



HAL
open science

Extended discussion on efficient train re-routing and rescheduling: valid inequalities and reformulation of RECIFE-MILP

Paola Pellegrini, Raffaele Pesenti, Joaquin Rodriguez

► **To cite this version:**

Paola Pellegrini, Raffaele Pesenti, Joaquin Rodriguez. Extended discussion on efficient train re-routing and rescheduling: valid inequalities and reformulation of RECIFE-MILP. [Research Report] IFSTTAR - Institut Français des Sciences et Technologies des Transports, de l'Aménagement et des Réseaux. 2017, 50p. hal-02046017

HAL Id: hal-02046017

<https://hal.science/hal-02046017>

Submitted on 22 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Extended discussion on efficient train re-routing and rescheduling: valid inequalities and reformulation of RECIFE-MILP

Paola Pellegrini^{1*}, Raffaele Pesenti² and Joaquin Rodriguez³

¹ Univ. Lille Nord de France, IFSTTAR, COSYS, LEOST,
20, rue Elisee Reclus - BP 70317, F-59666 Villeneuve d'Ascq, France

E-mail: paola.pellegrini@ifsttar.fr

² Dept. of Management, Università Ca' Foscari Venezia,
Cannaregio 873, 30121 Venice, Italy

E-Mail: pesenti@unive.it

³ Univ. Lille Nord de France, IFSTTAR, COSYS, ESTAS,
20, rue Elisee Reclus - BP 70317, F-59666 Villeneuve d'Ascq, France

E-mail: joaquin.rodriguez@ifsttar.fr

October 25, 2017

Abstract

The real-time Railway Traffic Management problem consists in finding suitable train routes and schedules to minimize delay propagation due to traffic perturbations. RECIFE-MILP is a mixed integer linear programming based heuristic for this problem which has proven to be effective in various contexts. However, when instances are very large or difficult, the performance of the algorithm may worsen. In this paper, we propose valid inequalities to boost the performance of RECIFE-MILP. These valid inequalities link the routing and scheduling binary variables and, at least in some cases, they are able to represent all the facets of the projection of the convex hull of the problem in the subspace of the binary variables. Moreover, they allow the definition of a model reformulation based on a reduced number of scheduling

*Corresponding author

binary variables. In an experimental analysis based on realistic instances representing traffic in four French infrastructures, we observe that the addition of valid inequalities must be done thoroughly not to add too much computational burden to the solution process, and that the reduction of the number of binary variables in general boosts the performance of RECIFE-MILP significantly.

Keywords: transportation; real-time railway traffic management problem; train re-routing and rescheduling; mixed-integer linear programming; valid inequalities

1 Introduction

Rail transport demand has been quite steadily increasing in recent years in many European countries. This is due to several reasons. One of them is the political efforts made to push users (passengers and freights) towards this green mode of transport. Another important one is the construction of high performance infrastructure (e.g., high speed passenger lines and dedicated freight corridors) which make railway more and more competitive with respect to other modes. To accommodate the increasing traffic, timetables extensively exploit the available infrastructure, especially at peak hours and at critical locations. This extensive exploitation often translates into many trains traveling through critical junctions within short time horizons, where junctions are physical areas in which multiple lines cross. Indeed, unexpected events, even of apparently negligible entity, may cause a relevant deviation with respect to the scheduled timetable. In fact, according to the timetable, trains may be scheduled to traverse the same track segment at a very short time distance. If one of them is delayed due to an unexpected event, conflicts may emerge: multiple trains traveling at the planned speed would claim one or more track segments concurrently, and hence some of them have to stop or slow-down for ensuring safety. Conflicts may generate a severe delay propagation. In the practice, conflicts are tackled by dispatchers, who decide how to locally route and schedule trains based on their experience and on quite basic visual support tools. Due to the absence of advanced decision support tools, the decisions made by dispatchers may often be of a rather low quality if compared to what may be possible thanks to optimization.

In the literature, the selection of the train routes and schedules for minimizing delay propagation has been formalized as the real-time Railway Traffic Management Problem (rtRTMP) (Pellegrini et al., 2014). Several algorithms have been presented to tackle this problem (Cacchiani et al., 2014; Corman and Meng, 2015). Among them, let us mention Caimi et al. (2011); Corman et al. (2012); D’Ariano et al. (2007); Lamorgese and Mannino (2015); Meng and Zhou (2014); Törnquist and Persson (2007), who apply linear programming approaches to this problem. All these studies propose specific formulations, modeling the infrastructure considering various levels of detail, aiming at different objectives and using commercial solvers or ad-hoc solution procedures. Recently, Fischetti and Monaci (2017) have shown that commercial solvers can be suitable to solve the rtRTMP despite the strong requirements on computational time due to the nature of this problem. Along with this branch of the literature, we have proposed RECIFE-MILP (Pellegrini et al., 2015). RECIFE-MILP is an algorithm based on the solution of a mixed-integer linear programming (MILP) formulation previously proposed (Pellegrini et al., 2014). It differentiates from the other cited approaches for some of the hypothesis considered (e.g., representation of train speed variation dynamics) and the model of the infrastructure (e.g., macroscopic or microscopic).

RECIFE-MILP models the infrastructure at the microscopic level and it implements the route-lock sectional-release interlocking system (Pachl, 2008). Although very often RECIFE-MILP quickly finds the optimal solution to realistic instances, it fails sometimes in delivering it within a computational time in line with real-time purposes (typically three minutes). If RECIFE-MILP finds and proves the optimal solution before this computational time elapses, then it stops the search process and returns the optimal solution. Otherwise, it stops the search process after the available time has elapsed and returns the best solution identified, together with the optimality gap. Since its proposal, RECIFE-MILP has been validated on several case-studies coming from France (Pellegrini et al., 2016; Samà et al., 2016), Sweden, the UK and the Netherlands (Quaglietta et al., 2016). Despite the good performance achieved in all these case-studies, it is indeed possible to define instances for which RECIFE-MILP finds it difficult to return a high quality solution in real-time. The difficulty is very often linked to the size of

the formulation describing the instances, which may easily include several tens of thousands of binary variables (Pellegrini et al., 2015).

The aim of this work is the strengthening of the RECIFE-MILP model¹, with the objective of improving the algorithm performance in difficult instances. To understand the general idea behind our proposal, consider that in the rtRTMP three different kinds of decisions are made. The route, the precedences and the timing of each train has to be decided. The first two sets of decisions are intrinsically binary. A train does or does not take a given route, a train does or does not take precedence over another one in accessing to some block section. Clearly, the different decisions are interdependent. As an example, a precedence decision has to be made only if two trains use routes sharing some track segments. Exploiting this interdependency, we define a set of valid inequalities. Specifically, we first show that the relations between the variables associated to route and precedence decisions in the RECIFE-MILP model have a bilinear nature. Then, starting from these bilinear relations, we generate a family of linear inequalities generalizing and customizing the McCormik approach (McCormick, 1976) in the light of the specific constraints that characterize the rtRTMP. The inequalities that we obtain are faces of the convex hull of the projection of the RECIFE-MILP feasible solution set in the space of the variables associated to route and precedence decisions. In particular, at least in some cases, they are sufficient to describe all the facets of this convex hull. We report the example of an instance representing traffic in a real-world control area where this holds. To try to attain the objective of strengthening the RECIFE-MILP model, we exploit these inequalities in a two-fold way. On the one hand, we use them to reduce the number of binary variables necessary to express all precedence relations between trains. On the other hand, we add some of them in the branch and bound procedure. To assess the performance of the boosted RECIFE-MILP, we run a thorough experimental analysis based on instances representing traffic in four French infrastructures to assess the performance of these two ways for exploiting the new inequalities. The results obtained are in general positive as regards the reduction of the number of binary variables, but not as regards the inequalities addition. A possible

¹In the rest of the paper we will use the wording “RECIFE-MILP model” to indicate the MILP formulation at the basis of RECIFE-MILP, for sake of readability.

reason for the non-encouraging results is that in this paper we do not consider the opportunity of dynamically generating the inequalities in a branch and cut fashion. Indeed, the problem of defining an oracle capable of determining which inequalities of the considered family can cut a non feasible solution is out of the scope of this paper as well as the possibility of piecewise tightening our inequalities as in Castro (2015). However, we reckon that it is possible that the conclusions on the merits of these inequalities might change if this was done.

The rest of the paper is organized as follows. Section 2 describes the standard RECIFE-MILP model. Section 3 reports some fundamental concepts that are necessary for the definition of the valid inequalities, which makes the object of Section 4. Section 5 analyzes some dominance relations between inequalities, and Section 6 presents our reformulation of RECIFE-MILP. Section 7 shows that the inequalities may represent all the facets of the projection of the convex hull of the problem in the subspace of the binary variables, at least for some instances. Section 8 reports the experimental analysis performed. Finally, Section 9 concludes the paper.

2 Standard RECIFE-MILP model

In this section, we describe the main characteristics of the RECIFE-MILP model introduced in Pellegrini et al. (2015), in the next sections referred as to standard RECIFE-MILP.

RECIFE-MILP considers the infrastructure in terms of track-circuits, that is, in terms of track sections on which the presence of a train is automatically detected. Thanks to this fine representation, the route-lock sectional-release interlocking system typically deployed in the practice can be modeled. Specifically, sequences of track-circuits are grouped into block sections, which are opened by a signal indicating their availability. Before a train can enter (start the occupation of) a block section, all the track-circuits belonging to the same block section must be reserved for the train itself. In the following, we will name *utilization* time the sum of reservation and occupation time.

