



HAL
open science

Global Semiclassical Limit from Hartree to Vlasov Equation for Concentrated Initial Data

Laurent Lafleche

► **To cite this version:**

Laurent Lafleche. Global Semiclassical Limit from Hartree to Vlasov Equation for Concentrated Initial Data. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 2021, 38 (6), pp.1739-1762. 10.1016/j.anihpc.2021.01.004 . hal-02046481v2

HAL Id: hal-02046481

<https://hal.science/hal-02046481v2>

Submitted on 14 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

GLOBAL SEMICLASSICAL LIMIT FROM HARTREE TO VLASOV EQUATION FOR CONCENTRATED INITIAL DATA

LAURENT LAFLECHE^{1,2}

ABSTRACT. We prove a quantitative and **global in time** semiclassical limit from the Hartree to the Vlasov equation in the case of a singular interaction potential in dimension $d \geq 3$, including the case of a Coulomb singularity in dimension $d = 3$. This result holds for initial data concentrated enough in the sense that some space moments are initially sufficiently small. As an intermediate result, we also obtain quantitative bounds on the space and velocity moments of even order and the asymptotic behavior of the spatial density due to dispersion effects, uniform in the Planck constant \hbar .

TABLE OF CONTENTS

1. Introduction	1
1.1. Main results	3
2. Free Transport	8
3. Propagation of moments	10
3.1. Classical case.	10
3.2. Boundedness of Eulerian moments	11
3.3. Application to the semiclassical limit	19
4. Acknowledgments	24
References	24

1. INTRODUCTION

The equation governing the dynamics of a large number of interacting particles of density $f = f(t, x, \xi)$ in the phase space is the Vlasov equation

$$\text{(Vlasov)} \quad \partial_t f + \xi \cdot \nabla_x f + E \cdot \nabla_\xi f = 0,$$

Date: October 15, 2020.

2010 Mathematics Subject Classification. 82C10, 35Q41, 35Q55 (82C05,35Q83).

Key words and phrases. Hartree equation, Nonlinear Schrödinger equation, Vlasov equation, Coulomb interaction, gravitational interaction, semiclassical limit.

¹CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, PSL Research University, Place du Maréchal de Lattre de Tassigny, 75775 Paris cedex 16 France, lafleche@ceremade.dauphine.fr.

²CMLS, École polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau cedex, France.

where $E = -\nabla V$ is the force field corresponding to the mean field potential $V = V(x)$ given by

$$V = K * \rho_f = \int_{\mathbb{R}^d} K(x-y) \rho_f(y) dy,$$

where we denote by $\rho_f(x) := \int_{\mathbb{R}^d} f(x,v) dv$ the spatial density and by $K(x)$ the pair interaction potential between two particles at distance $|x|$.

The counterpart of the [Vlasov](#) equation in quantum mechanics is the Hartree equation

$$(Hartree) \quad i\hbar \partial_t \rho = [H, \rho],$$

where ρ is a self-adjoint Hilbert-Schmidt operator called the density operator and the Hamiltonian is defined by

$$H = -\frac{\hbar^2}{2} \Delta + V.$$

In this formula, the potential is defined by $V = K * \rho$ where the spatial density ρ is defined as the diagonal of the kernel $\varrho(x,y)$ of the operator ρ , i.e. $\rho(x) = \varrho(x,x)$.

In this paper, we study in a quantitative way the limit when $\hbar \rightarrow 0$ of the [Hartree](#) equation which is known to converge to the [Vlasov](#) equation. The question of the derivation of this equation from the quantum mechanics is a very active topic of research. Non-constructive results in weak topologies have indeed already been proved, including the case of Coulomb interactions, starting from the work of Lions and Paul [36] and Markowich and Mauser [39]. See also [32, 23, 31, 2, 1].

Some more precise quantitative results have also more recently been proved for smooth forces which are always at least Lipschitz in [6, 3, 4, 12, 26]. In [27], Golse and Paul introduce a pseudo-distance on the model of the Wasserstein-(Monge-Kantorovitch) between classical phase space densities and quantum density operators to get a rate of convergence for the semiclassical limit for Lipschitz forces. This strategy has been used in the recent paper [35] of the present author to extend this result to more singular interactions, but only up to a fixed time in the case of potentials with a strong singularity such as the Coulomb interaction.

We also mention the work of Porta et al [48] and Saffirio [50] about the question of the mean-field limit for the Schrödinger equation to the [Hartree](#) equation for Fermions since this limit is coupled with a semiclassical limit. Results are proved for the Coulomb interaction under assumptions of propagation of regularity along the Hartree dynamics which is still an open problem. Other results about the mean-field limit can be found in [10, 20, 9] where non-quantitative results are established for the Coulomb potential, and more precise limits can be found in [49, 47, 26, 40, 27, 29, 30] for Bosons and in [22, 21, 13, 11, 7, 45, 48, 44] for Fermions.

Here, we extend the results of [35] by proving a global in time semiclassical limit under a smallness condition of space moments. We first prove a global in time bound on some modified space moments, from which we obtain the propagation of space and velocity moments. The same kind of results were already known for $\hbar = 1$ (see Remark 1.4), and the main novelty is the fact that the bounds we prove are uniform in \hbar . The bound on the velocity is then sufficient to use the theory already used in the above mentioned paper to get a global L^∞ bound on the spatial density and the quantitative semiclassical limit in the quantum Wasserstein pseudo-distance.

The fact that the time decay due to the dispersion properties gives global estimates for the Vlasov equation was already used in [8]. The modified space moments of order 2 are linked to a Lyapunov functional reminiscent of the conservation of energy, see [43, 19]. The propagation of modified space moments was investigated further in [16, 41, 42].

1.1. Main results. We adopt the same notational conventions as in [35]. In particular, $L^{p,\infty}$ denotes the weak Lebesgue spaces of functions on \mathbb{R}^d and we define the quantum version of the phase space Lebesgue and weighted Lebesgue spaces as

$$\begin{aligned}\mathcal{L}^p &:= \left\{ \rho \in \mathcal{B}(L^2), \|\rho\|_{\mathcal{L}^p} := h^{-d/p'} \operatorname{Tr}(|\rho|^p)^{\frac{1}{p}} < \infty \right\} \\ \mathcal{L}_+^p &:= \{ \rho \in \mathcal{L}^p, \rho = \rho^* \geq 0 \} \\ \mathcal{L}^p(m) &:= \{ \rho \in \mathcal{L}^p, \rho m \in \mathcal{L}^p \},\end{aligned}$$

where $\mathcal{B}(L^2)$ denotes the set of bounded linear operators on L^2 and Tr denotes the trace. We also define the quantum probability measures by

$$\mathcal{P} := \{ \rho \in \mathcal{L}_+^1, \operatorname{Tr}(\rho) = 1 \}.$$

Moreover, in order to ensure well-posedness

We will denote by $\mathbf{p} := -i\hbar\nabla$ the quantum impulsion, which is an unbounded operator on L^2 , and by M_0 the common total mass of the densities in both the quantum and the classical setting

$$M_0 := \operatorname{Tr}(\rho) = \iint_{\mathbb{R}^{2d}} f(t, x, v) \, dx \, dv.$$

Our first result states that if the spatial density is concentrated enough, then the Eulerian moments $\operatorname{Tr}(|x - t\mathbf{p}|^n \rho)$ are bounded globally in time.

Theorem 1. *Let $d \geq 3$, $n \in 2\mathbb{N} \setminus \{0\}$, $r \in [1, \infty]$ and define $\mathbf{b}_n := \frac{nr'+d}{n+1}$. Assume*

$$(1) \quad \nabla K \in L^{\mathbf{b},\infty} \text{ with } \mathbf{b} \in \left(\max\left(\frac{d}{3}, \mathbf{b}_4, \mathbf{b}_n\right), \frac{d}{2} \right),$$

and let ρ be a solution of the Hartree equation with initial condition

$$\rho^{\text{in}} \in \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^n + |\mathbf{p}|^n).$$

Then there exists an explicit constant $\mathcal{C} > 0$ depending on M_0 , $\operatorname{Tr}(|\mathbf{p}|^n \rho^{\text{in}})$, $\|\nabla K\|_{L^{\mathbf{b},\infty}}$, $\|\rho^{\text{in}}\|_{\mathcal{L}^r}$ and not on \hbar , such that if

$$(2) \quad \operatorname{Tr}(|x|^n \rho^{\text{in}}) < \mathcal{C},$$

then

$$\operatorname{Tr}(|x - t\mathbf{p}|^n \rho) \in L^\infty(\mathbb{R}_+),$$

uniformly in \hbar .

Remark 1.1. *The theorem applies in particular in the case of interaction kernels K with a singularity like the Coulomb interaction. For example for any $\varepsilon > 0$*

$$K(x) = \frac{\pm 1}{|x|} \mathbf{1}_{|x| \leq 1} + \frac{\pm 1}{|x|^{1+\varepsilon}} \mathbf{1}_{|x| > 1}.$$

An other interesting case of application is the case of the Yukawa potential that is commonly used as an approximation in the case when there are particles with positive and negative charge, and which is of the form

$$K(x) = \frac{e^{-|x|/\lambda_D}}{|x|},$$

where $\lambda_D > 0$ is the Debye length, which represents the characteristic size of the interaction.

Remark 1.2. An other good example of potentials verifying the assumptions of the theorem are potentials of the form

$$(3) \quad K(x) = \frac{\pm 1}{|x|^a},$$

with $a = \frac{d}{b} - 1 \in (1, \frac{8}{7})$ when $d = 3$. In dimension $d = 4$, $d = 5$ and $d \geq 6$, one can even better take respectively $a \in (1, \frac{3}{2})$, $a \in (1, \frac{16}{9})$ and $a \in (1, 2)$. Of course, regular potentials also enter the scope of this first theorem as long as they decay sufficiently at infinity, so that we can take for example

$$K(x) = \frac{\pm 1}{1 + |x|^a},$$

for any $a > 1$.

Remark 1.3. Since $\rho \in \mathcal{L}_+^1$, it is an Hilbert-Schmidt operator that can be written as a integral operator of kernel $\varrho(x, y)$ and it can also be diagonalized by the spectral theorem. Hence, we can write for any $\varphi \in L^2$

$$(4) \quad \rho \varphi(x) = \int_{\mathbb{R}^d} \varrho(x, y) \varphi(y) dy = \sum_{j \in J} \lambda_j |\psi_j\rangle \langle \psi_j | \varphi\rangle,$$

where $(\psi_j)_{j \in J} \in (L^2)^J$ with $J \subset \mathbb{N}$ is an orthogonal basis. The space density can then be written

$$\rho(x) := \varrho(x, x) = \sum_{j \in J} \lambda_j |\psi_j(x)|^2,$$

and the space moments

$$\mathrm{Tr}(|x|^n \rho) = \int_{\mathbb{R}^d} \rho(x) |x|^n dx.$$

For even integers $n \in 2\mathbb{N}$, the velocity moments can be written

$$(5) \quad \mathrm{Tr}(|\mathbf{p}|^n \rho) = \hbar^n \sum_{j \in J} \lambda_j \int_{\mathbb{R}^d} |\nabla^{\frac{n}{2}} \psi_j|^2.$$

Remark 1.4. Notice that the existence theory for both *Hartree* and *Vlasov* equations is already quite well understood, see for example [24, 25, 33, 14, 34, 15] for the *Hartree* equation and [37, 51, 46] for the *Vlasov* equation. For more singular potentials than the Coulomb potential, remark that by the real interpolation definition of Lorentz spaces, our hypothesis (1) on ∇K implies

$$\nabla K \in L^p + L^q \text{ for any } (p, q) \text{ such that } 0 < p < \mathfrak{b} < q < \infty,$$

where one can take $p = \frac{2d+8}{d+8}$ since $\mathfrak{b} > \mathfrak{b}_4$. Moreover, by Sobolev embeddings, since $\mathfrak{b} > \frac{d}{3}$,

$$K \in L^{r_0} + L^\infty \text{ with } r_0 > \frac{d}{2},$$

and also with $r_0 > \frac{d+4}{4}$ when $d \geq 3$, since $\mathfrak{b} > \mathfrak{b}_4$. Therefore, our assumptions implies hypotheses (90) and (91) in [36] and so the existence of solutions for both equations. Remark that, as in our previous paper [35], we are not trying to prove here the propagation of regularity for ψ . In particular the global in time propagation for $\hbar = 1$ of the multi-Sobolev norms defined by Formula (5) is proved in [15, Appendix] in the case of the Coulomb potential, where they are denoted by $\|\Psi\|_{H^n(\lambda)}$ for $\Psi = (\psi_j)_{j \in J}$. The same analysis can be performed for our class of potentials. Thus, since we assume initially bounded velocity moments of order $n \geq 2$, this implies that our solutions will always satisfy $\Psi \in L_{\text{loc}}^\infty((0, T), H^n(\lambda))$ but with a bound a priori not uniform in \hbar . Hence, the difficulty lies in the fact to obtain \hbar independent bounds, which prevent for example to estimate separately each part of the commutator appearing in *Hartree* equation.

