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Derivation of a Lode angle dependent model for ductile porous materials: an extension of the stress-based variational approach

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Abstract

Recently, a new methodology of macroscopic modeling for ductile porous media has been proposed by the authors, using a statical limit analysis approach (Cheng et al., 2013). Since this model has been derived by considering a hollow sphere under axisymmetric loadings, only the effect of the sign of the stress deviator third invariant (or two values of Lode angle) has been investigated. The aim of the present paper is to extend the above mentioned stress-based model to the general case of non-axisymmetric loadings by introducing a more general trial stress field. The established new yield locus explicitly depends on the effect of the third invariant (equivalently the Lode angle). Finally, non negligible effects of the third stress deviator invariant on the voids growth rate is fully demonstrated. The results are fully illustrated.

Keywords: Ductile Porous material, von Mises matrix, Statical Limit Analysis, Third invariant of stress deviator

1. Introduction

More than thirty years ago, Gurson (1977) proposed a kinematically-based limit analysis approach of a hollow sphere and hollow cylinder having a von Mises rigid plastic matrix. This approach delivered an upper bound of the macroscopic criterion which depends on the pressure and on the von Mises equivalent stress. Very recently, and in contrast to Gurson model, we have introduced in Cheng et al. (2013) a stress-based variational approach of the ductile porous material. The starting point is Hill's variational principle applied to the hollow sphere with rigid perfectly plastic matrix. Implementation of this variational approach requires the choice of a statically and plastically admissible trial stress field. For simplicity, we have resorted in Cheng et al. (2013) an axisymmetric trial stress field. Consequently, the mechanical loadings that can be considered are axisymmetric, and only the sign of the third invariant (or two values of Lode angle) of the stress deviator can then be accounted for the

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macroscopic criterion. Note also that the effects of the third invariant sign on the plastic flow rule and on the voids growth rate have been investigated. It is worth noticing that the importance of the above mentioned effects have been fully studied in (Cazacu et al., 2013; Garajeu et al., 2011) based on a kinematical limit analysis method.

The paper is devoted to an extension of the above mentioned stress-based model to the general case of non-axisymmetric loadings by introducing a more general trial stress field. Such extension is based on the consideration and implementation in the context of the SVM approach of an appropriate trial stress field. A macroscopic criterion which explicitly depends on the Load angle (or the third invariant of the stress deviator) is derived. It will be shown that the established new yield locus predicts a slight effect of the third invariant (equivalently the Lode angle). Moreover, the plastic flow rule and void growth functions are given as functions of the mean stress, equivalent stress and the third invariant of the stress deviator.

The outline of the paper is as follows. First, we briefly recall in Section 2 the statically limit analysis approach which aims at formulating a lower bound model, in contrast to the well-known kinematical one. Then, in Section 3, we propose for the hollow sphere model a non-axisymmetric trial stress field. The successful implementation of this new stress field in the stress-based method delivers a macroscopic criterion of the ductile porous medium which is first explicitly formulated in terms of the mean stress, the von Mises equivalent stress and the Lode angle. Finally, in Section 4 are formulated the plastic flow rule and the porosity evolution equation. Section 5 allows to conclude the study.

2. Brief recall of the stress variational limit analysis approach

In the framework of Limit Analysis for the ductile porous media, let us consider a cell Ω composed a void ω and a rigid perfectly plastic matrix $\Omega_M = \Omega - \omega$. The external boundary of the VER and the internal boundary of the void are respectively defined by $\partial\Omega$ and $\partial\omega$. The yield criterion for the matrix takes the form:

$$F(\boldsymbol{\sigma}) \leq 0 \quad (1)$$

where $\boldsymbol{\sigma}$ is the cauchy stress tensor.

The set of plastic admissible stress tensor is defined by

$$\mathcal{S}_p = \{\boldsymbol{\sigma} \quad s.t. \quad F(\boldsymbol{\sigma}) \leq 0\} \quad (2)$$

Considering the normality law, the plastically admissible velocity field reads

$$\mathcal{K}_p = \{\mathbf{d} \quad s.t. \quad \mathbf{d} = d_{eq} \cdot \frac{\partial F}{\partial \boldsymbol{\sigma}}\} \quad (3)$$

where \mathbf{d} is the strain rate tensor and d_{eq} the equivalent strain rate obtained from the strain deviator \mathbf{s} through $d_{eq} = \sqrt{\frac{2}{3} \mathbf{s} : \mathbf{s}}$.