We define the routes in terms of sequences of track-circuits and by the intermediate stops. Hence, as an example, a sequence of track-circuits defines two or more different routes if it can be traversed performing or not

intermediate stops. The running times for a route with intermediate stops include the appropriate deceleration and acceleration times, but not the dwell times.

In the MILP formulation, we use the following notation:

$T \equiv$ set of trains,

$w_t \equiv$ weight associated to train t 's delay,

$ty_t \equiv$ type corresponding to train t (indicating train characteristics),

$init_t, sched_t \equiv$ earliest time at which train t can be operated given the timetable and the primary delay, and earliest time at which train t can reach its destination given $init_t$, the route assigned to t in the timetable and the intermediate stops,

$i(t', t) \equiv$ indicator function: 1 if trains t' and t use the same rolling stock and t results from the turnaround, join or split of train t' , 0 otherwise,

$ms \equiv$ minimum separation time between the arrival of a train and the departure of another train which uses the same rolling stock,

$R_t, TC_t \equiv$ set of routes and track-circuits which can be used by train t ,

$TC^r \equiv$ set of track-circuits composing route r ,

$OTC_{ty_t, r, tc} \equiv$ set of consecutive track-circuits preceding tc which are occupied by t traveling along route r if its head is on tc , depending on t 's and tc 's length,

$TC(tc, tc', r) \equiv$ set of track-circuits between tc and tc' along route r ,

$pr_{r, tc}, sr_{r, tc} \equiv$ track-circuits preceding and following tc along route r ,

$rt_{ty_t, r, tc}, ct_{ty_t, r, tc} \equiv$ running and clearing time of tc along r for a train of type ty_t ,

$ref_{r, tc} \equiv$ reference track-circuit for the reservation of tc along route r ,

$e(tc, r) \equiv$ indicator function: 1 if track-circuit tc belongs to an extreme (either the first or the last) block section on route r , 0 otherwise,

$bs_{r, tc} \equiv$ block section including track-circuit tc along route r ,

$for_{bs}, rel_{bs} \equiv$ formation and release time for block section bs ,

$S_t, TCS_{t, s} \equiv$ set of stations where train t has a scheduled stop and set of track-circuits that can be used by t for stopping at station s ,

$dw_{t, s}, arr_{t, s}, dep_{t, s} \equiv$ minimum dwell time, scheduled arrival and scheduled departure times for train t at station s ,

$M \equiv$ large constant.

The MILP formulation includes the following non-negative continuous variables:

- for all triplets of train $t \in T$, route $r \in R_t$ and track-circuit $tc \in TC^r$:

$o_{t,r,tc}$: time at which t starts the occupation of tc along r ,

$l_{t,r,tc}$: longer stay of t 's head on tc along route r ,

due to dwell time and scheduling decisions (delay);

- for all pairs of train $t \in T$ and track-circuit $tc \in TC_t$:

$sU_{t,tc}$: time at which tc starts being utilized by t ;

$eU_{t,tc}$: time at which tc ends being utilized by t ;

- for all pairs of train $t \in T$ and track-circuit tc_∞

D_{t,tc_∞} : delay suffered by train t when exiting the infrastructure considered.

In addition the MILP formulation includes the following binary variables:

- for all pairs of train $t \in T$ and route $r \in R_t$:

$$x_{t,r} = \begin{cases} 1 & \text{if } t \text{ uses } r, \\ 0 & \text{otherwise,} \end{cases}$$

- for all triplets of train $t, t' \in T$ such that the index t is smaller than the index t' , and track-circuit $tc \in TC_t \cap TC_{t'}$:

$$y_{t,t',tc} = \begin{cases} 1 & \text{if } t \text{ utilizes } tc \text{ before } t' (t \prec t'), \\ 0 & \text{otherwise } (t \succ t'). \end{cases}$$

The objective function to be minimized is the total weighted delays suffered by trains at their exit from the infrastructure:

$$\min \sum_{t \in T} w_t D_{t,tc_\infty}. \quad (1)$$

The sets of constraints considered impose the following conditions:

- A train t cannot be operated earlier than $init_t$:

$$o_{t,r,tc} \geq init_t x_{t,r} \quad \forall t \in T, r \in R_t, tc \in TC^r. \quad (2)$$

- The start time of track-circuit occupation along a route is zero if the route itself is not used:

$$o_{t,r,tc} \leq Mx_{t,r} \quad \forall t \in T, r \in R_t, tc \in TC^r. \quad (3)$$

- A train starts occupying track-circuit tc along a route after spending in the preceding track-circuit its running time, if the route is used, and its longer stay.

$$o_{t,r,tc} = o_{t,r,p_{r,tc}} + l_{t,r,p_{r,tc}} + rt_{r,ty_{t,p_{r,tc}}} x_{t,r} \quad \forall t \in T, r \in R_t, tc \in TC^r; \quad (4)$$

remark that these constraints imply that $l_{t,r,p_{r,tc}}$ equals 0 if t does not use r .

- A train t with a scheduled stop at station s and using route r does not enter the track-circuit following tc before the scheduled departure time from s if tc is in $TCS_{t,s}$:

$$o_{t,r,s_r,tc} \geq \sum_{s \in S_t: tc \in TCS_{t,s} \cap TC^r} dep_{t,s} x_{t,r} \quad \forall t \in T, r \in R_t, tc \in \bigcup_{s \in S_t} TCS_{t,s}. \quad (5)$$

- A train t with a scheduled stop at station s and using route r has a longer stay in tc of at least $dw_{t,s}$ if tc is in $TCS_{t,s}$:

$$l_{t,r,s_r,tc} \geq \sum_{s \in S_t: tc \in TCS_{t,s} \cap TC^r} dw_{t,s} x_{t,r} \quad \forall t \in T, r \in R_t, tc \in \bigcup_{s \in S_t} TCS_{t,s}. \quad (6)$$

- A train t must use exactly one route:

$$\sum_{r \in R_t} x_{t,r} = 1 \quad \forall t \in T. \quad (7)$$

- The value of a delay D_{t,tc_∞} cannot be less than the difference between the actual and the scheduled arrival times at the exit of the infrastructure:

$$D_{t,tc_\infty} \geq \sum_{r \in R_t} o_{t,r,tc_\infty} - sched_t \quad \forall t \in T. \quad (8)$$

- A minimum separation time ms must separate the arrival and departure of trains using the same rolling stock:

$$\sum_{\substack{r \in R_t, tc \in TC^r: \\ p_{r,tc} = tc_0}} o_{t,r,tc} \geq \sum_{\substack{r \in R_{t'}, tc \in TC^r: \\ s_{r,tc} = tc_\infty}} o_{t',r,tc} + (ms + rt_{r,ty_{t',tc}}) x_{t',r} \\ \forall t, t' \in T : i(t', t) = 1. \quad (9)$$

- If trains t' and t use the same rolling stock and t results from the turnaround, join or split of train t' , the track-circuit tc where the turnaround, join or split takes place must be utilized for the whole time between t' 's arrival and t 's departure. Thus, tc starts being reserved by t at the latest when t' ends its utilization:

$$\sum_{tc \in TC_t : \exists r \in R_t, pr, tc = tc_0} sU_{t,tc} \leq \sum_{tc \in TC_{t'} : \exists r \in R_{t'}, sr, tc = tc_\infty} eU_{t',tc} \quad \forall t, t' \in T : i(t', t) = 1. \quad (10)$$

Here, the inequality must be imposed since, in case of a join, two trains arrive and are connected to become a single departing one. The utilization of the departing train must then immediately follow the utilization of the first train arriving, being strictly smaller than the one of the second train.

- A train's utilization of a track-circuit starts as soon as the train starts occupying the track-circuit $ref_{r,tc}$ along one of the routes including it, minus the formation time:

$$sU_{t,tc} = \sum_{r \in R_t : tc \in TC^r} \left(ot_{t,r,ref_{r,tc}} - for_{bs_{r,tc}} x_{t,r} \right) \quad (11)$$

$$\forall t \in T, tc \in TC_t : (\nexists t' \in T : i(t', t) = 1) \vee (\forall r \in R_t : ref_{r,tc} \neq s_{r,tc_0}).$$

Constraints (11) are imposed as inequalities (\leq) when they concern a track-circuit of the first block sections of the route ($ref_{r,tc} = s_{r,tc_0}$) and the train t results from the turnaround, join or split of one or more other trains. This fact is a consequence of the need of keeping platforms utilized. Indeed, if train t results from the turnaround of train t' , Constraints (10) ensure that the track-circuit where the turnaround takes place starts being reserved by t as soon as t' arrives. As a consequence, Constraints (10) impose that t 's reservation of the track-circuit starts much earlier than its occupation. Indeed, t needs to wait at least for a time ms before departing and, in order to guarantee the coherence with the constraints imposed by (4) on the occupation variables and the running time, the occupation of the track-circuit starts only on t actual departure.

