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Progress on the description of identifying code polyhedra for some families of split graphs

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Abstract

The identifying code problem is a newly emerging search problem, challenging both from a theoretical and a computational point of view, even for special graphs like bipartite graphs and split graphs. Hence, a typical line of attack for this problem is to determine minimum identifying codes of special graphs or to provide bounds for their size.

In this work we study the associated polyhedra for some families of split graphs: headless spiders and complete suns. We provide the according linear relaxations, discuss their combinatorial structure, and demonstrate how the associated polyhedra can be entirely described or polyhedral arguments can be applied to find minimum identifying codes for special split graphs. We discuss further lines of research in order to apply similar techniques to obtain strong lower bounds stemming from linear relaxations of the identifying code polyhedron, enhanced by suitable cutting planes to be used in a B&C framework.

Key words: identifying code problem, polyhedral approach, split graphs

1 Introduction

Many practical applications can be stated as set covering problems. For instance, some newly emerging search problems like fault detection in networks, fire detection in buildings, or performing group tests, can be modeled as special variant of domination problems: so-called identifying codes in graphs [17].

Consider a graph $G = (V, E)$ and denote by $N[i] = \{i\} \cup N(i)$ the closed neighborhood of i . A subset $C \subseteq V$ is *dominating* (resp. *identifying*) if $N[i] \cap C$ are non-empty (resp. distinct) sets for all $i \in V$. An *identifying code* of G is a node subset which is dominating and identifying, see Figure 1, and the *identifying code number* $\gamma^{ID}(G)$ of a graph G is the minimum cardinality of an identifying code of G .

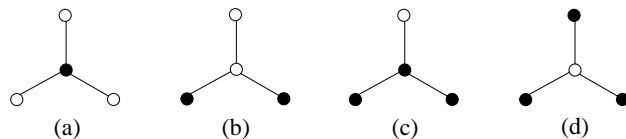


Fig. 1. A graph, where the subset of black nodes forms (a) a dominating (but not identifying) set, (b) an identifying (but not dominating) set, and (c), (d) minimum identifying codes.

Determining $\gamma^{ID}(G)$ is in general NP-complete [9]. It even remains hard for several graph classes where many other in general hard problems are easy to solve, including bipartite graphs [9], split graphs [11] and, thus, chordal graphs. From a graph theoretical point of view, the problem has been actively studied, see e.g. [6,7,12,15] for some recent papers in the area.

Polyhedral studies of the identifying code problem were initiated in [2,3]. For that, a suitable reformulation of the problem in terms of an integer linear program is in order. Determining a minimum identifying code in a graph $G = (V, E)$ can be formulated as set covering problem $\min \mathbf{1}^T x, M_{ID}(G) x \geq \mathbf{1}, x \in \{0, 1\}^{|V|}$ by:

$$\begin{aligned} \min \mathbf{1}^T x \\ x(N[j]) = \sum_{i \in N[j]} x_i \geq 1 \quad \forall j \in V \quad (\text{domination}) \\ x(N[j] \Delta N[k]) = \sum_{i \in N[j] \Delta N[k]} x_i \geq 1 \quad \forall j, k \in V, j \neq k \quad (\text{identification}) \\ x \in \{0, 1\}^{|V|}. \end{aligned}$$

We call

$$M_{ID}(G) = \begin{pmatrix} N[G] \\ \Delta[G] \end{pmatrix}$$

the *identifying code matrix* of G , encoding the closed neighborhoods of the nodes of G ($N[G]$) and their symmetric differences ($\Delta[G]$), and define the *identifying code polyhedron* of G as

$$P_{ID}(G) = \text{conv}\{x \in \mathbb{Z}_+^{|V|} : M_{ID}(G) x \geq \mathbf{1}\}.$$

It is clear by construction that $\gamma^{ID}(G)$ equals the covering number

$$\tau(M_{ID}(G)) := \min\{\mathbf{1}^T x : x \in P_{ID}(G)\}.$$

In addition, a graph G has an identifying code or is *identifiable* if and only if $M_{ID}(G)$ has no zero-row. As $N[G]$ has clearly no zero-row, G is identifiable if and only if $\Delta[G]$ has no zero-row which is equivalent to the known condition that G is identifiable if and only if it has no true twins, i.e., nodes i, j with $N[i] = N[j]$, see [17].

As $M_{ID}(G)$ may contain rows which are equal to or dominated by other rows in $M_{ID}(G)$, we define the corresponding clutter matrix, the *identifying code clutter* $C_{ID}(G)$ of G , obtained by removing repeated or dominated rows from $M_{ID}(G)$. We clearly have that

$$P_{ID}(G) = \text{conv}\{x \in \mathbb{Z}_+^{|V|} : C_{ID}(G) x \geq \mathbf{1}\},$$

and obtain as a linear relaxation the *fractional identifying code polyhedron*

$$Q_{ID}(G) = \{x \in \mathbb{R}_+^{|V|} : C_{ID}(G) x \geq \mathbf{1}\}.$$

In [2,3] we characterized when $P_{ID}(G)$ is full-dimensional and which constraints of $Q_{ID}(G)$ define facets of $P_{ID}(G)$:

Lemma 1 ([2,3]) *Let G be a graph without isolated nodes and let $V_1(G)$ be the set of nodes $k \in V(G)$ such that $\{k\} = N[i] \Delta N[j]$ for two different nodes i and j in $V(G)$. Then,*

- $P_{ID}(G)$ is full-dimensional if and only if $V_1(G) = \emptyset$.
- The constraint $x_i \geq 0$ defines a facet of $P_{ID}(G)$ if and only if $i \notin V_1(G)$.
- All constraints from $C_{ID}(G) x \geq \mathbf{1}$ define facets of $P_{ID}(G)$.

In this work we study the identifying code problem from a polyhedral point of view and focus on some families of split graphs: headless spiders and complete suns.

We provide the according linear relaxations, discuss their combinatorial structure, and demonstrate how the associated polyhedra can be entirely described or polyhedral arguments can be applied to find minimum identifying codes for special split graphs, see Section 2. Parts of the results appeared without proofs in [4].

We discuss further lines of research in order to apply similar techniques to obtain strong lower bounds stemming from linear relaxations of the identifying code polyhedron, enhanced by suitable cutting planes to be used in a B&C framework, see Section 3.

1.1 Preliminary definitions

Given a set F of vectors in $\{0,1\}^n$, we call $y \in F$ a dominating vector (of F) if there exists $x \in F$ with $x \leq y$. It can be also said that x is dominated by y . From now on, every matrix has 0,1-entries, no zero columns and no dominating rows.

As there is a one-to-one correspondence between a vector $x \in \{0,1\}^n$ and the subset $S_x \subset \{1, \dots, n\}$ having x as characteristic vector, we write x instead of S_x . Remind that a *cover* of a matrix M is a vector $x \in \{0,1\}^n$ such that $Mx \geq \mathbf{1}$. According to the previous convention, a cover of M is a subset of columns ($\{1, \dots, n\}$) that intersects all the rows of M .

In addition, the *cardinality* of a cover x is denoted by $|x|$ and equals $\mathbf{1}x$. A cover x is *minimum* if it has the minimum cardinality and in this case $|x|$ is called the *covering number* of the matrix M , denoted by $\tau(M)$.

Recall that the set covering polyhedron of M , denoted by $Q^*(M)$, is defined as the convex hull of its covers. The polytope $Q(M) = \{x \in [0,1]^n : Mx \geq \mathbf{1}\}$ is known as the *linear relaxation* of $Q^*(M)$. When $Q^*(M) = Q(M)$ the matrix M is *ideal* and the set covering problem can be solved in polynomial time (in the size of M).