Moreover, the sets of statically admissible stress field and the kinematically admissible velocity ones are respectively defined through

$$\mathcal{S}_a = \{ \boldsymbol{\sigma} \text{ s.t. } \operatorname{div} \boldsymbol{\sigma} = 0 \text{ in } \Omega, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \text{ on } \partial\omega, \quad \boldsymbol{\sigma} = 0 \text{ in } \omega \} . \quad (4)$$

$$\mathcal{K}_a = \{ \mathbf{v} \text{ s.t. } \mathbf{v}(\mathbf{x}) = \mathbf{D} \cdot \mathbf{x} \text{ on } \partial\Omega \} . \quad (5)$$

\mathbf{n} is the unit outward normal vector, and \mathbf{x} the position one.

When the cell is subjected to a kinematically admissible velocity field (5), the homogenization problem consists in calculating the minimizing, as implemented in Cheng et al. (2013) of the following functional which is involved to the Hill's variational principal (Hill, 1950)

$$\Psi = \min_{\boldsymbol{\sigma} \in \mathcal{S}_a} \left(\frac{1}{|\Omega|} \int_{\Omega_M} \psi(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right) , \quad (6)$$

where $\psi(\boldsymbol{\sigma})$ is an indicator function, which is convex and lower semicontinuous. It vanishes when the stress field is plastically admissible (2), $\psi(\boldsymbol{\sigma}) \mapsto \infty$, otherwise. While the second term on r.h.s. of (6) can be calculated from the microscopic counterpart through the Hill's lemma

$$\mathbf{D} : \boldsymbol{\Sigma} = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{d}(\mathbf{v}) : \boldsymbol{\sigma} dV = \frac{1}{|\Omega_M|} \int_{\Omega_M} \mathbf{d}(\mathbf{v}) : \boldsymbol{\sigma} dV . \quad (7)$$

For minimization of Ψ (6), in the context of the limit analysis requires to find a licit stress field satisfying

$$\mathcal{S}_l = \{ \boldsymbol{\sigma} \in \mathcal{S}_a \text{ and } \boldsymbol{\sigma} \in \mathcal{S}_p \text{ in } \Omega_M \} . \quad (8)$$

That is equivalent to the following saddle-point problem by introducing the multiplier of Lagrange $x \mapsto \dot{\Lambda}(\mathbf{x})$ and relaxing the yield criterion (1)

$$\max_{\dot{\Lambda} \geq 0} \min_{\boldsymbol{\sigma} \in \mathcal{S}_a} \left(\mathcal{L}(\boldsymbol{\sigma}, \dot{\Lambda}) = \frac{1}{|\Omega|} \int_{\Omega_M} \dot{\Lambda} F(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right) ,$$

Assuming $\dot{\Lambda}$ is uniform in the matrix, it follows

$$\max_{\dot{\Lambda} \geq 0} \min_{\boldsymbol{\sigma} \in \mathcal{S}_a} \left(\mathcal{L}(\boldsymbol{\sigma}, \dot{\Lambda}) = \frac{\dot{\Lambda}}{|\Omega|} \int_{\Omega_M} F(\boldsymbol{\sigma}) dV - \mathbf{D} : \boldsymbol{\Sigma} \right) . \quad (9)$$

For more information concerning about the statical limit analysis approach, the reader can referred to Salençon (1983); Save et al. (1997), etc..

The macroscopic limit load and the plastic flow can be respectively obtained from the stationarities of (9) with respect to $\dot{\Lambda}$ and $\boldsymbol{\Sigma}$,

$$\mathcal{F}(\boldsymbol{\Sigma}) = \frac{1}{|\Omega|} \int_{\Omega_M} F(\boldsymbol{\sigma}) dV = 0 . \quad (10)$$

$$\mathbf{D} = \dot{\Lambda} \frac{\partial \mathcal{F}}{\partial \boldsymbol{\Sigma}}(\boldsymbol{\Sigma}) . \quad (11)$$

Obviously, $\dot{\Lambda}$ is simultaneously proved to be the plastic multiplier.

3. Proposed non-axisymmetric trial stress field and 3D macroscopic criterion

In this section, we aim at deriving a new stress variational model for ductile porous media with a non-axisymmetric trial stress field. A macroscopic criterion depending not only on the macroscopic mean and equivalent stresses (Σ_m and Σ_e), but also on the Lode angle θ_L (or third invariant of the stress deviator J_3) is expressed ¹. This stress-based macroscopic modelling is achieved by taking the hollow sphere model having a porosity f . The matrix Ω_M is made up with a rigid perfectly plastic material obeying the von Mises yield criterion

$$F(\boldsymbol{\sigma}) = \sigma_e(\boldsymbol{\sigma}) - \sigma_0 \leq 0 \quad (12)$$

where $\sigma_e = \sqrt{\frac{3}{2} \mathbf{s} : \mathbf{s}}$ is the von Mises equivalent stress defined from the stress deviator part \mathbf{s} , while $\sigma_0 > 0$ the matrix shear cohesion.