- The utilization of a track-circuit tc lasts till the train utilizes it along

any route, plus the formation and the release time:

$$eU_{t,tc} = \sum_{r \in R_t: tc \in TC^r} o_{t,r,ref_{r,tc}} + (for_{bs_{r,tc}} + rel_{bs_{r,tc}}) x_{t,r} + ul_{t,r,tc} \quad \forall t \in T, tc \in TC_t. \quad (12)$$

Here $ul_{t,r,tc}$ is the total utilization time:

$$ul_{t,r,tc} = \sum_{tc' \in TC(ref_{r,tc}, tc, r)} (rt_{r,ty_t,tc'} x_{t,r} + l_{t,r,tc'}) + \sum_{tc' \in TC_t: tc \in OTC_{ty_t, r, tc'}} l_{t,r,tc'} + ct_{r,ty_t,tc} x_{t,r}.$$

It includes: the running time of all track-circuits between $ref_{r,tc}$ and tc , the longer stay of the train's head on each of these track-circuits $l_{t,r,tc}$ and the clearing time of tc . Moreover, it includes the longer stay on all track-circuits tc' such that $tc \in OTC_{ty_t, r, tc'}$. As mentioned in the definition of $OTC_{ty_t, r, tc'}$, if the head of the train is on one of these track-circuits, then its tail has not yet exited tc : the train is longer than tc' , or of the sequence of track-circuits between tc and tc' . Hence, if the train suffers a longer stay when its head is on one of these track-circuits, such a longer stay must be counted in the utilization time of tc .

- The track-circuit utilizations by two trains must not overlap.

$$eU_{t,tc} - M(1 - y_{t,t',tc}) \leq sU_{t',tc} \quad \forall t, t' \in T, \text{idx } t < \text{idx } t', tc \in TC_t \cap TC_{t'} : \quad (13)$$

$$i(t, t') \sum_{r \in R_t} e(tc, r) = 0 \wedge i(t', t) \sum_{r \in R_{t'}} e(tc, r) = 0,$$

$$eU_{t',tc} - My_{t,t',tc} \leq sU_{t,tc} \quad \forall t, t' \in T, \text{idx } t < \text{idx } t', tc \in TC_t \cap TC_{t'} : \quad (14)$$

$$i(t, t') \sum_{r \in R_t} e(tc, r) = 0 \wedge i(t', t) \sum_{r \in R_{t'}} e(tc, r) = 0.$$

3 Fundamentals of route and precedence decisions

Before proposing our ideas to strengthen the RECIFE-MILP model, let us introduce some concepts which will be useful in the rest of the paper.

First of all, given two trains t and t' and two routes $r \in R_t$ and $r' \in R_{t'}$, we define the following relation in the set of track-circuits $TC^r \cap TC^{r'}$ common to the two routes:

$tc \sim_{r,r'} \hat{t}c$ if $tc, \hat{t}c \in TC^r \cap TC^{r'}$ and $t \prec t'$ on $tc \Leftrightarrow t \prec t'$ on $\hat{t}c$ for any feasible schedule.

In words, in any feasible solution of RECIFE-MILP such that $x_{t,r} = x_{t',r'} = 1$, if $tc \sim_{r,r'} \hat{t}c$ then $y_{t,t',tc} = y_{t,t',\hat{t}c}$.

This relation is an equivalence one, i.e., it is reflexive, symmetric and transitive. Accordingly, it induces a partition of the set $TC^r \cap TC^{r'}$ into equivalence classes. Here, an equivalence class S is a maximal subset of $TC^r \cap TC^{r'}$ defined by the equivalence relation $\sim_{r,r'}$: it includes all track-circuits linked to each other by $\sim_{r,r'}$. Hereafter, we call *section* any of these equivalence classes S .

Example 1 *To show examples of sections, let us consider Figure 1. It depicts a simple infrastructure traversed by two trains t and t' , using $r \in R_t = \{r1, r2\}$ and $r' \in R_{t'} = \{r3, r4, r5\}$, respectively. There, relation $\sim_{r1,r3}$ defines section $S1 = \{tc1, tc2, tc3, tc4, tc5, tc6, tc7, tc8, tc9\}$. Indeed, in every feasible solution where t uses $r1$ and t' uses $r3$, if t precedes t' (respectively, t' precedes t) on $tc1$ then t (respectively, t') must necessarily traverse all the other track-circuits between $tc1$ and $tc9$ before t' (respectively, t). Relation $\sim_{r2,r5}$ defines $S2 = \{tc10, tc11, tc12\}$ and $S3 = \{tc15, tc16, tc17\}$. In this second situation, we have two sections as, if t uses $r2$ and t' uses $r5$, t may pass first through $S2$ and second through $S3$, if it is in $tc13$ while t' is in $tc5$. In the rest of the paper we will use the italics font to proceed with this example.*

Based on sections, we define the following sets for any pair of trains t and t' in T :

- $S^{r,r'} = \{S \subseteq TC^r \cap TC^{r'} : S \text{ equivalence class of } \sim_{r,r'}\}$ for all $r \in R_t$ and $r' \in R_{t'}$, i.e., $S^{r,r'}$ is the set of sections associated to the pair of routes r and r' ;
- $S_{t,t',tc} = \{S \in S^{r,r'} : tc \in S, r \in R_t, r' \in R_{t'}\}$ for all $tc \in TC_t \cap TC_{t'}$, i.e., $S_{t,t',tc}$ is the set of all the sections which include tc given all the possible pairs of routes r and r' that t and t' may choose, respectively;

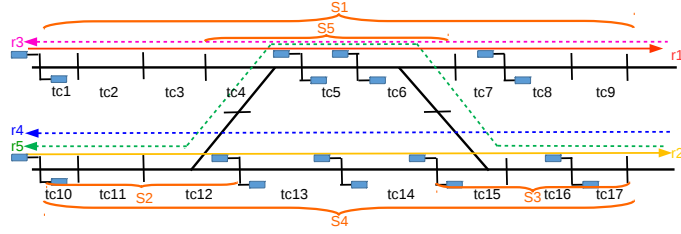


Figure 1: Two sets of routes for two trains on a simple infrastructure: t can use routes $r1$ and $r2$ and t' can use routes $r3$, $r4$ and $r5$. In orange, the sections for the pairs of routes of the two trains.

- $S_{t,t'} = \cup_{tc \in TC_t \cap TC_{t'}} S_{t,t',tc}$, i.e., $S_{t,t'}$ is the set of all the sections for the two trains.

Example 2 In Figure 1, we have already seen that set $S^{r1,r3}$ includes $S1$ and $S^{r2,r5}$ includes $S2$ and $S3$. The other non empty sets $S^{r,r'}$ are $S^{r1,r5} = \{S5 = \{tc4, tc5, tc6\}\}$ and $S^{r2,r4} = \{S4 = \{tc10, tc11, tc12, tc13, tc14, tc15, tc16, tc17\}\}$. Here, possible sets $S_{t,t',tc}$ are, for example, $S_{t,t',tc1} = \{S1\}$, $S_{t,t',tc5} = \{S1, S5\}$ and $S_{t,t',tc10} = \{S2, S4\}$. Finally, $S_{t,t'} = \{S1, S2, S3, S4, S5\}$.

Assumption 1 For a pair of trains t and t' and a track-circuit $tc \in TC_t \cap TC_{t'}$, let $y_{t,t',tc} = 1$ if t does not choose a route which includes tc . Formally,

$$\sum_{r \in R_t: tc \notin TC^r} x_{t,r} = 1 \Rightarrow y_{t,t',tc} = 1 \quad \forall t, t' \in T, idx t < idx t', tc \in TC_t \cap TC_{t'}.$$

With Assumption 1 we state a relation between x and y -variables. In particular, we state that we choose to set $y_{t,t',tc} = 1$ if no train uses tc , with no loss of optimality. Indeed, in the explanation of Constraints (13) and (14) of the standard RECIFE-MILP model, we discussed how, if only t' chooses a route which includes tc , then $y_{t,t',tc}$ must be equal to 1. Conversely, if no train uses tc , values 0 and 1 are indifferent as they are both feasible and induce the same objective function value.

Further relations between binary variables are underlined with the following lemma and corollaries. Specifically, Lemma 1 states sufficient conditions for which, depending on the route choices, the values of two y -variables y_{t,t',tc_1} and y_{t,t',tc_2} are not independent. Then, Corollary 1 goes one step

further, formalizing a relation between the y -variables concerning several track-circuits given some route choices. Finally, Corollary 2 generalizes the results for any possible choice of the train routes.

Lemma 1 *Consider two trains t and t' and two track-circuits $tc_1, tc_2 \in TC_t \cap TC_{t'}$. Let t choose route $r \in R_t$ and t' choose $r' \in R_{t'}$. The RECIFE-MILP model requires the inequality*

$$y_{t,t',tc_1} - y_{t,t',tc_2} \leq 0 \quad (15)$$

to hold if at least one of the following conditions is verified:

1. $tc_1 \in TC^r$, $tc_1 \notin TC^{r'}$;
2. $tc_2 \notin TC^r$;
3. $tc_1, tc_2 \in TC^r \cap TC^{r'}$ and they belong to a same section $S \in S^{r,r'}$;
4. $tc_1, tc_2 \in TC^r \cap TC^{r'}$ and the trains run in opposite directions, with $tc_2 \prec tc_1$ along r and $tc_1 \prec tc_2$ along r' .

