $\mathcal{C}^1$ function on $\mathbb{R}$ such that $F'(e_0(\theta,v))$ belongs to $L^1(\mathbb{T},\mathbb{R})$. Assume that $\kappa_0 > 1$, where κ_0 is given by* (1.7)*. Then there exists $\lambda > 0$ and a non-zero $f \in L^1(\mathbb{T}\times\mathbb{R})$ such that $e^{\lambda t}f$ is a nontrivial growing mode weak solution to the linearized HMF equation* (1.8)*.*

Our second result is the following nonlinear instability theorem.

Theorem 1.2 (Nonlinear instability)**.** *Let f_0 be a stationary solution of* (1.1) *of the form* (1.5)*, where F is a $\mathcal{C}^\infty$ function on $\mathbb{R}$, such that $F(e) > 0$ for $e < e_*$, $F(e) = 0$ for $e \geqslant e_*$, with $e_* < m_0$ and $|F'(e)| \leqslant C|e_* - e|^{-\alpha}F(e)$ in the neighborhood of e_*, for some $\alpha \geqslant 1$. Assume that $\kappa_0 > 1$, where κ_0 is given by* (1.7)*. Then f_0 is nonlinearly unstable in $L^1(\mathbb{T}\times\mathbb{R})$, namely, there exists $\delta_0 > 0$ such that for any $\delta > 0$ there exists a nonnegative solution $f(t)$ of* (1.1) *satisfying $\|f(0) - f_0\|_{L^1} \leqslant \delta$ and*

$$\|f(t_\delta) - f_0\|_{L^1} \geqslant \delta_0,$$

with $t_\delta = O(|\log\delta|)$ as $\delta \to 0$.

Remark 1.3. Note that in these two theorems we do not assume that the profile F is a decreasing function. Besides, the set of steady states satisfying the assumptions of these theorems is not empty, as proved in the Appendix (see Lemma A.1). Note also that the instability of Theorem 1.2 is not due to the usual orbital instability. Indeed the functional space of the pertubation can be restricted to the space of even functions in (θ, v).

The outline of the paper is as follows: Sections 2 and 3 are respectively devoted to the proofs of Theorem 1.1 and Theorem 1.2.

2 A linear instability result: proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. This proof will be done following the framework used by Lin for the study of periodic BGK waves in [19], which was generalized to the analysis of instabilities for the 3D Vlasov-Poisson system by Guo and Lin in [15]. We divide this proof into the three Lemmas 2.1, 2.2 and 2.3, respectively proved in Subsections 2.1, Subsection 2.2 and Subsection 2.3.

A growing mode of (1.8) is a solution of the form $e^{\lambda t}f$, where $f \in L^1(\mathbb{T},\mathbb{R})$ is an unstable eigenfunction of L, i.e. a nonzero function satisfying $Lf = \lambda f$ in the sense of distributions, with $\lambda \in \mathbb{R}_+^*$ and with L defined by (1.9). Note that the equation $Lf - \lambda f = 0$ is invariant by translation: if $f(\theta, v)$ is an eigenfunction, then for all θ_0, $f(\theta + \theta_0, v)$ is also an eigenfunction. Since for all $f \in L^1$ we can find a $\theta_0 \in \mathbb{T}$ such that $\int \rho_f(\theta + \theta_0)\sin\theta d\theta = 0$, we can assume that our eigenfunction of L always satisfy $\int \rho_f \sin\theta d\theta = 0$, i.e. $\phi_f = -m\cos\theta$ with $m = \int \rho_f \cos\theta d\theta$.

Let us first define $(\Theta(s,\theta,v), V(s,\theta,v))$ as the solution of the characteristics problem

$$\begin{cases} \dfrac{d\Theta(s,\theta,v)}{ds} = V(s,\theta,v) \\ \dfrac{dV(s,\theta,v)}{ds} = -\partial_\theta\phi_0\left(\Theta(s,\theta,v)\right) \end{cases} \tag{2.1}$$

with initial data $\Theta(0,\theta,v) = \theta$, $V(0,\theta,v) = v$. When there is no ambiguity, we denote simply $\Theta(s) = \Theta(s,\theta,v)$ and $V(s) = V(s,\theta,v)$. Since $\phi_0(\theta) = -m_0\cos\theta$, the solution (Θ, V) is globally defined and belongs to $\mathcal{C}^\infty(\mathbb{R}\times\mathbb{T}\times\mathbb{R})$. Note that the energy $e_0\left(\Theta(s), V(s)\right) = \frac{V(s)^2}{2} + \phi_0(\Theta(s))$ does not depend on s.

We shall reduce the existence of a growing mode of (1.8) to the existence of a zero of the following function, defined for all $\lambda \in \mathbb{R}_+^*$:

$$\begin{aligned} G(\lambda) =& 1 + \int_0^{2\pi}\int_{\mathbb{R}} F'\left(\frac{v^2}{2} - m_0\cos\theta\right)\cos^2\theta d\theta dv \\ &- \int_0^{2\pi}\int_{\mathbb{R}} F'\left(\frac{v^2}{2} - m_0\cos\theta\right)\left(\int_{-\infty}^0 \lambda e^{\lambda s}\cos\Theta(s,\theta,v)ds\right)\cos\theta d\theta dv. \end{aligned} \tag{2.2}$$

Lemma 2.1. *Let $f_0 \in L^1(\mathbb{T},\mathbb{R})$ be a stationary solution of (1.1) of the form (1.5), where F is a $\mathcal{C}^1$ function on $\mathbb{R}$ such that $F'(e_0(\theta,v))$ belongs to $L^1(\mathbb{T},\mathbb{R})$. Then the function G defined by (2.2) is well-defined and continuous on $\mathbb{R}_+^*$. Moreover, there exists a growing mode $e^{\lambda t}f$ solution to (1.8) associated with the eigenvalue $\lambda > 0$ if and only if $G(\lambda) = 0$. An unstable eigenfunction f of L is defined by*

$$f(\theta,v) = -F'(e_0(\theta,v))\cos\theta + F'(e_0(\theta,v))\int_{-\infty}^0 \lambda e^{\lambda s}\cos\left(\Theta(s,\theta,v)\right)ds. \tag{2.3}$$

Lemma 2.2. *Under the assumptions of Lemma 2.1, the function G defined by (2.2) satisfies*

$$\lim_{\lambda\to 0^+} G(\lambda) = 1 - \kappa_0, \tag{2.4}$$

where κ_0 is defined by (1.7).

Lemma 2.3. *Under the assumptions of Lemma 2.1, the function G defined by* (2.2) *satisfies*

$$\lim_{\lambda\to+\infty} G(\lambda) = 1. \tag{2.5}$$

Proof of Theorem 1.1. From these three lemmas, it is clear that if $\kappa_0 > 1$, we have $\lim_{\lambda\to 0^+} G(\lambda) < 0$ and $\lim_{\lambda\to+\infty} G(\lambda) > 0$ so by continuity of G, there exists $\lambda > 0$ such that $G(\lambda) = 0$. This means that there exists a growing mode to (1.8) and this proves Theorem 1.1. □

2.1 First properties of the function $G(\lambda)$: proof of Lemma 2.1

In this subsection, we prove Lemma 2.1. Let $\lambda \in \mathbb{R}_+^*$. Since, by assumption, the function $F'(e_0(\theta, v))$ belongs to $L^1(\mathbb{T}\times\mathbb{R})$, and since

$$\forall(\theta, v) \qquad \left|\int_{-\infty}^0 \lambda e^{\lambda s}\cos(\Theta(s))ds\right| \leqslant \int_{-\infty}^0 \lambda e^{\lambda s}ds = 1,$$

both functions

$$F'(e_0(\theta, v))\cos\theta \quad \text{and} \quad F'(e_0(\theta, v))\int_{-\infty}^0 \lambda e^{\lambda s}\cos\left(\Theta(s, \theta, v)\right)ds$$

belong to $L^1(\mathbb{T}\times\mathbb{R})$, so the function f defined by (2.3) also belongs to $L^1(\mathbb{T}\times\mathbb{R})$. Hence, by integrating with respect to $\cos\theta d\theta dv$, we deduce that $G(\lambda)$ is well-defined by (2.2). The continuity of G on $\mathbb{R}_+^*$ stems from dominated convergence.