We can state the analogue of this theorem for solutions of the *Vlasov* equation

Proposition 1.1. *Let $d \geq 3$, $n \in 2\mathbb{N}$, $r \in [1, \infty]$ and assume ∇K verifies Condition (1). Let f be a solution of the *Vlasov* equation with nonnegative initial condition*

$$f^{\text{in}} \in L_{x, \xi}^r \cap L^1(1 + |x|^n + |\xi|^n).$$

Then there exists an explicit constant $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\text{in}}(x, \xi) |x|^n dx d\xi < \mathcal{C},$$

then

$$\iint_{\mathbb{R}^{2d}} f(\cdot, x, \xi) |x - t\xi|^n dx d\xi \in L^\infty(\mathbb{R}_+).$$

Remark 1.5. *For the *Vlasov* equation, contrarily to the *Hartree* equation, we generally not have strong solutions in our setting (the velocity moments are not derivatives in the classical case). However, we still have global existence of renormalized solutions (see [17, 18]).*

We can use the first theorem to obtain good estimates on the space and velocity moments and on the spatial density that do not depend on \hbar .

Theorem 2. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N}) \setminus \{0, 2\}$ and assume*

$$(6) \quad \nabla K \in L^{\mathfrak{b}, \infty} \text{ with } \mathfrak{b} \in \left(\max\left(\mathfrak{b}_4, \frac{d}{3}\right), \frac{d}{2}\right),$$

and let ρ be a solution of the *Hartree* equation with initial condition

$$\rho^{\text{in}} \in \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^4 + |\mathbf{p}|^n),$$

for a given even integer $n \geq 4$. Then there exists a constant \mathcal{C} depending on M_0 , $\text{Tr}(|\mathbf{p}|^4 \rho^{\text{in}})$, $\|\nabla K\|_{L^{\mathfrak{b}, \infty}}$, and $\|\rho^{\text{in}}\|_{\mathcal{L}^r}$ such that if

$$\text{Tr}(|x|^4 \rho^{\text{in}}) \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} \geq 0$ and $C > 0$ depending on the initial conditions such that

$$(7) \quad \text{Tr}(|\mathbf{p}|^n \boldsymbol{\rho}) \leq C \langle t \rangle^{c_n}$$

$$(8) \quad \text{Tr}(|x|^n \boldsymbol{\rho}) \leq C \langle t \rangle^{n(c_n+1)}$$

$$(9) \quad \|\rho\|_{L^p} \leq C \langle t \rangle^{-d/p'},$$

where $p' = r' + \frac{d}{4}$ and $\langle t \rangle = \sqrt{1+t^2}$. Moreover, if $\text{Tr}(|x|^n \boldsymbol{\rho}^{\text{in}})$ is sufficiently small, then we can get more precise estimates

$$\text{Tr}(|\mathbf{p}|^n \boldsymbol{\rho}) \leq C$$

$$\text{Tr}(|x|^n \boldsymbol{\rho}) \leq C \langle t \rangle^n$$

$$\|\rho\|_{L^{p_n}} \leq C \langle t \rangle^{-d/p'_n},$$

where $p'_n = r' + \frac{d}{n}$.

We can once more state the analogue result for the Vlasov equation.

Proposition 1.2. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N}) \setminus \{0, 2\}$ and assume K verifies (6). Let f be a solution of the Vlasov equation with nonnegative initial condition*

$$f^{\text{in}} \in L^r_{x,\xi} \cap L^1_{x,\xi}(1 + |x|^4 + |\xi|^n),$$

for a given even integer $n \geq 4$. Then there exists $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\text{in}}(x, \xi) |x|^4 dx d\xi \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that

$$\iint_{\mathbb{R}^{2d}} f(t, x, \xi) |\xi|^n dx d\xi \leq C \langle t \rangle^{c_n}$$

$$\iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x|^n dx d\xi \leq C \langle t \rangle^{n(c_n+1)}$$

$$\|\rho_f\|_{L^p} \leq C \langle t \rangle^{-d/p'},$$

where $p' = r' + \frac{d}{4}$. As in the previous theorem, one can take $c_n = 0$ and $p' = r' + \frac{d}{n}$ if the initial space moments of order n are small.

Before stating the result about the semiclassical limit, we recall the definition of the semiclassical Wasserstein-(Monge-Kantorovitch) distance introduced by Golse and Paul in [27]. We say that $\gamma \in L^1(\mathbb{R}^{2d}, \mathcal{P})$ is a semiclassical coupling between a classical kinetic density $f \in L^1 \cap \mathcal{P}(\mathbb{R}^{2d})$ and a density operator $\rho \in \mathcal{P}$ and we write $\gamma \in \mathcal{C}_h(f, \rho)$ when

$$\text{Tr}(\gamma(z)) = f(z)$$

$$\int_{\mathbb{R}^{2d}} \gamma(z) dz = \rho.$$

Then we define the semiclassical Wasserstein-(Monge-Kantorovitch) pseudo-distance in the following way

$$(10) \quad W_{2,h}(f, \rho) := \left(\inf_{\gamma \in \mathcal{C}_h(f, \rho)} \int_{\mathbb{R}^{2d}} \text{Tr}(\mathbf{c}_h(z)\gamma(z)) dz \right)^{\frac{1}{2}},$$

where $\mathbf{c}_\hbar(z)\varphi(y) = (|x-y|^2 + |\xi - \mathbf{p}|^2)\varphi(y)$, $z = (x, \xi)$ and $\mathbf{p} = -i\hbar\nabla_y$. This is not a distance but it is comparable to the classical Wasserstein distance W_2 between the Wigner transform of the quantum density operator and the normal kinetic density, in the sense of the following theorem.

Theorem 3 (Golse & Paul [27]). *Let $\rho \in \mathcal{P}$ and $f \in \mathcal{P}(\mathbb{R}^{2d})$ be such that*

$$\int_{\mathbb{R}^{2d}} f(x, \xi) (|x|^2 + |\xi|^2) dx d\xi < \infty.$$

Then one has $W_{2, \hbar}(f, \rho)^2 \geq d\hbar$ and for the Husimi transform \tilde{f}_\hbar of ρ , it holds

$$(11) \quad W_2(f, \tilde{f}_\hbar)^2 \leq W_{2, \hbar}(f, \rho)^2 + d\hbar.$$

See [28] for more results about this pseudo-distance and the definition of the Husimi transform.

Our last theorem uses these results to obtain the semiclassical limit. We also recall the following theorem which will give us our assumptions on the classical solution of the **Vlasov** equation.

Theorem 4 (Lions & Perthame [37], Loeper [38]). *Assume $f^{\text{in}} \in L_{x, \xi}^\infty(\mathbb{R}^6)$ verify*

$$(12) \quad \iint_{\mathbb{R}^6} f^{\text{in}} |\xi|^{n_0} dx d\xi < C \text{ for a given } n_0 > 6,$$

and for all $R > 0$,

$$(13) \quad \sup_{(y, w) \in \mathbb{R}^6} \text{ess} \{f^{\text{in}}(y + t\xi, w), |x - y| \leq Rt^2, |\xi - w| \leq Rt\} \in L_{\text{loc}}^\infty(\mathbb{R}_+, L_x^\infty L_\xi^1).$$

*Then there exists a unique solution to the **Vlasov** equation with initial condition $f_{t=0} = f^{\text{in}}$. Moreover, in this case, the spatial density verifies $\rho_f \in L_{\text{loc}}^\infty(\mathbb{R}_+, L^\infty)$.*

Theorem 5. *Let $d \geq 3$ and assume*

$$\nabla K \in B_{1, \infty}^1 \cap L^{\mathbf{b}} \text{ with } \mathbf{b} \in (\max(\mathbf{b}_4, \frac{d}{3}), \frac{d}{2}).$$

*Let ρ be a solution of the **Hartree** equation with initial condition $\rho^{\text{in}} \in \mathcal{P}$ verifying*

$$\begin{aligned} \rho^{\text{in}} &\in \mathcal{L}^\infty \cap \mathcal{L}_+^1(1 + |\mathbf{p}|^{n_1}) \\ \forall i \in \llbracket 1, d \rrbracket, \mathbf{p}_i^{n_0} \rho^{\text{in}} &\in \mathcal{L}^\infty, \end{aligned}$$

where $\mathbf{p}_i := -i\hbar\partial_i$, $\llbracket 1, d \rrbracket = [1, d] \cap \mathbb{N}$, and $(n_0, n) \in (2\mathbb{N})^2$ is such that

$$\begin{aligned} n_0 &> d \\ n &\geq \frac{d + \mathbf{b}(n_0 - 1)}{\mathbf{b} - 1}. \end{aligned}$$

*Let f is a solution of the **Vlasov** equation with initial condition f^{in} verifying the hypotheses of Theorem 4 and of mass $M_0 = 1$. Then there exists a constant $C > 0$ depending on $\text{Tr}(|\mathbf{p}|^4 \rho^{\text{in}})$, $\|\nabla K\|_{L^{\mathbf{b}, \infty}}$ and $\|\rho^{\text{in}}\|_{\mathcal{L}^\infty}$ such that if*

$$\text{Tr}(|x|^4 \rho^{\text{in}}) \leq C,$$

then there exists a constant $C > 0$ depending on the initial conditions such that

$$(14) \quad \|\rho(t)\|_{L^\infty} \leq C \langle t \rangle^{n_0(1 + \frac{\sigma}{\mathbf{b}})}$$

where $c = c_n$ is given by (7). Again, if additionally $\text{Tr}(|x|^n \boldsymbol{\rho}^{\text{in}})$ is also sufficiently small, then one can take $c = 0$. Moreover, the following semiclassical estimate holds

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}(t)) \leq W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}^{\text{in}}) e^{\theta \lambda(t)} e^{e^{\lambda(t)}},$$

with

$$\begin{aligned} \theta &= \frac{\text{sign}(\ln W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}^{\text{in}}))}{2} \\ \lambda(t) &= C^{\text{in}} \langle t \rangle^{1+n_0(1+\frac{c}{b'})} \\ C^{\text{in}} &= 1 + C \|\nabla K\|_{B_{1,\infty}^1} \sup_t \frac{\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}}{\langle t \rangle^{n_0(1+c/b')}} \end{aligned}$$

for some constant $C > 0$ independent from the initial conditions.

Remark 1.6. Again, the additional assumption $\nabla K \in B_{1,\infty}^1$ is compatible with a kernel with a Coulomb singularity in dimension $d = 3$ such as the one given in Remark 1.1. However, higher local singularities such that the one from Equation (3) are not admissible for this result. This seems natural since even the uniqueness of solutions for the Vlasov equation is not known for such singular potentials.

Remark 1.7. This theorem implies a result of convergence in the classical Wasserstein distance at a rate $C_t \hbar^{C_t}$ as soon as the quantity $W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}^{\text{in}})$ is initially smaller than some power of \hbar . In particular, this implies weak convergence of the Wigner transform of the solution of the Hartree equation to the solution of the Vlasov equation. Remark that by [27, Theorem 2.4], $W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}^{\text{in}})$ is always larger than $\sqrt{d\hbar}$. The fact that $W_{2,\hbar}$ can be controlled by the classical Wasserstein distance up to an error term $\sqrt{d\hbar}$ can be proved for example when the initial states are superposition of coherent states and this leads to results that can be written uniquely in term of the classical Wasserstein distance as in [35, Section 7].

2. FREE TRANSPORT

We want to use the time decay properties of the kinetic free transport equation which writes for $f = f(t, x, \xi)$

$$\partial_t f + \xi \cdot \nabla_x f = 0.$$

In quantum mechanics, free transport is given by the free Schrödinger equation

$$i\hbar \partial_t \psi = H_0 \psi,$$

where $\hbar = \frac{h}{2\pi}$ and $H_0 = -\frac{\hbar^2 \Delta}{2}$ which can be written $H_0 = \frac{|\mathbf{p}|^2}{2}$ with the notation $\mathbf{p} = -i\hbar \nabla$. The solution corresponding to the initial condition ψ^{in} can be written $\mathbb{T}_t \psi^{\text{in}}$ where the semigroup \mathbb{T}_t is given by

$$(15) \quad \mathbb{T}_t \psi = e^{-itH_0/\hbar} \psi = \frac{e^{-i\pi|x|^2/(ht)}}{(iht)^{d/2}} * \psi.$$

The corresponding equation for density operators $\boldsymbol{\rho} \in \mathcal{P}$ is

$$(16) \quad i\hbar \partial_t \boldsymbol{\rho} = [H_0, \boldsymbol{\rho}],$$

whose solution is $S_t \boldsymbol{\rho}^{\text{in}}$ where the semigroup S_t is defined by

$$(17) \quad S_t \boldsymbol{\rho} := \mathbb{T}_t \boldsymbol{\rho} \mathbb{T}_{-t}.$$

As it can be easily noticed, it holds $\mathbb{T}_t^* = \mathbb{T}_t^{-1} = \mathbb{T}_{-t}$ and for any $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathcal{P}^2$, $\mathbb{S}(\boldsymbol{\rho}\boldsymbol{\rho}_2) = \mathbb{S}(\boldsymbol{\rho})\mathbb{S}(\boldsymbol{\rho}_2)$. Moreover, a straightforward computation shows that

$$(18) \quad \begin{aligned} \mathbb{S}_t \boldsymbol{p} &= \boldsymbol{p} \\ \mathbb{S}_t x &= x - t\boldsymbol{p}. \end{aligned}$$