Proof. If either Conditions 1 or 2 are verified, Inequality (15) is trivially satisfied. In fact, in the first case, $y_{t,t',tc_1} = 0$ as train t' does not traverse tc_1 but t does. In the second case, $y_{t,t',tc_2} = 1$ by Assumption 1 as train t does not use tc_2 along the route r chosen. In Condition 3, Inequality (15) becomes $y_{t,t',tc_2} = y_{t,t',tc_1}$ since the order in which the two trains traverse all the track-circuits in a section is necessarily the same (by definition of $S \in S^{r,r'}$). Finally, in Condition 4, Inequality (15) holds as $y_{t,t',tc_1} = 1$ implies $y_{t,t',tc_2} = 1$. Indeed, $y_{t,t',tc_1} = 1$ means that t traverses tc_1 before t' . Since, t must have traversed tc_2 to reach tc_1 , it must have done so before t' . Hence $y_{t,t',tc_2} = 1$. For the specular reasoning, $y_{t,t',tc_2} = 0$ implies $y_{t,t',tc_1} = 0$. □

Example 3 *In the example in Figure 1, consider $tc_1 = tc15$ and $tc_2 = tc10$. Condition 1 holds if t uses $r2$ and t' $r3$: only the former includes $tc15$. Condition 2 holds if t uses $r1$ (it does not include $tc10$), whatever route is used by t' . Conditions 3 and 4 hold if the trains use routes $r2$ and $r4$, respectively. Indeed, if t passes first on $tc10$, it will be first also on $tc15$, and vice-versa. Condition 4 holds also for routes $r2$ and $r5$, since $tc10 \prec tc15$ along $r2$ and $tc15 \prec tc10$ along $r5$.*

In summary, at least one condition of Lemma 1 verifies whatever pair of routes the trains use.

Next, we use the result of Lemma 1 and in particular that: Condition 1 implies $y_{t,t',tc_1} = 0$, Condition 2 implies $y_{t,t',tc_2} = 1$, Condition 3 implies $y_{t,t',tc_1} = y_{t,t',tc_2}$ and Condition 4 implies $y_{t,t',tc_1} \leq y_{t,t',tc_2}$. Specifically, we generalize Lemma 1, which deals with pairs of track-circuits tc_1 and tc_2 , to a condition on two sets of track-circuits Q^+ and Q^- . To do so, we introduce parameter $\gamma_{r,r'}$. It is the number of times the lemma holds when each track-circuit in Q^+ and Q^- is used to satisfy at most one condition and the two trains use routes r and r' , respectively. We also introduce the two sets

$$\begin{aligned}\Phi_1(r, r') &= \{tc_1 \in Q^+ : tc_1 \text{ satisfies Condition 1 of Lemma 1 for } (r, r')\}, \\ \Phi_2(r, r') &= \{tc_2 \in Q^- : tc_2 \text{ satisfies Condition 2 of Lemma 1 for } (r, r')\}.\end{aligned}$$

We can define the following corollary to formalize the generalized relation.

Corollary 1 Consider two trains t and t' and two subsets of track-circuits $Q^+, Q^- \subset TC_t \cap TC_{t'}, Q^+ \cap Q^- = \emptyset$. Let t choose route $r \in R_t$ and t' choose $r' \in R_{t'}$. The following inequality holds

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| - \gamma_{r,r'} \quad (16)$$

when: i) $|\Phi_1(r, r')| + |\Phi_2(r, r')| \leq \gamma_{r,r'}$ and ii) there are $\gamma_{r,r'} - (|\Phi_1(r, r')| + |\Phi_2(r, r')|)$ different pairs of track-circuits $(tc_1, tc_2) \in (Q^+ \setminus \Phi_1(r, r')) \times (Q^- \setminus \Phi_2(r, r'))$, that satisfy either Condition 3 or Condition 4 of Lemma 1 and such that each tc_1 and each tc_2 belong at most to one of these pairs.

Example 4 In the example in Figure 1, suppose t uses $r2$ and t' uses $r3$. Let $Q^+ = \{tc10\}$ and $Q^- = \{tc5\}$. We have $\Phi_1(r2, r3) = Q^+ \Rightarrow y_{t,t',tc10} = 0$: all the track-circuits in Q^+ , i.e., $tc10$, satisfy Condition 1 of Lemma 1, as $tc10$ belongs to $r2$ and not to $r3$, and this implies that the y -variable is equal to 0 due to the RECIFE-MILP constraints. Moreover, $\Phi_2(r2, r3) = Q^- \Rightarrow y_{t,t',tc5} = 1$, following a reasoning similar to the one mentioned for $tc10$. Since $|\Phi_1(r2, r3)| = |\Phi_2(r2, r3)| = 1$ and there cannot be pairs of track-circuits to satisfy point ii), then, we obtain the inequality of type (16)

$$y_{t,t',tc10} - y_{t,t',tc5} \leq -1$$

where the r.h.s. is equal to -1 as $|Q^+| = 1$ and $\gamma_{r2, r3} = 2$.

Note that (16) implies $\gamma_{r,r'} \leq |Q^+| + |Q^-|$. The equality holds when each track-circuit alone meets a condition of Lemma 1. This is possible when $\Phi_1(r, r') = Q^+$ and $\Phi_2(r, r') = Q^-$. Hereafter, when, given a pair of routes $(r, r') \in R_t \times R_{t'}$ and two sets of track-circuits Q^+ and $Q^- \subseteq TC_t \cap TC_{t'}$, conditions of Corollary 1 hold for the value $\gamma_{r,r'}$, we say that Lemma 1 holds $\gamma_{r,r'}$ times for (r, r') .

In Inequality (16) we consider a specific pair of routes for the two trains. We now generalize the reasoning to account that the two trains may choose their routes among several pairs of alternatives. This requires to compute a possibly different $\gamma_{r,r'}$ for each pair $(r, r') \in R_t \times R_{t'}$.

Example 5 *In the example in Figure 1, suppose $Q^+ = \{tc10\}$ and $Q^- = \{tc15\}$. The following matrices report, for all pairs of routes $(r, r') \in R_t \times R_{t'}$, the track-circuits or pairs satisfying at least one condition of Lemma 1 and the values of $\gamma_{r,r'}$ according to Corollary 1, i.e., the number of times that Lemma 1 holds for the pair of routes (r, r') :*

<i>tc satisfying at least one condition of Lemma 1</i>				$\gamma_{r,r'}$			
$R_t \downarrow R_{t'} \rightarrow$	$r3$	$r4$	$r5$	$R_t \downarrow R_{t'} \rightarrow$	$r3$	$r4$	$r5$
$r1$	$tc15$	$tc15$	$tc15$	$r1$	1	1	1
$r2$	$tc10$	$(tc10, tc15)$	$-$	$r2$	1	1	0

Indeed, consider first route $r1$ for train t , since $tc15$ does not belong to route $r1$ Condition 2 of the lemma is satisfied whatever route we consider for t' . Then consider route $r2$ for train t , Condition 1 of the lemma is satisfied if t' chooses $r3$, since $tc10$ belongs to $r2$ and not to $r3$. If t' chooses $r4$, then $tc10$ and $tc15$ belong to the same section $S4$ and Condition 3 of the lemma is satisfied. Finally, if t' chooses $r5$, no condition is satisfied and $\gamma_{r2,r5} = 0$: the track-circuits $tc10$ and $tc15$ belong to both routes, which excludes Conditions 1 and 2; they do not belong to the same section, which excludes Condition 3; $r2$ uses $tc10$ before $tc15$, which excludes Condition 4. Remark that in this example it never happens that both the pair of track-circuits and a single one of them satisfy conditions of Lemma 1. Hence the issue of each track-circuit being considered at most once for the counting of the number of times the lemma is satisfied ($\gamma_{r,r'}$) does not apply.

Corollary 2 *Consider two trains t and t' and two subsets of track-circuits Q^+ and Q^- of $TC_t \cap TC_{t'}$, $Q^+ \cap Q^- = \emptyset$, and a set of values $\gamma_{r,r'}$ for which*

Lemma 1 holds $\gamma_{r,r'}$ times for each pair (r, r') . The following inequality holds

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| - \sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'} x_{t,r} x_{t',r'}. \quad (17)$$

Example 6 Continuing Example 5, the matrix of $\gamma_{r,r'}$ reported there allows us to derive the following inequality of type (17):

$$y_{t,t',tc} 10^{-y_{t,t',tc} 15} \leq 1 - x_{t,r} x_{t',r} 3^{-x_{t,r} x_{t',r} 4} - x_{t,r} x_{t',r} 5^{-x_{t,r} x_{t',r} 3} - x_{t,r} x_{t',r} 4^{-x_{t,r} x_{t',r} 5}.$$

4 Valid inequalities

As pointed out in the Introduction, the aim of this work is to exploit the interdependence of the route and precedence decisions to try to strengthen the RECIFE-MILP model. In this section, we introduce a family of valid inequalities that describes how route choices may restrict the feasible precedence relations of pairs of trains on possibly common track-circuits, using the concepts introduced in Section 3.