Consider now a (nonzero) growing mode $e^{\lambda t}f$ of (1.8) associated to an eigenvalue $\lambda > 0$. Let us prove that $G(\lambda) = 0$. From $Lf = \lambda f$ and (2.1), we get, in the sense of distributions,

$$\frac{d}{ds}\left(e^{\lambda s}f\left(\Theta(s), V(s)\right)\right) = e^{\lambda s}\phi_f'\left(\Theta(s)\right)V(s)F'(e_0(\Theta, V)).$$

Integrating this equation from $-R$ to 0, we get, for almost all (θ, v) and all R,

$$f(\theta, v) = e^{-\lambda R}f\left(\Theta(-R), V(-R)\right) + F'(e_0)\int_{-R}^0 e^{\lambda s}\phi_f'\left(\Theta(s)\right)V(s)ds,$$

where we recall that $e_0(\Theta, V) = e_0(\theta, v)$. We multiply by a test function $\psi(\theta, v) \in C_0^\infty(\mathbb{T}\times\mathbb{R})$ and integrate with respect to (θ, v),

$$\begin{aligned}\int_0^{2\pi}\int_{\mathbb{R}} f(\theta, v)\psi(\theta, v)d\theta dv &= e^{-\lambda R}\int_0^{2\pi}\int_{\mathbb{R}} f\left(\theta, v\right)\psi\left(\Theta(R), V(R)\right)d\theta dv \\ &\quad + \int_{-R}^0\int_0^{2\pi}\int_{\mathbb{R}} e^{\lambda s}F'(e_0)\phi_f'\left(\Theta(s)\right)V(s)\psi\left(\theta, v\right)dsd\theta dv.\end{aligned}$$

In the first integral of the right-hand side, we have performed the change of variable $(\theta, v) = (\Theta(R, \theta', v'), V(R, \theta', v'))$. In the second integral, we remark that $|\phi_f'(\Theta(s))| \leqslant \|f\|_{L^1}$ and, the support of ψ being compact, v is bounded. Hence, by $\frac{v^2}{2} - m_0 \cos\theta = \frac{V^2}{2} - m_0 \cos\Theta$, $V(s)$ is bounded. Therefore, by dominated convergence (using that $F'(e_0) \in L^1$), as $R \to \infty$, we get

$$\int_0^{2\pi} \int_{\mathbb{R}} f(\theta, v)\psi(\theta, v) d\theta dv = \int_{-\infty}^0 \int_0^{2\pi} \int_{\mathbb{R}} e^{\lambda s} F'(e_0)\phi_f'(\Theta(s)) V(s)\psi(\theta, v)\, ds d\theta dv$$

i.e.

$$\begin{aligned} f(\theta, v) &= F'(e_0) \int_{-\infty}^0 e^{\lambda s}\phi_f'(\Theta(s)) V(s) ds = F'(e_0) \int_{-\infty}^0 e^{\lambda s} \frac{d}{ds}(\phi_f(\Theta(s)))\, ds \\ &= F'(e_0)\phi_f(\theta) - F'(e_0) \int_{-\infty}^0 \lambda e^{\lambda s}\phi_f(\Theta(s)) ds \end{aligned}$$

almost everywhere. Recall that $\phi_f(\theta) = -m\cos\theta$, with $m = \int_0^{2\pi} \rho_f(\theta)\cos\theta d\theta$. Then we can rewrite this expression of f as

$$f(\theta, v) = -mF'(e_0)\cos\theta - mF'(e_0) \int_{-\infty}^0 \lambda e^{\lambda s} \cos\Theta(s) ds.$$

Integrating both sides of this equation with respect to $\cos\theta d\theta dv$ we get

$$\begin{aligned} m = \int_0^{2\pi} \rho_f(\theta)\cos\theta d\theta = &- m \int_0^{2\pi} \int_{\mathbb{R}} F'(e_0)\cos^2\theta d\theta dv \\ &+ m \int_0^{2\pi} \int_{\mathbb{R}} F'(e_0) \left(\int_{-\infty}^0 \lambda e^{\lambda s} \cos\Theta(s) ds \right) \cos\theta d\theta dv, \end{aligned}$$

i.e. $mG(\lambda) = 0$. It is clear that $m \neq 0$, otherwise $f = 0$ a.e.. Finally, we get $G(\lambda) = 0$.

Reciprocally, assume that $G(\lambda) = 0$ for some $\lambda > 0$. Let f be given by (2.3). We have proved above that this function belongs to $L^1(\mathbb{T} \times \mathbb{R})$. Moreover, since $\Theta(s, -\theta, -v) = \Theta(s, \theta, v)$, we have $\int \rho_f \sin\theta d\theta = 0$. Multiplying (2.3) by $\cos\theta$ and integrating with respect to θ and v, and using that $G(\lambda) = 0$, we get $\int \rho_f \cos\theta d\theta = 1$, so f is not the zero function and we have $\phi_f(\theta) = -\cos\theta$.

We now check that the function f given by (2.3) is an eigenfunction of L associated with λ. From (2.3), we get

$$f(\Theta(t), V(t)) = -F'(e_0)\phi_f(\Theta(t)) + F'(e_0) \int_{-\infty}^0 \lambda e^{\lambda s}\phi_f(\Theta(s, \Theta(t), V(t)))\, ds.$$

Note that $\Theta(s, \Theta(t), V(t)) = \Theta(s+t, \theta, v)$. Therefore

$$\begin{aligned} f(\Theta(t), V(t)) &= -F'(e_0)\phi_f(\Theta(t)) + F'(e_0)\int_{-\infty}^{0} \lambda e^{\lambda s}\phi_f(\Theta(s+t))\,ds \\ &= -F'(e_0)\phi_f(\Theta(t)) + F'(e_0)e^{-\lambda t}\int_{-\infty}^{t} \lambda e^{\lambda s}\phi_f(\Theta(s))\,ds \\ &= F'(e_0)e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s}\phi_f'(\Theta(s))\,V(s)ds, \end{aligned}$$

where we integrated by parts. Then

$$\begin{aligned} e^{\lambda t} f(\Theta(t), V(t)) &= F'(e_0)\int_{-\infty}^{t} e^{\lambda s}\phi_f'(\Theta(s))\,V(s)ds \\ &= \int_{-\infty}^{t} e^{\lambda s}\phi_f'(\Theta(s))\,\partial_v f_0(\Theta(s), V(s))ds. \end{aligned}$$

Differentiating both sides with respect to t, we obtain in the sense of distributions that for all $t \in \mathbb{R}$,

$$\begin{aligned} e^{\lambda t}\left(\lambda f(\Theta(t), V(t)) + V(t)\partial_\theta f(\Theta(t), V(t)) - \phi_0'(\Theta(t))\partial_v f(\Theta(t), V(t))\right) \\ = e^{\lambda t}\phi_f'(\Theta(t))\,\partial_v f_0(\Theta(t), V(t)). \end{aligned}$$

By writing this equation at $t = 0$, we get $Lf = \lambda f$: f is an unstable eigenfunction of L. This ends the proof of Lemma 2.1. □

2.2 Limiting behavior of $G(\lambda)$ near $\lambda = 0$: proof of Lemma 2.2

To study $\lim_{\lambda \to 0^+} G(\lambda)$ we need to analyze the limit of the function

$$g_\lambda(\theta, v) = \int_{-\infty}^{0} \lambda e^{\lambda s}\cos\Theta(s, \theta, v)ds \tag{2.6}$$

as $\lambda \to 0$. We provide this result in the next lemma, where we also recall some well-known facts on the solution of the characteristics equations (2.1), which are nothing but the pendulum equations.