By the spectral theory, it implies that $\mathbb{S}_t(f(x)) = f(x - t\boldsymbol{p})$ for any nice function f . By analogy, we can define the operator of translation of the impulsion \boldsymbol{p} by

$$(19) \quad \begin{aligned} \tilde{\mathbb{T}}_t \psi(x) &:= e^{\frac{-\pi|x|^2 t}{i\hbar}} \psi(x) = G_t(x) \psi(x) \\ \tilde{\mathbb{S}}_t \boldsymbol{\rho} &:= \tilde{\mathbb{T}}_t \boldsymbol{\rho} \tilde{\mathbb{T}}_{-t}, \end{aligned}$$

which verifies the equation

$$i\hbar \partial_t (\tilde{\mathbb{S}}_t \boldsymbol{\rho}) = \left[\frac{-|x|^2}{2}, \tilde{\mathbb{S}}_t \boldsymbol{\rho} \right],$$

and the two following relations

$$(20) \quad \begin{aligned} \tilde{\mathbb{S}}_t x &= x \\ \tilde{\mathbb{S}}_t \boldsymbol{p} &= \boldsymbol{p} - t\boldsymbol{x}. \end{aligned}$$

We recall the quantum kinetic interpolation inequality that was already used in [35, Theorem 6]. For $k \in 2\mathbb{N}$ we define

$$\rho_k := \sum_{j \in J} \lambda_j |\boldsymbol{p}^{\frac{k}{2}} \psi_j|^2 = \text{diag} \left(\boldsymbol{p}^{\frac{k}{2}} \boldsymbol{\rho} \cdot \boldsymbol{p}^{\frac{k}{2}} \right),$$

and for $r \geq 1$ and $0 \leq k \leq n$, we define the exponent $p_{n,k}$ by its Hölder conjugate

$$(21) \quad p'_{n,k} = \binom{n}{k}' p'_n \text{ with } p'_n = \left(r' + \frac{d}{n} \right).$$

Then the following inequality holds

Proposition 2.1. *Let $(k, n) \in (2\mathbb{N})^2$ be such that $k \leq n$ and $\boldsymbol{\rho} \in \mathcal{L}^1(|\boldsymbol{\rho}|^n) \cap \mathcal{L}_+^r$ for a given $r \in [1, \infty]$. Then there exists $C = C_{d,r,n,k} > 0$ such that*

$$(22) \quad \|\rho_k\|_{L^{p_{n,k}}} \leq C \text{Tr}(|\boldsymbol{p}|^n \boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta,$$

where $\theta = \frac{r'}{p'_{n,k}}$.

From this result, we can get an inequality with an additional time decay if we replace the velocity moments by the Eulerian moments $\text{Tr}(|x - t\boldsymbol{p}|^n \boldsymbol{\rho})$.

Corollary 2.1. *Let $n \in 2\mathbb{N}$, $r \in [1, \infty]$, $p' := r' + \frac{d}{n}$ and $\theta = \frac{r'}{p'}$. Then*

$$\|\rho\|_{L^p} \leq \frac{1}{t^{d/p'}} \text{Tr}(|x - t\boldsymbol{p}|^n \boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta.$$

Proof of Corollary 2.1. We just remark that by Formula (20), we get

$$\begin{aligned} t^{-n} \text{Tr}(|x - t\boldsymbol{p}|^n \boldsymbol{\rho}) &= \text{Tr}(|\boldsymbol{p} - x/t|^n \boldsymbol{\rho}) \\ &= \text{Tr}(\tilde{\mathbb{S}}_{1/t}(|\boldsymbol{p}|^n \boldsymbol{\rho})) = \text{Tr}(|\boldsymbol{p}|^n \tilde{\mathbb{S}}_{-1/t}(\boldsymbol{\rho})). \end{aligned}$$

Moreover, since $\tilde{\mathbb{T}}$ is a unitary transformation, the following identities hold

$$\begin{aligned} \text{diag}(\tilde{\mathbb{S}}_t \boldsymbol{\rho}) &= G_t(x) \varrho(x, x) G_{-t}(x) = \rho(x) \\ \|\tilde{\mathbb{S}}_{-1/t} \boldsymbol{\rho}\|_{\mathcal{L}^r} &= \|\boldsymbol{\rho}\|_{\mathcal{L}^r}. \end{aligned}$$

Then by the interpolation Inequality (22) we get

$$\begin{aligned} \|\rho\|_{L^p} &= \|\text{diag}(\tilde{S}_{-1/t}\rho)\|_{L^p} \\ &\leq \text{Tr}(|\mathbf{p}|^n \tilde{S}_{-1/t}(\rho))^{1-\theta} \|\tilde{S}_{-1/t}\rho\|_{\mathcal{L}^r}^\theta \\ &\leq t^{-n(1-\theta)} \text{Tr}(|x - t\mathbf{p}|^n \rho)^{1-\theta} \|\rho\|_{\mathcal{L}^r}^\theta. \end{aligned}$$

Finally, we remark that $n(1-\theta) = \frac{n(r'+d/n-r')}{r'+d/n} = \frac{d}{p'}$ to get the result. \square

3. PROPAGATION OF MOMENTS

3.1. Classical case. In this section, we define the classical Eulerian, velocity and space moments by

$$\begin{aligned} L_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x - t\xi|^n dx d\xi \\ M_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |\xi|^n dx d\xi \\ N_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x|^n dx d\xi. \end{aligned}$$

Proposition 3.1 (Classical large time estimate). *Let $(r, \mathbf{b}) \in [1, \infty] \times [\mathbf{b}_n, \infty]$, $\nabla K \in L^{\mathbf{b}, \infty}$ and $f \in L^\infty(\mathbb{R}_+, L^r_{x, \xi} \cap L^1_{x, \xi})$ be a nonnegative solution of Vlasov equation. Then for any $n \in 2\mathbb{N}$, there exists a constant $C = C_{d, r, n} > 0$ such that*

$$\left| \frac{dL_n}{dt} \right| \leq C \|\nabla K\|_{L^{\mathbf{b}, \infty}} M_0^{\Theta_0} \|f^{\text{in}}\|_{L^r_{x, \xi}}^{\frac{r'}{\mathbf{b}}} \frac{L_n^{1+\frac{a}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathbf{b}} - 1$ and $\Theta_0 = 1 - \frac{a}{n} - \frac{r'}{\mathbf{b}}$.

Proof. We write $f = f(t, x, \xi)$, $E = E(x)$ and we compute

$$\begin{aligned} \frac{dL_n}{dt} &= \iint_{\mathbb{R}^{2d}} |x - t\xi|^n (-\xi \cdot \nabla_x f - E \cdot \nabla_\xi f) - |x - t\xi|^{n-2} (x - t\xi) \cdot \xi f dx d\xi \\ &= -t \iint_{\mathbb{R}^{2d}} f |x - t\xi|^{n-2} (x - t\xi) \cdot E d\xi dx. \end{aligned}$$

By Hölder's inequality, we deduce for $t \geq 0$ and any $p \in (1, \infty)$

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq t \left\| \int_{\mathbb{R}^d} f |x - t\xi|^{n-1} d\xi \right\|_{L_x^p} \|E\|_{L^{p'}} \\ &\leq t^n \left\| \int_{\mathbb{R}^d} f \left| \frac{x}{t} - \xi \right|^{n-1} d\xi \right\|_{L_x^p} \|\nabla K * \rho_f\|_{L^{p'}} \\ &\leq C_K t^n \left\| \int_{\mathbb{R}^d} f(t, x, \xi + \frac{x}{t}) |\xi|^{n-1} d\xi \right\|_{L_x^p} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^q}, \end{aligned}$$

where we used Hardy-Littlewood-Sobolev's inequality with $C_K = \|\nabla K\|_{L^{\mathbf{b}, \infty}}$ and $p \in (1, \infty)$ such that

$$(23) \quad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{\mathbf{b}}.$$

Then we want to use the classical kinetic interpolation inequality which tells that for $p'_{n,k} = \left(\frac{n}{k}\right)' \left(r' + \frac{d}{n}\right)$ and $g = g(x, \xi) \geq 0$, it holds

$$(24) \quad \left\| \int_{\mathbb{R}^d} g(x, \xi) |\xi|^k d\xi \right\|_{L^{p_{n,k}}} \leq C_{d,r,n} \left(\iint_{\mathbb{R}^{2d}} g(x, \xi) |\xi|^n d\xi dx \right)^{1-r'/p'_{n,k}} \|g\|_{L^r_{x,\xi}}^{r'/p'_{n,k}}.$$

Since

$$\frac{1}{p'_{n,n-1}} + \frac{1}{p'_n} = \frac{1}{r' + d/n} \left(1 - \frac{n-1}{n} + 1\right) = \frac{n+1}{nr' + d} = \frac{1}{\mathfrak{b}_n} \geq \frac{1}{\mathfrak{b}},$$

we can choose $p \leq p_{n,n-1}$ and $q \leq p_{n,0}$ verifying (23). Take $p := p_{n,n-1}$. Then $1 < q \leq p_n$ and by interpolation

$$\left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^q} \leq M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^{p_n}}^{p'_n/q'}.$$

Using the above inequality and then the interpolation inequality (24) for $k = 0$ and $k = n-1$ yields

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f(t, x, \xi + \frac{x}{t}) |\xi|^{n-1} d\xi \right\|_{L^p_x} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^{p_n}}^{p'_n/q'} \\ &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}} + \frac{1}{n}(1-\frac{d}{\mathfrak{b}})} \|f\|_{L^r_{x,\xi}}^{r'} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi + \frac{x}{t}) |\xi|^n d\xi \right)^{1+\frac{1}{n}(\frac{d}{\mathfrak{b}}-1)}. \end{aligned}$$

With $a = \frac{d}{\mathfrak{b}} - 1$ and by a change of variable, we get

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}} - \frac{a}{n}} \|f\|_{L^r_{x,\xi}}^{r'} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) \left| \xi - \frac{x}{t} \right|^n d\xi \right)^{1+\frac{a}{n}} \\ &\leq C_{d,r,n} C_K t^{-a} M_0^{1-\frac{r'}{\mathfrak{b}} - \frac{a}{n}} \|f\|_{L^r_{x,\xi}}^{r'} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x - t\xi|^n d\xi \right)^{1+\frac{a}{n}}, \end{aligned}$$

which is the expected inequality. \square

3.2. Boundedness of Eulerian moments. We define $\tilde{\rho} := \tilde{\mathfrak{S}}_{-1/t}(\rho)$ and for $n \in 2\mathbb{N}$

$$\begin{aligned} \tilde{\rho}_n &:= \text{diag}\left(\mathbf{p}^{n/2} \tilde{\rho} \cdot \mathbf{p}^{n/2}\right) \\ l_n &:= t^n \tilde{\rho}_n. \end{aligned}$$

We also introduce the following notations for the Eulerian, velocity and space moments

$$\begin{aligned} L_n &:= \text{Tr}(|x - t\mathbf{p}|^n \rho) \\ M_n &:= \text{Tr}(|\mathbf{p}|^n \rho) \\ N_n &:= \text{Tr}(|x|^n \rho), \end{aligned}$$

as well as the corresponding moments \tilde{M}_n and \tilde{N}_n for $\tilde{\rho}$. In particular, since we have

$$\begin{aligned} L_n &= \text{Tr}(|x - t\mathbf{p}|^n \rho) = t^n \text{Tr}(|\mathbf{p} - x/t|^n \rho) \\ &= t^n \text{Tr}(|\mathbf{p}|^n \tilde{\rho}) = t^n \text{Tr}(\mathbf{p}^{n/2} \tilde{\rho} \cdot \mathbf{p}^{n/2}), \end{aligned}$$

we obtain with these notations $L_n = \int_{\mathbb{R}^d} l_n$, $M_n = \int_{\mathbb{R}^d} \rho_n$ and $N_n = \int_{\mathbb{R}^d} \rho(x) |x|^n dx$.

3.2.1. *Long time estimate.* To obtain a differential inequality which will give us the long time behavior of the solution, we first need the following time dependent interpolation inequalities.