A cover of M is *minimal* if it does not dominate any other cover of M . The *blocker* of M , denoted by $b(M)$, is the matrix whose rows are the minimal covers of M . It is known that $b(b(M)) = M$ and also that a matrix M is ideal if and only if its blocker is (see [18]). In addition, since $b(b(M)) = M$ we can refer to $Q^*(M)$ and $Q(b(M))$ as a blocking pair of polyhedra. Moreover, \mathbf{a} is an extreme point of $Q(b(M))$ if and only if $\mathbf{a}^T x \geq 1$ is a facet defining inequality of $Q^*(M)$ (see [14]). In the sequel we will refer to this property as *blocking duality*.

Given a matrix M and $j \in \{1, \dots, n\}$, we introduce two matrix operations: the contraction of j , denoted by M/j , means that column j is removed from M as well as the resulting dominating rows and hence, corresponds to setting $x_j = 0$ in the constraints $Mx \geq \mathbf{1}$. The deletion of j , denoted by $M \setminus j$ means that column j is removed from M as well as all the rows with a 1 in column j and this corresponds to setting $x_j = 1$ in the constraints $Mx \geq \mathbf{1}$.

Given M and $V_1, V_2 \subset \{1, \dots, n\}$ disjoint, we will say that $M/V_1 \setminus V_2$ is a *minor* of M and this minor does not depend on the order of operations or elements in $\{1, \dots, n\}$. It is clear that M is always a minor of itself and we will say that a minor $M/V_1 \setminus V_2$ is *proper* if $V_1 \cup V_2 \neq \emptyset$. It is not hard to see that $b(M/j) = b(M) \setminus j$ and $b(M \setminus j) = b(M)/j$ for every $j \in \{1, \dots, n\}$. In addition, if a matrix is ideal then so are all its minors (see [10] for details).

A *rank* inequality is

$$\sum_{i \in M'} x_i \geq \tau(M') \quad (1)$$

associated with a minor $M' = M \setminus U$. If (1) is a facet of $Q^*(M')$, then it is also a facet of $Q^*(M)$ (see [19]).

In addition, if the rank constraint associated with some minor induces a facet defining inequality of $Q^*(M)$ then this inequality is also induced by a minor obtained by deletion (see [1] for further details).

2 Identifying code polyhedra of some split graphs

A graph $G = (C \cup S, E)$ is a *split graph* if its node set can be partitioned into a clique C and a stable set S . Hence, split graphs are closed under taking complements by definition.

Moreover, split graphs form the complementary core of chordal graphs (graphs without chordless cycles of length ≥ 4) since G is a split graph if and only if G and \overline{G} are chordal [13]. This is also reflected in terms of forbidden subgraphs since a graph is a split graph if and only if it is $(C_4, \overline{C}_4, C_5)$ -free [13] (note that C_5 is self-complementary and that \overline{C}_4 occurs as induced subgraph in any chordless cycle C_k with $k \geq 6$ such that all chordless cycles C_k with $k \geq 4$ are excluded in G as well as in \overline{G}). The relation between chordal and split graphs can also be interpreted in terms of intersection graphs: while chordal graphs are the intersection graphs of distinct subtrees of a tree, split graphs are the intersection graphs of distinct substars of a star, see e.g. [8].

Our aim is to study identifying codes in split graphs from a polyhedral point of view. First note that a split graph is identifiable if and only if no two nodes in C have the same neighbors in S . For instance, a complete split graph (i.e., a split graph where all edges between C and S are present) is not identifiable as soon as C contains 2 nodes (as any two nodes in C are true twins). For instance, crowns (the complete split graphs with $|C| = 2$) are not identifiable.

Next, recall that finding a minimum identifying code in split graphs is NP-hard [11]. So far, $\gamma^{ID}(G)$ is only known for two families: on the one hand, stars (the complete split graphs $G = (C \cup S, E)$ with $|C| = 1$) are the only identifiable complete split graphs and have $\gamma^{ID}(G) = |S|$; on the other hand, split graphs $G = (C \cup S, E)$ where every node in S is connected to a distinct 2-node subset of C have $\gamma^{ID}(G)$ of order $\log(|S| + |C|)$, see [11]. The two families show the wide range of the possible size of minimum identifying codes in split graphs: while the lowest possible lower bound of $\log n$ is attained for the latter, stars achieve almost the highest possible value n .

Moreover, a split graph is connected if and only if no node in S is isolated. Every non-connected split graph G contains a connected split graph G' and a non-empty subset $S' \subset S$ of isolated nodes, and clearly $\gamma^{ID}(G) = \gamma^{ID}(G') + |S'|$ holds.

This motivates the study of identifying codes in non-complete, connected split graphs G . We concentrate on three families of split graphs with a regular structure. This allows us to benefit from a certain combinatorial structure of the identifying code clutter $C_{ID}(G)$ of G and to draw conclusions for the polyhedra $P_{ID}(G)$ and the identifying code number $\gamma^{ID}(G)$ in a similar way as discussed for families of bipartite graphs in [2,3].

In particular, note that stars $K_{1,n}$ are bipartite graphs as well as split graphs. Their identifying code clutter is related to q -roses R_n^q , 0,1-matrices with n columns whose rows encode the incidence vectors of all the q -element subsets of $\{1, \dots, n\}$. We have:

Theorem 1 ([2,3]) For a star $K_{1,n} = (V, E)$ with $n \geq 3$, we have

- $C_{ID}(K_{1,n}) = R_{n+1}^2$;
- $P_{ID}(K_{1,n})$ is entirely described by the inequalities $x(V') \geq |V| - 1$ for all nonempty subsets $V' \subset V$;
- $\gamma^{ID}(K_{1,n}) = n$.

In this paper, we study three families of split graphs with $|S| = |C| \geq 2$ having a regular structure. A *headless spider* is a split graph $G = (C \cup S, E)$ with $S = \{s_1, \dots, s_n\}$, $C = \{c_1, \dots, c_n\}$, and $n \geq 2$. In a *thin headless spider*, s_i is adjacent to c_j if and only if $i = j$, and in a *thick headless spider*, s_i is adjacent to c_j if and only if $i \neq j$. It is straightforward to check that the complement of a thin spider is a thick spider, and vice-versa. Moreover, headless spiders where s_i is adjacent to exactly c_i and c_{i+1} for all $1 \leq i \leq n$ are *complete suns*.

It is easy to see that for $n = 2$, the path P_4 equals the thin and thick headless spider, whereas the complete sun is not identifiable. For $n = 3$, the thin headless spider equals the net, and thick headless spider and complete sun its complement, the 3-sun. We consider headless spiders with $n \geq 4$; Figure 2 illustrates all studied three families for $n = 4$. The partition (C, S) is called the *spider partition* and can be found in linear time [16].

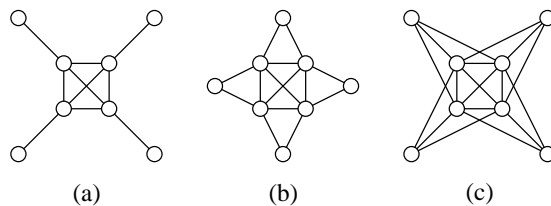


Fig. 2. (a) thin headless spider, (b) complete sun, (c) thick headless spider.

2.1 Thick headless spiders

For simplicity, we will denote thick headless spiders by their partition and we will consider that $C = \{1, \dots, n\}$ and $S = \{n+1, \dots, 2n\}$. Also, we denote by \mathbf{E} a matrix with all entries at value one and recall that R_n^q stands for a q -rose of order n .