Lemma 2.4. *Let $(\theta, v) \in \mathbb{T} \times \mathbb{R}$ and $e = \frac{v^2}{2} - m_0\cos\theta$. Consider the solution $(\Theta(s, \theta, v), V(s, \theta, v))$ to the characteristics equations (2.1). Then the following holds true.*

(i) If $e > m_0$ then, for all $s \in \mathbb{R}$, we have

$$\Theta(s + T_e) = \Theta(s) + 2\pi, \qquad V(s + T_e) = V(s), \tag{2.7}$$

with

$$T_e = \int_0^{2\pi} \frac{d\theta'}{\sqrt{2\,(e + m_0\cos\theta')}} > 0. \tag{2.8}$$

(ii) If $-m_0 < e < m_0$ then Θ and V are periodic with period given by

$$T_e = 4\int_0^{\theta_{m_0}} \frac{d\theta'}{\sqrt{2\,(e + m_0\cos\theta')}} > 0, \tag{2.9}$$

where $\theta_{m_0} = \arccos(-\frac{e}{m_0})$.

(iii) We have

$$\lim_{\lambda\to 0^+} g_\lambda(\theta, v) = \begin{cases} \dfrac{1}{T_e}\displaystyle\int_0^{2\pi} \frac{\cos\theta'}{\sqrt{2(e + m_0\cos\theta')}} d\theta' & \text{if } e > m_0, \\ \dfrac{4}{T_e}\displaystyle\int_0^{\theta_{m_0}} \frac{\cos\theta'}{\sqrt{2(e + m_0\cos\theta')}} d\theta' & \text{if } -m_0 < e < m_0. \end{cases} \tag{2.10}$$

Proof. (i) Let $e > m_0$. Without loss of generality, since $\Theta(s, -\theta, -v) = \Theta(s, \theta, v)$ and $V(s, -\theta, -v) = V(s, \theta, v)$, we can only treat the case $v > 0$. As we have

$$\frac{V(s)^2}{2} = e + m_0\cos\Theta(s) \geqslant e - m_0 > 0,$$

$V(s)$ does not vanish and remains positive. Hence $\Theta(s)$ is the solution of the following autonomous equation

$$\dot\Theta(s) = V(s) = \sqrt{2(e + m_0\cos\Theta(s))} \tag{2.11}$$

and is strictly increasing with $\Theta(s) \to +\infty$ as $s \to +\infty$. Let T_e be the unique time such that $\Theta(T_e) = \theta + 2\pi$. By Cauchy-Lipschitz's theorem, we have $\Theta(s + T_e) - 2\pi = \Theta(s)$ and (2.7) holds. Defining

$$P(\tau) = \int_0^\tau \frac{d\theta'}{\sqrt{2(e + m_0\cos\theta')}},$$

the solution of (2.11) satisfies $P(\Theta(s)) - P(\theta) = s$. Therefore, we have $T_e = P(\theta + 2\pi) - P(\theta)$, from which we get (2.8).

(ii) Let $-m_0 < e < m_0$. In this case, $\Theta(s)$ will oscillate between the two values $\theta_{m_0} = \arccos(-\frac{e}{m_0})$ and $-\theta_{m_0}$ with a period T_e given by (2.9). On the half-periods where Θ is increasing, we also have (2.11). We skip the details of the proof, which is classical.

(iii) We remark that $\cos\Theta(s + kT_e) = \cos\Theta(s)$ for all $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. Indeed, by *(i)*, for $e > m_0$ we have $\Theta(s + kT_e) = \Theta(s) + 2\pi k$ and, by *(ii)*, for $-m_0 < e < m_0$ we have

$\Theta(s + kT_e) = \Theta(s)$. Hence, we compute from (2.6)

$$\begin{aligned} g_\lambda(\theta, v) &= \sum_{k=0}^{+\infty} \int_{kT_e}^{(k+1)T_e} \lambda e^{-\lambda s} \cos\Theta(-s) ds \\ &= \sum_{k=0}^{+\infty} \int_0^{T_e} \lambda e^{-\lambda s - k\lambda T_e} \cos\Theta(-s) ds \\ &= \left(\sum_{k=0}^{+\infty} e^{-k\lambda T_e} \right) \int_0^{T_e} \lambda e^{-\lambda s} \cos\Theta(-s) ds \\ &= \frac{\lambda}{1 - e^{-\lambda T_e}} \int_0^{T_e} e^{-\lambda s} \cos\Theta(-s) ds. \end{aligned}$$

Therefore, clearly,

$$\lim_{\lambda \to 0^+} g_\lambda(\theta, v) = \frac{1}{T_e} \int_0^{T_e} \cos\Theta(-s) ds = \frac{1}{T_e} \int_0^{T_e} \cos\Theta(s) ds.$$

If $e > m_0$, we perform the change of variable $\theta' = \Theta(s)$ which is strictly increasing from $[0, T_e]$ to $[\theta, \theta + 2\pi]$. Using (2.11), we obtain

$$\lim_{\lambda \to 0^+} g_\lambda(\theta, v) = \frac{1}{T_e} \int_\theta^{\theta + 2\pi} \frac{\cos\theta'}{\sqrt{2(e + m_0 \cos\theta')}} d\theta' = \frac{1}{T_e} \int_0^{2\pi} \frac{\cos\theta'}{\sqrt{2(e + m_0 \cos\theta')}} d\theta'.$$

If $-m_0 < e < m_0$, we can always choose a time t_0 such that $\Theta(t_0) = -\theta_{m_0}$, $\Theta(t_0 + T_e/2) = \theta_{m_0}$, $\Theta(s)$ is strictly increasing on $[t_0, t_0 + T_e/2]$ and such that $\Theta(s) = \Theta(2t_0 + T_e - s)$ for $s \in [t_0 + T_e/2, t_0 + T_e]$. We have

$$\begin{aligned} \lim_{\lambda \to 0^+} g_\lambda(\theta, v) &= \frac{1}{T_e} \int_{t_0}^{t_0 + T_e/2} \cos\Theta(s) ds + \frac{1}{T_e} \int_{t_0 + T_e/2}^{t_0 + T_e} \cos\Theta(s) ds \\ &= \frac{1}{T_e} \int_{t_0}^{t_0 + T_e/2} \cos\Theta(s) ds + \frac{1}{T_e} \int_{t_0 + T_e/2}^{t_0 + T_e} \cos\Theta(2t_0 + T_e - s) ds \\ &= \frac{2}{T_e} \int_{t_0}^{t_0 + T_e/2} \cos\Theta(s) ds \\ &= \frac{2}{T_e} \int_{-\theta_{m_0}}^{\theta_{m_0}} \frac{\cos\theta'}{\sqrt{2(e + m_0 \cos\theta')}} d\theta' = \frac{4}{T_e} \int_0^{\theta_{m_0}} \frac{\cos\theta'}{\sqrt{2(e + m_0 \cos\theta')}} d\theta', \end{aligned}$$

where, on the time interval $[t_0, t_0 + T_e/2]$, we performed the change of variable $\theta' = \Theta(s)$. □

Proof of Lemma 2.2. Now we come back to the definition (2.2) of G_λ, which reads

$$\begin{aligned} G(\lambda) =& 1 + \int_0^{2\pi} \int_{\mathbb{R}} F'\left(\frac{v^2}{2} - m_0 \cos\theta \right) \cos^2\theta d\theta dv \\ &- \int_0^{2\pi} \int_{\mathbb{R}} F'\left(\frac{v^2}{2} - m_0 \cos\theta \right) g_\lambda(\theta, v) \cos\theta d\theta dv. \end{aligned} \tag{2.12}$$

