

Is an Inquiry-Based Approach Possible at the Elementary School?

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The value of problem solving to promote mathematical understanding and learning is recognized equally by mathematicians, teachers' trainers and teachers. However, in spite of this agreement to associate mathematical activity with problem solving, problem solving has had multiple and often contradictory meanings through the years (Schoenfeld, 1992, p. 337). This subject is regularly approached and questioned on international colloquiums such as ICME-13.

Santos-Trigo recognizes that “research in problem solving has generated interesting ideas and useful results to frame and discuss paths for students to develop mathematical knowledge and problem solving proficiency” (Santos-Trigo, 2013, p. 500). But he also notices that “it is not clear how teachers implement and assess their students' development of problem solving competencies” (ibid., 2013, p. 500). In this area, questions about *problem posing* especially emerge. As generation of new problems or reformulations of given problems (Silver, 1994) problem posing is epistemologically relevant for teaching and learning mathematics. Indeed, mathematicians, physicists and epistemologists like Hadamard, Einstein, Popper and Bachelard seem to agree that posing an interesting problem is more important than solving it. Following Singer, Ellerton, and Cai (2013), we can link problem posing experiences to “development of abilities, attitudes and creativity, and its interrelation with problem solving and studies on when and how problem solving sessions should take place” (Malaspina, 2016, p. 34). Likewise, it provides information about the ways to pose new problems and about the need for teachers to develop abilities to handle complex situations in problem solving contexts. So, problem posing seems to be an interesting topic to both study teaching and learning mathematics.

Inquiry is at the heart of problem posing (Singer et al., 2013). In Europe, over the last decade, the institutional willingness to promote Inquiry Based Learning (IBL) in mathematics revive interest on problem solving and posing to teach and learn mathematics at every level of education. But the conditions and constraints which might favor, or on the contrary hinder, implementation of IBL in mathematics and learning mathematics with IBL must still be specified. Dorier and Garcia (2013) considered that teachers play a central role in an institutional system and that attention should be paid to teachers' training, especially for primary school. They also mentioned the importance of the resources that should contained didactical comments.

In this context, we address the question of the possibilities of learning mathematics based on an inquiry approach at French elementary school, as far as most of elementary school teachers have a weak scientific background and therefore a weak experience on posing problems. To this end, we propose to study two well-contrasted case studies using IBL for

mathematics learning in ordinary teaching context and in didactic engineering context and, within, describe and characterize some conditions of possibilities of learning mathematics with inquiry at French elementary school. Taking into account previous studies about inquiry-based learning in mathematics and its implementation in the classroom, in the European context (Sect. 6.1), we hypothesize that these conditions are both determined by the problem and by the activity of the teacher as he supports students' activity. To characterize the conditions on the problem we introduce the notion of *potential of inquiry*. To identify conditions attached to the teachers' practices we rely on the analyze of the students' activity with the *leaning by problematization* framework (Fabre & Orange, 1997). This theoretical framework is strongly anchored in science epistemology (Popper, 1972; Bachelard, 1970) and in inquiry (Dewey, 1938; Fabre, 2005), as we will explain it (Sect. 6.2). Within this framework posing problem is as important as solving it and problem posing is considered as a way to explore what conditions and possibilities for problems or situations to engage students in problem posing activities. Indeed, we can identify within this framework whether teachers' in-class activity allowed problem posing and solving for the students.

6.1 Inquiry-Based Learning: An Inquiry Processes That Is Difficult to Implementation in Classroom

The promotion of a teaching of mathematics by IBL appears as a world movement even if the epistemological outlines and the didactic stakes are to be specified. Attempts to implement IBL in mathematics are made and numerous research projects concerning the inquiry in sciences and in mathematics have been carried out during the last 20 years (Erh-Tsung, & Fou-Lai, 2013; Engeln, Euler, & Maaß, 2013; O'Shea & Leavy, 2013). Our review of literature shows that the definition of IBL is not stabilized in the field of the international didactics. However, IBL appears strongly connected with inquiry as we will bring it out. Furthermore, this survey also highlights difficulties to implement IBL in day-to-day teaching.

6.1.1 IBL and Inquiry

IBL is anchored at the same time in the investigation and in the construction of knowledge about reasoning in a critical way, in reference to the philosopher Dewey (1859–1952) (Linn, Davis, & De Bell, 2004; Rocard et al., 2007). This inquiry progresses through the interactions between unknown elements, that raise questions, and known elements that allow to analyze these unknown elements and to form hypotheses or still to connect some elements in already lived experimentations. Thus, an essential function of IBL is to organize the field of experimentation of the students and the development of attitudes of learning based on the practice of reflexive activities like inquiry (Dewey, 2011; Hétier, 2008). The term “inquiry-based learning” refers to student-centered ways of teaching by posing questions, exploring situations and developing their own ways towards solutions. It thus reaffirms the link between inquiry and problem posing (Maaß & Artigue, 2013).

In Europe, an institutional context has intended to promote IBL for teaching and learning mathematics (Rocard et al., 2007) and several European research projects have been conducted to help in the development of new practices of mathematics teaching. We can especially mention the Fibonacci project (2010–2013; led by the Ecole Normale Supérieure, France and the University of Bayreuth, Germany), the LEMA project (2008, 2010; 6 European countries) and the PRIMAS project (2010–2013: 14 universities, 12 European countries) both rooted in the Pedagogical University of Freiburg.