3.1. Proposed non-axisymmetric trial stress field

Owing to the central symmetry of the hollow sphere model, we propose a trial non-axisymmetric trial stress field, which contains two part as follows,

- A heterogeneous part corresponding to the exact solution under pure hydrostatic loadings, it reads, in spherical coordinates with orthonormal frame $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta\}$:

$$\boldsymbol{\sigma}^{(1)} = -A \left[\ln \left(\frac{a}{r} \right) \mathbf{1} - \frac{1}{2} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi) \right], \quad (13)$$

where $\mathbf{1}$ is the second order unit tensor, A being a constant to be determined.

- A homogeneous part which is non axisymmetric and taken for capturing the shear effect:

$$\boldsymbol{\sigma}^{(2)} = \mathbf{B}, \quad \text{tr} \mathbf{B} = 0 \quad (14)$$

Hence, the final trial stress field in the matrix can be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} \quad (15)$$

which turns to be null in the void ω , and in cylindrical coordinates reads

It follows that the non-axisymmetric macroscopic stress field read:

$$\boldsymbol{\Sigma} = -\frac{1}{3} A \ln f \cdot \mathbf{1} + (1 - f) \mathbf{B} \quad (16)$$

Next, let us compute that in mechanics, there are three invariants for defining the plastic limit state. From (16) and (14), they can be respectively calculated:

¹In (Cheng et al., 2013), only the sign of the third invariant is obtained

- Macroscopic mean stress,

$$\Sigma_m = -\frac{1}{3}A \ln f \quad (17)$$

- Macroscopic equivalent stress,

$$\Sigma_e = (1 - f) B_{eq} \quad (18)$$

where B_{eq} is the equivalent quantity associated to the deviator \mathbf{B} (or microscopic stress deviator of $\boldsymbol{\sigma}^{(2)}$):

$$B_{eq} = \sqrt{\frac{3}{2}\mathbf{B} : \mathbf{B}} \quad (19)$$

- Third invariant of the macroscopic stress deviator,

$$J_3 = (1 - f)^3 \det(\mathbf{B}) \quad (20)$$

For convenience, Let us introduce the stress based quantities:

$$\begin{aligned} \tilde{\Sigma}_m &= -\frac{3\Sigma_m}{2 \ln f} = \frac{A}{2} \\ \tilde{\Sigma}_{eq} &= B_{eq} = \frac{\Sigma_e}{1 - f} \\ \tilde{J}_3 &= \frac{J_3}{(1 - f)^3} \end{aligned} \quad (21)$$

from which, the macroscopic Lode angle θ_L can be defined as:

$$\cos(3\theta_L) = \frac{27\tilde{J}_3}{2\tilde{\Sigma}_{eq}^3} = \frac{27J_3}{2\Sigma_e^3}, \quad 0 \leq \theta_L \leq 60^\circ \quad (22)$$

3.2. Macroscopic criterion

From Eqs.(13), (14) and (15) the deviator \mathbf{s} of the local stress field can be written as:

$$\mathbf{s} = \mathbf{s}^{(1)} + \mathbf{s}^{(2)} = \mathbf{s}^{(1)} + \mathbf{B} \quad (23)$$

where $\mathbf{s}^{(1)}$ is the deviator calculated from (13). Hence, the equivalent stress can be obtained from:

$$\sigma_e = \sqrt{\frac{3}{2}[\mathbf{s}^{(1)} : \mathbf{s}^{(1)} + 2\mathbf{s}^{(1)} : \mathbf{s}^{(2)} + \mathbf{s}^{(2)} : \mathbf{s}^{(2)}]} \quad (24)$$

It can be calculated from (23) and (24) that

$$\begin{aligned} \mathbf{s}^{(1)} : \mathbf{s}^{(1)} &= \frac{A^2}{6} \\ \mathbf{s}^{(1)} : \mathbf{s}^{(2)} &= -\frac{A}{2}\tilde{B} \\ \mathbf{s}^{(2)} : \mathbf{s}^{(2)} &= \frac{2}{3}B_{eq}^2 \end{aligned} \quad (25)$$

for which expression of \tilde{B} is determined in the following.