We remark that $|g_\lambda(\theta,v)| \leqslant 1$ and recall that the function $F'\left(\frac{v^2}{2} - m_0\cos\theta\right)$ belongs to $L^1(\mathbb{T}\times\mathbb{R})$. Therefore, we can pass to the limit in the second integral by dominated convergence and deduce from Lemma 2.4 *(iii)* (note that the set $\{(\theta,v) : e_0(\theta,v) \leqslant -m_0\}$ is of measure zero) that

$$\begin{aligned}
\lim_{\lambda\to 0^+} G(\lambda) = 1 &+ \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\cos^2\theta d\theta dv \\
&- \iint\limits_{e_0(\theta,v)>m_0} F'(e_0)\left(\frac{1}{T_e}\int_0^{2\pi}\frac{\cos\theta'}{\sqrt{2(e_0+m_0\cos\theta')}}d\theta'\right)\cos\theta d\theta dv \\
&- \iint\limits_{-m_0<e_0(\theta,v)<m_0} F'(e_0)\left(\frac{4}{T_e}\int_0^{\theta_{m_0}}\frac{\cos\theta'}{\sqrt{2(e_0+m_0\cos\theta')}}d\theta'\right)\cos\theta d\theta dv \\
=1 &+ \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\cos^2\theta d\theta dv \\
&- \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\left(\frac{\int_{\mathcal{D}_{e_0}}\cos\theta'(e_0+m_0\cos\theta')^{-1/2}d\theta'}{\int_{\mathcal{D}_{e_0}}(e_0+m_0\cos\theta')^{-1/2}d\theta'}\right)\cos\theta d\theta dv \\
=1 &+ \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\cos(\theta)^2 dv d\theta - \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\left(\Pi_{m_0}(\cos\theta)\right)^2 d\theta dv,
\end{aligned}$$

with

$$(\Pi_{m_0}h)(e) = \frac{\int_{\mathcal{D}_e}(e+m_0\cos\theta)^{-1/2}h(\theta)d\theta}{\int_{\mathcal{D}_e}(e+m_0\cos\theta)^{-1/2}d\theta}, \tag{2.13}$$

for all function $h(\theta)$ and

$$\mathcal{D}_e = \left\{\theta'\in\mathbb{T} :\ m_0\cos\theta' > -e\right\}.$$

Here the operator Π_{m_0} is a variant of the operator Π given by (3.8) in [21], this operator should be understood as the "projector"onto the functions which depend only on the microscopic energy $e_0(\theta,v)$. A projector of this type is also mentioned in the work by Guo and Lin [15].

Now we remark that straightforward calculations give

$$\begin{aligned}
1-\kappa_0 &= 1 + \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\left(\cos^2\theta - 2\cos\theta\Pi_{m_0}(\cos\theta) + \left(\Pi_{m_0}(\cos\theta)\right)^2\right)d\theta dv, \\
&= 1 + \int_0^{2\pi}\int_{\mathbb{R}} F'(e_0)\cos^2\theta d\theta dv - \iint F'(e_0)\left(\Pi_{m_0}(\cos\theta)\right)^2 d\theta dv,
\end{aligned}$$

where

$$-\kappa_0 := \int_0^{2\pi} \int_{\mathbb{R}} F'(e_0) \left(\frac{\displaystyle\int_{\mathcal{D}_e} (\cos\theta - \cos\theta')(e_0 + m_0\cos\theta')^{-1/2} d\theta'}{\displaystyle\int_{\mathcal{D}_e} (e_0 + m_0\cos\theta')^{-1/2} d\theta'} \right)^2 d\theta dv.$$

This calculation uses that Π_{m_0} is a projector. We finally get (2.4) and the proof of Lemma 2.2 is complete. □

2.3 Limiting behavior of $G(\lambda)$ as $\lambda \to \infty$: proof of Lemma 2.3

In this subsection, we prove Lemma 2.3. An integration by parts in (2.6) yields

$$g_\lambda(\theta, v) = \cos\theta + \int_{-\infty}^0 e^{\lambda s} V(s, \theta, v) \sin\Theta(s, \theta, v) ds.$$

The velocity can be bounded independently of s thanks to the conservation of the energy,

$$|V(s)| = \left(v^2 + 2m_0\cos\theta - 2m_0\cos\Theta(s)\right)^{1/2} \leqslant \left(v^2 + 4m_0\right)^{1/2}.$$

Thus

$$\left| \int_{-\infty}^0 e^{\lambda s} \sin\Theta(s, \theta, v) V(s, \theta, v) ds \right| \leqslant \left(v^2 + 4m_0\right)^{1/2} \int_{-\infty}^0 e^{\lambda s} ds = \frac{\left(v^2 + 4m_0\right)^{1/2}}{\lambda}$$

and, for all (θ, v),

$$\lim_{\lambda \to +\infty} g_\lambda(\theta, v) = \cos\theta.$$

Using again that $|g_\lambda(\theta, v)| \leqslant 1$ and that $F'(e_0(\theta, v))$ belongs to L^1, we deduce directly (2.5) from (2.12) and from dominated convergence. The proof of Lemma 2.3 is complete. □

3 A nonlinear instability result: proof of Theorem 1.2

We start by an analysis of the linearized HMF operator L around the inhomogeneous equilibrium state f_0, where L is given by (1.9). We write

$$L = L_0 + K, \tag{3.1}$$

where

$$L_0 f = -v\partial_\theta f - E_{f_0}\partial_v f, \qquad Kf = -E_f \partial_v f_0, \qquad E_f = -\partial_\theta \phi_f. \tag{3.2}$$

3.1 Estimates on the semigroup e^{tL}

Let us state some useful properties of the operator L_0 given by (3.2). Since $\phi_0(\theta) = -m_0 \cos\theta$ is smooth, the characteristics equations (2.1) admit a unique solution $\Theta(s,\theta,v), V(s,\theta,v)$, which is globally defined and $\mathcal{C}^\infty$ in the variables (s,θ,v). Moreover, this solution has bounded derivatives with respect to θ and v, locally in time. Let $k \in \mathbb{N}$. For any f in the Sobolev space $W^{k,1}(\mathbb{T},\mathbb{R})$, the function

$$e^{tL_0} f(t,\theta,v) := f\left(\Theta(-t,\theta,v), V(-t,\theta,v)\right), \quad \forall t \geqslant 0, \tag{3.3}$$

belongs to $\mathcal{C}^0(\mathbb{R}, W^{k,1}(\mathbb{T},\mathbb{R}))$ and is clearly a solution to $\partial_t g = L_0 g$ with initial data f. This means that the semigroup e^{tL_0} generated by the operator L_0 is strongly continuous on $W^{k,1}(\mathbb{T},\mathbb{R})$.

Our aim is to apply the abstract results in [25] concerning perturbation theory of linear operators. Hence, we need to prove the following estimate. For all $\beta > 0$, there exists a positive constant M_β such that

$$\|e^{tL_0} f\|_{W^{k,1}} \leqslant M_\beta e^{t\beta} \|f\|_{W^{k,1}} \qquad \forall f \in W^{k,1}(\mathbb{T},\mathbb{R}), \quad \forall t \geqslant 0, \tag{3.4}$$

where M_β depends on k. In fact a stronger estimate will be proved but for a subclass of functions f. From the assumptions of Theorem 1.2 on F, there exists $e_* < m_0$ such that support of F is $(-\infty, e_*]$. This means that the support of f_0 is contained in $\overline{\Omega_0}$, where Ω_0 is the smooth open set

$$\Omega_0 = \left\{ (\theta, v) : \ \frac{v^2}{2} - m_0 \cos\theta < e_* \right\}.$$

We then introduce the functional space

$$\mathscr{E}_k = \left\{ f \in W^{k,1}(\mathbb{T},\mathbb{R}) : \ \ \mathrm{Supp}(f) \subset \overline{\Omega_0} \right\},$$

and claim that for all $f \in \mathscr{E}_k$, we have $e^{tL_0} f \in \mathscr{E}_k$ and

$$\|e^{tL_0} f\|_{W^{k,1}} \leqslant M_k \|f\|_{W^{k,1}}, \qquad \forall f \in \mathscr{E}_k, \quad \forall t \geqslant 0, \tag{3.5}$$

M_k being a positive constant.

Assume for the moment that estimate (3.5) holds true. From the assumptions of Theorem 1.2, one deduces that $\partial_v f_0 \in \mathscr{E}_k$. It is then easy to check that $e^{tL_0} K$ is a compact operator on $\mathscr{E}_k$, for all $t \in \mathbb{R}$, and the map $t \mapsto e^{tL_0} K \in \mathscr{L}(\mathscr{E}_k)$ is continuous on $\mathbb{R}$. Hence, K is L_0-smoothing in the sense of [25] (page 707). Assumptions of Theorem 1.1 in [25] are therefore satisfied, which implies that L generates a strongly continuous semigroup e^{tL}. Now, from Theorem 1.2 in [25], for all $\beta > 0$, any point of the spectrum $\sigma(L)$ lying in the half plane $\mathrm{Re}\, z > \beta$ is an isolated eigenvalue with finite algebraic multiplicity. Furthermore, the set $\sigma(L) \cap \{\mathrm{Re}\, z > \beta\}$ is finite.

