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GLOBAL SEMICLASSICAL LIMIT FROM HARTREE TO VLASOV EQUATION FOR CONCENTRATED INITIAL DATA

LAURENT LAFLECHE^{1,2}

ABSTRACT. We prove a quantitative and **global in time** semiclassical limit from the Hartree to the Vlasov equation in the case of a singular interaction potential in dimension $d \geq 3$, including the case of a Coulomb singularity in dimension $d = 3$. This result holds for initial data concentrated enough in the sense that some space moments are initially sufficiently small. As an intermediate result, we also obtain quantitative semiclassical bounds on the space and velocity moments of even order and the asymptotic behaviour of the spatial density due to dispersion effects.

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1. INTRODUCTION

The equation governing the dynamics of a large number of interacting particles of density $f = f(t, x, \xi)$ in the phase space is the Vlasov equation

$$\text{(Vlasov)} \quad \partial_t f = -\xi \cdot \nabla_x f - E \cdot \nabla_\xi f,$$

where $E = -\nabla V$ is the force field corresponding to the mean field potential

$$V(x) = (K * \rho_f)(x) = \int_{\mathbb{R}^d} K(x-y) \rho_f(y) dy,$$

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where we denote by $\rho_f(x) := \int_{\mathbb{R}^d} f(x, v) dv$ the spatial density. Its counterpart in quantum mechanics is the following Hartree equation

$$\text{(Hartree)} \quad i\hbar \partial_t \rho = [H, \rho],$$

where ρ is a self-adjoint Hilbert-Schmidt operator called the density operator and the Hamiltonian is defined by

$$H = -\frac{\hbar^2}{2} \Delta + V.$$

In this formula, the potential is defined by $V = K * \rho$ where the spatial density ρ is defined as the diagonal of the kernel $\rho(x, y)$ of the operator ρ , i.e. $\rho(x) = \rho(x, x)$.

In this paper, we study in a quantitative way the limit when $\hbar \rightarrow 0$ of [\(Hartree\)](#) equation which is known to converge to the [\(Vlasov\)](#) equation. The question of the derivation of this equation from the quantum mechanics is a very active topic of research. Non-constructive results in weak topologies have indeed already been proved, including the case of Coulomb interactions, starting from the work of Lions and Paul [\[27\]](#) and Markowich and Mauser [\[30\]](#). See also [\[25, 18, 24, 2, 1\]](#).

Some more precise quantitative results have also more recently been proved for smooth forces which are always at least Lipschitz in [\[5, 3, 4, 11, 19\]](#). In [\[21\]](#), Golse and Paul introduce a pseudo-distance on the model of the Wasserstein-(Monge-Kantorovitch) between classical phase space densities and quantum density operators to get a rate of convergence for the semiclassical limit for Lipschitz forces. This strategy have been used in the recent paper [\[26\]](#) of the present author to extend this result to more singular interactions, but only up to a fixed time in the case of potentials with a strong singularity such as the Coulomb interaction.

We also mention the work of Porta et al [\[38\]](#) and Saffirio [\[40\]](#) about the question of the mean-field limit for the Schrödinger equation to the Hartree equation for Fermions since this limit is coupled with a semiclassical limit. Results are proved for the Coulomb interaction under assumptions of propagation of regularity along the Hartree dynamics which is still an open problem. Other results about the mean-field limit can be found in [\[9, 15, 8\]](#) where non-quantitative results are established for the Coulomb potential, and more precise limits can be found in [\[39, 37, 19, 31, 21, 20, 23\]](#) for Bosons and in [\[17, 16, 12, 10, 6, 36, 38, 35\]](#) for Fermions.

Here, we extends the results of [\[26\]](#) by proving a global in time semiclassical limit under a smallness condition of space moments. We first prove a global in time bound on some modified space moments, from which we obtain the propagation of space and velocity moments. The bound on the velocity is then sufficient to use the theory already used in the above mentioned paper to get a global L^∞ bound on the spatial density and the quantitative semiclassical limit in the quantum Wasserstein pseudo-distance.

The fact that the time decay due to the dispersion properties gives global estimates for the Vlasov equation was already used in [\[7\]](#). The modified space moments of order 2 are linked to a Lyapunov functional reminiscent of the conservation of energy, see [\[34, 14\]](#). The propagation of modified space moments was investigated further in [\[13, 32, 33\]](#).

1.1. **Main results.** We first define the quantum version of the phase space Lebesgue and weighted Lebesgue spaces as

$$\begin{aligned}\mathcal{L}^p &:= \left\{ \rho \in \mathcal{B}(L^2), \|\rho\|_{\mathcal{L}^p} := h^{-d/p'} \operatorname{Tr}(|\rho|^p)^{\frac{1}{p}} < \infty \right\}, \\ \mathcal{L}_+^p &:= \{ \rho \in \mathcal{L}^p, \rho = \rho^* \geq 0 \} \\ \mathcal{L}^p(m) &:= \{ \rho \in \mathcal{L}^p, \rho m \in \mathcal{L}^p \}.\end{aligned}$$

We also define the quantum probability measures by

$$\mathcal{P} := \{ \rho \in \mathcal{L}_+^1, \operatorname{Tr}(\rho) = 1 \}.$$

We will denote by $\mathbf{p} := -i\hbar\nabla$ the quantum impulsion, which is an unbounded operator on L^2 .

Our first result states that if the spatial density is concentrated enough, then some kinetic moments are bounded globally in time.

Theorem 1. *Let $d \geq 3$, $n \in 2\mathbb{N}$, $r \in [1, \infty]$ and define $\mathbf{b}_n := \frac{nr'+d}{n+1}$. Assume*

$$(1) \quad \nabla K \in L^{\mathbf{b}, \infty} \text{ with } \mathbf{b} \in \left(\max\left(\frac{d}{3}, \mathbf{b}_4, \mathbf{b}_n\right), \frac{d}{2} \right),$$

and let ρ be a solution of (Hartree) equation with initial condition

$$\rho^{\text{in}} \in \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^n + |\mathbf{p}|^n).$$

Then there exists an explicit constant $\mathcal{C} = \mathcal{C}(M_0, \operatorname{Tr}(|\mathbf{p}|^n \rho^{\text{in}}), \|\nabla K\|_{L^{\mathbf{b}, \infty}}, \|\rho^{\text{in}}\|_{\mathcal{L}^r}) > 0$ such that if

$$(2) \quad \operatorname{Tr}(|x|^n \rho^{\text{in}}) < \mathcal{C},$$

then

$$\operatorname{Tr}(|x - t\mathbf{p}|^n \rho) \in L^\infty(\mathbb{R}_+).$$

Remark 1.1. *The theorem applies in particular in the case of interaction kernels K with a singularity like the Coulomb interaction. For example for any $\varepsilon > 0$*

$$(3) \quad K(x) = \frac{\pm 1}{|x|} \mathbf{1}_{|x| \leq 1} + \frac{\pm 1}{|x|^{1+\varepsilon}} \mathbf{1}_{|x| > 1}.$$

Remark 1.2. *An other good example of potentials verifying the assumptions of the theorem are potentials of the form*

$$(4) \quad K(x) = \frac{\pm 1}{|x|^a},$$

with $a = \frac{d}{6} - 1 \in (1, \frac{8}{7})$. In dimension $d = 4$, $d = 5$ and $d \geq 6$, one can even better take respectively $a \in (1, \frac{3}{2})$, $a \in (1, \frac{16}{9})$ and $a \in (1, 2)$

Remark 1.3. *Since $\rho \in \mathcal{L}_+^1$, it is an Hilbert-Schmidt operator that can be written as a integral operator of kernel $\rho(x, y)$ and it can also be diagonalized by the spectral theorem. Hence, we can write for any $\varphi \in L^2$*

$$\rho \varphi(x) = \int_{\mathbb{R}^d} \rho(x, y) \varphi(y) dy = \sum_{j \in J} \lambda_j |\psi_j\rangle \langle \psi_j | \varphi\rangle,$$

where $(\psi_j)_{j \in J} \in (L^2)^J$ with $J \subset \mathbb{N}$ is an orthogonal basis. The space density can then be written

$$\rho(x) := \rho(x, x) = \sum_{j \in J} \lambda_j |\varphi_j(x)|^2,$$

and the space moments

$$\mathrm{Tr}(|x|^n \boldsymbol{\rho}^{\mathrm{in}}) = \int_{\mathbb{R}^d} \rho(x) |x|^n dx.$$

We can state the analogue of this theorem for solutions of the (Vlasov) equation

Proposition 1.1. *Let $d \geq 3$, $n \in 2\mathbb{N}$, $r \in [1, \infty]$ and assume ∇K verifies condition (1). Let f be a solution of (Vlasov) equation with nonnegative initial condition*

$$f^{\mathrm{in}} \in L_{x,\xi}^r \cap L^1(1 + |x|^n + |\xi|^n).$$

Then there exists an explicit constant $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}}(x, \xi) |x|^n dx d\xi < \mathcal{C},$$

then

$$\iint_{\mathbb{R}^{2d}} f(\cdot, x, \xi) |x - t\xi|^n dx d\xi \in L^\infty(\mathbb{R}_+).$$

We can use the first theorem to obtain good estimates on the space and velocity moments and on the spatial density that do not depend on \hbar .