Indeed, in order to compute the stress quantity \tilde{B} in spherical coordinates, let us first express the principal stress tensor of \mathbf{B} in cartesian coordinates with orthonormal frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$:

$$\mathbf{B} = B_1(\mathbf{e}_x \otimes \mathbf{e}_x - \mathbf{e}_z \otimes \mathbf{e}_z) + B_2(\mathbf{e}_y \otimes \mathbf{e}_y - \mathbf{e}_z \otimes \mathbf{e}_z), \quad (26)$$

for which the components can be expressed (without loss of generality) in the form:

$$\begin{aligned} B_1 &= -\frac{\Sigma_e}{3} \cos(\theta_L) + \frac{\Sigma_e}{\sqrt{3}} \sin(\theta_L) \\ B_2 &= -\frac{\Sigma_e}{3} \cos(\theta_L) - \frac{\Sigma_e}{\sqrt{3}} \sin(\theta_L) \end{aligned} \quad (27)$$

Consequently, one can then reexpress \mathbf{B} in spherical coordinates; it follows immediately that

$$\tilde{B} = \frac{1}{1-f} \left[\frac{\Sigma_e}{3} \cos(\theta_L)(3 \cos^2(\theta) - 1) + \frac{\Sigma_e}{\sqrt{3}} \sin(\theta_L) \sin^2(\theta) \cos(2\phi) \right] \quad (28)$$

where θ and ϕ are the polar angle and azimuthal one in spherical coordinates system.

As a result, the microscopic equivalent stress (24) can be written as

$$\sigma_e = \sqrt{\frac{A^2}{4} - \frac{3A}{2} \tilde{B} + B_{eq}^2} \quad (29)$$

Taking into account (17), (18) and (25), (29) can be recast in the form:

$$\sigma_e = \sqrt{\tilde{\Sigma}_m^2 - 3\tilde{\Sigma}_m \tilde{B} + \tilde{\Sigma}_e^2} = \sqrt{\frac{9\Sigma_m^2}{4 \ln^2 f} + \frac{9\Sigma_m \Sigma_L}{2(1-f) \ln f} + \frac{\Sigma_e^2}{(1-f)^2}} \quad (30)$$

where Σ_L is the macroscopic counterpart of \tilde{B} ; it reads:

$$\Sigma_L = (1-f)\tilde{B} = \frac{\Sigma_e}{3} \cos(\theta_L)(3 \cos^2(\theta) - 1) + \frac{\Sigma_e}{\sqrt{3}} \sin(\theta_L) \sin^2(\theta) \cos(2\phi) \quad (31)$$

Hence, the local von Mises yield criterion can be expressed as

$$F(\boldsymbol{\sigma}(\boldsymbol{\Sigma})) = \sigma_e - \sigma_0 = \sqrt{\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2 - \frac{3\tilde{\Sigma}_m \Sigma_L}{1-f}} - \sigma_0 \leq 0 \quad (32)$$

Let us recall that for obtaining the macroscopic criterion from (10), one need to integrate (32) over the matrix. However, due to the presence of the azimuth angle ϕ in the expression of Σ_L (31), there is no closed form solution. In order to overcome this difficulty, a simple idea consists in performing a Taylor series expansion (around 0) till the third order, this

leads to the following approximation:

$$\begin{aligned}\sigma_e &= \sqrt{\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2} \cdot \sqrt{1 - \frac{3\tilde{\Sigma}_m \Sigma_L}{(1-f)(\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2)}} \\ &\simeq \sqrt{\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2} \cdot \left[1 - \frac{3\tilde{\Sigma}_m \Sigma_L}{2(1-f)(\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2)} - \frac{9\tilde{\Sigma}_m^2 \Sigma_L^2}{8(1-f)^2(\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2)^2} - \frac{27\tilde{\Sigma}_m^3 \Sigma_L^3}{16(1-f)^3(\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2)^3} \right]\end{aligned}\quad (33)$$

Next, the final integration includes the computation of the following integrals:

$$\begin{aligned}\frac{1}{4\pi} \int_{S(r)} \Sigma_L dS &= 0 \\ \frac{1}{4\pi} \int_{S(r)} \Sigma_L^2 dS &= \frac{4\Sigma_e^2}{45} \\ \frac{1}{4\pi} \int_{S(r)} \Sigma_L^3 dS &= \frac{16\Sigma_e^3}{945} \cos(\theta_L)(4 \cos^2(\theta_L) - 3)\end{aligned}\quad (34)$$