The assumptions of Theorem 1.2 clearly imply those of Theorem 1.1. Hence L admits at least one eigenvalue $\lambda \in \mathbb{R}_+^*$ associated with an eigenfunction $\widetilde{f} \in L^1(\mathbb{T}\times\mathbb{R})$. We claim

that, in fact, $\widetilde{f} \in \mathscr{E}_k$ which will be proved below. This means that the set of eigenvalues of L on $\mathscr{E}_k$ with positive real part is not empty, and we therefore can choose an eigenvalue γ with positive maximal real part. Finally, we apply Theorem 1.3 in [25] and get that, for all $\beta > \mathrm{Re}\,\gamma$, there exists a positive constant $M_{\beta,k}$ such that

$$\|e^{tL} f\|_{W^{k,1}} \leqslant M_{\beta,k}\, e^{t\beta} \|f\|_{W^{k,1}} \qquad \forall f \in \mathscr{E}_k, \quad \forall t \geqslant 0. \tag{3.6}$$

Proof of (3.5) *and of the claim* $\widetilde{f} \in \mathscr{E}_k$. Let $f \in \mathscr{E}_k$. From (3.3), we clearly have

$$\|e^{tL_0} f\|_{L^1} = \|f\|_{L^1}, \quad \forall f \in L^1, \quad \forall t \geqslant 0.$$

Moreover, we know from the analysis of the characteristics problem (2.1) performed in Section 2, that by conservation of the energy, for all $(\theta, v) \in \Omega_0$, we have $(\Theta(t,\theta,v), V(t,\theta,v)) \in \Omega_0$. Thus

$$\mathrm{Supp}(e^{tL_0} f) \subset \overline{\Omega_0}.$$

Let $k \geqslant 1$. By (3.3), to get an estimate of $e^{tL_0} f$ in $L^\infty(\mathbb{R}_+, W^{k,1}(\mathbb{T},\mathbb{R}))$, it is sufficient to estimate Θ and V in $L^\infty(\mathbb{R}, W^{k,\infty}(\mathbb{T},\mathbb{R}))$ for $(\theta, v) \in \Omega_0$. Since the field $E_{f_0}(\theta) = -\partial_\theta \phi_0(\theta)$ and all its derivatives are bounded, one can apply Gronwall Lemma to get

$$|\partial_\theta^j \partial_v^\ell \Theta| + |\partial_\theta^j \partial_v^\ell V| \leqslant C_k e^{C_k t}, \quad \forall t \geqslant 0, \quad \forall (\theta, v) \in \Omega_0, \quad \text{for } j + \ell \leqslant k.$$

Now we observe that for $(\theta, v) \in \Omega_0$, we have $e = v^2/2 - m_0 \cos\theta < m_0$ and by Lemma 2.4, the solution (Θ, V) and its partial derivatives with respect to θ and v are periodic in time, with a period T_e given by (2.9). Hence

$$|\partial_\theta^j \partial_v^\ell \Theta| + |\partial_\theta^j \partial_v^\ell V| \leqslant C_k e^{C_k T_e}, \quad \forall t \geqslant 0, \quad \forall (\theta, v) \in \Omega_0, \quad \text{for } j + \ell \leqslant k.$$

By Lemma 2.2 *(ii)* in [22] the function $e \mapsto T_e$ is increasing on $(-m_0, m_0)$. Thus $T_e \leqslant T_{e_*}$ for all $(\theta, v) \in \Omega_0$. We conclude that Θ and V are bounded in $L^\infty(\mathbb{R}, W^{k,\infty}(\Omega_0))$, which yields the estimate (3.5).

Let us finally prove that $\widetilde{f} \in \mathscr{E}_k$. By Lemma 2.1, the function $\widetilde{f}$ is given by

$$\widetilde{f}(\theta, v) = -F'(e_0(\theta, v)) \cos\theta + F'(e_0(\theta, v)) \int_{-\infty}^0 \lambda e^{\lambda s} \cos\left(\Theta(s, \theta, v)\right) ds.$$

Hence, the support of $F'(e_0(\theta, v))$ being in $\overline{\Omega_0}$, the support on $\widetilde{f}$ will also be contained in $\overline{\Omega_0}$. Moreover, by using that Θ is bounded in $L^\infty(\mathbb{R}, W^{k,\infty}(\Omega_0))$, we obtain that, for some $C_k > 0$, we have

$$\forall j + \ell \leqslant k, \quad \forall (\theta, v) \in \mathbb{T} \times \mathbb{R}, \quad \left| \partial_\theta^j \partial_v^\ell \int_{-\infty}^0 \lambda e^{\lambda s} \cos\left(\Theta(s, \theta, v)\right) ds \right| \leqslant C_k.$$

This is sufficient to deduce from $F \in \mathcal{C}^\infty$ that $\widetilde{f} \in \mathscr{E}_k$.

3.2 An iterative scheme

In this part, we prove Theorem 1.2 by following the strategy developed by Grenier in [14], which has been also used in [17, 18] to analyse instabilities for homogeneous steady states of Vlasov-Poisson models. Let $N \geqslant 1$ be an integer to be fixed later. According to the previous subsection, we can consider an eigenvalue γ of L on $\mathscr{E}_N$ with maximal real part, $\operatorname{Re}\gamma > 0$. Let $g \in \mathscr{E}_N$ be an associated eigenfunction. With no loss of generality, we may assume that $\|\operatorname{Re} g\|_{L^1} = 1$. Let

$$f_1(t,\theta,v) = \operatorname{Re}\left(e^{\gamma t} g(\theta,v)\right) \chi_\delta(e_0(\theta,v)), \tag{3.7}$$

with $e_0(\theta,v) = \frac{v^2}{2} + \phi_0(\theta)$ and where $0 \leqslant \chi_\delta(e) \leqslant 1$ is a smooth real-valued truncation function to be defined further, in order to ensure the positivity of $f_0 + \delta f_1(0)$. Note that f_1 is almost a growing mode solution to the linearized HMF model (1.8) since we have

$$(\partial_t - L) f_1 = \operatorname{Re}(e^{\gamma t}\widetilde{R}_\delta),$$

where

$$\widetilde{R}_\delta = \left(-E_{(1-\chi_\delta)g} + (1-\chi_\delta)E_g\right)\partial_v f_0 \tag{3.8}$$

will be small. We now construct an approximate solution f^N_{app} to the HMF model (1.1) of the form

$$f^N_{app} = f_0 + \sum_{k=1}^{N} \delta^k f_k, \tag{3.9}$$

for sufficiently small $\delta > 0$, in which f_k ($k \geqslant 2$) solves inductively the linear problem

$$(\partial_t - L) f_k + \sum_{j=1}^{k-1} E_{f_j}\partial_v f_{k-j} = 0 \tag{3.10}$$

with $f_k(0) = 0$. Then f^N_{app} approximately solves the HMF model (1.1) in the sense that

$$\partial_t f^N_{app} + v\partial_\theta f^N_{app} - \partial_\theta \phi_{f^N_{app}} \partial_v f^N_{app} = R_N + \delta \operatorname{Re}(e^{\gamma t}\widetilde{R}), \tag{3.11}$$

where the remainder term R_N is given by

$$R_N = \sum_{1 \leqslant j,\ell \leqslant N;\, j+\ell \geqslant N+1} \delta^{j+\ell} E_{f_j}\partial_v f_\ell. \tag{3.12}$$

Step 1. Estimate of f_k. We claim that $f_k \in \mathscr{E}_{N-k+1}$ and, for all $1 \leqslant k \leqslant N$,

$$\|f_k\|_{W^{N-k+1,1}} \leqslant C_k e^{kt\operatorname{Re}\gamma}. \tag{3.13}$$