Lemma 2 *For a thick headless spider $G = (C \cup S, E)$ with $n \geq 4$, we have*

$$C_{ID}(G) = \begin{pmatrix} R_n^{n-1} & I \\ 0 & R_n^2 \end{pmatrix}.$$

Proof. Let $G = (C \cup S, E)$ be a thick headless spider. The neighborhood matrix of G can be written as

$$N[G] = \begin{pmatrix} \mathbf{E} & R_n^{n-1} \\ R_n^{n-1} & I \end{pmatrix}.$$

Now, in order to find $\Delta[G]$:

- (1) If $i, j \in C$, $N[i] \Delta N[j] = \{i+n, j+n\}$.
- (2) If $i, j \in S$, $N[i] \Delta N[j] = [\{i\} \cup (C - \{i-n\})] \Delta [\{j\} \cup (C - \{j-n\})] = \{i, j, i-n, j-n\}$ and are dominated by a row of the case (1).
- (3.a) If $i \in C$ and $j \in S$, $j \neq i+n$, $N[i] \Delta N[j] = [C \cup (S - \{i+n\})] \Delta [\{j\} \cup (C - \{j-n\})] = \{j-n\} \cup (S - \{j, i+n\})$ is dominated by $N[i]$.
- (3.b) If $i \in C$ and $j = i+n \in S$, $N[i] \Delta N[j] = [C \cup (S - \{i+n\})] \Delta [\{j\} \cup (C - \{j-n\})] = [C \cup (S - \{i+n\})] \Delta [\{i+n\} \cup (C - \{i\})] = \{i\} \cup S$ and are dominated by a row of the case (1).

As the first n rows of the matrix $N[G]$ above are also dominated, we have that the clutter matrix $C_{ID}(G)$ can be written as

$$C_{ID}(G) = \begin{pmatrix} R_n^{n-1} & I \\ 0 & R_n^2 \end{pmatrix}.$$

□

As an immediate consequence, we obtain:

Corollary 1 *Let $G = (C \cup S, E)$ be a thick headless spider. Then:*

- $P_{ID}(G)$ is full-dimensional.
- The constraint $x_i \geq 0$ defines a facet of $P_{ID}(G)$ for each $i \in C \cup S$.
- All constraints from $C_{ID}(G)$ $x \geq \mathbf{1}$ define facets of $P_{ID}(G)$.

Observe that $\tau(I, I) = n$. Then if $G = (C \cup S, E)$ is a thick headless spider, $\gamma^{ID}(G) \geq n$. In fact, we have:

Corollary 2 *Let $G = (C \cup S, E)$ be a thick headless spider. Then S is a minimum identifying code and, thus, $\gamma^{ID}(G) = n$.*

Proof. It is clear that every cover of $C_{ID}(G)$ must contain a subset of S having $n - 1$ elements. Nevertheless no subset of S having $n - 1$ elements covers the matrix $C_{ID}(G)$, but S does. Then, $\gamma^{ID}(G) = n$. \square

In [5], the set covering polyhedron $Q^*(R_n^q) = \text{conv}\{x \in \mathbb{Z}_+^n : R_n^q x \geq \mathbf{1}\}$ of complete q -roses was studied.

Theorem 2 ([5]) *Let $n \geq q \geq 2$. An inequality $ax \geq 1$ is a facet defining inequality for $Q^*(R_n^q)$ if and only if $ax \geq 1$ can be written as $x(A_s) \geq |A_s| - q + 1$ for some $A_s \subset \{1, \dots, n\}$ where $s \in \{0, \dots, n - q - 1\}$ and $|A_s| = n - s$.*

As R_n^2 is a minor of $C_{ID}(G)$ obtained from the deletion of the nodes of C in the underlying graph $G = (C \cup S, E)$, we have:

Corollary 3 *Let $G = (C \cup S, E)$ be a thick headless spider. Then, for all nonempty subsets $A \subset S$, the inequalities $x(A) \geq |A| - 1$ are facets of $P_{ID}(G)$.*

In order to study the remaining facets of $P_{ID}(G)$ we need a description of the blocker of $C_{ID}(G)$.

From now on we consider vectors in $\{0, 1\}^{l+k}$ of the form $e_i \oplus f_j$ where e_i for $i = 1, \dots, l$ is the unit vector in $\{0, 1\}^l$ and f_j for $j = 1, \dots, k$ is a vector in $\{0, 1\}^k$ such that $(f_j)_t = 0$ if $j = t$ and $(f_j)_t = 1$ if $j \neq t$.

Theorem 3 *Let $C_{ID}(G)$ be the clutter matrix of a thick headless spider $G = (C \cup S, E)$. Every minimal cover x of $C_{ID}(G)$ is minimum. Moreover, either $x = \mathbf{0} \oplus \mathbf{1}$ where $\mathbf{0}, \mathbf{1} \in \{0, 1\}^n$ or $x = e_i \oplus f_j$ where $e_i, f_j \in \{0, 1\}^n$ with $i \neq j$.*

Proof. From Corollary 2, S is a minimum cover of $C_{ID}(G)$. Now, let x be a minimal cover of $C_{ID}(G)$ such that $x_j = 0$ for some $j \in \{n + 1, \dots, 2n\}$. It is known that every row of R_n^{n-1} covers $(e_i + e_j)$ for every $i, j = 1, \dots, n$ [5], then any cover x with $x_{n+j} = 0$ for some $j \in \{n + 1, \dots, 2n\}$ must be of the form $x = y \oplus f_j$ with some $y \in \{0, 1\}^n$. In order to cover the rows of submatrix (R_n^{n-1}, I) it is enough to consider $x^{i,j} = e_i \oplus f_j$ for $i, j = 1, \dots, n$

and $i \neq j$. Then $|x| = |x^{i,j}| = n$ for every $i, j = 1, \dots, n$ and $i \neq j$ and they are all minimum covers. Now, let v be a cover of $C_{ID}(G)$, with $v_{n+j} = 0$. Then $v = y \oplus f_j$ and $y \in \{0, 1\}^n$. But y must be a cover of R_n^{n-1} , i.e., $y = e_i + h$ for some $i \neq j$ and some $h \in \{0, 1\}^n$. Then y is not minimal. \square

Now, we present some technical lemmas:

Lemma 3 *Let $l \geq k \geq 2$ and M be a matrix with $l + k$ columns and rows*

- $\mathbf{0} \oplus \mathbf{1}$ where $\mathbf{0} \in \{0, 1\}^l$ and $\mathbf{1} \in \{0, 1\}^k$,
- $e_1 \oplus f_j$, where $e_1 \in \{0, 1\}^l$, $f_j \in \{0, 1\}^k$ with $j = 2, \dots, k$,
- $e_i \oplus f_1$, where $e_i \in \{0, 1\}^l$, $f_1 \in \{0, 1\}^k$ and $i = 2, \dots, l$,
- $e_l \oplus f_k$

Then M has rank $l + k$.

Proof. The proof follows from the fact that the $l + k$ rows of matrix M are linearly independent. \square

As a consequence we can prove the following:

Corollary 4 *Consider $C_{ID}(G)$ of a thick headless spider $G = (C \cup S, E)$. If $\bar{x} \in \mathbb{R}^{2n}$ is an extreme point of $Q(b(C_{ID}(G)))$ such that $x_i \neq 0$ for all $i \in C \cup S$ then $\bar{x} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^{2n}$.*

Proof. From Theorem 3 it follows that every row of $b(C_{ID}(G))$ is either $\mathbf{0} \oplus \mathbf{1}$ where $\mathbf{0}, \mathbf{1} \in \{0, 1\}^n$ or $e_i \oplus f_j$ for $i, j = 1, \dots, n$ and $i \neq j$. Then they all have n ones per row. By Lemma 3 there are $2n$ linearly independent rows in $b(C_{ID}(G))$. It follows that if $\bar{x} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^{2n}$ then it satisfies $b(C_{ID}(G))\bar{x} = \mathbf{1}$. Hence \bar{x} is a fractional extreme point of $Q(b(C_{ID}(G)))$. Now, if \bar{y} is an extreme point of $Q(b(C_{ID}(G)))$ with all nonzero components then it must satisfy $2n$ linearly independent inequalities of $Q(b(C_{ID}(G)))$ at equality having n ones per row. It follows that $\bar{y} = \bar{x}$. \square

Using blocking duality it can be seen that Corollary 4 gives an alternative proof of $\gamma^{ID}(G) = n$ and states that the only facet of $P_{ID}(G)$ with full support is the rank inequality associated with G .