Theorem 2. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N}) \setminus \{0, 2\}$ and assume*

$$(5) \quad \nabla K \in L^{\mathbf{b}, \infty} \text{ with } \mathbf{b} \in \left(\max\left(\mathbf{b}_4, \frac{d}{3}\right), \frac{d}{2} \right),$$

and let $\boldsymbol{\rho}$ be a solution of (Hartree) equation with initial condition

$$\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^4 + |\mathbf{p}|^n),$$

for a given even integer $n \geq 4$. Then there exists $\mathcal{C} = \mathcal{C}(M_0, M_4^{\mathrm{in}}, \|\nabla K\|_{L^{\mathbf{b}, \infty}}, \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r})$ such that if

$$\mathrm{Tr}(|x|^4 \boldsymbol{\rho}^{\mathrm{in}}) \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that

$$(6) \quad \mathrm{Tr}(|\mathbf{p}|^n \boldsymbol{\rho}) \leq C(1 + t^{c_n})$$

$$(7) \quad \mathrm{Tr}(|x|^n \boldsymbol{\rho}) \leq C(\mathrm{Tr}(|x|^n \boldsymbol{\rho}^{\mathrm{in}}) + 1 + t^{n(c_n+1)})$$

$$(8) \quad \|\rho\|_{L^p} \leq C t^{-d/p'},$$

where $p' = r' + \frac{d}{4}$.

Remark 1.4. *The constant c_4 can be taken arbitrarily close to 0.*

Again, we can state the analogue result for the Vlasov equation.

Proposition 1.2. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N}) \setminus \{0, 2\}$ and assume K verifies (5). Let f be a solution of (Vlasov) equation with nonnegative initial condition*

$$f^{\mathrm{in}} \in L_{x,\xi}^r \cap L_{x,\xi}^1(1 + |x|^4 + |\mathbf{p}|^n),$$

for a given even integer $n \geq 4$. Then there exists $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}}(x, \xi) |x|^4 dx d\xi \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |\xi|^n dx d\xi &\leq C(1 + t^{c_n}) \\ \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x|^n dx d\xi &\leq C(1 + t^{n(c_n+1)}) \\ \|\rho_f\|_{L^p} &\leq Ct^{-d/p'}, \end{aligned}$$

where $p' = r' + \frac{d}{4}$.

Before stating the result about the semiclassical limit, we recall the definition of the semiclassical Wasserstein-(Monge-Kantorovitch) distance introduced by Golse and Paul in [21]. We say that $\gamma \in L^1(\mathbb{R}^{2d}, \mathcal{P})$ is a semiclassical coupling between a classical kinetic density $f \in L^1 \cap \mathcal{P}(\mathbb{R}^{2d})$ and a density operator $\rho \in \mathcal{P}$ and we write $\gamma \in \mathcal{C}(f, \rho)$ when

$$\begin{aligned} \text{Tr}(\gamma(z)) &= f(z) \\ \int_{\mathbb{R}^{2d}} \gamma(z) dz &= \rho. \end{aligned}$$

Then we define the semiclassical Wasserstein-(Monge-Kantorovitch) pseudo-distance in the following way

$$(9) \quad W_{2,\hbar}(f, \rho) := \left(\inf_{\gamma \in \mathcal{C}(f, \rho)} \int_{\mathbb{R}^{2d}} \text{Tr}(\mathbf{c}_\hbar(z)\gamma(z)) dz \right)^{\frac{1}{2}},$$

where $\mathbf{c}_\hbar(z)\varphi(y) = (|x-y|^2 + |\xi-\mathbf{p}|^2)\varphi(y)$, $z = (x, \xi)$ and $\mathbf{p} = -i\hbar\nabla_y$. This is not a distance but it is comparable to the classical Wasserstein distance W_2 between the Wigner transform of the quantum density operator and the normal kinetic density, in the sense of the following Theorem

Theorem 3 (Golse & Paul [21]). *Let $\rho \in \mathcal{P}$ and $f \in \mathcal{P}(\mathbb{R}^{2d})$ be such that*

$$\int_{\mathbb{R}^{2d}} f(x, \xi)(|x|^2 + |\xi|^2) dx d\xi < \infty.$$

Then one has $W_{2,\hbar}(f, \rho)^2 \geq d\hbar$ and for the Husimi transform \tilde{f}_\hbar of ρ , it holds

$$(10) \quad W_2(f, \tilde{f}_\hbar)^2 \leq W_{2,\hbar}(f, \rho)^2 + d\hbar.$$

See also [22] for more results about this pseudo-distance.

Our last theorem uses these results to obtain the semiclassical limit. We also recall the following theorem which will give us our assumptions on the classical solution of the (Vlasov) equation

Theorem 4 (Lions & Perthame [28], Loeper [29]). *Assume $f^{\text{in}} \in L_{x,\xi}^\infty(\mathbb{R}^6)$ verify*

$$(11) \quad \iint_{\mathbb{R}^6} f^{\text{in}} |\xi|^{n_0} dx d\xi < C \text{ for a given } n_0 > 6,$$

and for all $R > 0$,

$$(12) \quad \sup_{(y,w) \in \mathbb{R}^6} \text{ess} \{f^{\text{in}}(y + t\xi, w), |x-y| \leq Rt^2, |\xi-w| \leq Rt\} \in L_{\text{loc}}^\infty(\mathbb{R}_+, L_x^\infty L_\xi^1).$$

Then there exists a unique solution to the (Vlasov) equation with initial condition $f_{t=0} = f^{\text{in}}$. Moreover, in this case, the spatial density verifies $\rho_f \in L_{\text{loc}}^\infty(\mathbb{R}_+, L^\infty)$.

Theorem 5. *Let $d \geq 3$ and assume*

$$\nabla K \in B_{1,\infty}^1 \cap L^{\mathfrak{b}} \text{ with } \mathfrak{b} \in \left(\max\left(\mathfrak{b}_4, \frac{d}{3}\right), \frac{d}{2} \right).$$

Let ρ be a solution of (Hartree) equation with initial condition ρ^{in} verifying

$$\begin{aligned} \rho^{\text{in}} &\in \mathcal{L}^\infty \cap \mathcal{L}_+^1(1 + |\mathbf{p}|^{n_1}) \\ \forall i \in \llbracket 1, d \rrbracket, \mathbf{p}_i^{n_0} \rho^{\text{in}} &\in \mathcal{L}^\infty, \end{aligned}$$

where $\mathbf{p}_i := -i\hbar\partial_i$ and $(n_0, n) \in (2\mathbb{N})^2$ is such that

$$\begin{aligned} n_0 &> d \\ n &\geq \frac{d + \mathfrak{b}(n_0 - 1)}{\mathfrak{b} - 1}. \end{aligned}$$

Let f is a solution of the (Vlasov) equation with initial condition f^{in} verifying the hypotheses of Theorem 4 and with the same mass as ρ . Then there exists $\mathcal{C} = \mathcal{C}(M_0, \text{Tr}(|\mathbf{p}|^4 \rho^{\text{in}}), \|\nabla K\|_{L^{\mathfrak{b},\infty}}, \|\rho^{\text{in}}\|_{\mathcal{L}^\infty})$ such that if

$$\text{Tr}(|x|^4 \rho^{\text{in}}) \leq \mathcal{C},$$

then

$$(13) \quad \rho \in L_{\text{loc}}^\infty(\mathbb{R}_+, L^\infty).$$

Moreover, the following semiclassical estimate holds

$$W_{2,\hbar}(f(t), \rho(t)) \leq \max\left(\sqrt{d\hbar}, W_{2,\hbar}(f^{\text{in}}, \rho^{\text{in}}) e^{t/\sqrt{2}} e^{\lambda(e^{t/\sqrt{2}} - 1)}\right),$$

with

$$\lambda = C_d \left(1 + \frac{\|\nabla K\|_{B_{1,\infty}^1}}{1 + t^{n_0(1+c_n/\mathfrak{b}')}} \sup_{t \in \mathbb{R}_+} (\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}) \right),$$

where c_n is given by (6).