We proceed by induction. From (3.7), this estimate is a consequence, for $k = 1$, of

$$\|g\chi_\delta\|_{W^{N,1}} \leqslant C_1, \tag{3.14}$$

which is proved below in Step 5. Let $k \geqslant 2$. We have

$$f_k(t) = -\int_0^t e^{L(t-s)} \sum_{j=1}^{k-1} E_{f_j}(s)\partial_v f_{k-j}(s)ds.$$

Therefore, for $\operatorname{Re}\gamma < \beta < 2\operatorname{Re}\gamma$,

$$\begin{aligned}
\|f_k\|_{W^{N-k+1,1}} &\leqslant \sum_{j=1}^{k-1} \int_0^t \left\| e^{L(t-s)} \left(E_{f_j}(s)\partial_v f_{k-j}(s) \right) \right\|_{W^{N-k+1,1}} ds \\
&\leqslant M_{\beta,N-k+1} \sum_{j=1}^{k-1} \int_0^t e^{\beta(t-s)} \left\| E_{f_j}(s) \right\|_{W^{N-k+1,\infty}} \left\| \partial_v f_{k-j}(s) \right\|_{W^{N-k+1,1}} ds \\
&\leqslant M_{\beta,N-k+1} \sum_{j=1}^{k-1} \int_0^t e^{\beta(t-s)} \left\| f_j(s) \right\|_{L^1} \left\| f_{k-j}(s) \right\|_{W^{N-k+2,1}} ds \\
&\qquad \text{since } k-j \leqslant k-1, \\
&\leqslant M_{\beta,N-k+1} \left(\sum_{j=1}^{k-1} C_j C_{k-j} \right) \int_0^t e^{\beta(t-s)} e^{ks\operatorname{Re}\gamma} ds \\
&\leqslant \frac{M_{\beta,N-k+1}}{k\operatorname{Re}\gamma - \beta} \left(\sum_{j=1}^{k-1} C_j C_{k-j} \right) e^{kt\operatorname{Re}\gamma},
\end{aligned}$$

where we used (3.6) and the recursive assumption. This ends the proof of (3.13).

Step 2. Estimates of $f_{app}^N - f_0$ and R_N. The parameter δ and the time t will be such that

$$\delta e^{t\operatorname{Re}\gamma} \leqslant \min\left(\frac{1}{2}, \frac{1}{2K_N} \right), \qquad K_N = \max_{1\leqslant k\leqslant N} C_k. \tag{3.15}$$

Hence, from (3.13) we obtain

$$\|f_{app}^N - f_0\|_{W^{1,1}} \leqslant \sum_{k=1}^{N} \delta^k C_k e^{kt\operatorname{Re}\gamma} \leqslant K_N \frac{\delta e^{t\operatorname{Re}\gamma}}{1-\delta e^{t\operatorname{Re}\gamma}} \leqslant 1$$

and

$$\|R_N\|_{L^1} \leqslant \sum_{k=N+1}^{+\infty} \delta^k e^{kt\operatorname{Re}\gamma} \sum_{1\leqslant j,\ell\leqslant N; j+\ell=k} C_j C_\ell \leqslant \widetilde{C}_N \left(\delta e^{t\operatorname{Re}\gamma} \right)^{N+1}.$$

Step 3. Estimate of $f - f_{app}^N$. Let $f(t)$ be the solution of (1.1) with initial data $f_0 + \delta\operatorname{Re} g\chi_\delta$ and let $h = f - f_{app}^N$. Note that the positivity of $f(t)$ is ensured by $f_0 + \delta\operatorname{Re} g\chi_\delta \geqslant 0$ and that we have

$$\|f(0) - f_0\|_{L^1} \leqslant \delta.$$

The function h satisfies the following equation

$$\partial_t h + v\partial_\theta h + E_f \partial_v h = \left(E_{f_{app}^N} - E_f \right) \partial_v f_{app}^N - R_N - \delta\operatorname{Re}(e^{\gamma t}\widetilde{R}_\delta)$$

with $h(0) = 0$. To get a L^1-estimate of h, we multiply this equation by $\mathrm{sign}(h)$ and integrate in (θ, v). We get

$$\begin{aligned}\frac{d}{dt}\|h\|_{L^1} &\leqslant \left\|E_{f^N_{app}} - E_f\right\|_{L^\infty} \left\|\partial_v f^N_{app}\right\|_{L^1} + \|R_N\|_{L^1} + \delta e^{t\,\mathrm{Re}\,\gamma}\|\widetilde{R}_\delta\|_{L^1} \\ &\leqslant \|h\|_{L^1} \left\|\partial_v f^N_{app}\right\|_{L^1} + \|R_N\|_{L^1} + \delta e^{t\,\mathrm{Re}\,\gamma}\|\widetilde{R}_\delta\|_{L^1}.\end{aligned}$$

From Step 2 we have $\left\|\partial_v f^N_{app}\right\|_{L^1} \leqslant \|\partial_v f_0\|_{L^1} + 1$, which implies that

$$\|h(t)\|_{L^1} \leqslant \int_0^t e^{(t-s)(\|\partial_v f_0\|_{L^1}+1)} \left(\|R_N(s)\|_{L^1} + \delta e^{s\,\mathrm{Re}\,\gamma}\|\widetilde{R}_\delta\|_{L^1}\right) ds.$$

Again from Step 2, we then get

$$\|h(t)\|_{L^1} \leqslant \int_0^t e^{(t-s)(\|\partial_v f_0\|_{L^1}+1)} \left(\widetilde{C}_N \left(\delta e^{s\,\mathrm{Re}\,\gamma}\right)^{N+1} + \delta e^{s\,\mathrm{Re}\,\gamma}\|\widetilde{R}_\delta\|_{L^1}\right) ds$$

We now fix N as follows (with the notation $\lfloor\cdot\rfloor$ for the integer function)

$$N := \left\lfloor \frac{\|\partial_v f_0\|_{L^1} + 1}{\mathrm{Re}\,\gamma} \right\rfloor + 1 \geqslant 1$$

and claim that χ may be chosen such that

$$\|\widetilde{R}_\delta\|_{L^1} \leqslant \left(\delta e^{s\,\mathrm{Re}\,\gamma}\right)^N, \tag{3.16}$$

see Step 5 for the proof. This yields

$$\|f - f^N_{app}\|_{L^1}(t) \leqslant \breve{C}_N \left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N+1} \tag{3.17}$$

with $\breve{C}_N = \frac{1+\widetilde{C}_N}{3\,\mathrm{Re}\,\gamma}$.

Step 4. End of the proof. Since $\mathrm{Re}\, g$ is not zero, we can choose a real valued function $\varphi(\theta, v)$ in L^∞ such that $\|\varphi\|_{L^\infty}$=1 and

$$\mathrm{Re}\, z_g > 0 \quad \text{with} \quad z_g = \int_0^{2\pi}\int_{\mathbb{R}} g\varphi d\theta dv.$$

Denoting

$$z_{g,\delta} = \int_0^{2\pi}\int_{\mathbb{R}} g\chi_\delta\varphi d\theta dv,$$

we have

$$\begin{aligned}\iint f_1\varphi d\theta dv &= e^{t\,\mathrm{Re}\,\gamma}\,\mathrm{Re}\left(e^{it\,\mathrm{Im}\,\gamma} z_{g,\delta}\right) \\ &\geqslant e^{t\,\mathrm{Re}\,\gamma}\,\mathrm{Re}\left(e^{it\,\mathrm{Im}\,\gamma} z_g\right) - e^{t\,\mathrm{Re}\,\gamma}|z_g - z_{g,\delta}| \\ &\geqslant e^{t\,\mathrm{Re}\,\gamma}\,\mathrm{Re}\left(e^{it\,\mathrm{Im}\,\gamma} z_g\right) - e^{t\,\mathrm{Re}\,\gamma}\|g(1-\chi_\delta)\|_{L^1}\end{aligned}$$