In order to obtain the complete description of $P_{ID}(G)$, we need some further results. The proofs of the next two lemmas follow from straightforward computation of contraction of columns in matrices.

Lemma 4 *Consider $M = C_{ID}(G)$ of a thick headless spider $G = (C \cup S, E)$ and let $i \in C$. Then M/i is a matrix with $2n - 1$ columns and whose rows are*

- (1) $e_h \oplus f_i$ where $f_i \in \{0, 1\}^n$ and $e_h \in \{0, 1\}^{n-1}$ for $h = 1, \dots, n - 1$, $h \neq i$.
- (2) $\mathbf{0} \oplus f_j$ where $\mathbf{0} \in \{0, 1\}^{n-1}$ and $f_j \in \{0, 1\}^n$ for $j = 1, \dots, n$ and $j \neq i$.

Lemma 5 Consider $M = C_{ID}(G)$ of a thick headless spider $G = (C \cup S, E)$ and let $i \in S$. Then M/i is a matrix with $2n - 1$ columns, having $n - 1$ ones per row and whose rows are

- (1) $e_i \oplus f_j$ where $e_i \in \{0, 1\}^n$ and $f_j \in \{0, 1\}^{n-1}$ for $j = 1, \dots, n - 1$,
- (2) $e_h \oplus f_j$ where $e_h \in \{0, 1\}^n$ and $f_j \in \{0, 1\}^{n-1}$ for $j = 1, \dots, n - 1$, $j \neq h$ when $i > h$ and $j - 1 \neq h$ when $i < h$,
- (3) $\mathbf{0} \oplus \mathbf{1}$ where $\mathbf{0} \in \{0, 1\}^n$ and $\mathbf{1} \in \{0, 1\}^{n-1}$.

Lemma 6 Let M be a matrix with $2n - 1$ columns and $2(n - 1)$ rows such that the first $n - 1$ rows are of the form $e_i \oplus f_1$ where $e_i \in \{0, 1\}^{n-1}$ for $i = 1, \dots, n - 1$ and $f_1 \in \{0, 1\}^n$ and the remaining $n - 1$ rows are $\mathbf{0} \oplus f_i$ where $\mathbf{0} \in \{0, 1\}^{n-1}$ and $f_i \in \{0, 1\}^n$ for $i = 2, \dots, n$. Then every extreme point $x \in Q(M)$ having $x_l \neq 0$ for some $l = 1, \dots, n - 1$ is of the form $x_j = 1$ for all $j = 1, \dots, n$ and $x_j = 0$ if $j = n + 1, \dots, 2n - 1$.

Proof. Consider the system $Mx \geq \mathbf{1}$ with $2(n - 1)$ rows. Adding a slack variable s_i to each row i of the system, we get $Mx - Is = \mathbf{1}$ where I is the identity matrix of size $2(n - 1)$.

If \bar{x} is an extreme point of $Q(M)$ then there must be a basic feasible solution associated with it, i.e., there must be a subdivision of (M, I) of the form (B, N) where B is a regular submatrix of M of maximum rank. Also, if j is an index corresponding to a column in N then $x_j = 0$.

It is easy to check that the rows of M are linearly independent, so the rank of B is $2(n - 1)$. The total number of variables in $Mx - Is = \mathbf{1}$ including the original variables and the slack variables is $2n - 1 + 2(n - 1)$, but $2(n - 1)$ must be basic variables then there are at least $2n - 1$ variables at value zero. Then, there must be at least one index of a column in M , say j , such that $\bar{x}_j = 0$. Observe that for every $i = 1, \dots, n - 1$ the variable x_i and the slack s_i appear in the system $Mx - Is = \mathbf{1}$ only in row i , i.e., the columns corresponding to variables s_i and x_i are the same for all $i = 1, \dots, n - 1$. Hence, for all $i = 1, \dots, n - 1$ if x_i is a basic variable then s_i is a nonbasic variable.

From assumption $\bar{x}_l > 0$ we obtain $s_l = 0$ and then it holds that

$$\bar{x}_l + \sum_{i=1, \dots, n-1} \bar{x}_{n+i} = 1.$$

If $\sum_{i=1, \dots, n-1} \bar{x}_{n+i} = 0$ then $\bar{x}_i = 1$ for all $i = 1, \dots, n$ and the lemma follows.

Assume that $\sum_{i=1, \dots, n-1} \bar{x}_{n+i} = t > 0$. Then $\bar{x}_i = 1 - t + s_i = \bar{x}_l + s_i > 0$ for all $i = 1, \dots, n$ with $i \neq l$. Then x_i with $i = 1, \dots, n - 1$ are all basic variables and consequently s_i are nonbasic for $i = 1, \dots, n - 1$.

Now, consider the last $n - 1$ equalities of $Mx - Is = \mathbf{1}$. They are:

$$\bar{x}_n + \sum_{i=1, \dots, n-1, i \neq k} \bar{x}_{n+i} - s_{n-1+k} = 1 \text{ for all } k = 1, \dots, n-1.$$

Then it holds that

$$\bar{x}_n = 1 - t + \bar{x}_{n+k} + s_{n-1+k} \text{ for all } k = 1, \dots, n-1.$$

So, $\bar{x}_n > 0$ and then x_i is basic for $i = 1, \dots, n$. Remind that $\bar{x}_j = 0$ for some $j \in \{1, \dots, 2n - 1\}$ then $n + 1 \leq j \leq 2n - 1$. But, $\bar{x}_n = 1 - t + \bar{x}_{n+j} + s_{n-1+j}$ and if $s_{n-1+j} > 0$ then one of \bar{x}_{n+k} and s_{n-1+k} must be positive for every $k = 1, \dots, n-1$. This implies that we have $n - 1$ more basic variables, contradicting the fact that they are $2(n - 1)$.

Then the basic solution of $Mx - Is = \mathbf{1}$ corresponds to $\bar{x}_j = 1$ for $j = 1, \dots, n$, $s_j = 0$ for $j = 1, \dots, 2(n - 1)$ and $\bar{x}_j = 0$ if $j = n + 1, \dots, 2n - 1$. \square

Now, we can further show:

Theorem 4 *Let $b(C_{ID}(G))$ be the blocker of the identifying code clutter matrix of a thick headless spider $G = (C \cup S, E)$. Let $\bar{x} \in \mathbb{R}^{2n}$ be a fractional extreme point of $Q(b(C_{ID}(G)))$ such that the set $A = \{i : \bar{x}_i = 0\}$ is nonempty. Then either*

- (1) $A \subsetneq S$ and $\bar{x}_i = \frac{1}{n-|A|}$ when $i \notin A$ or
- (2) $C \subsetneq A$ and $|A| \leq 2(n - 1)$ and $\bar{x}_i = \frac{1}{2n-|A|-1}$ for all $i \notin A$.

Proof. Let \bar{x} be a fractional extreme point of $Q(b(C_{ID}(G)))$ and let $A = \{i : x_i = 0\}$ with A a nonempty set.