Proposition 3.2. *Let $0 \leq k \leq n$ and $p'_{n,k} := \left(\frac{n}{k}\right)' p'_n$ and $p \in [1, p_{n,k}]$. Then for any $\alpha \leq k$, there exists a constant $C = C_{d,r,n,k} > 0$ such that*

$$(25) \quad \|\rho_k\|_{L^p} \leq C M_\alpha^{1-\theta_{n,k,\alpha}} M_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}$$

$$(26) \quad \|l_k\|_{L^p} \leq C t^{-d/p'} L_\alpha^{1-\theta_{n,k,\alpha}} L_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}$$

where

$$\theta_{n,k,\alpha} = \frac{p'_{n,\alpha}}{p'} + \frac{k-\alpha}{n-\alpha}.$$

Proof. By the kinetic interpolation inequality (22),

$$\|\rho_k\|_{L^{p_{n,k}}} \leq C_{d,r,n,k} M_n^{1-\frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}.$$

Therefore, since $p \leq p_{n,k}$, by interpolation between L^p spaces, we get

$$\begin{aligned} \|\rho_k\|_{L^p} &\leq \|\rho_k\|_{L^1}^{1-\theta} \|\rho_k\|_{L^{p_{n,k}}}^\theta \\ &\leq C M_k^{1-\theta} M_n^{\theta - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}, \end{aligned}$$

where $\theta = \theta_{n,k,k} = \frac{p'_{n,k}}{p'}$ and we used the fact that $\|\rho_k\|_{L^1} = M_k$. It already proves Inequality (25) for $k = \alpha$. Since $k \in [\alpha, n]$, we can also bound M_k in the following way

$$M_k \leq M_\alpha^{1-\frac{k-\alpha}{n-\alpha}} M_n^{\frac{k-\alpha}{n-\alpha}},$$

which yields Inequality (25). To get (26), we follow the proof of Corollary 2.1. Since \tilde{S} preserves the Schatten norms, we can write

$$\|\tilde{\rho}\|_{\mathcal{L}^r} = \|\rho\|_{\mathcal{L}^r}.$$

Hence, by replacing ρ by $\tilde{\rho}$ in the kinetic interpolation inequality (22) and multiplying by t^k , we obtain

$$\begin{aligned} \|l_k\|_{L^{p_{n,k}}} &= t^k \|\tilde{\rho}_k\|_{L^{p_{n,k}}} \leq C t^k \left(\text{Tr}(|\mathbf{p}|^n \tilde{S}_{-1/t} \rho)\right)^{1-\frac{r'}{p'_{n,k}}} \|\tilde{S}_{-1/t} \rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq C t^k \left(\text{Tr}(|\mathbf{p} - x/t|^n \rho)\right)^{1-\frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq C t^{k-n+\frac{nr'}{p'_{n,k}}} L_n^{1-\frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}. \end{aligned}$$

Next we remark that

$$k - n + \frac{nr'}{p'_{n,k}} = k - n + \left(1 - \frac{k}{n}\right) \frac{nr'}{r' + d/n} = -(n-k) \left(\frac{d/n}{r' + d/n}\right) = -\frac{d}{p'_{n,k}},$$

and we deduce Inequality (26) again by interpolation of L^p between L^1 and $L^{p_{n,k}}$ and by interpolation of L_k between L_α and L_n . \square

Proposition 3.3 (Large time estimate). *Let $(r, \mathbf{b}) \in [1, \infty] \times [\mathbf{b}_n, \infty]$, $\nabla K \in L^{\mathbf{b}, \infty}$ and $\boldsymbol{\rho} \in L^\infty(\mathbb{R}_+, \mathcal{L}^r \cap \mathcal{L}_+^1)$ be a solution of the Hartree equation. Then for any $n \in 2\mathbb{N}$, there exists a constant $C = C_{d,r,n} > 0$ such that*

$$\left| \frac{dL_n}{dt} \right| \leq C \|\nabla K\|_{L^{\mathbf{b}, \infty}} M_0^{\Theta_0} \|\boldsymbol{\rho}^{\text{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathbf{b}}} \frac{L_n^{1+\frac{a}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathbf{b}} - 1$ and $\Theta_0 = 1 - \frac{a}{n} - \frac{r'}{\mathbf{b}}$.

Proof. We first remark that by Formula (18) and spectral theory, we deduce $|x - t\mathbf{p}|^n = \mathcal{S}_t(|x|^n)$. Therefore by defining $H_0 := \frac{|\mathbf{p}|^2}{2}$, by definition of \mathcal{S}_t

$$i\hbar \partial_t (\mathcal{S}_t(|x|^n)) = [H_0, \mathcal{S}_t(|x|^n)] = [H_0, |x - t\mathbf{p}|^n].$$

Hence, by differentiating L_n with respect to time, we obtain

$$\begin{aligned} i\hbar \partial_t L_n &= \text{Tr}([H_0, |x - t\mathbf{p}|^n] \boldsymbol{\rho} + |x - t\mathbf{p}|^n [H_0 + V, \boldsymbol{\rho}]) \\ &= \text{Tr}([H_0, |x - t\mathbf{p}|^n] \boldsymbol{\rho} + [|x - t\mathbf{p}|^n, H_0 + V] \boldsymbol{\rho}) \\ &= \text{Tr}([|x - t\mathbf{p}|^n, V] \boldsymbol{\rho}). \end{aligned}$$

Then we use the operator $\tilde{\mathcal{S}}_t$ of translation in the x direction defined in (19). By formulas (20) and spectral theory, we deduce that for any $t \in \mathbb{R}$, $\tilde{\mathcal{S}}_t V = V$. Therefore, we deduce

$$\begin{aligned} i\hbar \partial_t L_n &= t^n \text{Tr}([(|\mathbf{p} - x/t|^n), V] \boldsymbol{\rho}) \\ &= t^n \text{Tr}([\tilde{\mathcal{S}}_{1/t}(|\mathbf{p}|^n), V] \boldsymbol{\rho}) \\ &= t^n \text{Tr}([\tilde{\mathcal{S}}_{1/t}(|\mathbf{p}|^n), \tilde{\mathcal{S}}_{1/t}(V)] \boldsymbol{\rho}) \\ &= t^n \text{Tr}(\tilde{\mathcal{S}}_{1/t}([|\mathbf{p}|^n, V]) \boldsymbol{\rho}) \\ &= t^n \text{Tr}([|\mathbf{p}|^n, V] \tilde{\boldsymbol{\rho}}). \end{aligned}$$

As it has been proved in [35, Equation (38)], this expression can be bounded in the following way

$$|\text{Tr}([|\mathbf{p}|^n, V] \tilde{\boldsymbol{\rho}})| \leq C_K \hbar \tilde{M}_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|\tilde{\rho}_{2|a|}\|_{L^\alpha}^{\frac{1}{2}} \|\tilde{\rho}_{2|b|}\|_{L^\beta}^{\frac{1}{2}} \|\tilde{\rho}_{2|c|}\|_{L^\gamma}^{\frac{1}{2}},$$

where $(a, b, c) \in (\mathbb{N}^d)^3$ are multi-indices with $|a| = a_1 + \dots + a_d$ and

$$(27) \quad \frac{2}{\mathbf{b}} = \frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}$$

$$C_K = C_{d,n} \|\nabla K\|_{L^{\mathbf{b}, \infty}}.$$

As in [35, Proof of Theorem 3, Step 2], we remark that for the exponents $p_{n,k}$ defined in (21) and multi-indices such that $|a + b + c| = n/2 - 1$, we have

$$\frac{1}{p'_{n,2|a|}} + \frac{1}{p'_{n,2|b|}} + \frac{1}{p'_{n,2|c|}} = \frac{1}{p'_n} \left(3 - \frac{2|a| + 2|b| + 2|c|}{n} \right) = \frac{2(n+1)}{nr' + d} = \frac{2}{\mathbf{b}_n}.$$

Therefore, since $\mathbf{b} \geq \mathbf{b}_n$, we can find $(\alpha, \beta, \gamma) \in [1, p_{n,2|a|}] \times [1, p_{n,2|b|}] \times [1, p_{n,2|c|}]$ verifying (27) and use the interpolation inequality (26) for $\alpha = 0$. By the definition

of l_k and the fact that $L_0 = M_0$, we deduce

$$\begin{aligned} \partial_t L_n &\leq C_K t^{n-n/2-(|a|+|b|+|c|)} L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|l_{2|a}\|_{L^\alpha}^{\frac{1}{2}} \|l_{2|b}\|_{L^\beta}^{\frac{1}{2}} \|l_{2|c}\|_{L^\gamma}^{\frac{1}{2}} \\ &\leq C_K t L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|l_{2|a}\|_{L^\alpha}^{\frac{1}{2}} \|l_{2|b}\|_{L^\beta}^{\frac{1}{2}} \|l_{2|c}\|_{L^\gamma}^{\frac{1}{2}} \\ &\leq (C_{d,r,n} C_K M_0^{\Theta_0} \|\rho\|_{\mathcal{L}^r}^{\Theta_1}) t^{-a} L_n^\Theta, \end{aligned}$$

where

$$\begin{aligned} a &= \frac{d}{2} \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) - 1 = \frac{d}{\mathbf{b}} - 1 \\ \Theta_0 &= \frac{1}{2} \left(3 - p'_n \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) - \frac{2|a| + 2|b| + 2|c|}{n} \right) \\ &= 1 + \frac{1}{n} - \frac{p'_n}{\mathbf{b}} = 1 - \frac{a}{n} - \frac{r'}{\mathbf{b}} \\ \Theta_1 &= \frac{r'}{2} \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) = \frac{r'}{\mathbf{b}} \\ \Theta &= \frac{1}{2} \left(1 + p'_n \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) + \frac{2|a| + 2|b| + 2|c|}{n} - r' \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) \right) \\ &= 1 + \frac{1}{n} \left(\frac{d}{\mathbf{b}} - 1 \right) = 1 + \frac{a}{n}. \end{aligned}$$

We conclude by recalling that $\|\rho\|_{\mathcal{L}^r} = \|\rho^{\text{in}}\|_{\mathcal{L}^r}$ since the Hartree equation preserves the Schatten norm. \square

3.2.2. Short time estimate. To prove the short time estimate, we will use the boundedness of M_n and N_n for short times to get the boundedness of L_n . To achieve this, we first need some lemmas to bound traces expressions with products of x and \mathbf{p} by M_n and N_n .

Lemma 3.1 (Interpolation for weighted traces). *Let $0 \leq k \leq n$ and $A = |\mathbf{p}|$ or $A = |x|$. Then for any operator $\rho \geq 0$ we have the following inequalities*

$$(28) \quad \text{Tr}(A^k \rho) \leq \text{Tr}(A^n \rho)^{\frac{k}{n}} \text{Tr}(\rho)^{1-\frac{k}{n}}.$$

Proof. By our definition of the absolute value, for two positive operators we have $|AB|^2 = BA^2B$. Therefore, the lemma follows from Hölder's inequality for the trace and Araki-Lieb-Thirring inequality [5] since

$$\text{Tr}(A^k \rho) = \left\| A^{\frac{k}{2}} \rho^{\frac{1}{2}} \right\|_2^2 \leq \left\| A^{\frac{k}{2}} \rho^{\frac{k}{2n}} \right\|_{\frac{2n}{k}}^2 \left\| \rho^{\frac{1}{2}(1-\frac{k}{n})} \right\|_{\frac{2n}{2n-k}}^2 \leq \left\| A^{\frac{n}{2}} \rho^{\frac{1}{2}} \right\|_2^{\frac{2k}{n}} \text{Tr}(\rho)^{1-\frac{k}{n}},$$

and the right-hand side here is exactly the right-hand side of Inequality (28). \square

In the more general case of mixed product of x and \mathbf{p} , for any $\mathbf{i} \in \llbracket 1, d \rrbracket^n$ we can define the set

$$\mathcal{Z}_k^n(\mathbf{i}) = \{z_{i_1} \dots z_{i_n}, \forall j \in \llbracket 1, n \rrbracket, z_{i_j} = x_{i_j} \text{ or } z_{i_j} = \mathbf{p}_{i_j}, |\{j, z_j = \mathbf{p}_j\}| = k\}.$$

of operators \mathbf{z} consisting of a product of k partial derivatives and $n - k$ multiplications by a coordinate of x , and look at the following quantities

$$\text{Tr}(\mathbf{z} \rho) = \text{Tr}(z_{i_1} \dots z_{i_n} \rho).$$

We denote by $\mathcal{Z}_k^n := \bigcup_{i \in \llbracket 1, d \rrbracket^n} \mathcal{Z}_k^n(i)$. Then we have the following Lemma.

Lemma 3.2. *Let $n \in 2\mathbb{N}$, $k \in \llbracket 1, n-1 \rrbracket$ and*

$$Z_k^n = \sup_{\mathbf{z} \in \mathcal{Z}_k^n} |\mathrm{Tr}(\mathbf{z} \rho)|.$$

Then there exist an integer $\mathcal{I} \in \llbracket 0, n \rrbracket$ and a constant $C > 0$ depending only on d , n and k such that for any operator $\rho \geq 0$

$$(29) \quad Z_k^n \leq M_n^{\frac{k}{n}} N_n^{1-\frac{k}{n}} + C \sum_{i=0}^{\mathcal{I}} (\hbar Z_{k-1}^{n-2})^{2^{-i}} M_n^{(1-2^{-i})\frac{k}{n}} N_n^{(1-2^{-i})(1-\frac{k}{n})}.$$

Moreover, $Z_0^n = N_n$ and $Z_n^n = M_n$. This implies the fact that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$(30) \quad Z_k^n \leq \varepsilon N_n + C_\varepsilon (M_n + \hbar^{\frac{n}{2}} M_0).$$

Remark 3.1. *In the case $k = 1$, since $Z_0^{n-2} = N_{n-2} \leq N_n^{\frac{n-2}{n}} M_0^{\frac{2}{n}}$, this leads to the following inequalities*

$$(31) \quad \begin{aligned} Z_1^n &\leq M_n^{\frac{1}{n}} N_n^{1-\frac{1}{n}} + C \sum_{i=0}^{\mathcal{I}} \hbar^{2^{-i}} M_0^{\frac{2^{1-i}}{n}} M_n^{\frac{1-2^{-i}}{n}} N_n^{1-\frac{1+2^{-i}}{n}} \\ &\leq C \left(M_n^{\frac{1}{n}} N_n^{1-\frac{1}{n}} + \hbar M_0^{\frac{2}{n}} N_n^{1-\frac{2}{n}} \right) \end{aligned}$$

Remark 3.2. *For $n \in 2\mathbb{N}$, by expanding $\mathrm{Tr}(|x - \mathbf{p}|^n \rho)$ with the formula*

$$(32) \quad \mathrm{Tr}(|x - \mathbf{p}|^n \rho) = \mathrm{Tr}(|x|^n \rho) + \mathrm{Tr}(|\mathbf{p}|^n \rho) + \sum_{k=1}^{n-1} \sum_{\mathbf{z} \in \mathcal{Z}_k^n} C_{\mathbf{z}} \mathrm{Tr}(\mathbf{z} \rho),$$

using the above lemma and Young's inequality, and replacing \hbar by $\hbar t$, we obtain for any $\varepsilon > 0$

$$(33) \quad \mathrm{Tr}(|x - t\mathbf{p}|^n \rho) \leq (1 + \varepsilon) \mathrm{Tr}(|x|^n \rho) + C_\varepsilon \mathrm{Tr} \left(\left(|\mathbf{p}|^n + |\hbar t|^{\frac{n}{2}} \right) \rho \right).$$

Proof. The idea of the proof is to reiterate commutation of operators z_i and Hölder's inequality for the trace. We start with the simplest case $n = 2$.