We claim that

$$\lim_{\delta\to 0}\|(1-\chi_\delta)g\|_{L^1}=0, \tag{3.18}$$

which again will be proved in Step 5. In order to end the proof of Theorem 1.2, we estimate from below, using (3.17) and (3.13),

$$\begin{aligned}
\|f-f_0\|_{L^1} &\geqslant \iint (f-f_0)\varphi d\theta dv = \iint (f_{app}^N - f_0)\varphi d\theta dv + \iint (f-f_{app}^N)\varphi d\theta dv \\
&\geqslant \delta \iint f_1\varphi d\theta dv - \sum_{k=2}^{N}\delta^k\|f_k\|_{L^1} - \breve{C}_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N+1} \\
&\geqslant \delta \iint f_1\varphi d\theta dv - \sum_{k=2}^{N} C_k\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{k} - \breve{C}_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N+1} \\
&\geqslant \delta \iint f_1\varphi d\theta dv - 2K_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{2} - \breve{C}_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N+1} \\
&\geqslant \delta e^{t\,\mathrm{Re}\,\gamma}\left(\mathrm{Re}\left(e^{it\,\mathrm{Im}\,\gamma}z_g\right) - \|(1-\chi_\delta)g\|_{L^1} - 2K_N\delta e^{t\,\mathrm{Re}\,\gamma} - \breve{C}_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N}\right)
\end{aligned}$$

Assume for a while that

$$\mathrm{Re}\left(e^{it\,\mathrm{Im}\,\gamma}z_g\right) \geqslant \frac{\mathrm{Re}\,z_g}{2}. \tag{3.19}$$

We have

$$\|f-f_0\|_{L^1} \geqslant \frac{\delta e^{t\,\mathrm{Re}\,\gamma}\,\mathrm{Re}\,z_g}{2}\left(1-\frac{2\|(1-\chi_\delta)g\|_{L^1}}{\mathrm{Re}\,z_g} - \frac{4K_N\delta e^{t\,\mathrm{Re}\,\gamma}}{\mathrm{Re}\,z_g} - \frac{2\breve{C}_N\left(\delta e^{t\,\mathrm{Re}\,\gamma}\right)^{N}}{\mathrm{Re}\,z_g}\right)$$

Let $\delta_0>0$ be such that

$$\frac{32K_N}{(\mathrm{Re}\,z_g)^2}\delta_0 + \frac{2\breve{C}_N 8^N}{(\mathrm{Re}\,z_g)^{N+1}}\delta_0^N \leqslant \frac{1}{4} \quad\text{and}\quad \frac{8\delta_0}{\mathrm{Re}\,z_g} \leqslant \min\left(\frac{1}{2},\frac{1}{2K_N}\right)$$

(note that $N\geqslant 1$) and consider times t such that

$$\frac{4\delta_0}{\mathrm{Re}\,z_g} \leqslant \delta e^{t\,\mathrm{Re}\,\gamma} \leqslant \frac{8\delta_0}{\mathrm{Re}\,z_g}. \tag{3.20}$$

Owing to (3.18), we also choose δ small enough such that

$$\frac{2\|(1-\chi_\delta)g\|_{L^1}}{\mathrm{Re}\,z_g} \leqslant \frac{1}{4}.$$

We conclude from these inequalities that

$$\|f-f_0\|_{L^1}\geqslant \delta_0$$

and that (3.15) is satisfied.

To end the proof, it remains to fix the time t_δ and to choose the truncation function χ_δ. Let us show that, for δ small enough, there exists a time t_δ satisfying both (3.19) and (3.20). If $\operatorname{Im}\gamma = 0$, then (3.19) is clearly satisfied since $\operatorname{Re} z_g > 0$: a suitable t_δ is then

$$t_\delta = \frac{1}{\operatorname{Re}\gamma} \log\left(\frac{6\delta_0}{\delta \operatorname{Re} z_g}\right).$$

Assume now that $\operatorname{Im}\gamma \neq 0$. For δ small enough, the size of the interval of times t satisfying (3.20) becomes larger than $\frac{2\pi}{|\operatorname{Im}\gamma|}$. This means that it is possible to find a time t_δ in this interval satisfying (3.19).

Step 5. Choice of χ_δ. For all $\delta > 0$, we have to fix the function $\chi_\delta \in \mathcal{C}^\infty(\mathbb{R})$ such that (3.14), (3.16), (3.18) are satisfied and such that $f_0 + \delta f_1(0) \geqslant 0$. First of all, proceeding as in the proof of Lemma 2.1, we obtain that g takes the form

$$g(\theta, v) = -mF'(e_0)\cos\theta - mF'(e_0)\int_{-\infty}^{0} \gamma e^{\gamma s}\cos\Theta(s)ds, \tag{3.21}$$

with $m = \int_0^{2\pi} \rho_f(\theta)\cos\theta d\theta$. Hence,

$$|g(\theta, v)| \leqslant |m|\left(1 + \frac{|\gamma|}{\operatorname{Re}\gamma}\right)|F'(e_0)|. \tag{3.22}$$

The assumptions on F and F' in Theorem 1.2 imply that

$$\forall e < e_*, \qquad |F'(e)| \leqslant C(e_* - e)^{-\alpha}F(e) \tag{3.23}$$

with $\alpha \geqslant 1$. Since $F(e) > 0$ for $e < e_*$, the local assumption becomes global. Let χ be a $\mathcal{C}^\infty$ function such that $0 \leqslant \chi \leqslant 1$ and

$$\begin{cases} \chi(t) = 0 & \text{for } t \leqslant 0, \\ \chi(t) \leqslant 2t^\alpha & \text{for } t \geqslant 0, \\ \chi(t) = 1 & \text{for } t \geqslant 1 \end{cases}$$

and let

$$\chi_\delta(e) = \chi\left(\frac{e_* - e}{\delta^{1/(2\alpha)}}\right). \tag{3.24}$$

From

$$\|(1 - \chi_\delta)g\|_{L^1} \leqslant \|g 1_{e_* - \delta^{1/(2\alpha)} < e_0(\theta,v) < e_*}\|_{L^1}$$

and dominated convergence, we clearly have (3.18). By (3.22) and (3.23), we have, for all $(\theta, v) \in \mathbb{T} \times \mathbb{R}$,

$$\delta\,|\operatorname{Re} g(\theta, v)\chi_\delta(e_0(\theta, v))| \leqslant \delta C|e_* - e_0|^{-\alpha}F(e_0)\frac{|e_* - e_0|^\alpha}{\delta^{1/2}} = C\delta^{1/2}f_0(\theta, v),$$

so for δ small enough, we have $f_0 + \delta f_1(0) \geqslant 0$.

By differentiating (3.21) and using that Θ is bounded in $L^\infty(\mathbb{R}, W^{k,\infty}(\Omega_0))$ (see Subsection 3.1), we get

$$\forall j+\ell \leqslant N, \qquad |\partial_\theta^j \partial_v^\ell g(\theta, v)| \leqslant C \max_{k \leqslant N+1} F^{(k)}(e_0) \leqslant C|e_* - e_0|^{N-1},$$

where we used Taylor formulas and the fact that $F \in \mathcal{C}^\infty$ with $F(e) = 0$ for $e \geqslant e_*$. Besides, from (3.24), we obtain (if $\delta \leqslant 1$)

$$\forall\ 1 \leqslant j+\ell \leqslant N, \qquad |\partial_\theta^j \partial_v^\ell \chi_\delta(e_0(\theta, v))| \leqslant C\delta^{-N/(2\alpha)} 1_{e_* - \delta^{1/(2\alpha)} < e_0(\theta,v) < e_*}.$$

Therefore

$$\begin{aligned}
\|g\chi_\delta\|_{W^{N,1}} &\leqslant \|g\|_{L^1} + C\delta^{-N/(2\alpha)} \int_0^{2\pi} \int_{\mathbb{R}} |e_* - e_0(\theta, v)|^{N-1} 1_{e_* - \delta^{1/(2\alpha)} < e_0(\theta,v) < e_*} d\theta dv \\
&\leqslant \|g\|_{L^1} + C\delta^{-N/(2\alpha)} \int_{e_* - \delta^{1/(2\alpha)}}^{e_*} (e_* - e)^{N-1} \left(4 \int_0^{\theta_{m_0}} \frac{d\theta}{\sqrt{2\,(e + m_0 \cos\theta)}} \right) de \\
&= \|g\|_{L^1} + C\delta^{-N/(2\alpha)} \int_{e_* - \delta^{1/(2\alpha)}}^{e_*} (e_* - e)^{N-1}\, T_e\, de,
\end{aligned}$$

where $\theta_{m_0} = \arccos(-\frac{e}{m_0})$ and T_e is given by (2.9). Now we recall that for $e \leqslant e_*$ we have $T_e \leqslant T_{e_*}$. This yields

$$\|g\chi_\delta\|_{W^{N,1}} \leqslant \|g\|_{L^1} + CT_{e_*}\delta^{-N/(2\alpha)} \int_{e_* - \delta^{1/(2\alpha)}}^{e_*} (e_* - e)^{N-1}\, de = \|g\|_{L^1} + \frac{CT_{e_*}}{N}.$$

We have proved (3.14).