We notice that the cited above projects put forward the wealth of a work of modelling and then most of the time propose IBL from a modelling context. But we make assumption that some other kinds of problems can also lead to mathematical inquiry. In this paper, we illustrate this possibility with a discrete optimization problem (problem 2).

Outcomes of these projects include propositions of IBL situations that support the development of mathematics teaching practices and identification of difficulties of implementation for the teachers. This latter especially draws our attention for our work.

6.1.2 Difficulties of Implementation

Within PRIMAS project, which objective was the developing of devices of pre- service and in-service teachers' training, Dorier and Garcia (2013) identified the conditions and constraints that might favor, or on the contrary, hinder a large-scale implementation of inquiry-based mathematics and science education.

In most countries, it seems that teachers find it difficult to choose statements and to implement in class activities based on inquiry (Dorier & Garcia, 2013; Schoenfeld & Kilpatrick, 2013). It also seems difficult to define and to distribute the responsibilities between students and the teacher in front of proposed tasks. The teachers do not feel at ease with sharing students' results to compare their productions, with organizing them into a hierarchy and with implementing mathematical debates (Inoue & Buczynski, 2011). These difficulties that are related to the process of institutionalization (Choquet, 2014) do not seem specific to the IBL. They are well-known about problem solving.

Having identified and explained these difficulties, research proposed improvements in order to promote and develop IBL in rights conditions leading to students' learning. First, types of resources (textbooks, websites) are different among countries and it seems important to propose to teachers' resources promoting IBL "accompanied by didactical comments on how it can be efficiently implemented in class and embedded into a device to be used for professional development" (Dorier & Garcia, 2013, p. 849). Second, studies show the central role played by the use of digital technology in problem solving and in particular in the IBL:

"There is a need to develop or adjust current problem-solving frameworks [...] to characterize the ways of reasoning, including the use of new heuristics, for example, dragging in dynamic representations, with which students construct learning in a result of using digital tools in problem-solving approaches" (Santos-Trigo, 2013, p. 500). And it also seems necessary "to develop methodological tools to observe, analyze, and evaluate group's problem-solving behaviors that involve the use of digital technology" (Ibid., p. 500). This use should not be

reduced to the exploration of the problem to establish hypothesis but it has to be a part of all the resolution's process of the problem. (Artigue, 2012)

Third, even if curricula in all countries support IBL (Dorier & Garcia, 2013), the elementary teachers' mathematical and science competencies include a weak didactical qualification to implement an IBL approach in their class. That's why pre-service and in-service teachers' training might be increased especially on IBL.

In the French context, institutional aims greatly emphasized the fact that mathematics teaching should contribute to the development of students' inquiry competences. Low scores of French students to PISA problem solving samples, and disaffection of scientific programs at University are the two main reasons mentioned to support these injunctions. The mathematics curriculum of primary school (2016) affirms again the importance of developing problem solving students' competences, especially through the resolution of real-word problems. French elementary school teachers have to teach mathematics and sciences even if they are not specialist in these matters (Artigue, 2011). There exist strong links between IBL in sciences and in mathematics education, especially the fact that inquiry is the core of mathematical and scientific activities (Hersant & Orange-Ravachol, 2015). But, there also exist differences that make it difficult to implement and require a specific teachers' training.

In the following part, taking into account research results presented here, we present the theoretical framework of learning by problematization (Fabre & Orange, 1997) that allows to envisage mathematical problems solving in terms of inquiry, which "can spread and produce solutions to [these] problems but also establish necessities to which they are subjected" (Hersant & Orange-Ravachol, 2015, p. 100).

6.2 Theoretical References and Research Design

We propose to identify conditions of possibilities for learning mathematics with IBL at the elementary school. To this end, we use the learning by problematization framework (Fabre & Orange, 1997) and introduce the notion of inquiry potential of a problem.

6.2.1 Learning by Problematization: A Theoretical Framework to Analyze Students' Mathematical Activity

We use this framework to analyze the students' activity. So, it is important to precise that posing and constructing problem is here seen from the students' point of view. Indeed, even if the teacher posed a question it does not mean that the problem is posed for the students and least of all that they construct it. But when students make attempts, formulate sub-problems or conjectures we can say that they at least pose the problem.

Learning by problematization is a theoretical framework developed by Fabre and Orange (1997) for the didactic of biology. It is yet well known and used in many didactics, especially in didactic of mathematics (Hersant, 2010; Grau, 2017). In this framework, the position and the construction of the problem have a more important place than its

solution. This is connected with the importance of inquiring (Dewey, 1938), posing and constructing problems (Popper, 1972) in sciences.