It is clear that $\bar{x} \in Q(b(C_{ID}(G))) \cap \{x_i = 0 : i \in A\}$. Then \bar{x} can be partitioned into $(\bar{x}_A, \bar{x}_{\bar{A}})$ where \bar{A} is the complement of A in $\{1, \dots, 2n\}$. In order to ease notation let us call $\bar{y} = \bar{x}_{\bar{A}} \in \mathbb{R}^{n-|A|}$. Then it holds that \bar{y} is an extreme point of $Q(b(C_{ID}(G))/A)$ with all components different from zero.

First consider $A \subset S = \{n + 1, \dots, 2n\}$. If $n - |A| = 1$, it is easy to check that matrix $b(C_{ID}(G)/A)$ is the $n \times n$ identity matrix. Then $\bar{y} \in \{0, 1\}^{n-1}$ contradicting the fact that \bar{x} is a fractional extreme point.

If $n - |A| \geq 2$ we can iteratively apply Lemma 5 and then we get a matrix $b(C_{ID}(G)/A)$ whose rows all have $n - |A|$ ones and using Lemma 3 it holds that there are $n + n - |A|$ linearly independent rows of $b(C_{ID}(G))/A$ with $n - |A|$ ones per row. Then if \bar{y} is an extreme point of $Q(b(C_{ID}(G))/A)$ with all nonzero components then $\bar{y} = \frac{1}{n-|A|}\mathbf{1}$.

Now, let A be such that $i \in A$ and $j \notin A$ with $\{i, j\} \subset \{1, \dots, n\}$. W.l.o.g. we may assume that $i = 1$. Then, by Lemma 4 we can see that matrix $b(C_{ID}(G))/i$ is of the form of matrix M in Lemma 6 with $2(n-2)$ rows and $2n-1$ columns. Then using Lemma 6, the extreme point $\bar{y} \in \{0, 1\}^{2n-|A|}$ contradicting the fact that \bar{x} is a fractional extreme point. Hence if $i \in A$ for some $i \in \{1, \dots, n\}$ then $C \subset A$. In this case the rows of matrix $b(C_{ID}(G)/A)$ are of the form $f_i \in \{0, 1\}^{2n-|A|}$ for every $i = 1, \dots, 2n - |A|$. Then the only possible solution is $\bar{y}_i = \frac{1}{2n-|A|-1}$ for all $i = 1, \dots, 2n - |A|$. \square

As a consequence of Theorem 4 and blocking duality, we conclude:

Corollary 5 *Let $G = (C \cup S, E)$ be a thick headless spider and $S' \subset S$ nonempty. Then, the inequalities $x(C) + x(S') \geq n - |S - S'|$ when $2 \leq |S'| \leq n - 1$ and $x(S') \geq |S'| - 1$ when $2 \leq |S'| \leq n$ are facets of $P_{ID}(G)$.*

As a consequence of Corollary 5 and Theorem 4, we obtain the main result of this section:

Corollary 6 *Let $G = (C \cup S, E)$ be a thick headless spider. Then, the facets of $P_{ID}(G)$ are*

- *the constraint $x_i \geq 0$ for all $i \in C \cup S$,*
- *the constraints $C_{ID}(G) x \geq \mathbf{1}$,*
- *the constraints $x(C) + x(S') \geq n - |S - S'|$ and $x(S') \geq |S'| - 1$ for every $S' \subseteq S$ with $2 \leq |S'|$.*

2.2 Thin headless spiders

Lemma 7 *For a thin headless spider $G = (C \cup S, E)$ with $n \geq 4$, we have*

$$C_{ID}(G) = \begin{pmatrix} I & I \\ 0 & R_n^2 \\ R_n^{n-1} & 0 \end{pmatrix}.$$

Proof. Let $G = (C \cup S, E)$ be a thin headless spider with $C = \{1, \dots, n\}$ and $S = \{n+1, \dots, 2n\}$. The neighborhood matrix of (C, S) can be written as

$$N[G] = \begin{pmatrix} \mathbf{E} & I \\ I & I \end{pmatrix}.$$

Now, in order to find $\Delta[G]$:

- (1) If $i, j \in C$, $N[i] \Delta N[j] = \{i + n, j + n\}$.
- (2) If $i, j \in S$, $N[i] \Delta N[j] = \{i, i - n\} \Delta \{j, j - n\}$ and are dominated by $N[i]$.
- (3.a) If $i \in C$ and $j \in S$, $j \neq i + n$, $N[i] \Delta N[j] = [C \cup \{i + n\}] \Delta \{j, j - n\} = \{j, i + n\} \cup (C - \{j - n\})$ is dominated by $N[i]$.
- (3.b) If $i \in C$ and $j = i + n \in S$, $N[i] \Delta N[j] = [C \cup \{i + n\}] \Delta [\{i, i + n\} = C - \{i\}]$.

As the first n rows of the matrix N above are also dominated, we have that the clutter matrix $C_{ID}(G)$ can be written as claimed. \square

Corollary 7 *Let $G = (C \cup S, E)$ be a thin headless spider. Then we have:*

- $P_{ID}(G)$ is full-dimensional.
- The constraint $x_i \geq 0$ defines a facet of $P_{ID}(G)$ for all $i \in C \cup S$.
- All constraints from $C_{ID}(G)$ $x \geq \mathbf{1}$ define facets of $P_{ID}(G)$.

In [19], it is proved that $\tau(R_n^2) = n - 1$. Then if $G = (C \cup S, E)$ is a thin headless spider, $\gamma^{ID}(G) \geq n - 1$. In fact, we have:

Corollary 8 *Let $G = (C \cup S, E)$ be a thin headless spider. Then, $\gamma^{ID}(G) = n + 1$.*

Proof. Let $G = (C \cup S, E)$ be a thin headless spider with $C = \{1, \dots, n\}$ and $S = \{n + 1, \dots, 2n\}$. It is immediate to observe that $\gamma^{ID}(G) \geq \tau(R_n^{n-1}) + \tau(R_n^2) = 2 + n - 1 = n + 1$. In addition, if $T = \{1, 2, 2 + n, 3 + n, \dots, 2n\}$ then x_T satisfies $C_{ID}(G)x_T \geq \mathbf{1}$ and $|T| = n + 1$, we conclude $\gamma^{ID}(G) = n + 1$. \square

Moreover, we obtain:

Corollary 9 *Let $G = (C \cup S, E)$ be a thin headless spider. Then,*

- (1) *the inequalities $x(A) \geq |A| - 1$ for all nonempty subsets $A \subset S$ are facets of $P_{ID}(G)$,*
- (2) *the inequality $x(C) \geq 2$ is a facet of $P_{ID}(G)$.*

Proof. As R_n^2 is a minor of $C_{ID}(G)$ obtained after deletion of the nodes in C , as a consequence of Theorem 2 we have that the inequalities $x(A) \geq |A| - 1$ for all nonempty subsets $A \subset S$ are facets of $P_{ID}(G)$. Also, R_n^{n-1} is a minor of $C_{ID}(G)$ obtained after deletion of the nodes in S , and again using Theorem 2 we obtain $x(C) \geq 2$ as a facet of $P_{ID}(G)$. \square

In [4], it was conjectured that there are no further facets necessary to entirely describe the identifying code polyhedron of thin headless spiders. In the remaining part of this section, we present the proof of this conjecture.

As an immediate observation, the rank inequality $x(C \cup S) \geq n + 1$ is not a facet of $P_{ID}(G)$ since it can be obtained as the sum of the facets $x(S) \geq n - 1$ and $x(C) \geq 2$.