By (3.8), we have

$$\begin{aligned}
\|\widetilde{R}_\delta\|_{L^1} &\leqslant C\left(\|(1-\chi_\delta) g\|_{L^1} + \|(1-\chi_\delta)\partial_v f_0\|_{L^1} \right) \\
&\leqslant C\left(\|g 1_{e_* - \delta^{1/(2\alpha)} < e_0(\theta,v) < e_*}\|_{L^1} + \|\partial_v f_0 1_{e_* - \delta^{1/(2\alpha)} < e_0(\theta,v) < e_*}\|_{L^1} \right),
\end{aligned}$$

so by dominated convergence,

$$\lim_{\delta \to 0} \|\widetilde{R}_\delta\|_{L^1} = 0.$$

We now choose δ small enough such that

$$\|R_\delta\|_{L^1} \leqslant \left(\frac{4\delta_0}{\operatorname{Re} z_g} \right)^N.$$

From (3.20), we obtain (3.16), which ends the proof of Theorem 1.2. □

A Appendix. Existence of unstable steady states

In this section, we prove that the set of steady states satisfying the assumptions of Theorems 1.1 and 1.2 is not empty. More precisely, we prove the following

Lemma A.1. *Let $m > 0$. There exist $m > 0$, $e_* < m$ and there exists a nonincreasing function F, $\mathcal{C}^\infty$ on $\mathbb{R}$, such that $F(e) > 0$ for $e < e_*$, $F(e) = 0$ for $e \geqslant e_*$ and $|F'(e)| \leqslant C|e_* - e|^{-\alpha}F(e)$ in the neighborhood of e_*, for some $\alpha \geqslant 1$, and such that the function $f(\theta, v) = F(\frac{v^2}{2} - m\cos\theta)$ is a steady state solution to the HMF model (1.1) and such that $\kappa(m, F) > 1$, where $\kappa(m, F)$ is given by*

$$\kappa(m,F) = \int_0^{2\pi}\int_{-\infty}^{+\infty} \left|F'(e(\theta,v))\right| \left(\frac{\displaystyle\int_{\mathcal{D}_{e(\theta,v)}} (\cos\theta - \cos\theta')(e(\theta,v) + m\cos\theta')^{-1/2} d\theta'}{\displaystyle\int_{\mathcal{D}_{e(\theta,v)}} (e(\theta,v) + m\cos\theta')^{-1/2} d\theta'} \right)^2 d\theta dv,$$

with

$$e(\theta,v) = \frac{v^2}{2} - m\cos\theta, \qquad \mathcal{D}_e = \left\{\theta' \in \mathbb{T} \,:\, m\cos\theta' > -e\right\}.$$

Proof. Let $m > 0$ and F a nonincreasing $\mathcal{C}^\infty$ function on $\mathbb{R}$ supported in $(-\infty, m)$, which is not identically zero on $(-m, m)$. We first observe that $f(\theta, v) = F(\frac{v^2}{2} - m\cos\theta)$ is a steady state solution to the HMF model (1.1) if and only if m and F satisfy $\gamma(m, F) = m$ with

$$\gamma(m,F) := \int_0^{2\pi}\int_{\mathbb{R}} F\left(\frac{v^2}{2} - m\cos\theta\right)\cos\theta d\theta dv > 0.$$

By using the linearity of γ in F we deduce that $\frac{m}{\gamma(m,F)}F(\frac{v^2}{2} - m\cos\theta)$ is a steady state.

We proceed by a contradiction argument. Assume that

$$\kappa\left(m, \frac{m}{\gamma(m,F)}F\right) \leqslant 1$$

for all $m > 0$ and all nonincreasing $\mathcal{C}^\infty$ function F supported in $(-\infty, m)$ such that, denoting by $(-\infty, e_*]$ the support of F, we have $|F'(e)| \leqslant C|e_* - e|^{-\alpha}F(e)$ in the neighborhood of e_*, for some $\alpha \geqslant 1$. This is equivalent to

$$\kappa(m,F) \leqslant \frac{\gamma(m,F)}{m},$$

or, after straightforward calculation and an integration by parts,

$$-\iint F'\left(e(\theta,v)\right) g_m(e(\theta,v)) d\theta dv \leqslant 0 \tag{A.1}$$

with

$$g_m(e) = (\Pi_m \cos^2\theta)(e) - \left((\Pi_m\cos\theta)(e)\right)^2 - (\Pi_m \sin^2\theta)(e)$$

and for all function $h(\theta)$,

$$(\Pi_m h)(e) = \frac{\displaystyle\int_{\mathcal{D}_e} (e + m\cos\theta)^{-1/2} h(\theta) d\theta}{\displaystyle\int_{\mathcal{D}_e} (e + m\cos\theta)^{-1/2} d\theta}.$$

Now, we choose the functions F as follows. We first pick a nonincreasing $\mathcal{C}^\infty$ function Ψ on $\mathbb{R}$ with support $(\infty, e_\sharp] \subset (-\infty, m)$, then we set $e_* = \frac{e_\sharp + m}{2}$ and define

$$F_\varepsilon(e) = \Psi(e) + \varepsilon \exp\left(-(e_* - e)^{-1}\right), \quad \text{for } e < e_*,$$

the parameter $\varepsilon > 0$ being arbitrary. Since F_ε satisfies the assumptions, it satisfies (A.1). Then, letting $\varepsilon \to 0$, we get

$$-\iint \Psi'\left(e(\theta, v)\right) g_m(e(\theta, v)) d\theta dv \leqslant 0.$$

The function Ψ being arbitrary, this is equivalent to

$$g_m(e) \leqslant 0, \qquad \forall m > 0, \quad \forall e \in (-m, m),$$

or,

$$g_1(e) \leqslant 0, \qquad \forall e \in (-1, 1). \tag{A.2}$$

Let us now prove that the function $g_1(e)$ is in fact positive in the neighborhood of $e = 1$, which contradicts (A.2).

Indeed, we introduce

$$\alpha(e) = \int_{\mathcal{D}_e} (e + \cos\theta)^{-1/2} d\theta, \qquad \beta(e) = \int_{\mathcal{D}_e} (e + \cos\theta)^{-1/2} \sin^2\theta d\theta.$$

We have

$$\begin{aligned}
\alpha(e) g_1(e) &= \alpha(e) - 2\beta(e) - \frac{1}{\alpha(e)} \left(\int_{\mathcal{D}_e} (e + \cos\theta)^{1/2} d\theta - e\alpha(e) \right)^2 \\
&= (1 - e^2)\alpha(e) - 2\beta(e) + 2e \int_{\mathcal{D}_e} (e + \cos\theta)^{1/2} d\theta - \frac{1}{\alpha(e)} \left(\int_{\mathcal{D}_e} (e + \cos\theta)^{1/2} d\theta \right)^2.
\end{aligned}$$

From [22], we have

$$\alpha(e) \sim -\sqrt{2} \log(1 - e) \quad \text{as } e \to 1^-,$$

and direct calculations yield

$$\int_0^{2\pi} (1 + \cos\theta)^{1/2} d\theta = 4\sqrt{2}, \qquad \beta(1) = \frac{8\sqrt{2}}{3}.$$

This means that

$$\alpha(e) g_1(e) \to \frac{8\sqrt{2}}{3} > 0 \quad \text{as } e \to 1^-,$$

$$g_1(e) \sim \frac{8\sqrt{2}}{\alpha(e)} \text{as } e \to 1^-.$$

This proves the claim. □

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