Step 1. Case $n = 2$. In this case it is sufficient to look at $|\mathrm{Tr}(x_i \mathbf{p}_j \rho)|$ for any $(i, j) \in \llbracket 1, d \rrbracket^2$. By Hölder's and Young's inequalities, we get

$$(34) \quad |\mathrm{Tr}(x_i \mathbf{p}_j \rho)| = \left| \mathrm{Tr} \left(\rho^{\frac{1}{2}} x_i \mathbf{p}_j \rho^{\frac{1}{2}} \right) \right| \leq \left\| \rho^{\frac{1}{2}} x_i \right\|_2 \left\| \mathbf{p}_j \rho^{\frac{1}{2}} \right\|_2 \leq N_2^{\frac{1}{2}} M_2^{\frac{1}{2}}.$$

Step 2. Case $n > 2$. Let $\mathbf{z} \in \mathcal{Z}_k^n$. Since we have the following commutation relations for $j \neq k \in \llbracket 1, d \rrbracket^2$

$$(35) \quad [x_j, \mathbf{p}_k] = [\mathbf{p}_j, \mathbf{p}_k] = [x_j, x_k] = [x_j, x_j] = 0$$

$$(36) \quad [x_j, \mathbf{p}_j] = i\hbar,$$

any commutation operation of the form (36) of two adjacent z_j in $\mathrm{Tr}(\mathbf{z} \rho) = \mathrm{Tr}(z_{i_1} \dots z_{i_n} \rho)$ will add a term of the form $\mathrm{Tr}(\tilde{\mathbf{z}} \rho)$ with $\tilde{\mathbf{z}} \in \mathcal{Z}_{k-1}^{n-2}$. Therefore, we will use the notation \mathcal{C}_{n-2} in the following steps to denote a term of the form

$$\mathcal{C}_{n-2} = \hbar \sum_{\tilde{\mathbf{z}} \in \mathcal{Z}_{k-1}^{n-2}} C_{\tilde{\mathbf{z}}} \mathrm{Tr}(\tilde{\mathbf{z}} \rho),$$

for some constants $C_{\bar{z}} > 0$. Hence, we can write

$$\mathrm{Tr}(\mathbf{z} \rho) \leq \mathrm{Tr}(\mathbf{p}^\alpha x^\beta \rho) + C_{n-2} \leq \mathrm{Tr}\left(\rho^{\frac{1}{2}} \mathbf{p}^\alpha x^\beta \rho^{\frac{1}{2}}\right) + C_{n-2}$$

with $(\alpha, \beta) \in (\mathbb{N}^d)^2$ such that $|\alpha| = k$ and $|\beta| = n - k$, where we use the multi-index notations $|\alpha| = \alpha_1 + \dots + \alpha_d$, $x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d}$, and $\mathbf{p}^\alpha = (-i\hbar)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$.

• If $|\alpha| \leq \frac{n}{2}$, then we can write $\beta = (\gamma, \delta)$ with $|\gamma| = \frac{n}{2} - |\alpha| \geq 0$ and $|\delta| = \frac{n}{2}$, so that by Hölder's inequality

$$\begin{aligned} \mathrm{Tr}(\mathbf{p}^\alpha x^\beta \rho) &= \mathrm{Tr}\left(\rho^{\frac{1}{2}} \mathbf{p}^\alpha x^\gamma x^\delta \rho^{\frac{1}{2}}\right) \leq \mathrm{Tr}\left(\left|\rho^{\frac{1}{2}} \mathbf{p}^\alpha x^\gamma\right|^2\right)^{\frac{1}{2}} \mathrm{Tr}\left(\left|x^\delta \rho^{\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \\ &\leq \mathrm{Tr}(\mathbf{p}^\alpha x^{2\gamma} \mathbf{p}^\alpha \rho)^{\frac{1}{2}} N_n^{\frac{1}{2}} \\ &\leq \mathrm{Tr}(\mathbf{p}^{2\alpha} x^{2\gamma} \rho)^{\frac{1}{2}} N_n^{\frac{1}{2}} + C_{n-2}^{\frac{1}{2}} N_n^{\frac{1}{2}} \end{aligned}$$

where we used the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. If $|\alpha| = \frac{n}{2}$, then $|\gamma| = 0$ and $\mathrm{Tr}(\mathbf{p}^{2\alpha} x^{2\gamma} \rho) \leq M_n$. If $|\alpha| < \frac{n}{2}$, remark that $\mathrm{Tr}(\mathbf{p}^{2\alpha} x^{2\gamma} \rho) = \mathrm{Tr}(\mathbf{p}^{\alpha_{(1)}} x^{\beta_{(1)}} \rho)$ with $\alpha_{(1)} = 2\alpha$ and $\beta_{(1)} = 2\gamma$, which is of the same form as the term $\mathrm{Tr}(x^\alpha \mathbf{p}^\beta \rho)$ we started from, but with $|\alpha_{(1)}| = 2|\alpha| > |\alpha|$. We can continue repeating this process creating in this way a sequence $(\alpha_{(i)})_{i \in \{1, \dots, j\}}$ as long as $|\alpha_{(i-1)}| < \frac{n}{2}$, and with the property that $|\alpha_{(i)}| = 2^i |\alpha|$ is increasing and $|\beta_{(i)}| = n - |\alpha_{(i)}|$ is decreasing. If the last term of this sequence $\alpha_{(j)}$ verifies $|\alpha_{(j)}| = \frac{n}{2}$, then $|\beta_{(j)}| = \frac{n}{2}$ and we get

$$\begin{aligned} \mathrm{Tr}(\mathbf{z} \rho) &\leq \mathrm{Tr}(\mathbf{p}^{\alpha_{(j)}} x^{\beta_{(j)}} \rho)^{\frac{1}{2^j}} N_n^{1 - \frac{1}{2^j}} + \sum_{i=0}^j C_{n-2}^{\frac{1}{2^i}} N_n^{1 - \frac{1}{2^i}} \\ &\leq M_n^{\frac{1}{2^{j+1}}} N_n^{1 - \frac{1}{2^{j+1}}} + \sum_{i=0}^{j+1} C_{n-2}^{\frac{1}{2^i}} N_n^{1 - \frac{1}{2^i}}, \end{aligned}$$

where C_{n-2} depends on i .

• If this is not the case, then we get $|\alpha_{(j)}| \in (\frac{n}{2}, n)$, so we are in the case of terms of the form $\mathrm{Tr}(\mathbf{p}^\alpha x^\beta \rho)$ with $|\alpha| > \frac{n}{2}$. For such a term, we do the same reasoning as in the case $|\alpha| < \frac{n}{2}$ but replacing the role of x and \mathbf{p} . This gives us a sequence $(\alpha_{(i)}, \beta_{(i)})_{i \in [j, j_2]}$ with $|\beta_{(i)}| < \frac{n}{2}$ strictly increasing, $|\alpha_{(i)}| > \frac{n}{2}$ strictly decreasing and such that

$$\mathrm{Tr}(\mathbf{p}^{\alpha_{(j)}} x^{\beta_{(j)}} \rho) \leq \mathrm{Tr}(\mathbf{p}^{\alpha_{(j_2)}} x^{\beta_{(j_2)}} \rho)^{\frac{1}{2^{j_2-j}}} M_n^{1 - \frac{1}{2^{j_2-j}}} + \sum_{i=1}^{j_2-j} C_{n-2}^{\frac{1}{2^i}} M_n^{1 - \frac{1}{2^i}}.$$

One more time, the last term verifies either $|\alpha_{(j_2)}| = \frac{n}{2}$, either $|\alpha_{(j_2)}| < \frac{n}{2}$, in which case we continue to create the sequence as we did for $i \leq j$.

• If the sequence $|\alpha_{(i)}|_{i \in \mathbb{N}}$ never converges to $\frac{n}{2}$, there is a periodic orbit. More precisely, since the sequence $(|\alpha_{(i)}|)_{i \in \mathbb{N}}$ takes values in the finite set $[1, n-1] \setminus \{\frac{n}{2}\}$, we deduce that it will come back twice at the same point, i.e. there exists $(i_0, i_1) \in \mathbb{N}^2$ such that $0 < i_0 < i_1 < n$, $\alpha_{(i_0)} = \alpha_{(i_1)}$ and

$$(37) \quad \mathrm{Tr}(\mathbf{p}^{\alpha_{(i_0)}} x^{\beta_{(i_0)}} \rho) \leq \mathrm{Tr}(\mathbf{p}^{\alpha_{(i_1)}} x^{\beta_{(i_1)}} \rho)^{2^{-i_2}} M_n^{\theta_{i_2}} N_n^{\varepsilon_{i_2}} + \sum_{i=1}^{i_2} C_{n-2}^{2^{-i}} M_n^{\theta_i} N_n^{\varepsilon_i},$$

where $i_2 = i_1 - i_0$ and $(\theta_i, \varepsilon_i) \in (0, 1)^2$ are such that

$$\theta_i + \varepsilon_i = 1 - 2^{-i} =: \tau_i.$$

From Inequality (37), using Young's inequality on the first term of the right-hand side and removing $2^{-i_2} \text{Tr}(\mathbf{p}^{\alpha(i_0)} x^{\beta(i_0)} \boldsymbol{\rho})$ on both side yields

$$\tau_{i_2} \text{Tr}(\mathbf{p}^{\alpha(i_0)} x^{\beta(i_0)} \boldsymbol{\rho}) \leq \tau_{i_2} M_n^{\frac{\theta_{i_2}}{\tau_{i_2}}} N_n^{\frac{\varepsilon_{i_2}}{\tau_{i_2}}} + \sum_{i=1}^{i_2} C_{n-2}^{2^{-i}} M_n^{\theta_i} N_n^{\varepsilon_i},$$

and this leads to

$$\begin{aligned} \text{Tr}(\mathbf{z} \boldsymbol{\rho}) &\leq \text{Tr}(\mathbf{p}^{\alpha(i_0)} x^{\beta(i_0)} \boldsymbol{\rho})^{2^{-i_0}} M_n^{\Theta_{i_0}} N_n^{\mathcal{E}_{i_0}} + \sum_{i=0}^{i_0} C_{n-2}^{2^{-i}} M_n^{\Theta_i} N_n^{\mathcal{E}_i} \\ &\leq M_n^{\Theta} N_n^{\mathcal{E}} + \sum_{i=0}^{i_1} C_{n-2}^{2^{-i}} M_n^{\Theta_i} N_n^{\mathcal{E}_i}, \end{aligned}$$

where $(\Theta_i, \mathcal{E}_i) = (\Theta_{i_0}, \mathcal{E}_{i_0}) + 2^{-i_0}(\theta_{i-i_0}, \varepsilon_{i-i_0})$ if $i > i_0$ and $\Theta = \frac{2^{-i_0} \theta_{i_2}}{\tau_{i_2}} + \Theta_{i_0}$. In particular, since $\Theta_{i_1} = \Theta_{i_0} + 2^{-i_0} \theta_{i_2}$, we deduce that

$$\Theta = \Theta_{i_0} + \frac{1}{\tau_{i_2}} (\Theta_{i_1} - \Theta_{i_0}) = \Theta_{i_1} + \frac{2^{i_0}}{2^{i_1} - 2^{i_0}} (\Theta_{i_1} - \Theta_{i_0}) > \Theta_{i_1},$$

and similarly $\mathcal{E} = 1 - \Theta > \mathcal{E}_{i_1}$.