Specifically, given two trains t and t' in T , we are interested in inequalities where the algebraic sum of precedence decision variables $\{y_{t,t',tc} : tc \in TC_t \cap TC_{t'}\}$ are bounded by an affine combination of route decision variables $\{x_{t,r} : r \in R_t\}$ and $\{x_{t',r'} : r' \in R_{t'}\}$ as the following ones:

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| - \left(\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k \right). \quad (18)$$

In (18):

- Q^+ and Q^- are two disjoint subsets of the sets of track-circuits that may be traversed by both trains, i.e., $Q^+, Q^- \subseteq TC_t \cap TC_{t'}$ such that $Q^+ \cap Q^- = \emptyset$
- $\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k$ is an affine combination of the variables $\{x_{t,r} : r \in R_t\}$ and $\{x_{t',r'} : r' \in R_{t'}\}$, whose constant factor k and coefficients l_r^1 and $l_{r'}^2$ may depend on Q^+ and Q^- .

Inequalities (18) are of interest only if for some values of $x_{t,r}$ and $x_{t',r'}$

$$\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k > 0.$$

If this condition does not apply, Inequalities (18) are trivially dominated by conditions $0 \leq y_{t,t',tc} \leq 1$, for $tc \in Q^- \cup Q^+$.

Hereinafter, we will simply refer to a pair of trains t and t' to indicate a generic pair of trains in T that share some track-circuits tc 's in $TC_t \cap TC_{t'}$.

Example 7 *In the example in Figure 1, a possible valid inequality for RECIFE-MILP of type (18), not dominated by condition $0 \leq y_{t,t',tc} \leq 1$, is:*

$$y_{t,t',tc10} - y_{t,t',tc15} \leq 1 - x_{t',r3} - x_{t',r4}. \quad (19)$$

This condition, where $Q^+ = \{tc10\}$ and $Q^- = \{tc15\}$ and $\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k = x_{t',r3} + x_{t',r4}$, states that the values of $y_{t,t',tc10}$ and $y_{t,t',tc15}$ are not independent when t' uses either $r3$ or $r4$. In the former case ($x_{t',r3} = 1$), $y_{t,t',tc10} = y_{t,t',tc15} = 0$ because t' is not using either track-circuit. Instead, if t' uses $r4$, the y -variables must assume identical values: if t uses $r2$, the tc 's are in the same section ($y_{t,t',tc10} = y_{t,t',tc15} = 0$ or 1); if t uses $r1$, it does not pass through the tc 's ($y_{t,t',tc10} = y_{t,t',tc15} = 1$).

In the definition of the valid inequalities of type (18), Lemma 1 and Corollaries 1 and 2 turn useful for determining the values that should be assumed by the parameters l_r^1 , $l_{r'}^2$ and k . In particular, Lemma 1 states sufficient conditions for which two y -variables y_{t,t',tc_1} and y_{t,t',tc_2} are linked by a stronger condition than $y_{t,t',tc_1} - y_{t,t',tc_2} \leq 1$. Then, Corollary 1 helps us finding an upper bound of the value for $\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k$ once the routes r and r' are chosen by the two trains and the sets Q^+ and Q^- are fixed. Finally, Corollary 2 provides a bilinear function that upper envelopes this value for any possible choice of the train routes.

In the following we exploit Equation (17) of Corollary 2 to derive a family of valid inequalities for the RECIFE-MILP model by setting coefficients l 's such that the bilinear function $\sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'} x_{t,r} x_{t',r'}$ upper envelopes $\sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k$. To do so, we require that inequality

$$\sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'} x_{t,r} x_{t',r'} \geq \sum_{r \in R_t} l_r^1 x_{t,r} + \sum_{r' \in R_{t'}} l_{r'}^2 x_{t',r'} - k \quad (20)$$

i) holds for all $(r, r') \in R_t \times R_{t'}$ and ii) is tight for some choice of routes.

Note that, from McCormick conditions (Castro, 2015; McCormick, 1976),

$$\gamma_{r,r'} x_{t,r} x_{t',r'} \geq \gamma_{r,r'} x_{t,r} + \gamma_{r,r'} x_{t',r'} - \gamma_{r,r'} \quad \forall (r, r') \in R_t \times R_{t'}. \quad (21)$$

Then, consider that Constraints (7) on the uniqueness of the route choice for each train impose that

$$\gamma_{r,r'}x_{t,r}x_{t',r'} = \begin{cases} \gamma_{r,r'} & \text{if } r \text{ is chosen by } t \text{ and } r' \text{ is chosen by } t', \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

Then, for each k such that $\min_{r \in R_t, r' \in R_{t'}}(\gamma_{r,r'}) \leq k \leq \max_{r \in R_t, r' \in R_{t'}}(\gamma_{r,r'}) \leq |Q^+| + |Q^-|$, we can define two sets $\Psi_t^k \subseteq R_t$ and $\Psi_{t'}^k \subseteq R_{t'}$, such that $\gamma_{r,r'} \geq k$ for all pairs of routes $(r, r') \in \Psi_t^k \times \Psi_{t'}^k$, and we can write

$$\begin{aligned} \sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'}x_{t,r}x_{t',r'} &\geq \sum_{r \in \Psi_t^k} \sum_{r' \in \Psi_{t'}^k} \gamma_{r,r'}x_{t,r}x_{t',r'} \geq \\ &\geq \sum_{r \in \Psi_t^k} kx_{t,r} + \sum_{r' \in \Psi_{t'}^k} kx_{t',r'} - k. \end{aligned} \quad (23)$$

The first inequality depends on the sets on which the sums are made, which are larger in the l.h.s. than in the r.h.s.. The second one derives from (21) and (22), and from the fact that $k \geq \gamma_{r,r'}$ by definition.

To strengthen Condition (23), we aim to include additional $x_{t,r}$ and $x_{t',r'}$ to the r.h.s.. To do so, we loosen the requirements on the associated $\gamma_{r,r'}$. Indeed, given the definition of k , the requirement for all the pairs of routes in $\Psi_t^k \times \Psi_{t'}^k$ is that $\gamma_{r,r'}$ is at least k . As a first step in this requirements loosening, we define two additional sets $\Psi_t^{k-1} \subseteq R_t \setminus \Psi_t^k$ and $\Psi_{t'}^{k-1} \subseteq R_{t'} \setminus \Psi_{t'}^k$, such that for all pairs of routes $(r, r') \in \Psi_t^{l_1} \times \Psi_{t'}^{l_2}$ ($l_1, l_2 \in \{k-1, k\}$) it holds that $\gamma_{r,r'} \geq \max\{0, l_1 + l_2 - k\}$. With these sets we obtain

$$\begin{aligned} \sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'}x_{t,r}x_{t',r'} &\geq \sum_{r \in \Psi_t^{k-1} \cup \Psi_t^k} \sum_{r' \in \Psi_{t'}^{k-1} \cup \Psi_{t'}^k} \gamma_{r,r'}x_{t,r}x_{t',r'} \geq \\ &\geq \sum_{r \in \Psi_t^k} kx_{t,r} + \sum_{r' \in \Psi_{t'}^k} kx_{t',r'} + \sum_{r \in \Psi_t^{k-1}} (k-1)x_{t,r} + \sum_{r' \in \Psi_{t'}^{k-1}} (k-1)x_{t',r'} - k. \end{aligned} \quad (24)$$

Indeed the above inequalities holds as, whenever trains t and t' choose a pair of routes $(r, r') \in \Psi_t^k \times \Psi_{t'}^k$ we have $\gamma_{r,r'} \geq k$ by definition of k , and the r.h.s. of (24) assumes value k . Differently, whenever trains t and t' choose a pair of routes $(r, r') \in \Psi_t^{k-1} \times \Psi_{t'}^k$ or $\Psi_t^k \times \Psi_{t'}^{k-1}$ we have $\gamma_{r,r'} \geq k-1$ and the r.h.s. of (24) assumes value $k-1$. Finally, whenever trains t and t' choose a pair of routes $(r, r') \in \Psi_t^{k-1} \times \Psi_{t'}^{k-1}$, we have $\gamma_{r,r'} \geq k-2$ and the r.h.s. of (24) assumes value $k-2$.

As further steps in the strengthening of the conditions, and hence in the $\gamma_{r,r'}$ requirements loosening, we can iterate the above argument by defining two additional sets for each $0 \leq l \leq k - 2$ as $\Psi_t^l \subseteq R_t \setminus \bigcup_{i=l+1}^k \Psi_t^{\hat{i}}$ and $\Psi_{t'}^l \subseteq R_{t'} \setminus \bigcup_{i=l+1}^k \Psi_{t'}^{\hat{i}}$, such that for all pairs of routes $(r, r') \in \Psi_t^{l_1} \times \Psi_{t'}^{l_2}$ ($l_1, l_2 \in \{l, \dots, k - 1, k\}$) it holds that $\gamma_{r,r'} \geq \max\{0, l_1 + l_2 - k\}$. We then obtain

$$\begin{aligned} \sum_{r \in R_t} \sum_{r' \in R_{t'}} \gamma_{r,r'} x_{t,r} x_{t',r'} &\geq \sum_{r \in \bigcup_{i=l}^k \Psi_t^i} \sum_{r' \in \bigcup_{i=l}^k \Psi_{t'}^i} \gamma_{r,r'} x_{t,r} x_{t',r'} \geq \\ &\geq \sum_{l_1=0}^k l_1 \sum_{r \in \Psi_t^{l_1}} x_{t,r} + \sum_{l_2=0}^k l_2 \sum_{r' \in \Psi_{t'}^{l_2}} x_{t',r'} - k. \end{aligned} \quad (25)$$