Again, the additional assumption $\nabla K \in B_{1,\infty}^1$ is compatible with a kernel with a Coulomb singularity in dimension $d = 3$ such as the one given in Remark 1.1.

2. FREE TRANSPORT

We want to use the time decay properties of the kinetic free transport equation which writes for $f = f(t, x, \xi)$

$$\partial_t f + \xi \cdot \nabla_x f = 0.$$

In quantum mechanics, free transport is given by the free Schrödinger equation

$$i\hbar \partial_t \psi = H_0 \psi,$$

where $\hbar = \frac{\hbar}{2\pi}$ and $H_0 = -\frac{\hbar^2 \Delta}{2}$ which can be written $H_0 = \frac{|\mathbf{p}|^2}{2}$ with the notation $\mathbf{p} = -i\hbar \nabla$. The solution corresponding to the initial condition ψ^{in} can be written $T_t \psi^{\text{in}}$ where the semigroup T_t is given by

$$(14) \quad T_t \psi = e^{-itH_0/\hbar} \psi = \frac{e^{-i\pi|x|^2/(ht)}}{(iht)^{d/2}} * \psi.$$

The corresponding equation for density operators $\rho \in \mathcal{P}$ is

$$(15) \quad i\hbar \partial_t \rho = [H_0, \rho],$$

whose solution is $S_t \boldsymbol{\rho}^{\text{in}}$ where the semigroup S_t is defined by

$$(16) \quad S_t \boldsymbol{\rho} := T_t \boldsymbol{\rho} T_{-t}.$$

As it can be easily noticed, it holds $T_t^* = T_t^{-1} = T_{-t}$ and for any $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathcal{P}^2$, $S(\boldsymbol{\rho}_1 \boldsymbol{\rho}_2) = S(\boldsymbol{\rho}_1) S(\boldsymbol{\rho}_2)$. Moreover, a straightforward computation shows that

$$(17) \quad \begin{aligned} S_t \mathbf{p} &= \mathbf{p} \\ S_t x &= x - t\mathbf{p}. \end{aligned}$$

By the spectral theory, it implies that $S_t(f(x)) = f(x - t\mathbf{p})$ for any nice function f . By analogy, we can define the operator of translation of the impulsion \mathbf{p} by

$$(18) \quad \begin{aligned} \tilde{T}_t \psi(x) &:= e^{\frac{-\pi|x|^2 t}{i\hbar}} \psi(x) = G_t(x) \psi(x) \\ \tilde{S}_t \boldsymbol{\rho} &:= \tilde{T}_t \boldsymbol{\rho} \tilde{T}_{-t}, \end{aligned}$$

which verifies the equation

$$i\hbar \partial_t (\tilde{S}_t \boldsymbol{\rho}) = \left[\frac{-|x|^2}{2}, \tilde{S}_t \boldsymbol{\rho} \right],$$

and the two following relations

$$(19) \quad \begin{aligned} \tilde{S}_t x &= x \\ \tilde{S}_t \mathbf{p} &= \mathbf{p} - tx. \end{aligned}$$

We recall the quantum kinetic interpolation inequality that was already used in [26, Theorem 6]. For $k \in 2\mathbb{N}$ we define

$$\rho_k := \sum_{j \in J} \lambda_j |\mathbf{p}^{\frac{k}{2}} \psi_j|^2 = \text{diag}(\mathbf{p}^{\frac{k}{2}} \boldsymbol{\rho} \cdot \mathbf{p}^{\frac{k}{2}}),$$

and for $r \geq 1$ and $0 \leq k \leq n$, we define the exponent $p_{n,k}$ by its Hölder conjugate

$$(20) \quad p'_{n,k} = \binom{n}{k}' p'_n \text{ with } p'_n = \left(r' + \frac{d}{n} \right).$$

Then the following inequality holds

Proposition 2.1. *Let $(k, n) \in (2\mathbb{N})^2$ be such that $k \leq n$ and $\boldsymbol{\rho} \in \mathcal{L}^1(|\boldsymbol{\rho}|^n) \cap \mathcal{L}_+^r$ for a given $r \in [1, \infty]$. Then there exists $C = C_{d,r,n,k} > 0$ such that*

$$(21) \quad \|\rho_k\|_{L^{p_{n,k}}} \leq C \text{Tr}(|\mathbf{p}|^n \boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta,$$

where $\theta = \frac{r'}{p'_{n,k}}$.

Corollary 2.1. *Let $n \in 2\mathbb{N}$, $r \in [1, \infty]$, $p' := r' + \frac{d}{n}$ and $\theta = \frac{r'}{p'}$. Then*

$$\|\rho\|_{L^p} \leq \frac{1}{t^{d/p'}} \text{Tr}(|x - t\mathbf{p}|^n \boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta.$$

Proof of Corollary 2.1. We just remark that by formula (19), we get

$$\begin{aligned} t^{-n} \text{Tr}(|x - t\mathbf{p}|^n \boldsymbol{\rho}) &= \text{Tr}(|\mathbf{p} - x/t|^n \boldsymbol{\rho}) \\ &= \text{Tr}(\tilde{S}_{1/t}(|\mathbf{p}|^n) \boldsymbol{\rho}) = \text{Tr}(|\mathbf{p}|^n \tilde{S}_{-1/t}(\boldsymbol{\rho})). \end{aligned}$$

Moreover, the following also holds

$$\begin{aligned} \text{diag}(\tilde{S}_t \boldsymbol{\rho}) &= G_t(x) \boldsymbol{\rho}(x, x) G_{-t}(x) = \boldsymbol{\rho}(x) \\ \|\tilde{S}_{-1/t} \boldsymbol{\rho}\|_{\mathcal{L}^r} &= \|\boldsymbol{\rho}\|_{\mathcal{L}^r}. \end{aligned}$$

Then by the interpolation inequality (21) we get

$$\begin{aligned} \|\rho\|_{L^p} &= \|\text{diag}(\tilde{S}_{-1/t}\boldsymbol{\rho})\|_{L^p} \\ &\leq \text{Tr}(|\boldsymbol{p}|^n \tilde{S}_{-1/t}(\boldsymbol{\rho}))^{1-\theta} \|\tilde{S}_{-1/t}\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta \\ &\leq t^{-n(1-\theta)} \text{Tr}(|x - t\boldsymbol{p}|^n \boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^\theta. \end{aligned}$$

Finally, we remark that $n(1-\theta) = \frac{n(r'+d/n-r')}{r'+d/n} = \frac{d}{p'}$ to get the result. \square

3. PROPAGATION OF MOMENTS

3.1. Classical case. In this section, we define the classical kinetic, velocity and space moments by

$$\begin{aligned} L_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x - t\xi|^n dx d\xi \\ M_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |\xi|^n dx d\xi \\ N_n &:= \iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x|^n dx d\xi. \end{aligned}$$

Proposition 3.1 (Classical large time estimate). *Let $(r, \mathbf{b}) \in [1, \infty] \times [\mathbf{b}_n, \infty]$, $\nabla K \in L^{\mathbf{b}, \infty}$ and $f \in L^\infty(\mathbb{R}_+, L^r_{x, \xi} \cap L^1_{x, \xi})$ be a nonnegative solution of (Vlasov) equation. Then for any $n \in 2\mathbb{N}$, there exists a constant $C = C_{d, r, n} > 0$ such that*

$$\left| \frac{dL_n}{dt} \right| \leq C \|\nabla K\|_{L^{\mathbf{b}, \infty}} M_0^{\Theta_0} \|f^{\text{in}}\|_{L^r_{x, \xi}}^{\frac{r'}{\mathbf{b}}} \frac{L_n^{1+\frac{a}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathbf{b}} - 1$ and $\Theta_0 = 1 - \frac{a}{n} - \frac{r'}{\mathbf{b}}$.