- In all the cases, we end up with an inequality of the form

$$(38) \quad \text{Tr}(\mathbf{z} \boldsymbol{\rho}) \leq M_n^{\Theta} N_n^{\mathcal{E}} + \sum_{i=0}^{\mathcal{I}} C_{n-2}^{2^{-i}} M_n^{\Theta_i} N_n^{\mathcal{E}_i},$$

where $(\Theta_i)_{i \in [0, \mathcal{I}]}$ and $(\mathcal{E}_i)_{i \in [0, \mathcal{I}]}$ are two increasing sequences with values in $[0, 1]$ such that $\mathcal{E}_i + \Theta_i = \tau_i$, $\Theta \geq \Theta_{\mathcal{I}}$ and $\mathcal{E} \geq \mathcal{E}_{\mathcal{I}}$.

Step 3. Rescaling. Replacing \hbar by $\lambda \hbar$ for some $\lambda > 0$ and dividing Formula (38) by λ^k yields

$$\text{Tr}(\mathbf{z} \boldsymbol{\rho}) \leq \lambda^{\Theta n - k} M_n^{\Theta} N_n^{1 - \Theta} + \sum_{i=0}^{\mathcal{I}} \lambda^{\Theta_i n - \tau_i k} C_{n-2}^{2^{-i}} M_n^{\Theta_i} N_n^{\tau_i - \Theta_i}.$$

Taking $\lambda = N_n^{\frac{1}{n}} M_n^{-\frac{1}{n}}$, we arrive at the following formula

$$\text{Tr}(\mathbf{z} \boldsymbol{\rho}) \leq M_n^{\frac{k}{n}} N_n^{1 - \frac{k}{n}} + \sum_{i=0}^{\mathcal{I}} C_{n-2}^{2^{-i}} M_n^{\tau_i \frac{k}{n}} N_n^{\tau_i (1 - \frac{k}{n})}.$$

from which we deduce (29). Formula (30) then follows by an induction on n , Lemma 3.1 and Young's inequality. \square

To get a short time Eulerian moment estimate, we use [35, Theorem 3] which tells us that for any $n \in 2\mathbb{N}$ and $\mathbf{b} > \max(\mathbf{b}_4, \mathbf{b}_n)$, there exists a time

$$(39) \quad T = T_{\|\nabla K\|_{L^{\mathbf{b}, \infty}}, \|\boldsymbol{\rho}^{\text{in}}\|_{\mathcal{L}^r}, M_0, M_n^{\text{in}}, d, r, n},$$

and a positive constant m depending on ∇K , $\|\boldsymbol{\rho}^{\text{in}}\|_{\mathcal{L}^r}$, M_0 , M_n^{in} , d , r and n such that

$$(40) \quad \forall (k, t) \in [0, n] \times [0, T], \quad M_k(t) \leq m.$$

Proposition 3.4 (Short time estimate). *Assume $\hbar \in (0, 1)$ and let $n \in 2\mathbb{N} \setminus \{0\}$, $r \in [1, \infty]$,*

$$\nabla K \in L^{\mathbf{b}, \infty} \text{ for } \mathbf{b} \in (\max(\mathbf{b}_4, \mathbf{b}_n), \infty],$$

and $\rho \in L^\infty([0, T], \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^n + |\mathbf{p}|^n))$ be a solution of the Hartree equation. Then for any $t \in [0, T]$ it holds

$$L_n \leq 2^n \text{Tr}(|x|^n \rho^{\text{in}}) + C_{d,n,T} m (1 + \hbar^{\frac{n}{2}}) t^{\frac{n}{2}},$$

where T is given by (39).

Proof. We first remark that

$$\begin{aligned} [|x|^n, |\mathbf{p}|^2] &= 2i\hbar \nabla(|x|^n) \cdot \mathbf{p} - (-i\hbar)^2 \Delta(|x|^n) \\ &= ni\hbar |x|^{n-2} (2x \cdot \mathbf{p} - i\hbar(d+n-2)). \end{aligned}$$

Therefore, recalling the notation $N_n := \text{Tr}(|x|^n \rho)$ and using the fact that $|x|^n$ commutes with $V(x)$, we can compute

$$\begin{aligned} \frac{dN_n}{dt} &= \frac{1}{i\hbar} \text{Tr} \left(\left[|x|^n, \frac{|\mathbf{p}|^2}{2} + V \right] \rho \right) \\ &= \frac{n}{2} \text{Tr} \left(|x|^{n-2} (2x \cdot \mathbf{p} - i\hbar(d+n-2)) \rho \right) \\ &= n \text{Tr} \left(|x|^{n-2} x \cdot \mathbf{p} \rho \right) - \frac{n i \hbar (d+n-2)}{2} \text{Tr} \left(|x|^{n-2} \rho \right). \end{aligned}$$

But since N_n remains a real number, its derivative is the same as the derivative of its real part and we finally obtain

$$(41) \quad \frac{dN_n}{dt} = n \text{Tr} \left(|x|^{n-2} x \cdot \mathbf{p} \rho \right).$$

This quantity can be bounded using Inequality (31), leading to

$$(42) \quad \frac{dN_n}{dt} \leq C \hbar M_0^{\frac{2}{n}} N_n^{1-\frac{2}{n}} + C M_n^{\frac{1}{n}} N_n^{1-\frac{1}{n}}.$$

Since M_n is uniformly bounded on $[0, T]$ by the bound (40), we deduce by Gronwall's Lemma that N_n is also bounded uniformly on $[0, T]$. Therefore, since $\frac{n}{2} \leq n$, we can write $N_n^{\frac{n-1}{n}} \leq C_T N_n^{\frac{n-2}{n}}$ on $[0, T]$ with $C_T^n = \|N_n\|_{L^\infty(0,T)}$ depending only on the initial conditions. Thus, Inequality (42) implies $\partial_t N_n \leq C_T N_n^{\frac{n-2}{n}}$ with $C_T = \hbar M_0^{\frac{2}{n}} + C C_T m^{\frac{1}{n}}$. Therefore, using once more Gronwall's Lemma

$$(43) \quad N_n(t) \leq \left((N_n^{\text{in}})^{\frac{2}{n}} + \frac{2}{n} C_T t \right)^{\frac{n}{2}} \leq 2^{\frac{n}{2}-1} (N_n^{\text{in}} + (C_T t)^{\frac{n}{2}}).$$

Finally, by Inequality (33) with $1 + \varepsilon = 2^{1/2}$, and the fact that $t \leq T$, we obtain

$$\begin{aligned} L_n &\leq 2^{\frac{n}{2}} N_n + C_n \left(t^n M_n + |\hbar t|^{\frac{n}{2}} M_0 \right) \\ &\leq 2^n \left(N_n^{\text{in}} + t^{\frac{n}{2}} \left(C_T^{\frac{n}{2}} + C_n T^{\frac{n}{2}} m + C_n \hbar^{\frac{n}{2}} M_0 \right) \right), \end{aligned}$$

which yields the result. \square

3.2.3. *Global estimate.* To prove the first theorem, we will now combine the short time estimate, which tells that L_n is not growing fast initially, with the long time estimate which works only when L_n is not too large after a short time. Since by assumption L_n is small initially, the combination of these estimates will give us a global bound on L_n .

Proof of Theorem 1. Since $\mathfrak{b} < \frac{d}{2}$, we have $a := \frac{d}{\mathfrak{b}} - 1 > 1$. Thus, by Gronwall's Lemma and Proposition 3.3, for any $t > \tau > 0$ we obtain

$$\begin{aligned} L_n(t)^{-a/n} &\geq L_n(\tau)^{-a/n} + \frac{1}{A} \left(\frac{1}{t^{a-1}} - \frac{1}{\tau^{a-1}} \right) \\ &\geq L_n(\tau)^{-a/n} - \frac{1}{A\tau^{a-1}}, \end{aligned}$$

where

$$A = \left(1 - \frac{1}{a} \right) \frac{n}{C \|\nabla K\|_{L^{\mathfrak{b}, \infty}} M_0^{\Theta_0} \|\rho^{\text{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathfrak{b}}}}.$$

Combining the above inequality with Proposition 3.4, we know that there exists T such that for any $\tau \in (0, T]$ and $t > 0$, it holds

$$L_n(t) \leq \left((2^n N_n^{\text{in}} + C_T \tau^{\frac{n}{2}})^{-\frac{a}{n}} - \frac{1}{A\tau^{a-1}} \right)^{-n/a},$$

as soon as the right-hand side is positive and with $C_T = C_{T, N_n^{\text{in}}, M_4^{\text{in}}, M_0}$. Now we choose τ and N^{in} so that this positivity property holds. Since $\mathfrak{b} > \frac{d}{3}$, we have $2 < a' = \frac{a}{a-1}$. Thus we can define $\tau_0 := \min\left(T, (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}}\right)$ and $N := 2^{-n} \left(A^{\frac{n}{a}} \tau_0^{\frac{n}{2a'}} - C_T \tau_0^{\frac{n}{4}} \right)$. We remark that $N \geq 0$ since

$$\tau_0 \leq (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}} \implies C_T \tau_0^{\frac{n}{4} - \frac{n}{2a'}} \leq A^{\frac{n}{a}} \implies N \geq 0.$$

Taking $\tau = \tau_0$ and $N_n^{\text{in}} < N$, we obtain that

$$\begin{aligned} C_{T, N_n^{\text{in}}, M_4^{\text{in}}, M_0} &:= (2^n N_n^{\text{in}} + C_T \tau_0^{\frac{n}{4}})^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} \\ &> (N + C_T \tau_0)^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} = 0. \end{aligned}$$

We deduce that for any $t > 0$

$$L_n(t) < C_{T, N_n^{\text{in}}, M_4^{\text{in}}, M_0}^{-\frac{n}{a}},$$

which proves the result. \square

3.3. Application to the semiclassical limit. We fix now $\rho \geq 0$ a solution of the Hartree equation and we assume $\hbar \leq 1$ (or equivalently, we do not write the \hbar dependence of constants that are bounded when $\hbar \rightarrow 0$). We will now use the uniform in time estimate on L_n and again the differential inequality for N_n to obtain bounds on N_n and M_n .

To implement this strategy, first remark that by defining $\tilde{\rho} := \tilde{S}_{-1/t} \rho$ and $\tilde{M}_n := \text{Tr}(|\mathbf{p}|^n \tilde{\rho})$, then as in the proof of Corollary 2.1, it holds

$$(44) \quad t^n \tilde{M}_n = t^n \text{Tr}(|\mathbf{p}|^n \tilde{S}_{-1/t}(\rho)) = \text{Tr}(|x - t\mathbf{p}|^n \rho) = L_n,$$

while $\text{Tr}(|x|^n \tilde{\rho}) = N_n$ and $\text{Tr}(\tilde{\rho}) = M_0$ are unchanged by this transformation. Similarly, we also have

$$t^n M_n = \text{Tr}(|x + t\mathbf{p}|^n \tilde{\rho}),$$

Expanding the right-hand side of this inequality as in Equation (32) and then using Formula (44), we obtain

$$(45) \quad t^n M_n \leq N_n + L_n + C \sum_{k=1}^{n-1} t^k \tilde{Z}_k^n,$$

where $\tilde{Z}_k^n = \sup_{\mathbf{z} \in \mathcal{Z}_k^n} |\text{Tr}(\mathbf{z} \tilde{\rho})|$. From this inequality and the result of Theorem 1, we obtain the following bounds.

Proposition 3.5. *Under the hypotheses of Theorem 1, it holds*

$$\begin{aligned} N_n &\leq N_n^{\text{in}} + C(t^n + t^{\frac{n}{2}}) \\ M_n &\leq C, \end{aligned}$$

and for any $k \in \llbracket 0, n \rrbracket$, $\tilde{Z}_k^n \leq C t^{-k} \langle t \rangle^{n-k}$, where the constants $C > 0$ involved depends on $\|\nabla K\|_{L^{\mathbf{b}, \infty}}$, M_0 , M_n^{in} , $\|\rho^{\text{in}}\|_{\mathcal{L}^r}$, d , n and r .

Proof. Once again, we will proceed by induction on n .

Step 1. Case $n = 2$. In this case, conservation of energy together with the boundedness of the potential energy by M_2 as soon as $K \in L^{\frac{d+2r'}{4}, \infty} + L^\infty$ implies that M_2 is uniformly bounded independently from \hbar and t (see e.g. [35, Remark 3.1] or [36]). By Sobolev's embedding, this is always true if $\nabla K \in L^{\mathbf{b}^4}$. Then by Formula (41) and Inequality (34), we deduce

$$\frac{dN_2}{dt} = 2 \text{Tr}(x \cdot \mathbf{p} \rho) \leq 2 M_2^{\frac{1}{2}} N_2^{\frac{1}{2}},$$

which by Gronwall's inequality yields

$$N_2(t) \leq \left(N_2(0)^{\frac{1}{2}} + m_2 t \right)^2,$$

where $m_2 = \|M_2\|_{L^\infty(\mathbb{R}_+)}$. Then, by Formula (44), we obtain

$$\text{Tr}(x \cdot \mathbf{p} \tilde{\rho}) \leq 2 \tilde{M}_2^{\frac{1}{2}} \tilde{N}_2^{\frac{1}{2}} = 2 t^{-1} L_2^{\frac{1}{2}} N_2^{\frac{1}{2}} \leq C t^{-1} \langle t \rangle,$$

from which we easily deduce that $\tilde{Z}_1^2 \leq C t^{-1} \langle t \rangle$. Finally, $\tilde{Z}_0^2 = N_2 \leq C \langle t \rangle^2$ and $\tilde{Z}_2^2 = \tilde{M}_2 = L_2 t^{-2}$.