Taking also into account Bachelard's epistemology, this framework considers that knowing is not "knowing that" but "knowing that it cannot be otherwise". Indeed, this framework makes a distinction between facts that come under opinions, and necessities that come under scientific constructions built into a scientific paradigm (Kuhn, 1962). Therefore, problematization is defined as a multidimensional process involving posing, building and solving problem in a dialectic of facts and of ideas (Orange, 2000). From an epistemological point of view, this above feature of the problematization process which deals with scientific activity is also relevant for mathematics. Let's refer to mathematicians to explain this specific point. Regarding the multidimensional process, we can first refer to Poincaré (1905, 1970) who noticed the strong link between intuition as "an instrument of creation" (p. 37) and logical as an "instrument of proof" (p. 37). So, from his point of view, intuition plays a key role in posing and building problem, whereas logical, and especially deduction, mainly intervenes in solving problem. Moreover, this multidimensional process deals with an experimental dimension. Perrin who is a mathematician asserts this experimental dimension of the mathematics (2007) when he explains that "mathematics is also an experimental science". For Pólya (1954, 1965) this experimental dimension underlies a similar dialectic of facts and ideas in biology and in mathematics. Indeed, for him (1965, pp. 110–111), "specific examples" (facts) suggests "new significations" (ideas) that lead to hypothesis and then proof (Pólya, 1965, p. 111).

These distinctions between facts and ideas lead Orange to consider three structures of thinking summoned up during the search of a problem (Orange, 2000, 2005). The first one is the *empirical register* that corresponds to relevant facts for the problem, established during the search of the problem. The second one deals with the *register of necessities* that are established into an "epistemic structure" that Orange calls the *explicative register* (Orange, 2000, 2005). We shall explain these registers with an example. If the problem is to know if 46 is the sum of three consecutive numbers, 45 $14 + 15 + 16$ and 48 $15 + 16 + 17$ are relevant facts. Then relating and confronting these facts, in an "induction" and "more general statement" process (Pólya, 1954), make it possible to establish necessities: as $14 + 15 + 16 = 45$ and $15 + 16 + 17 = 48$, there is no other possibility to sum three consecutive numbers and to obtain 46; indeed 46 will never be the sum of three consecutive numbers. These latter propositions are not facts, nor opinions. But they are not only conclusions: they are built necessities. And building them we ensure that it cannot be otherwise. So, these elements come under the *register of necessities*. These necessities are established into a model. Indeed, to put up these necessities we consider arithmetic domain. But we could also envision the problem in a functional way (with a discreet function). Then the necessities will have to do with surjection function. In a way, this model matches with Piaget's "epistemic structure". Orange (2000, 2005) call it the *explicative register*. For Scientifics or mathematicians, excepted during paradigm shifts, models are well shared and known. But, for students who are in process of learning what sciences or mathematics are, these explicative models are in construction. And we have to take into account this in-process-building in our analysis of students' search of problem. For the previous problem, for example, at the end of the primary school many students

think that 46 cannot be the sum of three consecutive numbers because, even if they try a lot, they do not find any such sum. Their model corresponds to “naïve empiricism” (Balacheff, 1987). It is not an acceptable model in mathematics, regarding to proof criteria but it explains the way they envision the solution of the problem that is in an empirical model (Hersant, 2010).

The space of constraints and necessities (Orange, 2000, 2005) is a way to represent the construction of the problem. It accounts for tensions between empirical facts and necessities into an epistemic structure that are realized by one student or a group of students. The pertinent facts and the tensions established by the students can be indeed inferred from their productions and the verbal interactions observed in class. Then these tensions are represented by linking facts and necessities (for examples, see Fig. 6.1). In these diagrams connections between the elements of the three registers are not represented by arrows but only by segments. Indeed they indicate no direction, nor logical or chronological links but mean putting in tense relations.

6.2.2 Inquiry Potential of a Problem

We suppose that conditions of learning by inquiry both depend on the way the teacher posed the problem—especially the problem’s writing as the setting of the search—and on the activity of the teacher as a help for the inquiry. Therefore, with respect to the first condition, in the first step of our study we look for the *inquiry potential* of the problem. This empowers us to estimate the possibilities, for the students, to pose and construct the problem from the question posed by the teacher. To define the inquiry potential of a problem we use the following questions:

- (i) is the problem likely to engage students in a research activity for considerable time?
- (ii) in particular, does the problem engage students in making attempts?
- (iii) does the problem support the formulation of sub-problems? Of conjectures?

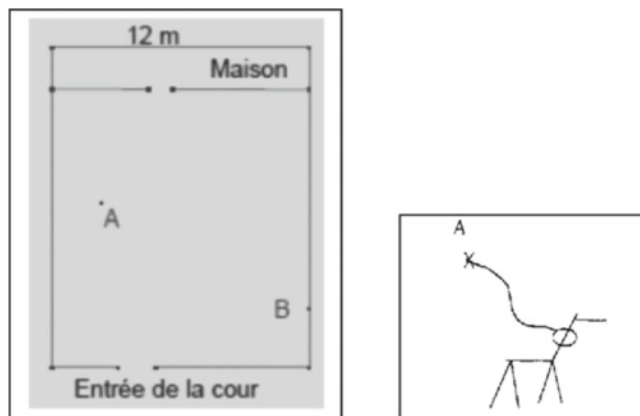


Fig. 6.1 Drawing at the scale given by the teacher for problem P0 (left) and for problem P1 (right)

The first and second concern the possibility for the students to explore the problem and, therefore, to have great conditions to construct it. The third concerns the construction of the problem as the formulation of sub-problem helps it.