We observe that, without loss of generality, we can assume that Ψ_t^0 and $\Psi_{t'}^0$ include all the routes of R_t and $R_{t'}$, respectively, which are not included in other sets $\Psi_t^{l_1}$ and $\Psi_{t'}^{l_2}$. Hence, by their definition, for any fixed value of k , the sets $\Psi_t^{l_1}$, for $0 \leq l_1 \leq k$, partition R_t ; respectively, the sets $\Psi_{t'}^{l_2}$, for $0 \leq l_2 \leq k$, partition $R_{t'}$. Hereinafter, we say that the sets $\Psi_t^{l_1}$ and $\Psi_{t'}^{l_2}$, for $0 \leq l_1 \leq k$ and $0 \leq l_2 \leq k$, define a k -double partition of $R_t \times R_{t'}$. In addition, we say that a k -double partition is *maximal* if there is no route $r \in \Psi_t^{l_1}$, $0 \leq l_1 \leq k$, such that $\gamma_{r,r'} \geq \max\{0, l_1 + 1 + l_2 - k\}$ for all $0 \leq l_2 \leq k$ and $r' \in \Psi_{t'}^{l_2}$. Symmetrical conditions must hold for all routes $r' \in \Psi_{t'}^{l_2}$, $0 \leq l_2 \leq k$. In words, all routes are in the set with the highest possible index l_1 or l_2 , respectively for t and t' . Note that different maximal k -double partitions may exist for the same value of k and the same pair of sets Q^+ and Q^- .

The next theorem formalizes the above arguments.

Theorem 1 *Consider two trains t and t' and two subsets of track-circuits Q^+ and Q^- of $TC_t \cap TC_{t'}$. The following inequality is valid for the RECIFE-MILP model, for any $k \geq 0$ and k -double partition composed of $\Psi_t^{l_1}$ and $\Psi_{t'}^{l_2}$, with $0 \leq l_1 \leq k$ and $0 \leq l_2 \leq k$:*

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| - \left(\sum_{l_1=0}^k l_1 \sum_{r \in \Psi_t^{l_1}} x_{t,r} + \sum_{l_2=0}^k l_2 \sum_{r' \in \Psi_{t'}^{l_2}} x_{t',r'} - k \right). \quad (26)$$

Proof. Assume, without loss of generality, that train t chooses route $\hat{r} \in \Psi_t^{\hat{l}_1}$ ($x_{t,\hat{r}}$ is the only x -variable valued 1 for t) and train t' route $\hat{r}' \in \Psi_{t'}^{\hat{l}_2}$ ($x_{t',\hat{r}'}$

is the only x -variable valued 1 for t' . Indeed, (26) r.h.s. is equal to

$$|Q^+| - (\hat{l}_1 + \hat{l}_2 - k).$$

Then, the theorem is proven as (26) l.h.s. satisfies the following chain of inequalities:

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| - \gamma_{\hat{r},\hat{r}'} \leq |Q^+| - (\hat{l}_1 + \hat{l}_2 - k).$$

The first one is due to Corollary 2. The second one follows the definition of the k -double partition since $\gamma_{\hat{r},\hat{r}'} \geq \max\{0, \hat{l}_1 + \hat{l}_2 - k\}$.

□

Example 8 Consider Figure 1 and $Q^+ = \{tc10\}$ and $Q^- = \{tc15\}$. The matrix reported in Example 7 induces two possible 1-double partitions:

- the 1-double partition $\Psi_t^1 = \{r1, r2\}$, $\Psi_t^0 = \emptyset$, $\Psi_{t'}^1 = \{r3, r4\}$, $\Psi_{t'}^0 = \{r5\}$ that induces valid inequality (19):

$$y_{t,t',tc10} - y_{t,t',tc15} \leq 1 - (x_{t,r1} + x_{t,r2} + x_{t',r3} + x_{t',r4} - 1) = 1 - x_{t',r3} - x_{t',r4};$$

- the 1-double partition $\Psi_t^1 = \{r1\}$, $\Psi_t^0 = \{r2\}$, $\Psi_{t'}^1 = \{r3, r4, r5\}$, $\Psi_{t'}^0 = \emptyset$ that induces the valid inequality

$$y_{t,t',tc10} - y_{t,t',tc15} \leq 1 - (x_{t,r1} + x_{t',r3} + x_{t',r4} + x_{t',r5} - 1) = 1 - x_{t,r1}.$$

Observe that the results of Theorem 1 can be generalized considering Q^+ and Q^- as multisets, including multiple instances of the same track-circuits. In this case, two or more instances of a same track-circuit tc^* have to be regarded as they were different track-circuits when computing how many times Lemma 1 holds for each pair of routes $(r, r') \in R_t \times R_{t'}$. Indeed, these tc^* 's will appear with coefficient two or more in the valid inequality.

5 Cases of dominance between valid inequalities

Inequalities (26) are valid for any possible choice of the sets Q^+ and Q^- and k -double partition. Then, an exponential number of these inequalities can be generated. Indeed, we have $3^{|TC_t \cap TC_{t'}|} - 1$ different choices for the

subsets of track-circuits Q^+ and Q^- included in $TC_t \cap TC_{t'}$, $Q^+ = Q^- = \emptyset$ excluded.

For given sets Q^+ and Q^- , Inequalities (26) are of interest only in case of maximal k -double partitions, being dominated otherwise. Indeed, if for some $0 \leq \hat{l}_1 \leq k$ there is a route $\hat{r} \in \Psi_t^{\hat{l}_1}$ such that $\gamma_{\hat{r}, r'} \geq \max\{0, \hat{l}_1 + 1 + l_2 - k\}$ for all $r' \in \Psi_{t'}^{l_2}$ and $0 \leq l_2 \leq k$, i.e., which can be moved to $\Psi_t^{\hat{l}_1+1}$, we can obtain a stronger inequality by actually moving \hat{r} to this set. Remark that, in general, the inequality obtained for a maximal k -double partition does not dominate the one obtained for a different maximal \hat{k} -double partition. However, some conditions which imply a dominance relation can be identified.

The next lemma that states a sufficient condition under which the dominance relation holds when $\hat{k} = k - 1$.

Lemma 2 *Consider a maximal k -double partition made of the sets $\Psi_t^{l_1}$ and $\Psi_{t'}^{l_2}$ for $l_1, l_2 \in [0, k]$ and a maximal $(k-1)$ -double partition made of the sets $\hat{\Psi}_t^{l_1}$ and $\hat{\Psi}_{t'}^{l_2}$ for $l_1, l_2 \in [0, k-1]$. If they are such that: i) $\Psi_{t'}^0 = \emptyset$, ii) $\hat{\Psi}_{t'}^{l_2} = \Psi_{t'}^{l_2+1}$ for $l_2 \in [0, k-1]$ and iii) $\hat{\Psi}_t^{l_1} \subseteq \bigcup_{l \in [l_1, k]} \Psi_t^l$ for $l_1 \in [0, k-1]$, then the Inequality (26) based on the k -double partition dominates the one based on the $(k-1)$ -double partition.*

Proof. Indeed, the inequalities based on the two double partitions are formulated as

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| + k - \sum_{l=0}^k l \left(\sum_{r \in \Psi_t^l} x_{t,r} + \sum_{r' \in \Psi_{t'}^l} x_{t',r'} \right) \quad (27)$$

and

$$\sum_{tc \in Q^+} y_{t,t',tc} - \sum_{tc \in Q^-} y_{t,t',tc} \leq |Q^+| + (k-1) - \sum_{l=0}^{k-1} l \left(\sum_{r \in \hat{\Psi}_t^l} x_{t,r} + \sum_{r' \in \hat{\Psi}_{t'}^l} x_{t',r'} \right). \quad (28)$$

Inequality (27) dominates (28) if the r.h.s. of the former is less than or equal to the r.h.s. of the latter. By hypothesis ii), in (28),

$$\sum_{l=0}^{k-1} l \sum_{r' \in \hat{\Psi}_{t'}^l} x_{t',r'} = \sum_{l=1}^k (l-1) \sum_{r' \in \Psi_{t'}^l} x_{t',r'} = \sum_{l=1}^k l \sum_{r' \in \Psi_{t'}^l} x_{t',r'} - \sum_{l=1}^k \sum_{r' \in \Psi_{t'}^l} x_{t',r'}.$$

Hence the comparison between the two r.h.s.'s reduces to

$$-\sum_{l=0}^k l \sum_{r \in \Psi_t^l} x_{t,r} \leq -1 + \underbrace{\sum_{l=1}^k \sum_{r' \in \Psi_{t'}^l} x_{t',r'}}_{1 - \sum_{r' \in \Psi_{t'}^0} x_{t',r'} = 1 \text{ for hypothesis i)} - \sum_{l=0}^{k-1} l \sum_{r \in \hat{\Psi}_t^l} x_{t,r}$$

that is

$$\sum_{l=0}^k l \sum_{r \in \Psi_t^l} x_{t,r} \geq \sum_{l=0}^{k-1} l \sum_{r \in \hat{\Psi}_t^l} x_{t,r}. \quad (29)$$

Inequality (29) holds as for all $r \in R_t$ as the coefficient of variable $x_{t,r}$ in the r.h.s. is by hypothesis iii) less than or equal to its coefficient in the l.h.s.. \square

Next, we present some situations of dominance between instances of Inequalities (26) characterized by the same value of k , but possibly considering different sets Q^+ and Q^- .