Step 2. Case $n > 2$. We go back to Equation (41) which together with Inequality (29) with $k = 1$ and the fact that by our induction hypothesis $Z_0^{n-2} = N_{n-2} \leq C \langle t \rangle^{n-2}$ implies that

$$(46) \quad \begin{aligned} \frac{dN_n}{dt} &\leq n \left(M_n^{\frac{1}{n}} N_n^{1-\frac{1}{n}} + C \sum_{i=0}^n \left(\hbar \langle t \rangle^{n-2} \right)^{2^{-i}} M_n^{\frac{1-2^{-i}}{n}} N_n^{(1-2^{-i})(1-\frac{1}{n})} \right) \\ &\leq n M_n^{\frac{1}{n}} N_n^{1-\frac{1}{n}} + C \left(\langle t \rangle^{n-2} + \langle t \rangle^{\varepsilon(n-2)} M_n^{\frac{1-\varepsilon}{n}} N_n^{(1-\varepsilon)(1-\frac{1}{n})} \right), \end{aligned}$$

with $\varepsilon = 2^{-n}$. Remark that we can replace N_n by $N := 1 + N_n$ in Inequality (46). From Theorem 1, we know that L_n is bounded by a constant, hence by Inequality (45), we can bound M_n by

$$(47) \quad M_n \leq \frac{N}{t^n} + \frac{C}{t^n} + C \sum_{k=1}^{n-1} t^{k-n} \tilde{Z}_k^n.$$

To control \tilde{Z}_k^n , we use Inequality (29) with $\tilde{\rho}$. Together with the the induction hypothesis on \tilde{Z}_{k-1}^{n-2} , the fact that $\tilde{M}_n \leq C t^{-n}$ and the fact that $\tilde{N}_n = N_n \leq N$, we get

$$(48) \quad \begin{aligned} \tilde{Z}_k^n &\leq C t^{-k} N^{1-\frac{k}{n}} + C \sum_{i=0}^n t^{2^{-i}-k} \langle t \rangle^{2^{-i}(n-1-k)} N^{(1-2^{-i})(1-\frac{k}{n})} \\ &\leq C \left(t^{-k} N^{1-\frac{k}{n}} + t^{1-k} \langle t \rangle^{n-1-k} + t^{\varepsilon-k} \langle t \rangle^{\varepsilon(n-1-k)} N^{(1-\varepsilon)(1-\frac{k}{n})} \right). \end{aligned}$$

where the second inequality follows from Young's inequality for the product. Putting these inequalities back into Formula (47) and using again Young's inequality, we obtain for any $t \geq \tau > 0$

$$(49) \quad M_n \leq \frac{N}{t^n} + C_\tau \left(\frac{N^{1-\frac{1}{n}}}{t^n} + \frac{1}{t} \right)$$

for some constant C_τ depending on τ . Together with (46) and the fact that $t \geq \tau$ and $N \geq 1$, this implies

$$\frac{dN}{dt} \leq n \frac{N}{t} + C_\tau \left(\frac{N^{1-\frac{1}{n^2}}}{t} + \frac{N^{1-\frac{1}{n}}}{t^{\frac{1}{n}}} + t^{n-2} + \frac{N^{1-\varepsilon}}{t^{1-\varepsilon(n-1)}} + \frac{N^{(1-\varepsilon)(1-\frac{1}{n})}}{t^{\frac{1}{n}-\varepsilon(n-2-\frac{1}{n})}} \right).$$

or equivalently, for $\bar{N} = N t^{-n}$,

$$\frac{d\bar{N}}{dt} \leq C_\tau \left(\frac{\bar{N}^{1-\frac{1}{n^2}}}{t^{1+\frac{1}{n}}} + \frac{\bar{N}^{1-\frac{1}{n}}}{t^{1+\frac{1}{n}}} + t^{-2} + \frac{\bar{N}^{1-\varepsilon}}{t^{1+\varepsilon}} + \frac{\bar{N}^{(1-\varepsilon)(1-\frac{1}{n})}}{t^{(1+\varepsilon)(1+\frac{1}{n})}} \right) =: F(\bar{N}, t)$$

with $F \in C^1((0, \infty) \times [\tau, \infty))$. Taking $y(t)$ as the solution of $y'(t) = F(y(t), t)$ with initial condition $y(\tau) = \bar{N}(\tau)$, we see that $y'(t) \geq 0$, hence $y(t) \geq y(\tau) > 0$. Therefore, up to multiplying by a constant depending on τ , we can keep only the biggest powers of y and t in $F(y, t)$ if we just want to bound y' by above. This leads to

$$y'(t) \leq C_\tau \frac{y(t)^{1-\varepsilon}}{t^{1+\varepsilon}}$$

where we used the fact that $\varepsilon^{-1} = 2^n \geq n^2 \geq n$ since $n \geq 4$. This implies that for any $t > \tau$, $y(t)^\varepsilon \leq y(\tau)^\varepsilon + C_\tau$ is bounded if $y(\tau)$ is bounded. However, we already know by Inequality (43) that $N_n(t) \leq N_n^{\text{in}} + C_T t^{\frac{n}{2}}$ for any $t \in [0, T]$ for some $T > 0$. Therefore, taking for example $\tau = T$, we obtain that $y(\tau) = (N_n(T) + 1) T^{-n}$ is bounded. Therefore, y is bounded for $t \geq T$ and also N_n since

$$\begin{aligned} N_n(t) &\leq N_n^{\text{in}} + C_T t^{\frac{n}{2}} && \text{if } t \leq T \\ N_n(t) &= \bar{N}(t) t^n - 1 \leq y(t) t^n \leq C_\tau t^n && \text{if } t \geq T. \end{aligned}$$

The bound on M_n is then an immediate consequence of Inequality (49) for large times and the fact that M_n is bounded on $[0, T]$, while the bound on \tilde{Z}_k^n is a consequence of Formula (48). \square

Actually, it is sufficient to use the condition of smallness of moments for $n = 4$ to get a global propagation of higher moments as soon as $\mathfrak{b}_4 > \mathfrak{b}_n$ (which corresponds to $r > \frac{d}{d-1}$). This leads to the following proposition.

Proposition 3.6. *Under the condition of Theorem 2, $M_n \in L_{\text{loc}}^\infty(\mathbb{R}_+)$ and more precisely, there exists $c_n = c_{d,n,r} \geq 0$ and $C > 0$ depending on the initial conditions such that*

$$M_n \leq C \langle t \rangle^{c_n}.$$

Proof of the proposition and of Theorem 2. Since $\mathfrak{b} \geq \mathfrak{b}_4$ and $\mathfrak{b} \geq \frac{d}{3}$, we can use Proposition 3.5 for $n = 4$, and deduce

$$M_4 \leq C,$$

for a given $C > 0$. This already proves the result in the case $n = 4$, so that we assume now that $n \geq 6$. Then, we use Formula (44) from [35], which reads

$$(50) \quad \frac{dM_n}{dt} \leq C_{d,r,n} C_K \|\rho^{\text{in}}\|_{\mathcal{L}^r}^{\Theta_2} M_{n-2}^{\Theta_0} M_n^{\Theta},$$

with

$$\begin{aligned} \Theta &= 1 + \frac{n-1}{2} \left(\frac{\mathfrak{b}_{n-2}}{\mathfrak{b}} - 1 \right), \\ \Theta_0 &= (1-\varepsilon) \left(\frac{3}{2} - \frac{r'}{p'_{n-2}} \right) \\ \Theta_2 &= \frac{3}{2} - \Theta_1 - \Theta_0, \end{aligned}$$

where

$$\varepsilon = \frac{nr' + d}{(n-2)r' + 3d} \left(\frac{(n-2)r' + d}{\mathfrak{b}} - (n-2) \right).$$

In particular, since $r \geq d'$, then \mathfrak{b}_n is a non-increasing sequence and we deduce that for any $n \geq 6$, $\mathfrak{b} > \mathfrak{b}_4 \geq \mathfrak{b}_{n-2}$, which implies that $\Theta < 1$. We then obtain Inequality (7) by Gronwall's Lemma and by induction over $n \in 2\mathbb{N}$. From this bound, Formula (8) about N_n can be deduced by using again Inequality (42) and Gronwall's Lemma. Finally, since we know by Theorem 1 that L_4 is bounded, the asymptotic behavior of ρ in Formula (9) is a consequence of Corollary 2.1. The other inequalities follows from Proposition 3.5. \square

Proof of Theorem 5. The hypotheses of Theorem 2 are fulfilled with $r = \infty$, thus we deduce the existence of nonnegative constants c and ν_M such that

$$M_n \leq \nu_M \langle t \rangle^c.$$

Therefore, we can use [35, Proposition 5.3], which tells us that for any $(n_0, n) \in (2\mathbb{N})^2$ verifying $d < n_0 \leq (1 - \frac{1}{\mathfrak{b}})n + 1 - \frac{d}{\mathfrak{b}}$, it holds

$$(51) \quad \begin{aligned} c_{d,n_0} \|\rho(t)\|_{L^\infty} &\leq \|\rho(t)\|_{\mathcal{L}^\infty(m)} \\ &\leq 2^{n_0} \left(\|\rho^{\text{in}}\|_{\mathcal{L}^\infty(m)} + C_{\rho^{\text{in}}} \left(t + \int_0^t M_n^{1-\frac{1}{\mathfrak{b}}} \right)^{n_0} \right) \\ &\leq 2^{n_0} \left(\|\rho^{\text{in}}\|_{\mathcal{L}^\infty(m)} + C_{\nu_M, \rho^{\text{in}}} \langle t \rangle^{n_0(1+\frac{c}{\mathfrak{b}'})} \right), \end{aligned}$$

where $m = 1 + \sum_{i=1}^d \mathbf{p}_i^{n_0}$ and $C_{\rho^{\text{in}}} = (C_{d,n,n_0} \|\nabla K\|_{L^b} (1 + M_0))^{n_0} \|\rho^{\text{in}}\|_{\mathcal{L}^\infty}^{1 + \frac{n_0}{b}}$. This proves (14). As in [27, Section 4], we then define the time dependent coupling $\gamma = \gamma(t, z)$ with $z = (x, \xi)$ as the solution to the Cauchy problem

$$\partial_t \gamma = (-v \cdot \nabla_x - E \cdot \nabla_\xi) \gamma + \frac{1}{i\hbar} [H, \gamma],$$

with initial condition $\gamma^{\text{in}} \in \mathcal{C}_\hbar(f^{\text{in}}, \rho^{\text{in}})$. As proved in [27, Lemma 4.2], the coupling property is preserved by the dynamics, and so $\gamma \in \mathcal{C}(f(t), \rho(t))$. We also define with the notations of (10) the quantity

$$\mathcal{E}_\hbar = \mathcal{E}_\hbar(t) := \int_{\mathbb{R}^{2d}} \text{Tr}(\mathbf{c}_\hbar(z) \gamma(z)) dz,$$

so that by the definition (10) of $W_{2,\hbar}$, we have

$$(52) \quad W_{2,\hbar}(f(t), \rho(t))^2 \leq \mathcal{E}_\hbar.$$

Moreover, remark that by [27, Theorem 2.4], the left-hand side here is bigger or equal to $d\hbar$, so that $\mathcal{E}_\hbar \geq d\hbar$. Then, as in [35, Proof of Proposition 6.3], we obtain

$$\frac{d\mathcal{E}_\hbar}{dt} \leq \mathcal{E}_\hbar + \sqrt{2\mathcal{E}_\hbar} \left(C_1^2 (\mathcal{E}_\hbar + d\hbar) + C_2^2 \left(\mathcal{E}_\hbar + \frac{1}{4} \mathcal{E}_\hbar \ln(\mathcal{E}_\hbar)^2 \right) \right)^{\frac{1}{2}}$$

with $C_1 = C_K \max(\|\rho(t)\|_{L^\infty}, \|\rho_f(t)\|_{L^\infty})^{\frac{1}{2}} \|\rho(t)\|_{L^\infty}^{\frac{1}{2}}$ and $C_2 = C_K \|\rho(t)\|_{L^\infty}$ with $C_K = C_d \|\nabla K\|_{B_{1,\infty}^1}$. Using the facts that $\mathcal{E}_\hbar \geq d\hbar$ and $C_2 \leq C_1$, and dividing the inequality by \mathcal{E}_\hbar , this leads to

$$\frac{d \ln(\mathcal{E}_\hbar)}{dt} \leq 1 + \left(6C_1^2 + \frac{C_2^2}{2} \ln(\mathcal{E}_\hbar)^2 \right)^{\frac{1}{2}} \leq 1 + \sqrt{6} C_1 + C_2 |\ln(\mathcal{E}_\hbar)|.$$

Remark that the right-hand side is a Lipschitz function of $\ln(\mathcal{E}_\hbar)$. Denoting by $y(t, y_0)$ the solution of $\partial_t y = 1 + \sqrt{6} C_1 + C_2 |y|$ with initial condition y_0 , we deduce by Inequality (52) and comparison of ordinary differential equations that

$$W_{2,\hbar}(f(t), \rho(t))^2 \leq \mathcal{E}_\hbar(t) \leq e^{y(t, \ln(\mathcal{E}_\hbar(0)))}.$$