Proof. We write $f = f(t, x, \xi)$, $E = E(x)$ and we compute

$$\begin{aligned} \frac{dL_n}{dt} &= \iint_{\mathbb{R}^{2d}} |x - t\xi|^n (-\xi \cdot \nabla_x f - E \cdot \nabla_\xi f) - |x - t\xi|^{n-2} (x - t\xi) \cdot \xi f dx d\xi \\ &= -t \iint_{\mathbb{R}^{2d}} f |x - t\xi|^{n-2} (x - t\xi) \cdot E d\xi dx. \end{aligned}$$

By Hölder's inequality, we deduce for $t \geq 0$ and any $p \in [1, \infty]$

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq t \left\| \int_{\mathbb{R}^d} f |x - t\xi|^{n-1} d\xi \right\|_{L^p_x} \|E\|_{L^{p'}} \\ &\leq t^n \left\| \int_{\mathbb{R}^d} f \left| \frac{x}{t} - \xi \right|^{n-1} d\xi \right\|_{L^p_x} \|\nabla K * \rho_f\|_{L^{p'}} \\ &\leq C_K t^n \left\| \int_{\mathbb{R}^d} f(t, x, \xi + \frac{x}{t}) |\xi|^{n-1} d\xi \right\|_{L^p_x} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^q}, \end{aligned}$$

where we used Hardy-Littlewood-Sobolev's inequality with $C_K = \|\nabla K\|_{L^{\mathbf{b}, \infty}}$ and

$$(22) \quad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{\mathbf{b}}.$$

Then we want to use the classical kinetic interpolation inequality which tells that for $p'_{n,k} = \left(\frac{n}{k}\right)' \left(r' + \frac{d}{n}\right)$ and $g = g(x, \xi) \geq 0$, it holds

$$(23) \quad \left\| \int_{\mathbb{R}^d} g(x, \xi) |\xi|^k d\xi \right\|_{L^{p_{n,k}}} \leq C_{d,r,n} \left(\iint_{\mathbb{R}^{2d}} g(x, \xi) |\xi|^n d\xi dx \right)^{1-r'/p'_{n,k}} \|g\|_{L^\infty_{x,\xi}}^{r'/p'_{n,k}}.$$

Since

$$\frac{1}{p'_{n,n-1}} + \frac{1}{p'_n} = \frac{1}{r' + d/n} \left(1 - \frac{n-1}{n} + 1\right) = \frac{n+1}{nr' + d} = \frac{1}{\mathfrak{b}_n} > \frac{1}{\mathfrak{b}},$$

we can choose $p \leq p_{n,n-1}$ and $q \leq p_{n,0}$ verifying (22). Take $p := p_{n,n-1}$. Then $1 < q < p_n$ and by interpolation

$$\left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^q} \leq M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^{p_n}}^{p'_n/q'}.$$

Using the above inequality and then the interpolation inequality (23) for $k = 0$ and $k = n-1$ yields

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f(t, x, \xi + \frac{x}{t}) |\xi|^{n-1} d\xi \right\|_{L^p_x} \left\| \int_{\mathbb{R}^d} f d\xi \right\|_{L^{p_n}}^{p'_n/q'} \\ &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}} + \frac{1}{n}(1-\frac{d}{\mathfrak{b}})} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi + \frac{x}{t}) |\xi|^n d\xi \right)^{1+\frac{1}{n}(\frac{d}{\mathfrak{b}}-1)}. \end{aligned}$$

With $a = \frac{d}{\mathfrak{b}} - 1$ and by a change of variable, we get

$$\begin{aligned} \left| \frac{dL_n}{dt} \right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}} - \frac{a}{n}} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) \left| \xi - \frac{x}{t} \right|^n d\xi \right)^{1+\frac{a}{n}} \\ &\leq C_{d,r,n} C_K t^{-a} M_0^{1-\frac{r'}{\mathfrak{b}} - \frac{a}{n}} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) |x - t\xi|^n d\xi \right)^{1+\frac{a}{n}}, \end{aligned}$$

which is the expected inequality. \square

3.2. Boundedness of kinetic moments. We define $\tilde{\rho} := \tilde{S}_{-1/t}(\rho)$ and for $n \in 2\mathbb{N}$

$$\begin{aligned} \tilde{\rho}_n &:= \text{diag}(\mathbf{p}^{n/2} \tilde{\rho} \cdot \mathbf{p}^{n/2}) \\ l_n &:= t^n \tilde{\rho}_n. \end{aligned}$$

We also introduce the following notations for the kinetic, velocity and space moments

$$\begin{aligned} L_n &:= \text{Tr}(|x - t\mathbf{p}|^n \rho) \\ M_n &:= \text{Tr}(|\mathbf{p}|^n \rho) \\ N_n &:= \text{Tr}(|x|^n \rho), \end{aligned}$$

as well as the corresponding moments \tilde{M}_n and \tilde{N}_n for $\tilde{\rho}$. In particular, since we have

$$\begin{aligned} L_n &= \text{Tr}(|x - t\mathbf{p}|^n \rho) = t^n \text{Tr}(|\mathbf{p} - x/t|^n \rho) \\ &= t^n \text{Tr}(|\mathbf{p}|^n \tilde{\rho}) = t^n \text{Tr}(\mathbf{p}^{n/2} \tilde{\rho} \cdot \mathbf{p}^{n/2}), \end{aligned}$$

we obtain with these notations $L_n = \int_{\mathbb{R}^d} l_n$, $M_n = \int_{\mathbb{R}^d} \rho_n$ and $N_n = \int_{\mathbb{R}^d} \rho(x) |x|^n dx$. The following interpolation inequalities hold.

Proposition 3.2. *Let $0 \leq k \leq n$ and $p'_{n,k} := \left(\frac{n}{k}\right)' p'_n$ and $p \in [1, p_{n,k}]$. Then for any $\alpha \leq k$, there exists a constant $C = C_{d,r,n,k} > 0$ such that*

$$(24) \quad \|\rho_k\|_{L^p} \leq CM_\alpha^{1-\theta_{n,k,\alpha}} M_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}$$

$$(25) \quad \|l_k\|_{L^p} \leq Ct^{-d/p'} L_\alpha^{1-\theta_{n,k,\alpha}} L_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}$$

where

$$\theta_{n,k,\alpha} = \frac{p'_{n,\alpha}}{p'} + \frac{k-\alpha}{n-\alpha}.$$

Proof. By the kinetic interpolation inequality (21)

$$\|\rho_k\|_{L^{p_{n,k}}} \leq C_{d,r,n,k} M_n^{1 - \frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}.$$

Therefore, since $p \leq p_{n,k}$, by interpolation between L^p spaces, we get

$$\begin{aligned} \|\rho_k\|_{L^p} &\leq \|\rho_k\|_{L^1}^{1-\theta} \|\rho_k\|_{L^{p_{n,k}}}^\theta \\ &\leq CM_k^{1-\theta} M_n^{\theta - \frac{r'}{p'}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'}}, \end{aligned}$$

where $\theta = \theta_{n,k,k} = \frac{p'_{n,k}}{p'}$ and we used the fact that $\|\rho_k\|_{L^1} = M_k$. It already proves inequality (24) for $k = \alpha$. Since $k \in [\alpha, n]$, we can also bound M_k in the following way

$$M_k \leq M_\alpha^{1 - \frac{k-\alpha}{n-\alpha}} M_n^{\frac{k-\alpha}{n-\alpha}},$$

which yields inequality (24). To get (25), we follow the proof of Corollary 2.1. Since $\tilde{\mathcal{S}}$ preserves the Schatten norms, we can write

$$\|\tilde{\rho}\|_{\mathcal{L}^r} = \|\rho\|_{\mathcal{L}^r}.$$

Hence, by replacing ρ by $\tilde{\rho}$ in the kinetic interpolation inequality (21) and multiplying by t^k , we obtain

$$\begin{aligned} \|l_k\|_{L^{p_{n,k}}} &= t^k \|\tilde{\rho}_k\|_{L^{p_{n,k}}} \leq Ct^k (\text{Tr}(|\mathbf{p}|^n \tilde{\mathcal{S}}_{-1/t} \rho))^{1 - \frac{r'}{p'_{n,k}}} \|\tilde{\mathcal{S}}_{-1/t} \rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq Ct^k (\text{Tr}(|\mathbf{p} - x/t|^n \rho))^{1 - \frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq Ct^{k-n + \frac{nr'}{p'_{n,k}}} L_n^{1 - \frac{r'}{p'_{n,k}}} \|\rho\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}. \end{aligned}$$