6.2.3 Research Design

We analyze the activity of students who are between 8 and 11 years old while they try to solve a mathematical problem. To this end, we use a corpus extracted from previous projects (Choquet, 2014; Hersant, 2010) and take at it a fresh look with new theoretical tools. The first case study deals with modelling. It focuses on the learning of the concept of circle—as the set of all points in a plane that are at a given distance from a fixed point—and of disc—as the set of points that are at a smaller than or equal distance to a fixed point. It corresponds to an ordinary classroom situation (Laborde, Perrin-Glorian, & Sierpiska 2005) as the researcher does not intervene in the choice of the problem nor in its management in the classroom.

From the teacher's point of view this situation corresponds to an investigation situation. The second case study is extracted from a didactic engineering. Its goal was to overcome a widespread misconception among the young students about the impossibility in mathematics: “it is impossible because I did not succeed in doing it”. Hersant showed that discreet optimization problems are suitable to overcome this misconception and therefore proposed a set of didactical situations about discreet optimization. The design of these situations both relied on the theory of didactical situations in mathematics (Brousseau, 1997)—importance of a retroactive milieu and of the didactical contract—and on learning by problematization—importance of posing and constructing problem and of building necessities. These situations can also be considered as inquiry situations.

For these both cases, our analyze consists of two steps. The first one deals with doing a priori analyze and the second with a posteriori one. In the first step, we determine the inquiry potential of the problem and then, as a minimal investigation exists, we establish a priori space of constraints and necessities. That means an ideal space of constraints taking into account students' knowledge when they have to solve the problem. In the second step, we confront these results with the students' productions and the teacher's intervention. This brings elements on the role of the teacher in the inquiring process.

To study the possibilities to learn mathematics by inquiring at the elementary school, we will look into the way these students construct the problem, that means the pertinent facts they consider, the necessities they establish and the epistemic structure they summon up. We will represent this activity with a space of constraints and necessities. This space will help us to characterize their activity as a problematization or not. But it will also help us to understand the conditions that permit or not this problematization.

6.3 First Case Study: Modeling a Situation to Learn About Disc in an Ordinary Teaching Practice

Let's consider the two following problems. The first one is part of the French official instructions. The second one is part of a textbook, it's the problem that the teacher chose to submit to his students.

Problem P0: Sophie has to fetch milk from the farm whose yard is shown below. In A and B are leashed two dogs. In A, Azor has a 6 m long leash; in B, Baltazar has a 5 m long leash. Can Sophie go to the door of the farm without being bitten?

Problem P1: A pet is leashed to a post. The leash is 8 m long. Draw a picture of the area where the pet can move.

None of these problems has an immediate answer for a pupil who ignores the definition of a circle as the set of all points in a plane equidistant from a fixed point. For both of these problems, we can analyze as follows the inquiry potential.

6.3.1 Inquiry Potential of the Problem

Concerning the problem P0, the question allows students to easily propose an answer—that may not be the expected answer (i). Indeed, the wording of the question of P0 is non-mathematical but the figure introduces the geometric framework as the posts of both dogs are represented by points. In an inquiry way, as Dewey describes it, they will make attempts (can Sophie go straight ahead? can she take this way?) (ii). Students can easily formulate a conjecture (it is possible or not) and then have to find a way to prove it (iii). Owing to the effect of the didactical contract (Brousseau, 1997; Hersant & Perrin-Glorian, 2005, Hersant, 2014), they will surely try to find one. As this task appears as a mathematical one, students know the teacher will not accept an unjustified answer and, therefore, they won't themselves accept it because in the mathematics classroom answers must be justified in accordance with the epistemological side of the didactical contract (Hersant, 2014). They can identify sub-problems to increase their understanding of the situation (when she is there, what happens?). Their tests will certainly lead them to conjecture that the border of the "unbitten" zone is made of circles (iii). So, this problem is likely to generate doubts and implication in the task to remove these doubts.

Concerning the problem P1, it is more difficult for students to have an idea of the expected answer (it is neither «yes because» nor «no, because») (i). Indeed, the answer matches with staked knowledge that students are supposed to ignore. This significant difference with the previous problem is due to the wording of the problem: the students have to «draw a picture of the area» and not to decide to the possibility of plowing a path. Moreover, counter to the previous problem, this one is not clearly a mathematical one. Indeed, even if the word "area" is used in the wording of the problem, the task and the draw accompanying this wording suggest the expected answer is not mathematical nor geometrical. We can suppose that the schema (Fig. 6.1, right) is given to help children to imagine the situation but it hinders the setup of a suitable didactical contract, especially a geometrical contract. Nevertheless, students can make attempts (the dog can be here, he cannot be there, etc.) (ii). They can identify sub-problems (can the dog go here?). But they probably will be satisfied with the identification of some places and will not seek further (iii).

6.3.2 A Priori Space of Constraints and Necessities

Figures 6.2 and 6.3 respectively picture the a priori spaces of constraints and necessities for P0 and P1.