Finally, from (10), the macroscopic criterion is obtained as:

$$\mathcal{F} \simeq \frac{1}{4\pi} \int_{S(r)} \sigma_e^2 dS - \sigma_0 = D \left(1 - \frac{C^2 \Sigma_e^2}{90D^4} + \frac{C^3 \Sigma_e^3}{945D^6} \cos(\theta_L)(4 \cos^2(\theta_L) - 3) \right) - \sigma_0 \leq 0 \quad (35)$$

where we have denoted C and D the following functions of Σ

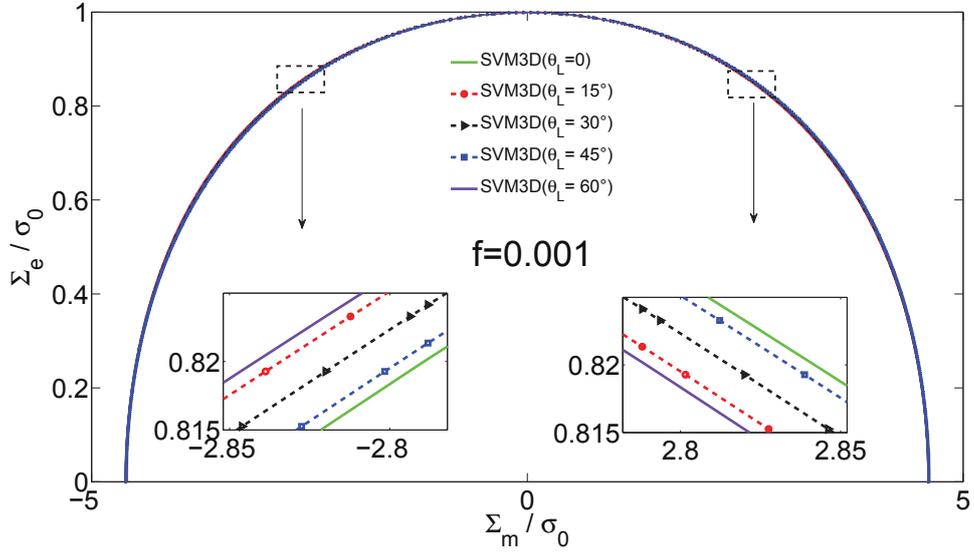
$$\begin{aligned}D(\Sigma) &= \sqrt{\tilde{\Sigma}_m^2 + \tilde{\Sigma}_e^2} = \sqrt{\frac{9\Sigma_m^2}{4 \ln^2 f} + \frac{\Sigma_e^2}{(1-f)^2}} \\ C(\Sigma) &= -\frac{3\tilde{\Sigma}_m}{1-f} = \frac{9\Sigma_m}{2(1-f) \ln f}\end{aligned}\quad (36)$$

It should be underlined that the established criterion (35) depends not only on the the macroscopic mean stress and equivalent stress, but also explicitly on the Lode angle (or the third invariant of the stress deviator).

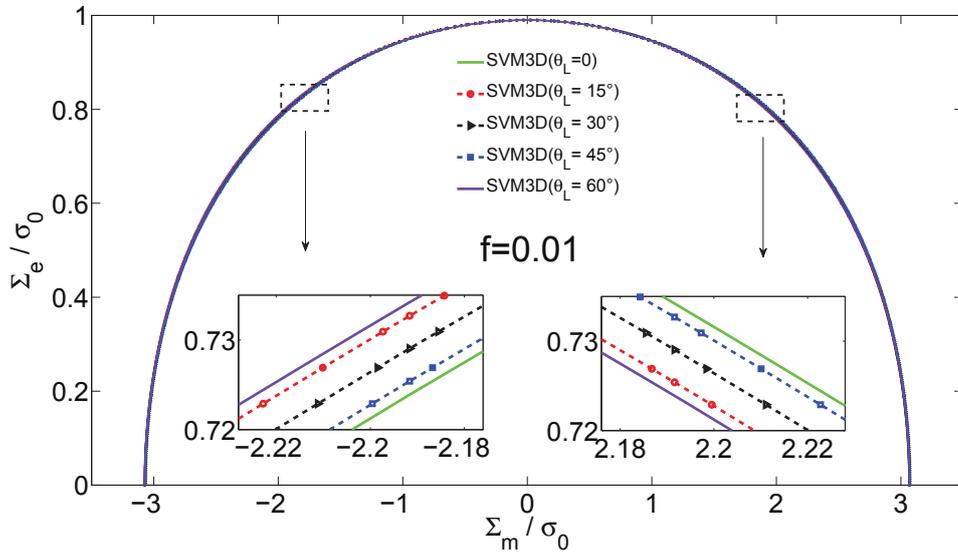
3.3. Illustration of the established macroscopic criterion

Next, we provide in this subsection the illustration of the established criterion (35) and its comparison with Gurson model and the Stress Variational Model (SVM) (Cheng et al., 2013). It is worthy to note that the later one has been derived from a closed-form formulation. Two values of porosity $f = 0.001$ and $f = 0.01$ are adopted for the later illustration and comparisons.