Theorem 5 *Let $C_{ID}(G)$ be the clutter matrix of a thin headless spider $G = (C \cup S, E)$. Every minimal cover x of $C_{ID}(G)$ is minimum. Moreover, $x = (e_i + e_j) \oplus f_j$ where $e_i, f_j \in \{0, 1\}^n$ with $i \neq j$.*

Proof. Let x be a minimal cover of $C_{ID}(G)$. It is clear that $x = v \oplus u$ where $v \in \{0, 1\}^n$ covers the rows of matrix R_n^{n-1} and $u \in \{0, 1\}^n$ covers the rows of R_n^2 . Hence $v = e_i + e_j$ with $i, j \in \{1, \dots, n\}$ and $i \neq j$ and $u = f_l$ with $l \in \{1, \dots, n\}$. Also, x must cover the first n rows of matrix $C_{ID}(G)$, then if $x = v \oplus u$ with $v = e_i + e_j$ then either $u = f_i$ or $u = f_j$. Then $|x| = n + 1$ for every $i, j = 1, \dots, n$ and $i \neq j$ and they are all minimum covers. \square

In order to identify the complete description of $P_{ID}(G)$ when $G = (C \cup S, E)$ is a thin headless spider, we use blocking duality again. Consider the polyhedron $Q(b(C_{ID}(G)))$.

As a consequence of the previous theorem all the rows in matrix $b(C_{ID}(G))$ have $n + 1$ ones, then the point $\bar{x} = \mathbf{1}_{\frac{1}{n+1}} \in Q(b(C_{ID}(G)))$. But \bar{x} cannot be an extreme point of $Q(b(C_{ID}(G)))$ since we have already mentioned that the rank inequality $x(C \cup S) \geq n + 1$ is not a facet of $P_{ID}(G)$. So, every fractional extreme point of $Q(b(C_{ID}(G)))$ must have some of its components at value zero. From Corollary 9 we know that the point x of the form $x_i = \frac{1}{2}$ for all $i \in C$ and $x_i = 0$ for all $i \in S$ is an extreme point as well as all the points of the form $x_i = \frac{1}{|A|-1}$ for all $i \in A \subset S$ with A nonempty and $x_i = 0$ for $i \notin A$. The following results show that these are the only fractional extreme points of $Q(b(C_{ID}(G)))$.

Lemma 8 *Let M be the $n \times 2n$ matrix whose rows are the characteristic vectors of $e_i \oplus f_i$ for $i \in \{1, \dots, n\}$ and $e_i, f_i \in \{0, 1\}^n$ for every $i \in \{1, \dots, n\}$. Let $\bar{x} \in Q(M)$ such that $\bar{x}_l \neq 0$ for some $l \in \{1, \dots, n\}$. If \bar{x} is an extreme point of $Q(M)$ then $\bar{x} \in \{0, 1\}^{2n}$.*

Proof. It is easy to check that the nontrivial inequalities describing $Q(M)$ are

$$x_i + \sum_{k=1, \dots, n; k \neq i} x_{n+k} \geq 1 \text{ for all } i = 1, \dots, n.$$

After adding the slack variables to the inequalities above we obtain

$$x_i + \sum_{k=1, \dots, n; k \neq i} x_{n+k} - s_i = 1 \text{ for all } i = 1, \dots, n.$$

For every $i = 1, \dots, n$ it holds that variable x_i is basic if and only if variable s_i is nonbasic, since their respective columns are the same. By assumption there

is $l \in \{1, \dots, n\}$ such that $\bar{x}_l > 0$, then $s_l = 0$ and $\bar{x}_l = 1 - \sum_{k=1, \dots, n; k \neq l} \bar{x}_{n+k}$. If $\bar{x}_i > 0$ for all $i \in \{1, \dots, n\}$ then all basic variables are x_i with $i \in \{1, \dots, n\}$. Then $\bar{x}_{n+i} = 0$ for $i \in \{1, \dots, n\}$ and $\bar{x}_1 = 1$ for $i \in \{1, \dots, n\}$ and the lemma follows.

Observe that there is $i \in \{1, \dots, n\}$ such that $\bar{x}_{n+i} = 0$, otherwise variables x_{n+i} with $i = 1, \dots, n$ are basic contradicting the fact that $x_l > 0$ for some $l \in \{0, \dots, n\}$. So, let $I = \{i : 1 \leq i \leq n \text{ and } \bar{x}_i = 0\}$. By assumption $I \neq \{1, \dots, n\}$ and consider I nonempty. Recall that $\bar{x}_l > 0$ and $\bar{x}_l + \sum_{k=1, \dots, n; k \neq l} \bar{x}_{n+k} = 1$. Let $i \in I$ then $\sum_{k=1, \dots, n; k \neq i} \bar{x}_{n+k} - s_i = 1$. Thus $x_{n+l} - x_l - x_{n+i} = s_i$ and then $x_{n+l} \geq x_l > 0$. Therefore, if \bar{x} is an extreme point then x_i, x_{i+n} is a pair of basic variables for every $i \notin I$. Since there are n basic variables, then $n - |I| \leq \frac{n}{2}$.

W.l.o.g assume that $I = \{1, \dots, l\}$ with $l \leq \frac{n}{2}$. If $\bar{x}_{n+i} > 0$ for some $i \in I$ then variable s_i is nonbasic, since adding the columns corresponding to the variables x_n and x_{2n} yields the sum of columns corresponding to the variables s_i and x_{n+i} . Let us call $J = \{i \in I : \bar{x}_{n+i} = 0\}$.

Again, we can assume that $J = \{1, \dots, m\}$ with $m < l \leq \frac{n}{2}$. Hence we have

$$\sum_{k=m+1, \dots, n} \bar{x}_{n+k} - s_i = 1 \text{ for all } i = 1, \dots, m$$

and

$$\sum_{k=m+1, \dots, n; k \neq i} \bar{x}_{n+k} = 1 \text{ for all } i = m+1, \dots, l.$$

It holds that for every $j \in J$, $s_j > 0$ follows and is thus basic. But, we have that for every $i = 1, \dots, n$ either $x_{n+i} > 0$ and is thus basic or $s_i > 0$ and is thus basic. These variables together with every x_i with $i \notin I$ make more than n positive variables. It follows that \bar{x} is not an extreme point. \square

Lemma 9 *Let $\bar{x} \in Q(b(C_{ID}(G)))$ such that $\bar{x}_l \neq 0$ and $\bar{x}_j = 0$ for $l, j \in \{1, \dots, n\}$. If \bar{x} is an extreme point of $Q(b(C_{ID}(G)))$ then $\bar{x} \in \{0, 1\}^{2n}$.*

Proof. W.l.o.g. assume $j = 1$ and consider an extreme point \bar{x} of $Q(b(C_{ID}(G)))$ with $\bar{x}_1 = 0$ and $\bar{x}_l > 0$. If $\bar{x}_{n+1} = 0$ we can study the extreme points of $Q(b(C_{ID}(G))) \cap \{x \in \mathbb{R}^{2n} : x_1 = x_{n+1} = 0\}$. It is easy to see that the matrix $b(C_{ID}(G))/1, n+1$ corresponds to a $(n-1) \times 2(n-1)$ matrix whose rows are the characteristic vectors of $e_i \oplus f_i$ for $i \in \{1, \dots, n-1\}$ and $e_i, f_i \in \{0, 1\}^n$ for every $i \in \{1, \dots, n-1\}$. Hence the result follows from Lemma 8.