Initially consider two instances of Inequalities (26) associated to the same k -double partition and respectively to the sets Q_1^+, Q_1^- and Q_2^+, Q_2^- , with $Q_1^+ \supseteq Q_2^+$ or $Q_1^- \supseteq Q_2^-$. The second inequality dominates the first one. Indeed, the difference of the r.h.s.'s of the two inequalities is $|Q_1^+| - |Q_2^+|$, whereas the differences of the l.h.s.'s is $\sum_{tc \in Q_1^+ \setminus Q_2^+} y_{t,t',tc} - \sum_{tc \in Q_1^- \setminus Q_2^-} y_{t,t',tc}$, which is certainly less than or equal to $|Q_1^+| - |Q_2^+|$.

When $k = l_1 = l_2 = 0$ Inequality (26) becomes trivially satisfied. Indeed, for $Q^+ = \emptyset$ and $|Q^-| = 1$, it corresponds to $-y_{t,t',tc} \leq 0$, and for $|Q^+| = 1$ and $Q^- = \emptyset$, it corresponds to $y_{t,t',tc} \leq 1$. These two inequalities dominate all the other instances defined by a 0-double partition.

Consider now Inequality (26) associated to $Q^+ = \emptyset$, but for $k \geq 1$. Lemma 1 may hold only with respect to Condition 2 for all $tc \in Q^-$: all the other conditions of the lemma require the presence of at least a track-circuit in Q^+ . As a consequence, $\Psi_{t'}^k = R_{t'}$ and $\Psi_{t'}^l = \emptyset$ for $0 \leq l \leq k-1$. In turn, this implies $\sum_{r' \in \Psi_{t'}^k} x_{t',r'} = 1$ and $\sum_{r' \in \Psi_{t'}^l} x_{t',r'} = 0$ for $0 \leq l \leq k-1$. Also, each Ψ_t^l is made up of routes of t that do not include at least l track-circuits of Q^- , for $0 \leq l \leq k$. Then, if $Q^+ = \emptyset$, the instances of Inequality (26) have

the following structure:

$$-\sum_{tc \in Q^-} y_{t,t',tc} \leq -\sum_{l=0}^k l \sum_{r \in \Psi_t^l} x_{t,r}. \quad (30)$$

In particular, when $|Q^-| = 1$ and $k = 1$, (30) becomes

$$-y_{t,t',tc} \leq -\sum_{r \in \Psi_t^1 = \{r \in R_t : tc \notin TC^r\}} x_{t,r}. \quad (31)$$

These last instances define the circumstances for which $y_{t,t',tc}$ is necessarily equal to one. Similarly, for $|Q^-| > 1$, (30) imposes that for $r \in \Psi_t^l$ the sum of $y_{t,t',tc}$ is necessarily greater than or equal to l , for $0 \leq l \leq k$. Hence, each inequality obtained for a set Q^- such that $|Q^-| > 1$ is dominated by the collection of Inequalities (31), one for each $tc \in Q^-$.

However, consider the symmetrical situation, i.e., instances of Inequality (26) with $Q^- = \emptyset$: for all $tc \in Q^+$ Lemma 1 may hold only with respect to Condition 1. All the other conditions of the lemma require the presence of at least a track-circuit in Q^- . Then, for $|Q^+| = 1$ and $k = 1$, we have:

$$y_{t,t',tc} \leq 2 - \sum_{r \in \Psi_t^1 = \{r \in R_t : tc \in TC^r\}} x_{t,r} - \sum_{r' \in \Psi_{t'}^1 = \{r' \in R_{t'} : tc \notin TC^{r'}\}} x_{t',r'}. \quad (32)$$

These instances define the circumstances for which $y_{t,t',tc}$ is necessarily equal to 0. Differently from the previous case, they do not dominate inequalities for $|Q^+| > 1$ and $Q^- = \emptyset$, when $k < |Q^+|$.

Finally, consider instances of Inequality (26) with $k = |Q^+| + |Q^-| \geq 2$. Indeed, conditions $\Phi_1(r, r') = Q^+$ and $\Phi_2(r, r') = Q^-$ must hold for any pair of routes $(r, r') \in \Psi_t^k \times \Psi_{t'}^k$, as any track-circuit in $Q^+ \cup Q^-$ can be used to satisfy at most one condition of Lemma 1. Following what mentioned before the definition of $\Phi_1(r, r')$ and $\Phi_2(r, r')$, the value of the y -variables is determined (0 and 1 respectively) if a pair of routes in $\Psi_t^k \times \Psi_{t'}^k$ is chosen. This implies that these instances are dominated by instances of (31) and (32).

6 Binary variables reduction

In this section, we show how inequalities of type (26), together with the equivalence between track-circuits defined in Section 3, may allow a reduction of the number of variables $y_{t,t',tc}$ in the RECIFE-MILP model. Specifically, in Section 3, after the introduction of the track-circuit equivalence

relation, we have observed that, if $tc \sim_{r,r'} \hat{tc}$, variable $y_{t,t',\hat{tc}}$ can play the role of variable $y_{t,t',tc}$ provided that the two trains t and t' respectively choose routes r and r' . Next, we extend the concept of equivalence to cover all the possible route choices.

To do so, we first introduce the concept of representative track-circuit. Then we illustrate the reformulation of the RECIFE-MILP model which allows the reduction of the number of binary variables exploiting representativeness, and we discuss how this number can be minimized.

6.1 Representative track-circuit

For any $tc \in TC_t \cap TC_{t'}$, we say that $\hat{tc} \in TC_t \cap TC_{t'}$ is *representative* of tc if it satisfies the following conditions:

- there exists at least a pair of routes $r \in R_t$ and $r' \in R_{t'}$ such that $tc, \hat{tc} \in TC^r \cap TC^{r'}$;
- for all pairs of routes $r \in R_t$ and $r' \in R_{t'}$ such that $tc, \hat{tc} \in TC^r \cap TC^{r'}$, there exists $S \in S^{r,r'}$ such that $tc, \hat{tc} \in S$, or equivalently $tc \sim_{r,r'} \hat{tc}$.

In words, both track-circuits are included in one or more pairs of routes and, when this happens, the precedence relation on the two must be the same. Remark that any tc is always representative of itself, and that it may have several different representatives.

Example 9 *In Figure 1, for example, consider $tc12$, $tc14$ and $tc17$ as candidate representatives of $tc10$. They all belong to $TC^{r2} \cap TC^{r4}$ and $tc10 \sim_{r2,r4} tc12$, $tc10 \sim_{r2,r4} tc14$, $tc10 \sim_{r2,r4} tc17$. Then, observe that $tc10$, $tc12$, and $tc17$ also belong to $TC^{r2} \cap TC^{r5}$, but it only holds that $tc10 \sim_{r2,r5} tc12$ ($tc10 \not\sim_{r2,r5} tc17$). In addition, note that there is no other pair of routes r and r' such that both $tc10$ and any of the other track-circuits belong to $TC^r \cap TC^{r'}$. Hence, $tc12$ and $tc14$ are representative of $tc10$ but $tc17$ is not. Extending this reasoning to all track-circuits, we can say that all the representatives of $tc10$ are: $tc10$, $tc11$, $tc12$, $tc13$, $tc14$.*

Give the above definition, if \hat{tc} is representative of tc then $y_{t,t',\hat{tc}}$ can play the role of $y_{t,t',tc}$ for any pair of routes $r \in R_t$ $r' \in R_{t'}$ such that $tc, \hat{tc} \in TC^r \cap TC^{r'}$. In what follows, we show how inequalities of type (26)

Table 1: Conditions of Lemma 1 that hold when $Q^+ = \{tc\}$ and $Q^- = \{\hat{tc}\}$ where \hat{tc} is representative of tc .

	$tc \notin TC^{r'}$	$tc \in TC^{r'} \wedge \hat{tc} \in TC^{r'}$	$tc \in TC^{r'} \wedge \hat{tc} \notin TC^{r'}$
$\hat{tc} \notin TC^r$	Condition 2	Condition 2	Condition 2
$tc \in TC^r \wedge \hat{tc} \in TC^r$	Condition 1	Condition 3	-
$tc \notin TC^r \wedge \hat{tc} \in TC^r$	-	-	-

for $k = 1$ allow the substitution of $y_{t,t',tc}$ with $y_{t,t',\hat{tc}}$ whatever pair of routes is used by the two trains.