Next we remark that

$$\begin{aligned} k - n + \frac{nr'}{p'_{n,k}} &= k - n + \left(1 - \frac{k}{n}\right) \frac{nr'}{r' + d/n} = -(n-k) \left(1 - \frac{r'}{r' + d/n}\right) \\ &= -(n-k) \left(\frac{d/n}{r' + d/n}\right) = -\frac{d}{p'_{n,k}}, \end{aligned}$$

and we deduce inequality (25) again by interpolation of L^p between L^1 and $L^{p_{n,k}}$ and by interpolation of L_k between L_α and L_n . \square

Proposition 3.3 (Large time estimate). *Let $(r, \mathbf{b}) \in [1, \infty] \times [\mathbf{b}_n, \infty]$, $\nabla K \in L^{\mathbf{b}, \infty}$ and $\rho \in L^\infty(\mathbb{R}_+, \mathcal{L}^r \cap \mathcal{L}_+^1)$ be a solution of (Hartree) equation. Then for any $n \in 2\mathbb{N}$, there exists a constant $C = C_{d,r,n} > 0$ such that*

$$\left| \frac{dL_n}{dt} \right| \leq C \|\nabla K\|_{L^{\mathbf{b}, \infty}} M_0^{\Theta_0} \|\rho^{\text{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathbf{b}}} \frac{L_n^{1+\frac{\mathbf{a}}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathbf{b}} - 1$ and $\Theta_0 = 1 - \frac{\mathbf{a}}{n} - \frac{r'}{\mathbf{b}}$.

Proof. We first remark that by formula (17) and spectral theory, we deduce $|x - t\mathbf{p}|^n = S_t(|x|^n)$. Therefore by defining $H_0 := \frac{|\mathbf{p}|^2}{2}$, by definition of S_t

$$i\hbar \partial_t (S_t(|x|^n)) = [H_0, S_t(|x|^n)] = [H_0, |x - t\mathbf{p}|^n].$$

Hence, by differentiating L_n with respect to time, we obtain

$$\begin{aligned} i\hbar \partial_t L_n &= \text{Tr}([H_0, |x - t\mathbf{p}|^n] \rho + |x - t\mathbf{p}|^n [H_0 + V, \rho]) \\ &= \text{Tr}([H_0, |x - t\mathbf{p}|^n] \rho + [|x - t\mathbf{p}|^n, H_0 + V] \rho) \\ &= \text{Tr}([|x - t\mathbf{p}|^n, V] \rho). \end{aligned}$$

Then we use the operator \tilde{S}_t of translation in the x direction defined in (18). By formulas (19) and spectral theory, we deduce that for any $t \in \mathbb{R}$, $\tilde{S}_t V = V$. Therefore, we deduce

$$\begin{aligned} i\hbar \partial_t L_n &= t^n \text{Tr}([(|\mathbf{p} - x/t|^n), V] \rho) \\ &= t^n \text{Tr}([\tilde{S}_{1/t}(|\mathbf{p}|^n), V] \rho) \\ &= t^n \text{Tr}([\tilde{S}_{1/t}(|\mathbf{p}|^n), \tilde{S}_{1/t}(V)] \rho) \\ &= t^n \text{Tr}(\tilde{S}_{1/t}([|\mathbf{p}|^n, V]) \rho) \\ &= t^n \text{Tr}([|\mathbf{p}|^n, V] \tilde{\rho}). \end{aligned}$$

As it has been proved in [26, Proof of Theorem 3, Step 1], this expression can be bounded in the following way

$$\frac{1}{i\hbar} \text{Tr}([|\mathbf{p}|^n, V] \tilde{\rho}) \leq C_K \tilde{M}_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|\tilde{\rho}_{2|a|}\|_{L^\alpha}^{\frac{1}{2}} \|\tilde{\rho}_{2|b|}\|_{L^\beta}^{\frac{1}{2}} \|\tilde{\rho}_{2|c|}\|_{L^\gamma}^{\frac{1}{2}},$$

where $(a, b, c) \in (\mathbb{N}^d)^3$ are multi-indices with $|a| = a_1 + \dots + a_d$ and

$$(26) \quad \begin{aligned} \frac{2}{\mathbf{b}} &= \frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \\ C_K &= C_{d,n} \|\nabla K\|_{L^{\mathbf{b}, \infty}}. \end{aligned}$$

As in [26, Proof of Theorem 3, Step 2], we remark that for the exponents $p_{n,k}$ defined in (20) and multi-indices such that $|a + b + c| = n/2 - 1$, we have

$$\frac{1}{p'_{n,2|a|}} + \frac{1}{p'_{n,2|b|}} + \frac{1}{p'_{n,2|c|}} = \frac{1}{p'_n} \left(3 - \frac{2|a| + 2|b| + 2|c|}{n} \right) = \frac{2(n+1)}{nr' + d} = \frac{2}{\mathbf{b}_n}.$$

Therefore, since $\mathbf{b} \geq \mathbf{b}_n$, we can find $(\alpha, \beta, \gamma) \in [1, p_{n,2|a|}] \times [1, p_{n,2|b|}] \times [1, p_{n,2|c|}]$ verifying (26) and use the interpolation inequality (25) for $\alpha = 0$. By the definition

of l_k and the fact that $L_0 = M_0$, we deduce

$$\begin{aligned} \partial_t L_n &\leq C_K t^{n-n/2-(|a|+|b|+|c|)} L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|l_{2|a}| \|_{L^\alpha}^{\frac{1}{2}} \|l_{2|b}| \|_{L^\beta}^{\frac{1}{2}} \|l_{2|c}| \|_{L^\gamma}^{\frac{1}{2}} \\ &\leq C_K t L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \|l_{2|a}| \|_{L^\alpha}^{\frac{1}{2}} \|l_{2|b}| \|_{L^\beta}^{\frac{1}{2}} \|l_{2|c}| \|_{L^\gamma}^{\frac{1}{2}} \\ &\leq (C_{d,r,n} C_K M_0^{\Theta_0} \|\rho\|_{\mathcal{L}^r}^{\Theta_1}) t^{-a} L_n^\Theta, \end{aligned}$$

where

$$\begin{aligned} a &= \frac{d}{2} \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) - 1 = \frac{d}{\mathbf{b}} - 1 \\ \Theta_0 &= \frac{1}{2} \left(3 - p'_n \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) - \frac{2|a| + 2|b| + 2|c|}{n} \right) \\ &= 1 + \frac{1}{n} - \frac{p'_n}{\mathbf{b}} = 1 - \frac{a}{n} - \frac{r'}{\mathbf{b}} \\ \Theta_1 &= \frac{r'}{2} \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) = \frac{r'}{\mathbf{b}} \\ \Theta &= \frac{1}{2} \left(1 + p'_n \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) + \frac{2|a| + 2|b| + 2|c|}{n} - r' \left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \right) \right) \\ &= 1 + \frac{1}{n} \left(\frac{d}{\mathbf{b}} - 1 \right) = 1 + \frac{a}{n}. \end{aligned}$$

We conclude by recalling that $\|\rho\|_{\mathcal{L}^r} = \|\rho^{\text{in}}\|_{\mathcal{L}^r}$ since the Hartree equation preserves the Schatten norm. \square

To prove the short time estimate, we first need the following lemma.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon = C_{n,\varepsilon}$ such that for any operator $\rho \geq 0$ and any $t \geq 0$*

$$\text{Tr}(|x - t\mathbf{p}|^{2n} \rho) \leq (1 + \varepsilon) \text{Tr}(|x|^{2n} \rho) + C_\varepsilon \text{Tr}((|t\mathbf{p}|^{2n} + |\hbar t|^n) \rho).$$

Proof. We can assume that $n \geq 1$. We first write

$$\begin{aligned} \text{Tr}(|x - t\mathbf{p}|^{2n} \rho) &= \text{Tr}((|x|^2 - t\mathbf{p} \cdot x - tx \cdot \mathbf{p} + t^2|\mathbf{p}|^2)^n \rho) \\ (27) \quad &= \text{Tr}(|x|^{2n} \rho) + \text{Tr}(|t\mathbf{p}|^{2n} \rho) \\ &\quad + \sum_{k=1}^{2n-1} \sum_{i \in \llbracket 1, d \rrbracket^{2n}} \sum_{a \in \mathcal{A}_k^{2n}(i)} C_a \text{Tr}(a_{i_1} \dots a_{i_{2n}} \rho). \end{aligned}$$

where

$$\mathcal{A}_k^n(i) = \{a = (a_{i_1}, \dots, a_{i_n}), \forall j \in \llbracket 1, n \rrbracket, a_{i_j} = x_{i_j} \text{ or } a_{i_j} = t\mathbf{p}_{i_j}, |\{j, a_j = t\mathbf{p}_j\}| = k\}.$$

Then for any $\varepsilon > 0$ we proceed by recurrence to prove that for any $m \leq 2n$ it holds

$$(28) \quad \forall k \in (1, m-1), \forall i \in \llbracket 1, d \rrbracket^{2n}, \forall a \in \mathcal{A}_k^m, \\ |\hbar t|^r |\text{Tr}(a_{i_1} \dots a_{i_m} \rho)| \leq \varepsilon \text{Tr}(|x|^{2n} \rho) + C_\varepsilon \text{Tr}((|t\mathbf{p}|^{2n} + |\hbar t|^n) \rho),$$

where $2r = 2n - m$.