Now, assume that $\bar{x}_{n+1} > 0$. Moreover, we assume that any time if $\bar{x}_i = 0$ then $\bar{x}_{n+i} > 0$. Observe that the rows of matrix $b(C_{ID}(G))/1$ correspond to the characteristic vectors of $e_i \oplus f_1$ and $e_i \oplus f_{i+1}$ for $i \in \{1, \dots, n-1\}$ where $e_i \in \{0, 1\}^{n-1}$ and $f_i \in \{0, 1\}^n$. So the inequalities describing $Q(b(C_{ID}(G))/1)$

are

$$x_i + \sum_{k=2, \dots, n} x_{n+k} \geq 1 \text{ for all } i = 2, \dots, n$$

and

$$x_i + \sum_{k=1, \dots, n; k \neq i} x_{n+k} \geq 1 \text{ for all } i = 2, \dots, n.$$

Consider the system of equalities obtained after adding the slack variables to the inequalities above as follows:

$$x_i + \sum_{k=2, \dots, n} x_{n+k} - s_i = 1 \text{ for all } i = 2, \dots, n$$

and

$$x_i + \sum_{k=1, \dots, n; k \neq i} x_{n+k} - s_{n+i-1} = 1 \text{ for all } i = 2, \dots, n.$$

So we have $2n - 2$ basic variables and $2n - 1$ nonbasic variables. Suppose that $\bar{x}_i > 0$ for every $i \in 2, \dots, n$, then $\bar{x}_{n+i} > 0$ for $i \in 1, \dots, n$. This implies that variables x_i for every $i \in \{2, \dots, n\}$ and x_{n+i} for $i \in \{1, \dots, n\}$ are basic variables, a contradiction.

There must be some $l \in \{2, \dots, n\}$ such that $x_l = 0$. Therefore, our set of basic variables includes variables x_i for every i such that $\bar{x}_i > 0$ and every x_{n+i} for which $\bar{x}_i = 0$. These are n basic variables. W.l.o.g assume $I = \{1, \dots, m\}$ such that $\bar{x}_i = 0$ when $i \in I$ and $\bar{x}_i > 0$ otherwise. Then let $l \notin I$. Then

$$\bar{x}_l + \sum_{k=2, \dots, m} \bar{x}_{n+k} - s_l = 1,$$

but also

$$\bar{x}_l + \sum_{k=1, \dots, m} \bar{x}_{n+k} - s_{l+n-1} = 1.$$

Observe that one between s_l and s_{l+n-1} must be nonbasic, since x_l is a basic variable. It follows that $s_{l+n-1} > 0$ and $s_l = 0$. Also $\bar{x}_l + \sum_{k=2, \dots, m} \bar{x}_{n+k} = 1$. So we have that for every $j \notin I$, s_{j+n-1} is a basic variable. Now, let $i \in I$ it holds that

$$\sum_{k=2, \dots, m} \bar{x}_{n+k} - s_j = 1.$$

But as we have already seen $\sum_{k=2, \dots, m} \bar{x}_{n+k} < 1$ and hence $s_j > 0$ for all $j \in I$. So we have $2n$ basic variables which is a contradiction. \square

In summary, we obtain the main result of this section:

Corollary 10 *Let $G = (C \cup S, E)$ be a thin headless spider. Then, the facets of $P_{ID}(G)$ are*

- the constraint $x_i \geq 0$ for all $i \in C \cup S$;
- all constraints from $C_{ID}(G)$ $x \geq \mathbf{1}$;
- the constraint $x(C) \geq 2$;
- the constraints $x(A) \geq |A| - 1$ for all nonempty subsets $A \subset S$.

2.3 Complete suns

As third family of headless spiders $G = (C \cup S, E)$ having a regular structure, we consider complete suns, where $S = \{s_1, \dots, s_n\}$, $C = \{c_1, \dots, c_n\}$ and s_i is adjacent to exactly c_i and c_{i+1} for all $1 \leq i \leq n$ (indices are taken modulo n).

In contrary to thin and thick headless spiders whose identifying code clutters are composed by few q -roses, the identifying code clutters of complete suns have a more complex structure, involving different combinations of submatrices with a circular structure, where some submatrices occur for all $n \geq 4$, others not (depending on the parity of n and the size of the graph).

A circulant matrix is a square matrix where each row vector is shifted one element to the right relative to the preceding row vector. We denote by C_n^k a circulant matrix in $\{0, 1\}^{n \times n}$ having as first row the vector starting with k 1-entries and having 0-entries otherwise. Moreover, we denote by C_n^{k+k} a matrix in $\{0, 1\}^{n \times n}$ with $n \geq 2k + 2$ having as first row the vector starting with k 1-entries, then having at least two 0-entries, again k 1-entries, and 0-entries otherwise (at least two again).

Lemma 10 *For a complete sun $G = (C \cup S, E)$ with $n \geq 4$, the identifying code clutter $C_{ID}(G)$ is composed by the following submatrices*

$$\begin{array}{ll}
 (C_n^2 \mid I) & \forall n \geq 4 \\
 (0 \mid C_n^{1+1}) & \forall n \geq 5 \\
 (0 \mid C_{\frac{n}{2}}^1, C_{\frac{n}{2}}^1) & \text{for } n = 4 \\
 (0 \mid C_n^{2+2}) & \forall n \geq 9 \\
 (0 \mid C_{\frac{n}{2}}^2, C_{\frac{n}{2}}^2) & \forall n \geq 8, n \text{ even} \\
 (C_n^{n-2} \mid I) & \forall n \geq 4 \\
 (C_n^{1+1} \mid C_n^2) & \forall n \geq 4
 \end{array}$$

where the first part refers to C , the second part of the matrices to S .

Proof. Let $G = (C \cup S, E)$ be a complete sun. The neighborhood matrix $N[G]$ of (C, S) is composed by

$$\begin{array}{ll}
 (\mathbf{E} \mid C_n^2) & \text{for } N[C], \\
 (C_n^2 \mid I) & \text{for } N[S].
 \end{array}$$

Thus, only $N[S]$ is in $C_{ID}(G)$. To find $\Delta[G]$, we distinguish three cases.

Case 1: the symmetric differences between two nodes in C have the form

- $N[c_i] \Delta N[c_{i+1}] = \{s_{i-1}, s_{i+1}\}$;
- $N[c_i] \Delta N[c_{i+j}] = \{s_{i-1}, s_i, s_{i+j-1}, s_{i+j}\}$ for $1 < j \leq \frac{n}{2}$.

For all $n \geq 4$, the former symmetric differences remain in $C_{ID}(G)$ as submatrix

$$(0 \mid C_n^{1+1}) \quad \forall n \geq 5$$

(but yield for $n = 4$ not the whole circulant matrix). The latter symmetric differences are dominated by the former if $j = 2, 3$. Thus, for each $4 \leq j \leq \frac{n}{2}$, the symmetric differences $N[c_i] \Delta N[c_{i+j}]$ remain in $C_{ID}(G)$ as submatrix

$$(0 \mid C_n^{2+2}) \quad \forall n \geq 9$$

(but yield for $j = \frac{n}{2}$ not the whole circulant matrix).

Case 2: the symmetric differences between nodes in C and S have the form

- $N[c_i] \Delta N[s_i] = (C - \{c_{i-1}, c_i\}) \cup \{s_{i-1}\}$;
- $N[c_i] \Delta N[s_{i-1}] = (C - \{c_{i-2}, c_{i-1}\}) \cup \{s_i\}$;
- $N[c_i] \Delta N[s_j] = (C - \{c_{j-1}, c_j\}) \cup \{s_{i-1}, s_i\}$ for $j \neq i, i-1$.

Thus, $N[c_i] \Delta N[s_j]$ is dominated by $N[s_i]$ if $j \neq i, i-1$, and remains in $C_{ID}(G)$ for $j = i, i-1$, forming two submatrices of the form

$$(C_n^{n-2} \mid I) \quad \forall n \geq 4.$$

Case 3: the symmetric differences between two nodes in S have the form

- $N[s_i] \Delta N[s_{i+1}] = \{c_i, c_{i+2}\} \cup \{s_i, s_{i+1}\}$;
- $N[s_i] \Delta N[s_j] = \{c_i, c_{i+1}, c_j, c_{j+1}\} \cup \{s_i, s_j\}$ for $j \neq i$.