Since $y(t, \cdot)$ is a non-decreasing function, minimizing over $\mathcal{E}_\hbar(0)$ yields

$$(53) \quad W_{2,\hbar}(f(t), \rho(t))^2 \leq e^{y(t, 2 \ln(W_{2,\hbar}(f(t), \rho(t))))}.$$

Recalling that $1 + \sqrt{6} C_1$ and C_2 are bounded by above by a function of the form $b(t) = C^{\text{in}} \langle t \rangle^{n_0(1 + \frac{c}{b'})}$, we find by solving the equation $y' = b(t)(1 + |y|)$ that

$$\begin{aligned} y(t) &\leq (y(0) + 1) e^{B(t)} - 1 && \text{if } y(0) \geq 0 \\ y(t) &\leq y(0) e^{-B(t)} + 1 - e^{-B(t)} && \text{if } y(0) \leq 0 \text{ and } y(t) \leq 0 \\ y(t) &\leq \frac{e^{B(t)}}{1 - y(0)} - 1 && \text{if } y(0) \leq 0 \text{ and } y(t) \geq 0 \end{aligned}$$

where $B(t) = \int_0^t b(s) ds$ and $C^{\text{in}} = 1 + C_d \|\nabla K\|_{B_{1,\infty}^1} \sup_t \frac{\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}}{\langle t \rangle^{n_0(1 + c/b')}}$ depend only on the initial conditions by Inequality (51). Remark that in the last case, we have $0 \leq (y(0) - 1) e^{-B(t)} + 1$, thus we obtain that $y(t) \leq e^{B(t)} - 1 \leq (y(0) - 1) e^{-B(t)} + e^{B(t)}$. Therefore, we can summarize the inequalities for any values of $y(0)$ by

$$y(t) \leq y(0) e^{\text{sign}(y(0))B(t)} + e^{B(t)}.$$

We conclude by combining this inequality with Formula (53). \square

4. ACKNOWLEDGMENTS

This work has been supported by Université Paris-Dauphine, PSL Research University.

REFERENCES

- [1] L. Ambrosio, A. Figalli, G. Friesecke, J. Giannoulis, and T. Paul. Semiclassical Limit of Quantum Dynamics with Rough Potentials and Well-posedness of Transport Equations with Measure Initial Data. *Communications on Pure and Applied Mathematics*, 64(9):1199–1242, 2011.
- [2] L. Ambrosio, G. Friesecke, and J. Giannoulis. Passage from Quantum to Classical Molecular Dynamics in the Presence of Coulomb Interactions. *Communications in Partial Differential Equations*, 35(8):1490–1515, July 2010.
- [3] L. Amour, M. Khodja, and J. Nourrigat. The classical limit of the Heisenberg and time-dependent Hartree–Fock equations: the Wick symbol of the solution. *Mathematical Research Letters*, 20(1):119–139, Jan. 2013.
- [4] L. Amour, M. Khodja, and J. Nourrigat. The Semiclassical Limit of the Time Dependent Hartree–Fock Equation: the Weyl Symbol of the Solution. *Analysis & PDE*, 6(7):1649–1674, 2013.
- [5] H. Araki. On an inequality of Lieb and Thirring. *Letters in Mathematical Physics*, 19(2):167–170, 1990.
- [6] A. Athanassoulis, T. Paul, F. Pezzotti, and M. Pulvirenti. Strong Semiclassical Approximation of Wigner Functions for the Hartree Dynamics. *Rendiconti Lincei - Matematica e Applicazioni*, 22(4):525–552, 2011. arXiv: 1009.0470.
- [7] V. Bach, S. Breteaux, S. Petrat, P. Pickl, and T. Tzaneteas. Kinetic Energy Estimates for the Accuracy of the Time-Dependent Hartree-Fock Approximation with Coulomb Interaction. *Journal de Mathématiques Pures et Appliquées*, 105(1):1–30, Jan. 2016.
- [8] C. Bardos and P. Degond. Global Existence for the Vlasov-Poisson Equation in 3 Space Variables with Small Initial Data. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, 2(2):101–118, 1985.
- [9] C. Bardos, L. Erdős, F. Golse, N. J. Mauser, and H.-T. Yau. Derivation of the Schrödinger–Poisson Equation from the Quantum N-body Problem. *Comptes Rendus Mathématique*, 334(6):515–520, Jan. 2002.
- [10] C. Bardos, F. Golse, and N. J. Mauser. Weak Coupling Limit of the N-particle Schrödinger Equation. *Methods and Applications of Analysis*, 7(2):275–294, 2000.
- [11] N. Benedikter, V. Jaksic, M. Porta, C. Saffirio, and B. Schlein. Mean-Field Evolution of Fermionic Mixed States. *Communications on Pure and Applied Mathematics*, 69(12):2250–2303, 2016.
- [12] N. Benedikter, M. Porta, C. Saffirio, and B. Schlein. From the Hartree Dynamics to the Vlasov Equation. *Archive for Rational Mechanics and Analysis*, 221(1):273–334, 2016.
- [13] N. Benedikter, M. Porta, and B. Schlein. Mean-field Evolution of Fermionic Systems. *Communications in Mathematical Physics*, 331(3):1087–1131, Nov. 2014.
- [14] F. Brezzi and P. A. Markowich. The Three-Dimensional Wigner-Poisson Problem: Existence, Uniqueness and Approximation. *Mathematical Methods in the Applied Sciences*, 14(1):35–61, Jan. 1991.
- [15] F. Castella. L2 Solutions to the Schrödinger–Poisson System: Existence, Uniqueness, Time Behaviour, and Smoothing Effects. *Mathematical Models and Methods in Applied Sciences*, 7(08):1051–1083, 1997.
- [16] F. Castella. Propagation of space moments in the Vlasov-Poisson Equation and further results. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 16(4):503–533, July 1999.
- [17] R. DiPerna and P.-L. Lions. Solutions globales d'équations du type Vlasov-Poisson. *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, 307(12):655–658, 1988.
- [18] R. J. DiPerna and P.-L. Lions. Global weak solutions of kinetic equations. *Rend. Sem. Mat. Univ. Politec. Torino*, 46(3):259–288, 1988. Publisher: Citeseer.

- [19] J. Dolbeault and G. Rein. Time-dependent rescalings and Lyapunov functionals for the Vlasov-Poisson and Euler-Poisson systems, and for related models of kinetic equations, fluid dynamics and quantum physics. *Mathematical Models and Methods in Applied Sciences*, 11(03):407–432, Apr. 2001.
- [20] L. Erdős and H.-T. Yau. Derivation of the Nonlinear Schrödinger Equation from a Many Body Coulomb System. *Advances in Theoretical and Mathematical Physics*, 5(6):1169–1205, 2001.
- [21] J. Fröhlich and A. Knowles. A Microscopic Derivation of the Time-Dependent Hartree-Fock Equation with Coulomb Two-Body Interaction. *Journal of Statistical Physics*, 145(1):23, Oct. 2011. arXiv: 0810.4282.
- [22] J. Fröhlich, A. Knowles, and S. Schwarz. On the Mean-Field Limit of Bosons with Coulomb Two-Body Interaction. *Communications in Mathematical Physics*, 288(3):1023–1059, June 2009. arXiv: 0805.4299.
- [23] I. Gasser, R. Illner, P. A. Markowich, and C. Schmeiser. Semiclassical, $t \rightarrow \infty$ asymptotics and dispersive effects for Hartree-Fock systems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 32(6):699–713, 1998.
- [24] J. Ginibre and G. Velo. On a Class of non Linear Schrödinger Equations with non Local Interaction. *Mathematische Zeitschrift*, 170(2):109–136, 1980.
- [25] J. Ginibre and G. Velo. The Global Cauchy Problem for the non Linear Schrödinger Equation Revisited. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 2(4):309–327, July 1985.
- [26] F. Golse, C. Mouhot, and T. Paul. On the Mean Field and Classical Limits of Quantum Mechanics. *Communications in Mathematical Physics*, 343(1):165–205, 2016. arXiv: 1502.06143.
- [27] F. Golse and T. Paul. The Schrödinger Equation in the Mean-Field and Semiclassical Regime. *Archive for Rational Mechanics and Analysis*, 223(1):57–94, 2017.
- [28] F. Golse and T. Paul. Wave Packets and the Quadratic Monge-Kantorovich Distance in Quantum Mechanics. *Comptes Rendus Mathématique*, 356(2):177–197, Feb. 2018.
- [29] F. Golse and T. Paul. Empirical Measures and Quantum Mechanics: Applications to the Mean-Field Limit. *Communications in Mathematical Physics*, 1711:arXiv:1711.08350, Mar. 2019.
- [30] F. Golse, T. Paul, and M. Pulvirenti. On the Derivation of the Hartree Equation from the N-Body Schrödinger Equation: Uniformity in the Planck Constant. *Journal of Functional Analysis*, 275(7):1603–1649, Oct. 2018.
- [31] S. Graffi, A. Martinez, and M. Pulvirenti. Mean-Field Approximation of Quantum Systems and Classical Limit. *Mathematical Models and Methods in Applied Sciences*, 13(01):59–73, Jan. 2003. arXiv: math-ph/0205033.
- [32] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud. Homogenization Limits and Wigner Transforms. *Communications on Pure and Applied Mathematics*, 50(4):323–379, 1997.
- [33] N. Hayashi and T. Ozawa. Smoothing Effect for Some Schrödinger Equations. *Journal of Functional Analysis*, 85(2):307–348, Aug. 1989.
- [34] R. Illner, P. F. Zweifel, and H. Lange. Global Existence, Uniqueness and Asymptotic Behaviour of Solutions of the Wigner-Poisson and Schrödinger-Poisson Systems. *Mathematical Methods in the Applied Sciences*, 17(5):349–376, Apr. 1994.
- [35] L. Lafèche. Propagation of Moments and Semiclassical Limit from Hartree to Vlasov Equation. *Journal of Statistical Physics*, 177(1):20–60, July 2019.
- [36] P.-L. Lions and T. Paul. Sur les mesures de Wigner. *Revista Matemática Iberoamericana*, 9(3):553–618, 1993.
- [37] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Inventiones Mathematicae*, 105(2):415–430, 1991.
- [38] G. Loeper. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *Journal de Mathématiques Pures et Appliquées*, 86(1):68–79, July 2006.
- [39] P. A. Markowich and N. J. Mauser. The Classical Limit of a Self-Consistent Quantum Vlasov Equation. *Mathematical Models and Methods in Applied Sciences*, 3(01):109–124, Feb. 1993.
- [40] D. Mitrouskas, S. Petrat, and P. Pickl. Bogoliubov corrections and trace norm convergence for the Hartree dynamics. *Reviews in Mathematical Physics*, page 1950024, Feb. 2019. arXiv: 1609.06264.

- [41] C. Pallard. Moment Propagation for Weak Solutions to the Vlasov–Poisson System. *Communications in Partial Differential Equations*, 37(7):1273–1285, July 2012.
- [42] C. Pallard. Space Moments of the Vlasov–Poisson System: Propagation and Regularity. *SIAM Journal on Mathematical Analysis*, 46(3):1754–1770, Jan. 2014.
- [43] B. Perthame and P.-L. Lions. Time decay, propagation of low moments and dispersive effects for kinetic equations. *Communications in Partial Differential Equations*, 21(3-4):801–806, Jan. 1996.
- [44] S. Petrat. Hartree Corrections in a Mean-field Limit for Fermions with Coulomb Interaction. *Journal of Physics A: Mathematical and Theoretical*, 50(24):244004, 2017.
- [45] S. Petrat and P. Pickl. A New Method and a New Scaling for Deriving Fermionic Mean-Field Dynamics. *Mathematical Physics, Analysis and Geometry*, 19(1):3, Mar. 2016.
- [46] K. Pfaffelmoser. Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data. *Journal of Differential Equations*, 95(2):281–303, Feb. 1992.
- [47] P. Pickl. A Simple Derivation of Mean Field Limits for Quantum Systems. *Letters in Mathematical Physics*, 97(2):151–164, Aug. 2011.
- [48] M. Porta, S. Rademacher, C. Saffirio, and B. Schlein. Mean Field Evolution of Fermions with Coulomb Interaction. *Journal of Statistical Physics*, 166(6):1345–1364, 2017.
- [49] I. Rodnianski and B. Schlein. Quantum Fluctuations and Rate of Convergence Towards Mean Field Dynamics. *Communications in Mathematical Physics*, 291(1):31–61, Oct. 2009. arXiv: 0711.3087.
- [50] C. Saffirio. Mean-Field Evolution of Fermions with Singular Interaction. In D. Cadamuro, M. Duell, W. Dybalski, and S. Simonella, editors, *Macroscopic Limits of Quantum Systems*, Springer Proceedings in Mathematics & Statistics, pages 81–99, Cham, 2018. Springer International Publishing. arXiv: 1801.02883.
- [51] J. Schaeffer. Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. *Communications in Partial Differential Equations*, 16(8-9):1313–1335, Jan. 1991.