Step 1. Case $m = 2$. In this case for any $\epsilon > 0$, by Hölder's and Young's inequalities, there exists $C_\epsilon > 0$ such that

$$\begin{aligned} |\mathrm{Tr}(x_i t \mathbf{p}_j \rho)| &\leq |\hbar t|^{n-1} \mathrm{Tr}(|x|^2 \rho)^{\frac{1}{2}} \mathrm{Tr}(|t \mathbf{p}|^2 \rho)^{\frac{1}{2}} \\ &\leq \epsilon \mathrm{Tr}(|x|^2 \rho) + C_\epsilon \mathrm{Tr}(|t \mathbf{p}|^2 \rho), \end{aligned}$$

where we used the fact that $\mathrm{Tr}(|x_j|^2 \rho) \leq \mathrm{Tr}(|x|^2 \rho)$ and $\mathrm{Tr}(|\mathbf{p}_j|^2 \rho) \leq \mathrm{Tr}(|\mathbf{p}|^2 \rho)$. Then using again Hölder's and Young's inequalities, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |\hbar t|^{n-1} |\mathrm{Tr}(x_i t \mathbf{p}_j \rho)| &\leq \epsilon \mathrm{Tr}(|x|^{2n} \rho)^{\frac{1}{n}} \mathrm{Tr}(|\hbar t|^n \rho)^{\frac{n-1}{n}} + C_\epsilon \mathrm{Tr}(|t \mathbf{p}|^{2n} \rho)^{\frac{1}{n}} \mathrm{Tr}(|\hbar t|^n \rho)^{\frac{n-1}{n}} \\ &\leq \varepsilon \mathrm{Tr}(|x|^{2n} \rho) + C_\varepsilon \mathrm{Tr}((|t \mathbf{p}|^{2n} + |\hbar t|^n) \rho). \end{aligned}$$

Step 2. Case $m > 2$. Since we have the following commutation relations for $j \neq k \in \llbracket 1, d \rrbracket^2$

$$(29) \quad [x_j, t \mathbf{p}_k] = [\mathbf{p}_j, \mathbf{p}_k] = [x_j, x_k] = [x_j, x_j] = 0$$

$$(30) \quad [x_j, t \mathbf{p}_j] = i \hbar t,$$

any commutation operation of a_j in $(i \hbar t)^{r_0} \mathrm{Tr}(a_{i_1} \dots a_{i_m} \rho)$ involving r commutations of the form (30) will create terms of the form

$$\pm (i \hbar t)^{r_0+r} \mathrm{Tr}(a_{i_1} \dots a_{i_{m-2r}} \rho),$$

which will be bounded using the recurrence hypothesis, so that we can assume that all the operators commute. Let $k \in (1, m-1)$ and $a \in \mathcal{A}_k^m$. Then, by using m times Hölder's inequality and then Young's inequality, we get

$$\begin{aligned} |\mathrm{Tr}(a_{i_1} \dots a_{i_m} \rho)| &\leq \mathrm{Tr}(|a_{i_1}|^m \rho)^{\frac{1}{m}} \dots \mathrm{Tr}(|a_{i_m}|^m \rho)^{\frac{1}{m}} \\ &\leq \mathrm{Tr}(|t \mathbf{p}|^m \rho)^{\frac{k}{m}} \mathrm{Tr}(|x|^m \rho)^{\frac{m-k}{m}} \\ &\leq \epsilon \mathrm{Tr}(|t \mathbf{p}|^m \rho) + C_\epsilon \mathrm{Tr}(|x|^m \rho) \end{aligned}$$

Then using again Hölder's and Young's inequalities and the fact that $\frac{r}{n} = \frac{2n-m}{2n}$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |\hbar t|^r |\mathrm{Tr}(x_i t \mathbf{p}_j \rho)| &\leq \epsilon \mathrm{Tr}(|x|^{2n} \rho)^{\frac{m}{2n}} \mathrm{Tr}(|\hbar t|^n \rho)^{\frac{2n-m}{2n}} + C_\epsilon \mathrm{Tr}(|t \mathbf{p}|^{2n} \rho)^{\frac{m}{2n}} \mathrm{Tr}(|\hbar t|^n \rho)^{\frac{2n-m}{2n}} \\ &\leq \varepsilon \mathrm{Tr}(|x|^{2n} \rho) + C_\varepsilon \mathrm{Tr}((|t \mathbf{p}|^{2n} + |\hbar t|^n) \rho). \end{aligned}$$

Step 3. Conclusion. Thus, coming back to formula (27), we obtain for any $\varepsilon > 0$ the existence of $C_\varepsilon > 0$ such that

$$\begin{aligned} \mathrm{Tr}(|x - t \mathbf{p}|^{2n} \rho) &\leq \mathrm{Tr}(|x|^{2n} \rho) + \mathrm{Tr}(|t \mathbf{p}|^{2n} \rho) \\ &\quad + \varepsilon \mathrm{Tr}(|x|^{2n} \rho) + C_\varepsilon \mathrm{Tr}((|t \mathbf{p}|^{2n} + |\hbar t|^n) \rho), \end{aligned}$$

which proves the result. \square

To get a short time kinetic moment estimate, we use [26, Theorem 3] which tells us that for any $n \in 2\mathbb{N}$ and $\mathbf{b} > \max(\mathbf{b}_4, \mathbf{b}_n)$, there exists a time

$$(31) \quad T = T_{\|\nabla K\|_{L^{\mathbf{b}, \infty}}, \|\rho^{\mathrm{in}}\|_{\mathcal{L}^r}, M_0, M_n^{\mathrm{in}}, d, r, n},$$

and a positive constant m depending on ∇K , $\|\rho^{\mathrm{in}}\|_{\mathcal{L}^r}$, M_0 , M_n^{in} , d , r and n such that

$$(32) \quad \forall (k, t) \in [0, n] \times [0, T], \quad M_k(t) \leq m.$$

Proposition 3.4 (Short time estimate). *Let $n \in 2\mathbb{N} \setminus \{0\}$, $r \in [1, \infty]$,*

$$\nabla K \in L^{\mathfrak{b}, \infty} \text{ for } \mathfrak{b} \in [\max(\mathfrak{b}_4, \mathfrak{b}_n), \infty],$$

and $\rho \in L^\infty([0, T], \mathcal{L}^r \cap \mathcal{L}_+^1(1 + |x|^n + |\rho|^n))$ be a solution of (Hartree) equation. Then for any $t \in [0, T]$ it holds

$$L_n \leq 2^n \text{Tr}(|x|^n \rho^{\text{in}}) + C_{d,n,T}(m + \hbar)t^{\frac{n}{2}},$$

where T is given by (31).