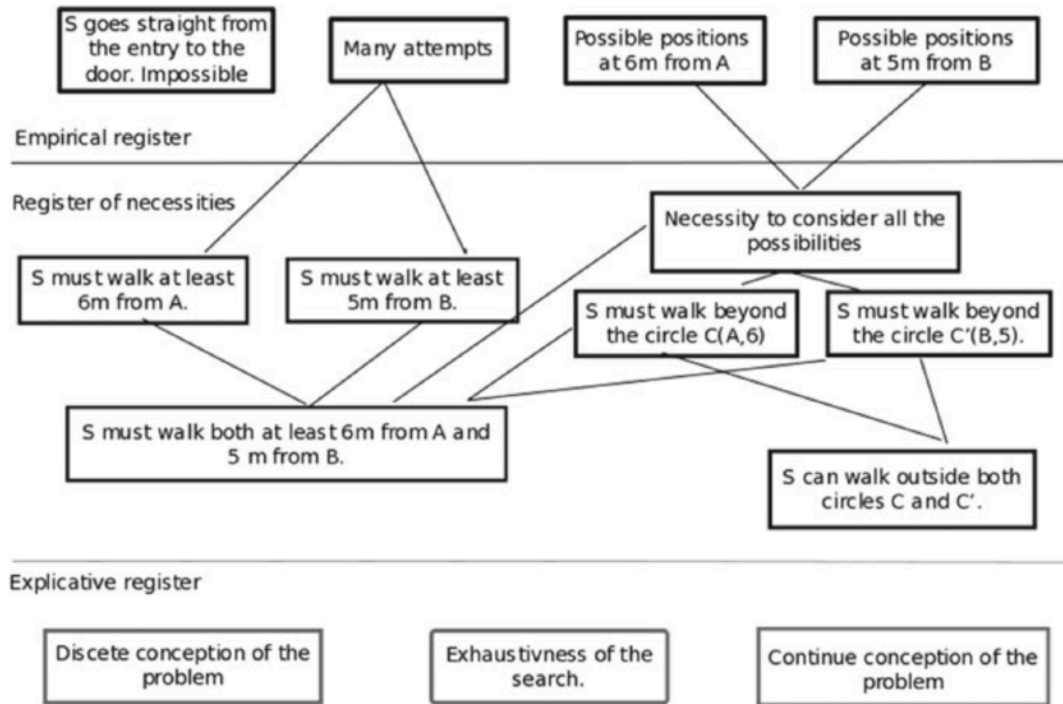


Fig. 6.2 A priori space of constraints and necessities for P0. "S" means Sophie, "C(A, 6)" means circle which center is A and which radius is 6; D(A,6) means the disc

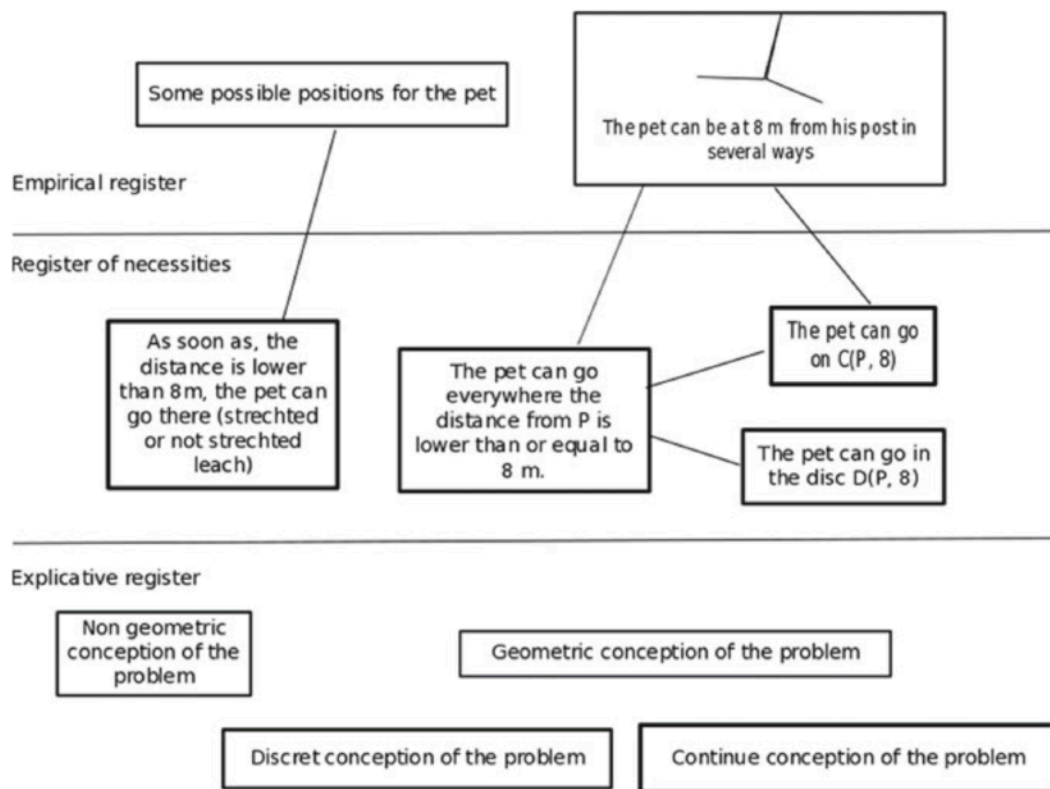


Fig. 6.3 A priori space of constraints and necessities for P1. “P” matches with the post

For each problem, students can construct sub-problems and this leads them to do some tests. These tests allow them to constitute a corpus of possible and impossible ways either for Sophie or the pet. Indeed, for P0, students use drawing to scale and, so they can see on their drawing if the circles intersect. Therefore, the discussion about the number of intersections is moot. These new elements about the problem are facts and match with the empirical register. Dually, these empirical elements allow students to give off necessities of the problem (for example: the pet must go beyond 8 m; he can go everywhere the distance to the post is less than 8 m; he always can go at 3 m of his post).