Thus, $N[s_i] \Delta N[s_j]$ is dominated by $N[s_i]$ if $j \neq i+1$, and remains in $C_{ID}(G)$ for $j = i+1$, forming a submatrix of the form

$$(C_n^{1+1} \mid C_n^2) \quad \forall n \geq 4.$$

This together completely describes the identifying code clutter $C_{ID}(G)$. \square

Corollary 11 *Let $G = (C \cup S, E)$ be a complete sun with $n \geq 4$.*

- $P_{ID}(G)$ is full-dimensional.
- The constraint $x_v \geq 0$ defines a facet of $P_{ID}(G)$ for each $v \in C \cup S$.
- All constraints from $C_{ID}(G)$ $x \geq \mathbf{1}$ define facets of $P_{ID}(G)$.

Unfortunately, the whole system of facet-defining inequalities for the identifying code polyhedra $P_{ID}(G)$ of complete suns is not easy to describe since non-rank facets are required for all cases $n \geq 4$ (in fact, most facets of $P_{ID}(G)$ are non-rank and involve large coefficients). However, from a careful analysis of the constraints involved in the identifying code clutter $C_{ID}(G)$ of complete suns, we derive at the following conjecture:

Conjecture 1 *For a complete sun $G = (C \cup S, E)$ with $n \geq 4$, the stable set S is a minimum identifying code.*

Note that it is easy to see that S is always an identifying code for a complete sun $G = (C \cup S, E)$, since all rows of $C_{ID}(G)$ have at least one 1-entry in S . Hence, $\gamma^{ID}(G) \leq |S| = n$ follows. On the other hand, for some cases, it has been already verified that S is a minimum identifying code, by generating the full rank constraint $x(C) + x(S) \geq |S| = n$ by means of the Chátal-Gomory procedure. This implies $\gamma^{ID}(G) \geq |S| = n$, and together equality follows for these cases. Our goal is to find a general construction of this type for all $n \geq 4$.

3 Concluding remarks

The identifying code problem is hard in general and challenging both from a theoretical and a computational point of view, even for special graphs like split graphs [11]. Hence, a typical line of attack for this problem is to determine minimum identifying codes of special graphs, or to provide lower and upper bounds.

In this paper, we studied three families of split graphs with $|S| = |C| \geq 2$ having a regular structure: thin headless spiders, thick headless spiders, and complete suns. For all three families, we determined the identifying code clutter and discussed according consequences.

In the case of thin and thick spiders G , $C_{ID}(G)$ is composed from certain q -roses. Based on related results from [5,19], we could give the complete description of $P_{ID}(G)$ for thick and thin spiders. As a consequence for both classes, we found the exact value for $\gamma^{ID}(G)$: $|S|$ for thick spiders and $|S| + 1$ for thin spiders. Moreover, the complete descriptions of $P_{ID}(G)$ also allow us to compute minimum weight identifying codes of thin and thick spiders for all possible node weights.

It turned out that the identifying code clutters of complete suns have a more complex structure involving different circulant matrices and, accordingly, more involved facets are required to describe $P_{ID}(G)$. For this class, we showed $\gamma^{ID}(G) \leq |S|$ and conjecture that $\gamma^{ID}(G) = |S|$ holds, too. So, all three

families seem to have small minimum identifying codes close to the lower bound of order $\log(|S| + |C|)$.

This demonstrates how the polyhedral approach can be applied to find identifying codes of minimum size for special graphs G , just by determining and analyzing the identifying code clutter $C_{ID}(G)$, even in cases where no complete description of $P_{ID}(G)$ is known yet.

As future lines of research, we plan to apply similar and more advanced techniques to obtain either the identifying code of minimum size or strong lower bounds stemming from linear relaxations of the identifying code polyhedron, enhanced by suitable cutting planes. For that, note that facets associated with deletion minors of $C_{ID}(G)$ remain facets in $P_{ID}(G)$, so according facets identified for special graphs are relevant for every graph having such subgraphs.

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References

- [1] Argiroffo, G., Bianchi, S.: On the set covering polyhedron of circulant matrices, *Discrete Optimization* 6(2), 162–173 (2009)
- [2] Argiroffo, G., Bianchi, S., Lucarini, Y., Wagler, A.: Polyhedra associated with identifying codes, submitted to *Discrete Applied Mathematics*
- [3] Argiroffo, G., Bianchi, S., Wagler, A.: Polyhedra associated to identifying codes (extended abstract), In: *Proc. of LAGOS 2013, Electronic Notes in Discrete Mathematics* 44, 175–180 (2013)
- [4] Argiroffo, G., Bianchi, S., Wagler, A.: Study of identifying code polyhedra for some families of split graphs, In: *Proc. of ISCO 2014, Lecture Notes in Computer Science* 8596, 13–25 (2014)
- [5] Argiroffo, G., Carr, M.: On the set covering polyhedron of q -roses, In: *Proc. of ALIO/EURO 2008, Buenos Aires, Argentina* (2008).
- [6] Auger, D.: Minimal identifying codes in trees and planar graphs with large girth, *European Journal of Combinatorics* 31, 1372–1384 (2010)
- [7] Bertrand, N., Charon, I., Hudry, O., Lobstein, A.: Identifying and locating dominating codes on chains and cycles, *European Journal of Combinatorics* 25, 969–987 (2004)
- [8] Brandstädt, A., Le, V.B., Spinrad, J.: *Graph Classes: A Survey*, SIAM Monographs on Discrete Mathematics and Applications (1999)

- [9] Charon, I., Hudry, O., Lobstein, A.: Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard, *Theoretical Computer Science* 290, 2109–2120 (2003)
- [10] Cornuéjols, G.: *Combinatorial Optimization: Packing and Covering*, SIAM, CBMS 74 (2001)
- [11] Foucaud, F.: The complexity of the identifying code problem in restricted graph classes, In: *Proc. of IWOCA 2013*, LNCS, vol. 8288, pp. 150–163. Springer, Heidelberg (2013)
- [12] Foucaud, F., Gravier, S., Naserasr, R., Parreau, A., Valicov, P.: Identifying codes in line graphs, *Journal of Graph Theory* 73, 4 (2013) 425–448
- [13] Földes, S., Hammer, P.: Split graphs, In: *Proc. of the VIII Southeastern Conference on Combinatorics, Graph Theory and Computing* (Baton Rouge, La.), *Congressus Numerantium XIX*, Winnipeg: Utilitas Math., pp. 311–315, (1977)
- [14] Fulkerson, D.: Blocking polyhedra, In: B. Haris (Ed.), *Graph Theory and its Applications*, pp. 93–112. Academic Press, New York (1970)
- [15] Gravier, S., Moncel, J., Semri, A.: Identifying codes of cycles, *European Journal of Combinatorics* 27, 767–776 (2006)
- [16] Jamison, B., Olariu, S.: Recognizing P_4 -tidy graphs in linear time, *SIAM Journal on Comput.* 21, 381–406 (1992)
- [17] Karpovsky, M.G., Chakrabarty, K., Levitin, L.B.: On a new class of codes for identifying vertices in graphs, *IEEE Transactions on Information Theory* 44, 599–611 (1998)
- [18] Lehman, A.: On the width-length inequality, *Mathematical Programming* 17, 403–417 (1979)
- [19] Sassano, A.: On the facial structure of the set covering polytope, *Mathematical Programming* 44, 181–202 (1989)