First, five yield loci obtained from (35) are illustrated on Fig.1 with different values of Lode angle: $\theta_L = 0, 15^\circ, 30^\circ, 45^\circ$ and 60° , while the first and the last ones are corresponding to the macroscopic model obtained from the axisymmetric trial stress field. Consequently,



(a) Porosity: $f = 0.001$



(b) Porosity: $f = 0.01$

Figure 1: Illustrations of the yield surfaces obtained from the new established criterion SVM3D (35) with five values of Lode angle: $\theta_L = 0, 15^\circ, 30^\circ, 45^\circ$ and 60° . Porosity: $f = 0.001$ and $f = 0.01$.

the yield surface displays an asymmetry due to the sign of the third invariant (Cheng et al., 2013). It can be observed that the yield surfaces obtained from other values of the Lode angle are absolutely between the above two ones.

For completeness, due to the approximation taken for the derivation of the macroscopic criterion, the new established criterion (35) derived from the axisymmetric trial stress field should perform a difference with respect to the closed-form criterion SVM. To this end, Fig.2 is contributed to the comparison between the SVM, the axisymmetric case of SVM3D and the Gurson model. Finally, slight differences between the SVM and the SVM3D is obtained.

4. Plastic flow rule and void growth rate

We aim now at deriving the plastic strain rate from the normality rule. Unlike the conventional modeling, the three invariants of the macroscopic criterion (35) are taken into account. Not only the mean strain rate D_m and the equivalent one D_e have to be computed, but also the contribution D_{III} related to the third invariant of deviator J_3 will be provided. It is worthy to interpret that D_{III} can indicated the influence of the Lode angle upon the π -plane of principal stress space to the macroscopic plastic flow rule. Let us first define the macroscopic stress,

$$\Sigma_{III} = \sqrt[3]{J_3} \quad (37)$$

Hence, the dissipation power Π can be written as

$$\Pi = \mathbf{D} : \mathbf{\Sigma} = 3\Sigma_m D_m + \Sigma_e D_e + \Sigma_{III} D_{III} \quad (38)$$

Moreover, considering the macroscopic criterion (35), the macroscopic strain rate can be obtained from the associated flow rule

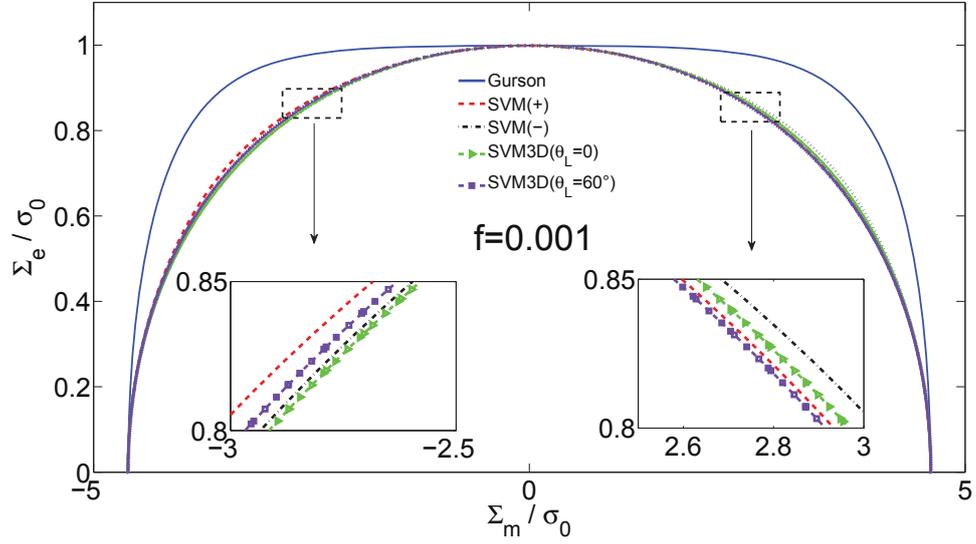
$$\begin{aligned} D_m &= \frac{1}{3} \dot{\Lambda} \frac{\partial \mathcal{F}_{3D}}{\partial \Sigma_m} \\ &= \frac{1}{3} \dot{\Lambda} \left[\frac{\partial D}{\partial \Sigma_m} - \frac{\Sigma_e^2}{90} \cdot \frac{2CD \frac{\partial C}{\partial \Sigma_m} - 3C^2 \frac{\partial D}{\partial \Sigma_m}}{D^4} + \frac{J_3}{70} \left(\frac{729J_3^2}{\Sigma_e^6} - 3 \right) \left(\frac{3C^2 D^2 \frac{\partial C}{\partial \Sigma_m} - 5D^4 \frac{\partial D}{\partial \Sigma_m}}{D^7} \right) \right] \\ D_e &= \dot{\Lambda} \frac{\partial \mathcal{F}_{3D}}{\partial \Sigma_e} = \dot{\Lambda} \left[\frac{\partial D}{\partial \Sigma_e} - \frac{C^2 2D \Sigma_e - 3 \frac{\partial D}{\partial \Sigma_e} \Sigma_e^2}{90 D^4} - \frac{C^3 J_3}{70} \left(\frac{5}{D^6} \frac{\partial D}{\partial \Sigma_e} \left(\frac{729J_3^2}{\Sigma_e^6} - 3 \right) - \frac{6}{D^5} \frac{729J_3^2}{\Sigma_e^7} \right) \right] \\ D_{III} &= \dot{\Lambda} \frac{\partial \mathcal{F}_{3D}}{\partial \Sigma_{III}} = \dot{\Lambda} \left[\frac{C^3}{70D^5} \left(\frac{729J_3^2}{\Sigma_e^6} - 1 \right) J_3^{-\frac{2}{3}} \right] \end{aligned} \quad (39)$$