These two spaces highlight the necessity to move from a discrete representation (search for punctual solutions) of the problem to a continuous one (search for all solutions). This moving is also a crossing from the one-dimensional geometry to a two-dimensional geometry. For P0, the problem takes charge of this cognitive gap. Indeed, the two following necessities “Sophie has to walk at least at 6 m from A” and “Sophie has to walk at least at 5 m from B” only give positions where Sophie is not bitten. But they do not tell a possible way to the door. So, it does not close the problem and maintains the doubt to the possibility to reach the door. This doubt keeps the necessity to browse all the possibilities in an exhaustive manner. And thereby, it leads to encounter the move from the discrete to the continuous. The effect of an epistemological clause of the didactical contract also play a part (Hersant, 2014): students know that it is a mathematical task and especially a geometrical task (points are represented on the diagram) and that, therefore, they cannot be satisfied with a “yes” or “no” answer.

At the opposite, in problem 1, the expected answer is a drawing and the didactical contract is not clearly a mathematics one (the drawing indicates a drawing pet and a point) or a geometric representation. The epistemological clause of the contract cannot play for inducing an exhaustive research and moving from a discrete to a continuous envision of the problem. Moreover, the situation itself does not generate many doubts. In other words, the situation does not have the potential to lead the students to the research of every possibility. In this case the move from the discrete to the continuous envisions the problem that is not supported by the situation.

6.3.3 Problem P1: Implementation, Students' Productions and Sharing

One of the observed teachers turned the problem P0 into the P1. Its realization has been observed by Choquet (2014) in a 21 students' classroom at the end of the elementary school (10–11 years old students). They knew the signification of common vocabulary associated with a circle (ray, diameter, center, chord). They also knew how to draw a circle with a compass. But, they did not yet know the mathematical definition of a circle.

The teacher presents the activity and let students search individually for ten minutes. Then they work in small groups during fifteen minutes: each student has to search the problem and to give a written solution, but students are allowed to speak about the problem. After the students' research, the teacher selects three productions to be collectively discussed for ten minutes. Indeed, five students turn in a blank page and the productions of the sixteen other students can be split into three categories.



Fig. 6.4 Productions of students from first category (left), from second category (middle) and from third category (right)

The first category ties with students who draw the leash and a dog in a vertical plan (Fig. 6.4, left). They are seven in this case. Their production is closely linked to the drawing proposed by the teacher in the presentation of the problem (Fig. 6.1, right). Only one position of the dog is considered; they do not represent any area. These answers indicate that these students did not achieve to pose the mathematical problem, nor construct the problem, nor explored the field of possibilities. Indeed, they probably did not establish any necessity about this problem. Why? They did not consider the problem as a mathematical one because the statements of the problem implemented a didactical contract that is beside mathematics. In the second category, we gather two students who begin to schematize and envision several positions for the dog (Fig. 6.4, middle). We can suppose that these students lead a mathematical inquiry but they come up against the difficulty to move from discrete to continuous. These students most probably do not strike the problem of exhaustiveness of the answer. Their activity may correspond to our a priori space of constraints (Fig. 6.3). In the last category, there are seven students who draw the

circle that bounds the zone without prior trials (Fig. 6.4, right). The observation suggests that these students already knew the definition of a circle as the set of all points in a plane that are at a given distance from a given point. So it is difficult to say that they have posed and constructed the problem.

Finally, it seems that the students have no approach of inquiry: those who have well conceptualized the notion of circle already reinvest it, probably without considering any sub-problems; the others stumbled on the exploration of a one-dimension problem.

6.3.4 What Can We Learn from This Case?

In this case, we think that the teacher changes the wording of the problem P0 without enough considering the effects of these modifications on the inquiry potentialities of the problem. That reveals a critical point for teaching by inquiry at the elementary school in France: most teachers have a literature Baccalaureate and it is not easy for them to fashion problems for their students. Moreover, this example mainly shows the limits of using real-world problems to impulse an inquiry process. Indeed, despite of an attempt of class discussion after group work, most of the students do not identify the task as a mathematical one. The didactical contract is not clear enough and the teacher does not intervene to make this contract explicit. So, we can imagine that the students will not learn about geometry with this problem. Indeed, during the research phase their activity do not allow them to build up a mathematical problem. Then, the solution given by the teacher will not be anchored in a problem research. In these conditions, can we still consider that these students learn by solving problem?

6.4 Second Case Study: Searching the Optimal Solution in a Discrete Optimization Problem in a Didactic Engineering

The situation has been designed within the framework of a didactic engineering involving a researcher (Hersant) and teachers (Hersant, 2010). It refers to the called “no three in line problem”. The wording of the problem is the following:

Problem 2: How many points can we put at most on this grid without forming any alignment of three points (see Fig. 6.5)?

6.4.1 Inquiry Potential of the Problem

This situation has been realized in several classes by teachers who all contributed to the design of the situation; the researcher did not intervene in the class management. The scenario was the following. First each student has to test possibilities, respecting constraints of the problem. This is the enumeration phase. As the task demands to make tests, all the students can do something and start to explore the problem (i and ii). So, the problem is likely to engage students in search. Moreover, this engagement in the problem is durable because students take to the game (i).