Proof. We first remark that

$$\begin{aligned} [|\mathbf{p}|^2, |x|^n] &= -2i\hbar \nabla(|x|^n) \cdot \mathbf{p} + (-i\hbar)^2 \Delta(|x|^n) \\ &= -ni\hbar |x|^{n-2} (2x \cdot \mathbf{p} - i\hbar(d + n - 2)). \end{aligned}$$

Therefore, by defining $N_n := \text{Tr}(|x|^n \rho)$, we can compute

$$\begin{aligned} \frac{dN_n}{dt} &= \frac{1}{i\hbar} \text{Tr} \left(\left[|x|^n, \frac{|\mathbf{p}|^2}{2} + V \right] \rho \right) \\ &= \frac{1}{2i\hbar} \text{Tr}([|x|^n, |\mathbf{p}|^2] \rho) \\ &= n \text{Tr}(|x|^{n-2} (2x \cdot \mathbf{p} - i\hbar(d + n - 2)) \rho). \end{aligned}$$

Then, by Hölder's inequality for the trace, it holds

$$(33) \quad \frac{dN_n}{dt} \leq nN_n^{1-\frac{1}{n}} \left(M_n^{\frac{1}{n}} + \hbar n(d + n - 2)M_0^{\frac{1}{n}} \right).$$

Hence by using the bound (32), we deduce that for any $t \in [0, T]$, it holds

$$\frac{dN_n}{dt} \leq nC_T N_n^{1-\frac{1}{n}}.$$

where $C_T = m^{\frac{1}{n}} + \hbar n(d + n - 2)M_0^{\frac{1}{n}}$. By Gronwall's Lemma, we deduce

$$N_n(t) \leq \left((N_n^{\text{in}})^{\frac{1}{n}} + C_T t \right)^n.$$

Finally, by Lemma 3.1 and convexity of $x \mapsto |x|^n$, we obtain

$$\begin{aligned} L_n &\leq 2N_n + C_n(t^n M_n + |\hbar t|^{\frac{n}{2}} M_0) \\ &\leq 2^n \left(N_n^{\text{in}} + (C_T^n + C_n m)t^n + C_n |\hbar t|^{\frac{n}{2}} M_0 \right) \\ &\leq 2^n \left(N_n^{\text{in}} + t^{\frac{n}{2}} \left((C_T^n + C_n m)T^{\frac{n}{2}} + C_n \hbar M_0 \right) \right), \end{aligned}$$

which yields the result. \square

Proof of Theorem 1. Since $\mathfrak{b} < \frac{d}{2}$, we have $a := \frac{d}{\mathfrak{b}} - 1 > 1$. Thus, by Gronwall's Lemma and Proposition 3.3, for any $t > \tau > 0$ we obtain

$$\begin{aligned} L_n(t)^{-a/n} &\geq L_n(\tau)^{-a/n} + \frac{1}{A} \left(\frac{1}{t^{a-1}} - \frac{1}{\tau^{a-1}} \right) \\ &\geq L_n(\tau)^{-a/n} - \frac{1}{A\tau^{a-1}}, \end{aligned}$$

where

$$A = \left(1 - \frac{1}{a} \right) \frac{n}{C \|\nabla K\|_{L^{\mathfrak{b}, \infty}} M_0^{\Theta_0} \|\rho^{\text{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathfrak{b}}}}.$$

Combining the above inequality with Proposition 3.4, we know that there exists T such that for any $\tau \in (0, T]$ and $t > 0$, it holds

$$L_n(t) \leq \left((2^n N_n^{\text{in}} + C_T \tau^{\frac{n}{2}})^{-\frac{a}{n}} - \frac{1}{A\tau^{a-1}} \right)^{-n/a},$$

where $C_T = C_{T, M_4^{\text{in}}, M_0}$. Since $\mathbf{b} > \frac{d}{3}$, we have $2 < a' = \frac{a}{a-1}$. Now let $\tau_0 := \min\left(T, (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}}\right)$ and $\mathcal{C} := 2^{-n}(A^{\frac{n}{a}}\tau_0^{\frac{n}{a'}} - C_T\tau_0^{\frac{n}{2}})$. We remark that $\mathcal{C} \geq 0$ since

$$\tau_0 \geq (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}} \implies C_T\tau_0^{\frac{n}{2} - \frac{n}{a'}} \leq A^{\frac{n}{a}} \implies \mathcal{C} \geq 0.$$

Taking $\tau = \tau_0$ and $N_n^{\text{in}} < \mathcal{C}$, we obtain that

$$\begin{aligned} C_{T, N_n^{\text{in}}, M_4^{\text{in}}, M_0} &:= (2^n N_n^{\text{in}} + C_T \tau_0^{\frac{n}{2}})^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} \\ &> (\mathcal{C} + C_T \tau_0)^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} = 0. \end{aligned}$$

We deduce that for any $t > 0$

$$L_n(t) < C_{T, N_n^{\text{in}}, M_4^{\text{in}}, M_0}^{-\frac{n}{a}},$$

which proves the result. \square

3.3. Application to the semiclassical limit. Remark that we have the following corollary of Lemma 3.1, which proves in particular that for a fixed $(t, \hbar) \in \mathbb{R}_+^* \times \mathbb{R}_+$, $\mathcal{L}^1(1 + |x|^n + |t\mathbf{p}|^n)$ and $\mathcal{L}^1(1 + |x|^n + |x - t\mathbf{p}|^n)$ have equivalent norms.

Corollary 3.1. *Let $n \in 2\mathbb{N}$ and assume $\hbar < 1$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon = C_{n, \varepsilon} > 0$ such that for any operator $\rho \geq 0$ and any $t \geq 0$*

$$t^n \text{Tr}(|\mathbf{p}|^n \rho) \leq (1 + \varepsilon) \text{Tr}(|x|^n \rho) + C_\varepsilon \text{Tr}(|x - t\mathbf{p}|^n + |\hbar t|^{\frac{n}{2}}) \rho).$$

Proof. We just remark that since $\tilde{S}_t^{-1} = \tilde{S}_{-t}$ and by the properties (19) of \tilde{S} , we have for any $t \in \mathbb{R}$

$$|t|^n \text{Tr}(|\mathbf{p}|^n \rho) = |t|^n \text{Tr}(\tilde{S}_{1/t}(|\mathbf{p}|^n) \tilde{S}_{1/t} \rho) = \text{Tr}(|x - t\mathbf{p}|^n \tilde{S}_{1/t} \rho).$$

Therefore, using Lemma 3.1, we obtain

$$\begin{aligned} |t|^n \text{Tr}(|\mathbf{p}|^n \rho) &\leq \text{Tr}(((1 + \varepsilon)|x|^n + C_\varepsilon(|t\mathbf{p}|^n + |\hbar t|^{\frac{n}{2}})) \tilde{S}_{1/t} \rho) \\ &\leq \text{Tr}(\tilde{S}_{-1/t}((1 + \varepsilon)|x|^n + C_\varepsilon(|t\mathbf{p}|^n + |\hbar t|^{\frac{n}{2}})) \rho) \\ &\leq \text{Tr}(((1 + \varepsilon)|x|^n + C_\varepsilon(|x + t\mathbf{p}|^n + |\hbar t|^{\frac{n}{2}})) \rho). \end{aligned}$$

Replacing t by $-t$ and taking $t \geq 0$ yields the result. \square

From the above corollary and the result of Theorem 1, we obtain the following bounds

Proposition 3.5. *Under the hypotheses of Theorem 1, it holds*

$$\begin{aligned} N_n &\leq N_n^{\text{in}} + C(t^{n+\varepsilon} + t) \\ M_n &\leq C(1 + t^\varepsilon), \end{aligned}$$

where the constants $C > 0$ involved depends on ε , $\|\nabla K\|_{L^{\mathbf{b}, \infty}}$, M_0 , M_n^{in} , $\|\rho^{\text{in}}\|_{\mathcal{L}^r}$, d , n and r .

Proof. We go back to equation (33) which says that for $N_n = N_n(t)$ we have

$$(34) \quad \frac{dN_n}{dt} \leq nN_n^{1-\frac{1}{n}} \left(M_n^{\frac{1}{n}} + \hbar C_{d,n} M_0^{\frac{1}{n}} \right).$$

where $C_{d,n} = n(d+n-2)$. Using Corollary 3.1 to bound M_n yields for any $\varepsilon > 0$ and any $t > \tau > 0$,

$$\begin{aligned} \frac{dN_n}{dt} &\leq nN_n^{1-\frac{1}{n}} \left(((1+\varepsilon^n)N_n + C_\varepsilon(L_n + |\hbar t|^{\frac{n}{2}}))^{\frac{1}{n}} t^{-1} + \hbar C_{d,n} M_0^{\frac{1}{n}} \right) \\ &\leq n(1+\varepsilon)N_n t^{-1} + nN_n^{1-\frac{1}{n}} \left((C_\varepsilon(L_n \tau^{-n} + |\hbar \tau^{-1}|^{\frac{n}{2}}))^{\frac{1}{n}} + \hbar C_{d,n} M_0^{\frac{1}{n}} \right) \\ &\leq n(1+2\varepsilon)N_n t^{-1} + C_{d,n,\varepsilon} \left(L_n^{\frac{1}{n}} \tau^{-1} + |\hbar \tau^{-1}|^{\frac{1}{2}} + \hbar M_0^{\frac{1}{n}} \right)^n, \end{aligned}$$

where we used the triangle inequality for $x \mapsto |x|^{\frac{1}{n}}$ and Young's inequality $ab \leq \varepsilon a^p + C_\varepsilon b^{p'}$. Since L_n is uniformly bounded in time by Theorem 1, we obtain that

$$B := C_{d,n,\varepsilon} \left(L_n^{\frac{1}{n}} \tau^{-1} + |\hbar \tau^{-1}|^{\frac{1}{2}} + \hbar M_0^{\frac{1}{n}} \right)^n,$$

is also uniformly bounded in time. Therefore, for any $\varepsilon > 0$ and $t > \tau$, Gronwall's inequality yields

$$N_n(t) \leq N_n(\tau) + \frac{B\tau^{1-n-\varepsilon}}{n+\varepsilon-1} t^{n+\varepsilon}.$$