where

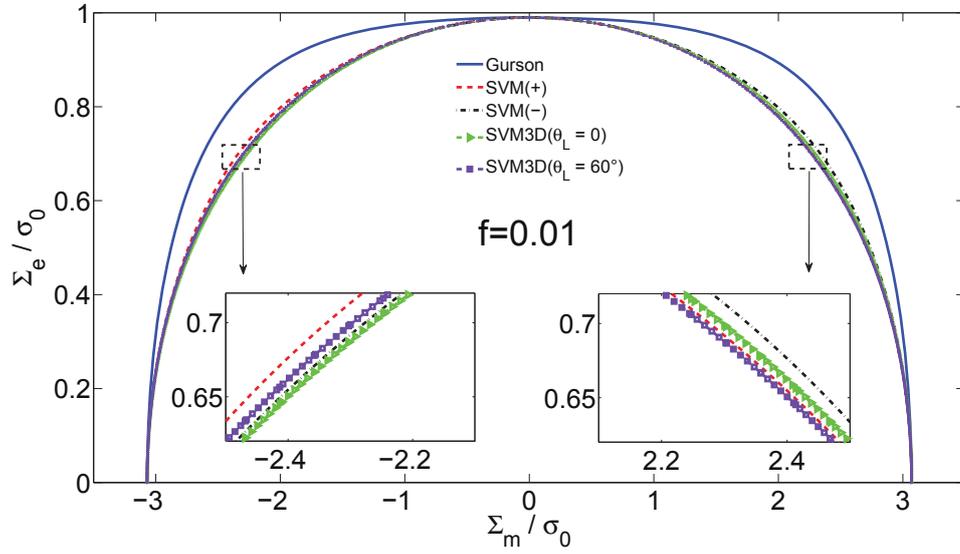
$$\frac{\partial C}{\partial \Sigma_m} = \frac{9}{2(1-f) \ln f}, \quad \frac{\partial D}{\partial \Sigma_m} = \frac{9\Sigma_m}{2D \ln^2 f}, \quad \frac{\partial D}{\partial \Sigma_e} = \frac{2\Sigma_e}{D(1-f)^2}$$

Finally, the plastic void growth rate can be obtained from the mass balance equation. Taking into account the plastic flow rule (39) and eliminating the plastic multiplier $\dot{\Lambda}$, it follows that

$$\frac{\dot{f}}{D_e} = 3(1-f) \frac{D_m}{D_e} \quad (40)$$



(a) Porosity: $f = 0.001$



(b) Porosity: $f = 0.01$

Figure 2: Comparison between the yield surfaces obtained from the established criterion SVM3D (35) with axisymmetric trial stress field, the closed form criterion of Stress Variational Model (SVM(+)) and SVM(-) (Cheng et al., 2013) and the Gurson criterion (Gurson, 1977). Porosity: $f = 0.001$ and $f = 0.01$.

It is readily seen from Eqs.(40) and (39) that the void growth rate depends on the third invariant of the stress deviator J_3 (or the Lode angle θ_L).