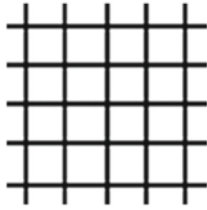


Fig. 6.5 The grid given to the students

If necessary, the teacher takes examples of putting points and asks students if the case correspond to an alignment. Then students can work by group and start to construct sub-problems (iii) like: is 7 the best solution? For example, the researcher often observed the following situation. A student comes against the possibility to put more than six points on the grid (ii), he is convinced that it is impossible to put seven points on it. Indeed, in a naïve empirical explicative register, he thinks mathematical impossibility matches with pragmatic impossibility: “it’s impossible because I search hard but I do not succeed”. But, when one of his classmate succeed in putting seven points, he wonders: is seven the solution? how can I be sure? All the students will not doubt the same, indeed teacher can help them to construct these sub-problems (iii).

When many students have a solution that they can’t improve, the teacher stops the research and ask each group to realize a poster with its best or one of its best solution(s). The solutions of each group are post up on the board and are collectively examined to check that they respect the no three in line constraint. The best solution(s) of the class is (are) identified. At this time, the question is to know if it is worth to keep searching, that means if we can improve the best solution of the class, or not. Then students are led to construct sub-problems: how can we know if this solution is the best? Is it possible to put n points on the grid?

So, with this problem, students are engaged in the search, they make attempts and construct sub-problems. The inquiry potential of the problem is real.

6.4.2 A Priori *Space of Constraints and Necessities*

The a priori space of constraints and necessities for this problem is the following (Fig. 6.6).

This figure highlights the empirical facts that can be built up by the students during the search of the problem, especially the enumeration phase. It also shows the possible conflict that can exist between an in progress-necessity based on an empirical conception of the impossible and a fact like “I can put 9 points on the grid”. These contradictions will lead students to search for necessities and to evolve their point of view on the problem: searching how to put points on the grid without any alignment of three points will never bring the solution, the proof of the problem also needs to mobilize short-cuts and proofs of impossibilities. Especially they will have to establish the following necessity: there are at most two points on a line (or on a column). Then, if a ten points solution has been found by some students, then they will conclude that the solution is ten.

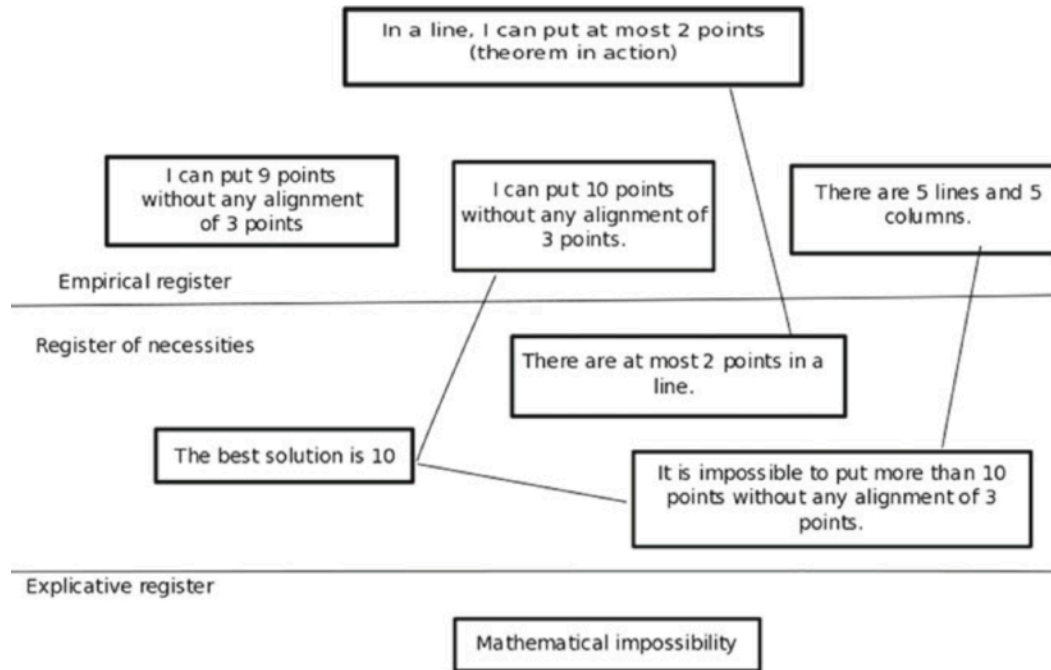


Fig. 6.6 A priori space of constraints and necessities for this problem

6.4.3 Students' Productions

We propose this problem in several classes of students of 8–11 years old (Hersant, 2010). We will here especially be interested in the first class where the situation has been tested (8 and 9 years old students). The first hour was dedicated to enumeration on an arrangement of 4 lines and 4 columns grid. The students found eight points solutions. The second session was dedicated to an enumeration of five lines and five columns grid. At the end of this session, students summarize the state of the search in the class. They formulate the conjectures:

“Our record is 9, but I am sure than we can put 10 on the grid”; “If we concentrate more, we can put more points on the grid”; “There are two in each column”.