However, since as previously stated we know by [26, Proof of Theorem 3] that M_n is bounded on $[0, T]$ for a short time T depending of M_n^{in} . By inequality (34), it implies that $N_n(t) \leq N_n^{\text{in}} + C_T t$ for any $t \in [0, T]$. Therefore, taking $\tau = T$ finally yields for any $\varepsilon \in (0, 1)$ and $t \geq 0$

$$N_n(t) \leq N_n^{\text{in}} + C_{B,T,n,\varepsilon} (t^{n+\varepsilon} + t).$$

The bound on M_n is then an immediate consequence of Corollary 3.1 for large times and the fact that M_n is bounded on $[0, T]$. \square

In fact, it is sufficient to use the condition of smallness of moments for $n = 4$ to get a global propagation of higher moments as soon as $\mathfrak{b}_4 > \mathfrak{b}_n$ (which corresponds to $r > \frac{d}{d-1}$), which leads to the following

Proposition 3.6. *Under the condition of Theorem 2, $M_n \in L_{\text{loc}}^\infty(\mathbb{R}_+)$ and more precisely there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that*

$$M_n \leq C(1 + t^{c_n}).$$

Proof of theorems 2. Since $\mathfrak{b} \geq \mathfrak{b}_4$ and $\mathfrak{b} \geq \frac{d}{3}$, we can use Proposition 3.5 for $n = 4$, and deduce

$$M_4 \leq C(1 + t^\varepsilon),$$

for a given $C > 0$. This already proves the result in the case $n = 4$, so that we assume now that $n \geq 6$. Then, we use formula (44) from [26], which reads

$$(35) \quad \frac{dM_n}{dt} \leq C_{d,r,n} C_K \|\rho^{\text{in}}\|_{\mathcal{L}^r}^{\Theta_2} M_{n-2}^{\Theta_0} M_n^{\Theta},$$

with

$$\begin{aligned}\Theta &= 1 + \frac{n-1}{2} \left(\frac{\mathbf{b}_{n-2}}{\mathbf{b}} - 1 \right). \\ \Theta_0 &= (1-\varepsilon) \left(\frac{3}{2} - \frac{r'}{p'_{n-2}} \right) \\ \Theta_2 &= \frac{3}{2} - \Theta_1 - \Theta_0,\end{aligned}$$

where

$$\varepsilon = \frac{nr' + d}{(n-2)r' + 3d} \left(\frac{(n-2)r' + d}{\mathbf{b}} - (n-2) \right).$$

In particular, since $r \geq d'$, \mathbf{b}_n is a non-increasing sequence and we deduce that for any $n \geq 6$, $\mathbf{b} \geq \mathbf{b}_4 \geq \mathbf{b}_{n-2}$, which implies that $\Theta \leq 1$. We then obtain inequality (6) by Gronwall's Lemma and by recurrence over $n \in 2\mathbb{N}$. From this bound, formula (7) about N_n can be deduced by using again inequality (33) and Gronwall's Lemma. Finally, since we know by Theorem 1 that L_4 is bounded, the asymptotic behaviour of ρ in formula (8) is a consequence of Corollary 2.1. \square

Proof of Theorem 5. The hypotheses of Theorem 2 are fulfilled with $r = \infty$, thus we deduce that

$$M_n \leq C(1 + t^c).$$

Therefore, we can use [26, Proposition 5.3], which tells us that for any $(n_0, n) \in (2\mathbb{N})^2$ verifying $d < n_0 \leq (1 - \frac{1}{\mathbf{b}})n + 1 - \frac{d}{\mathbf{b}}$, it holds

$$\begin{aligned}c_{d,n_0} \|\rho(t)\|_{L^\infty} &\leq \|\rho(t)\|_{\mathcal{L}^\infty(\mathbf{p}_i^{n_0})} \\ &\leq 2^{n_0} \left(\|\rho^{\text{in}}\|_{\mathcal{L}^\infty(\mathbf{p}_i^{n_0})} + \tilde{C}_{\rho^{\text{in}}} \left(t + \int_0^t M_n^{1-\frac{1}{\mathbf{b}}} \right)^{n_0} \right) \\ &\leq 2^{n_0} \left(\|\rho^{\text{in}}\|_{\mathcal{L}^\infty(\mathbf{p}_i^{n_0})} + \tilde{C}_{\rho^{\text{in}}} \left((1+C)t + Ct^{1+\frac{c}{\mathbf{b}'}} \right)^{n_0} \right) \\ &\leq 2^{n_0} \left(\|\rho^{\text{in}}\|_{\mathcal{L}^\infty(\mathbf{p}_i^{n_0})} + \tilde{C}_{\rho^{\text{in}}} (1+C)^{n_0} 2^{n_0-1} t^{n_0} \left(1 + t^{\frac{cn_0}{\mathbf{b}'}} \right) \right),\end{aligned}$$

where $\tilde{C}_{\rho^{\text{in}}} = (4^{n_0} C_{d,n} \|\nabla K\|_{L^{\mathbf{b}}} (1 + M_0))^{n_0} \|\rho^{\text{in}}\|_{\mathcal{L}^\infty}^{1+\frac{n_0}{\mathbf{b}}}$. This proves (13). As in [21, Section 4], we then define the time dependent coupling $\gamma = \gamma(t, z)$ with $z = (x, \xi)$ as the solution to the Cauchy problem

$$\partial_t \gamma = (-v \cdot \nabla_x - E \cdot \nabla_\xi) \gamma + \frac{1}{i\hbar} [H, \gamma],$$

with initial condition $\gamma^{\text{in}} \in \mathcal{C}(f^{\text{in}}, \rho^{\text{in}})$. As proved in [21, Lemma 4.2], $\gamma \in \mathcal{C}(f(t), \rho(t))$. We also define

$$\mathcal{E}_\hbar = \mathcal{E}_\hbar(t) := \int_{\mathbb{R}^{2d}} \text{Tr}(\mathbf{c}_\hbar(z) \gamma(z)) \, dz.$$

Then, by [26, Proof of Proposition 6.3], we obtain

$$\begin{aligned}W_{2,\hbar}(f(t), \rho(t)) &\leq \max\left(\sqrt{d\hbar}, \mathcal{E}_\hbar\right) \\ \frac{d \ln(\mathcal{E}_\hbar)}{dt} &\leq 2\lambda + \ln(\mathcal{E}_\hbar)/\sqrt{2},\end{aligned}$$

with

$$\begin{aligned}\lambda &= 1 + C_d \|\nabla K\|_{B_{1,\infty}^1} (\|\rho_f\|_{L^\infty} + \|\rho\|_{L^\infty}) \\ &\leq C^{\text{in}} (1 + t^{n_0(1+c/b')}),\end{aligned}$$

where $C^{\text{in}} = 1 + C_d \|\nabla K\|_{B_{1,\infty}^1} \sup_t (\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty}) / (1 + t^{n_0(1+c/b')})$. This yields

$$\begin{aligned}\ln(\mathcal{E}_\hbar) &\leq \ln(\mathcal{E}_\hbar(0))e^{t/\sqrt{2}} + 2 \int_0^t \lambda(s)e^{(t-s)/\sqrt{2}} ds \\ &\leq \ln(\mathcal{E}_\hbar(0))e^{t/\sqrt{2}} + \lambda(e^{t/\sqrt{2}} - 1),\end{aligned}$$

with $\lambda = C_d C^{\text{in}}$. Therefore, as in [26, Proof of Proposition 6.3], we obtain

$$W_{2,\hbar}(f(t), \rho(t)) \leq \max\left(\sqrt{d\hbar}, W_{2,\hbar}(f^{\text{in}}, \rho^{\text{in}})e^{t/\sqrt{2}} e^{\lambda(e^{t/\sqrt{2}}-1)}\right),$$

which ends the proof. \square

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