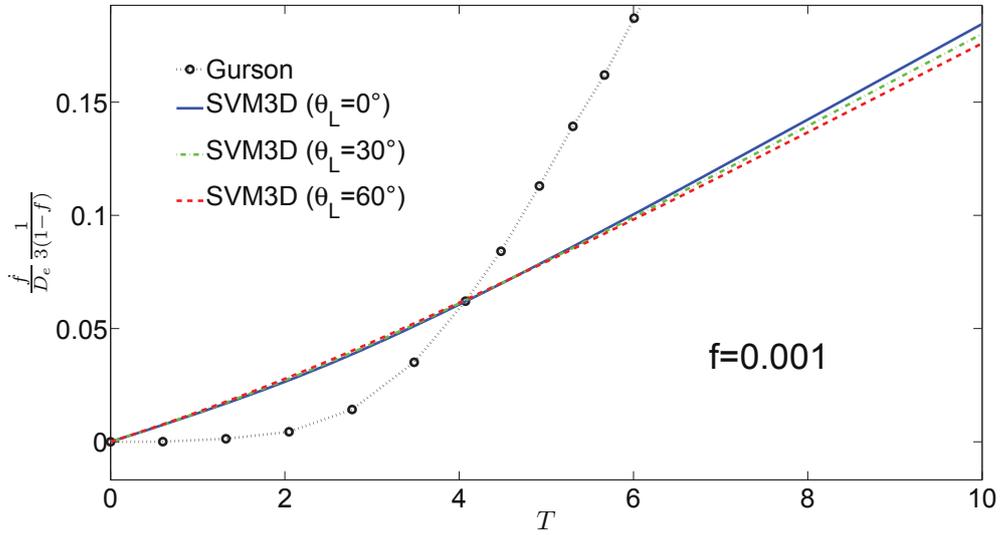
Fig.3 illustrates the evolution of porosity given as function of stress triaxiality $T = \frac{\Sigma_m}{\Sigma_e}$ for two values of initial porosity. It can be observed that, the ones with axisymmetric loadings ($\theta_L = 0^\circ$ and 60°) give two extremal values of the void evolution for a fixed value of triaxiality, while for another case with non axisymmetric state ($\theta_L = 30^\circ$) is exactly between the two extremal ones. Slight differences due to the Lode angle (or the third invariant of stress deviator) can be observed.

5. Conclusion

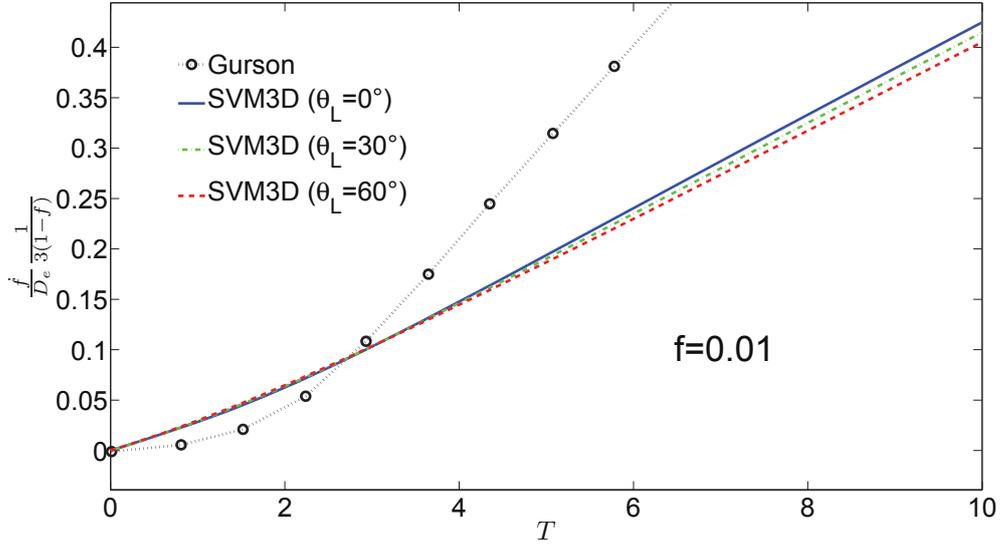
In this study, we applied the statically limit analysis method to derive a Lode angle dependent macroscopic model. To this end, we have proposed a non axisymmetric trial stress field for the porous media whose matrix obeys the von Mises yield criterion. Unlike the conventional macroscopic modelling in literature, the new criterion shows an effect of the three macroscopic invariants: mean stress, equivalent stress and Lode coefficient (or the third invariant of stress deviator). The influence of the last one was specially discussed not only for the macroscopic criterion, but also for the plastic flow rule and the void growth rate.

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(a) Porosity: $f = 0.001$



(b) Porosity: $f = 0.01$

Figure 3: Evolution of porosity as function of the stress triaxiality for initial porosity $f = 0.001$ and $f = 0.01$ with three values of Lode angle: $\theta_L = 0, 30^\circ$ and 60° .