The third session was devoted to the search of the solution of the problem. For this we introduce the following question: is it possible to put more than 10 points on the grid? By introducing this question, our aim was to oblige students to work on short-cuts necessities. We knew that if the teacher does not take this initiative, students will keep on enumerating the problem and have no chance to establish necessities. But we observe during this session that students have difficulties to envision the proposition «we can at most put two points on a line without forming any alignment of three points» as a mathematical necessity. Indeed, they envision it only as a theorem in action (Vergnaud, 1998) stemming from their experience of the enumeration. So only 2 pupils in 16 who expressed their views in an individual productions think that it is impossible because we can only put 2 points on a line. The other produced arguments such as the following:

“it’s impossible because Jean said it” (1 in 16 students)

“it’s impossible because we have already well searched and we did not find more” (4 in 16 students)

“because each time we try to add one more point on a 10 points grid there are three in a line” (3 in 16 students)

“it’s impossible, the grid is to small” or “there are not enough crosses” (6 in 16 view students)

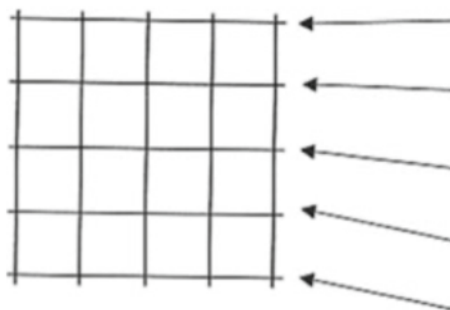


Fig. 6.7 The box of points

So, we decide to introduce a box of points (see Fig. 6.7) and to ask them «You disagree about the reason. Now, using this table and without putting any points on the grid, tell us if it is possible to put more than ten points on the grid». This box of points involves switching from a geometrical setting (Douady, 1986) to an arithmetic one. Indeed, thus, implicitly, the question is: can we make n with a sum of 5 terms small or equal to 2? This helps students without killing the inquiry. It is also a great support to anchor the proof of the problem in the pupils’ activity.

6.4.4 What Do We Learn from This Case?

This brief analysis of the students’ activity shows that, in this case, students have posed and constructed the mathematical problem; they have carried out an investigation with the help of the teacher. So, this case study shows that inquiry is possible at the elementary school in France. It also empowers us to identify some conditions for its existence. First, it seems that the inquiry potential of the problem plays an important role. Here, he is mainly due to the doubt generated by the enumeration phase. Indeed, during this phase each student build certainties that could be destabilized by one of his classmate. Here, the milieu (Brousseau, 1997) of the situation plays an important role in problem posing. Moreover, the problem is easily identified as a mathematical problem by the students and it seems to facilitate the process of inquiry inside the mathematics field as the didactical contract is clear for the students. Finally, the engineering process provided the interventions of the teacher in case of students’ difficulties and these interventions seem to effectively empowered students’ inquiry.

6.5 Conclusion

We studied two cases to identify some conditions of possibilities of learning mathematics with inquiry in elementary French school. For this, we first characterized inquiry for learning mathematics as a problem posing and constructing issue that leads to the establishment of necessities of the problem, according to our framework. Taking this

point of view allows to broaden our vision of inquiry-based learning in mathematics beyond modeling. Indeed, problem posing activity and establishing necessities is at the core of mathematics activity, whatever the statement of the problem is.

Our two cases are well contrasted considering their objectives of learning. Problem P1 is inspired from problem P0. Both of them deal with a real world problem and modelling. They both aim to learn about circle. But, P0 does not lead to investigation by the students. Problem P2 deals with the meaning of impossible in mathematics without any ambition about learning curricular knowledge. It nevertheless brings students to an investigation.

Furthermore, we also highlight that the three problems P0, P1 and P2 have a different inquiry potential. We can explain these differences in the following way. Comparing problem P0 and problem P1, we emphasize the importance of designing the statement of the problem and making available didactical comments to teachers as Dorier and Garcia (2013) proposed it. Indeed, we can suppose that from P0 to P1, we lost a part of inquiry potential because of the lack of didactical comments associated to P0. For the problem P2, instead, we can suppose that the didactic engineering insures the inquiry potential of the problem.

Moreover, important difference between the two cases concerns the possible link between the effective activity of the students during the search of the problem and the solution of the problem. In problem P1, we saw that for many students there may not exist such a link. Therefore, it is very difficult for the teacher to explain it. On the other hand, for problem P2, these links exist and it is easy for the teacher to highlight them. Thus, our study shows differences between the way each teacher manages the students' research and highlights the crucial role of supporting teachers in students' inquiry activity, through didactical comments on the management of the situation, either through their participation in didactic engineering that could contribute to their professional development.

Finally, which conditions of possibility of inquiry at elementary school can we identify? Two of the conditions presented by the second case appear especially important. First of all, the design of the situation conducts students to doubt and therefore enrolls them on inquiry. This property that is related with the potentiality of the problem seems to play a crucial role. Then, the teacher is able to support the students' inquiry activity taking into account their questions and introducing new sub-problem. This allows students to establish the necessities of the problem in direct link with their own exploration of the problem. Regarding this point, we can suppose that the teacher's participation in the modelling of the situation with the researcher have a significant influence that remains to be determined.

In conclusion, this study asks us new questions concerning knowledge at stake in inquiry based learning situations and also teachers' training to manage problem- posing and to use resources cautiously.

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