Consider $Q^+ = \{tc\}$ and $Q^- = \{\hat{tc}\}$ where \hat{tc} is representative of tc and let us define a 1-double partition composed by

$$\begin{aligned}\Psi_t^1 &= \{r \in R_t : tc \in TC^r \vee \hat{tc} \notin TC^r\} \\ \Psi_{t'}^1 &= \{r' \in R_{t'} : tc \notin TC^{r'} \vee \hat{tc} \in TC^{r'}\} \\ \Psi_t^0 &= \{r \in R_t : (tc \notin TC^r \wedge \hat{tc} \in TC^r)\} \\ \Psi_{t'}^0 &= \{r' \in R_{t'} : (tc \in TC^{r'} \wedge \hat{tc} \notin TC^{r'})\}.\end{aligned}$$

Indeed, as summarized in Table 1,

- for (r, r') such that $\hat{tc} \notin TC^r$, Condition 2 of Lemma 1 holds;
- for (r, r') such that $tc \in TC^r \wedge \hat{tc} \in TC^r$ and $tc \notin TC^{r'}$, Condition 1 of Lemma 1 holds;
- for (r, r') such that $tc \in TC^r \wedge \hat{tc} \in TC^r$ and $tc \in TC^{r'} \wedge \hat{tc} \in TC^{r'}$, Condition 3 of Lemma 1 holds since, by definition of representative track-circuit, tc and \hat{tc} are in the same section;
- for (r, r') such that $tc \in TC^r \wedge \hat{tc} \in TC^r$ and $tc \in TC^{r'} \wedge \hat{tc} \notin TC^{r'}$ no condition of Lemma 1 applies;
- for (r, r') such that $tc \notin TC^r \wedge \hat{tc} \in TC^r$, no condition of Lemma 1 applies.

With this 1-double partition, Inequality (26) becomes:

$$y_{t,t',tc} - y_{t,t',\hat{tc}} \leq 2 - \sum_{r \in \Psi_t^1} x_{t,r} - \sum_{r' \in \Psi_{t'}^1} x_{t',r'}. \quad (33)$$

Then, we observe that the l.h.s. of (33) can be transformed as:

$$\begin{aligned}
2 - \sum_{r \in \Psi_t^1} x_{t,r} - \sum_{r' \in \Psi_{t'}^1} x_{t',r'} &= (1 - \sum_{r \in \Psi_t^1} x_{t,r}) + (1 - \sum_{r' \in \Psi_{t'}^1} x_{t',r'}) = \\
&= \sum_{r \in R_t \setminus \Psi_t^1} x_{t,r} + \sum_{r' \in R_{t'} \setminus \Psi_{t'}^1} x_{t',r'} = \sum_{r \in \Psi_t^0} x_{t,r} + \sum_{r' \in \Psi_{t'}^0} x_{t',r'} = \\
&= \sum_{r \in R_t: tc \notin TC^r \wedge \hat{tc} \in TC^r} x_{t,r} + \sum_{r' \in R_{t'}: tc \in TC^{r'} \wedge \hat{tc} \notin TC^{r'}} x_{t',r'}
\end{aligned}$$

and, hence we can rewrite the whole Inequality (33) as

$$y_{t,t',tc} \leq y_{t,t',\hat{tc}} + \sum_{r \in R_t: tc \notin TC^r \wedge \hat{tc} \in TC^r} x_{t,r} + \sum_{r' \in R_{t'}: tc \in TC^{r'} \wedge \hat{tc} \notin TC^{r'}} x_{t',r'}. \quad (34)$$

By setting $Q^+ = \{\hat{tc}\}$ and $Q^- = \{tc\}$ and using symmetric arguments, we can derive also the following inequality from (26):

$$y_{t,t',\hat{tc}} - \sum_{r \in R_t: tc \in TC^r \wedge \hat{tc} \notin TC^r} x_{t,r} - \sum_{r' \in R_{t'}: tc \notin TC^{r'} \wedge \hat{tc} \in TC^{r'}} x_{t',r'} \leq y_{t,t',tc} \quad (35)$$

Remark that a track-circuit tc may have several representative \hat{tc} 's. Each \hat{tc} covers a specific set of route pairs for t and t' , i.e., all the pairs of routes (r, r') such that both tc and \hat{tc} belong to both r and r' . We say that a set of track circuits is a *representative set* of tc if it is a minimal set that includes at least one representative \hat{tc} in $TC^r \cap TC^{r'}$ for any pair of routes $r \in R_t$ and $r' \in R_{t'}$ that include tc . In other words, let $Rep_{t,t'}(tc)$ be the set of all possible representative sets of tc , the generic i -th representative set $Rep_{t,t'}^i(tc) \in Rep_{t,t'}(tc)$ is a minimal set such that $Rep_{t,t'}^i(tc) \cap S \neq \emptyset$ for any $S \in \mathcal{S}_{t,t',tc}$. Consequently, for any pair of routes, we can find a $\hat{tc} \in Rep_{t,t'}^i(tc)$ such that $y_{t,t',\hat{tc}}$ plays the role of $y_{t,t',tc}$. Remark that at least the representative set $\{tc\}$ exists for any $tc \in TC_t \cap TC_{t'}$. Let $Rep_{t,t'}(tc)$ be the set of all possible representative sets of tc .

Example 10 In Figure 1, for example, possible representative sets are $Rep_{t,t'}^0(tc10) = \{tc10, tc11, tc12, tc13, tc14\}$, $Rep_{t,t'}^1(tc10) = \{tc11, tc13, tc14\}$ and $Rep_{t,t'}^2(tc10) = \{tc10\}$. Indeed, $tc10$ belongs to two sections: S_2 and S_4 . To be sure of covering both, any representative set of $tc10$ needs to include at least one of $tc10$, $tc11$ or $tc12$.

6.2 Constraints reformulation and variables reduction

Now observe that the combination of Constraints (13) concerning a track-circuit tc and Inequalities (35) implies

$$eU_{t,tc} - M(1 - y_{t,t',\hat{t}c} + \sum_{\substack{r \in R_t: \\ tc \in TC^r \\ \wedge \hat{t}c \notin TC^r}} x_{t,r} + \sum_{\substack{r' \in R_{t'}: \\ tc \notin TC^{r'} \\ \wedge \hat{t}c \in TC^{r'}}} x_{t',r'}) \leq sU_{t',tc}. \quad (36)$$

Analogously, the combination of Constraints (14) and Inequalities (34) implies

$$eU_{t',tc} - M(y_{t,t',\hat{t}c} + \sum_{\substack{r \in R_t: \\ tc \notin TC^r \\ \wedge \hat{t}c \in TC^r}} x_{t,r} + \sum_{\substack{r' \in R_{t'}: \\ tc \in TC^{r'} \\ \wedge \hat{t}c \notin TC^{r'}}} x_{t',r'}) \leq sU_{t,tc}. \quad (37)$$

The following theorem holds

Theorem 2 *Let two trains t and t' in T be given. Consider track-circuit $tc \in TC_t \cap TC_{t'}$ and a track-circuit $\hat{t}c$ representative of tc . Then, Constraints (36) and (37) are logically equivalent to Constraints (13) and (14). The former constraints can substitute the latter ones in imposing the precedences between times $eU_{t,tc}$ and $sU_{t',tc}$, respectively $eU_{t',tc}$ and $sU_{t,tc}$, if the routes chosen by t and t' include both track-circuits. Otherwise, Constraints (36) and (37) are trivially satisfied.*

Proof. Throughout this proof, let r and r' be the routes chosen by trains t and t' , respectively, i.e., the routes for which $x_{t,r} = 1$ and $x_{t',r'} = 1$.

First, consider the case in which both r and r' include tc and $\hat{t}c$. In this case, the theorem holds for the following reasons. By definition of representative track-circuit, $y_{t,t',tc} = y_{t,t',\hat{t}c}$ since both tc and $\hat{t}c$ are used. In addition, the sum of x -variables in (36) and (37) cancels out. Indeed, in Constraints (36), the sum of the x -variables for train t on the routes which include tc and not $\hat{t}c$ is zero, since the chosen r (for which the x -variable is set to 1) includes both. The same holds for t' and r' : the sum of the x -variables on the routes including $\hat{t}c$ and not tc is zero:

$$\sum_{r \in R_t: tc \in TC^r \wedge \hat{t}c \notin TC^r} x_{t,r} + \sum_{r' \in R_{t'}: tc \notin TC^{r'} \wedge \hat{t}c \in TC^{r'}} x_{t',r'} = 0.$$

Analogous argument holds for Constraints (37).

Then, suppose that the route chosen by at least one train does not include both tc and $\hat{t}c$. In this case, we show that Constraints (36) and (37) become negligible: they are either equivalent or dominated by $sU_{t',tc} \geq 0$ and $sU_{t,tc} \geq 0$, respectively. We discuss only Constraints (36), as symmetric arguments hold for Constraints (37).

- If $tc \notin TC^r$, i.e., t does not use tc , then (36) is negligible as $eU_{t,tc} = 0$ and the rest of the l.h.s. of the inequality is at most 0, whether t uses or not $\hat{t}c$ and t' uses or not tc or $\hat{t}c$.
- If $tc \in TC^r$, $\hat{t}c \notin TC^r$, then (36) is negligible as $\sum_{r \in R_t: tc \in TC^r \wedge \hat{t}c \notin TC^r} x_{t,r} = 1$ and the second part of the l.h.s. of the inequality is for sure negative, whether t' uses or not tc or $\hat{t}c$.
- If $tc \in TC^r$, $\hat{t}c \in TC^r$, $\hat{t}c \notin TC^{r'}$, then (36) is negligible as $y_{t,t',\hat{t}c} = 0$ as remarked right above this theorem.
- If $tc \in TC^r$, $\hat{t}c \in TC^r$, $\hat{t}c \in TC^{r'}$, $tc \notin TC^{r'}$, then (36) is negligible as $\sum_{r' \in R_{t'}: tc \notin TC^{r'} \wedge \hat{t}c \in TC^{r'}} x_{t',r'} = 1$.

□

If Constraints (36) and (37) are replicated for all $\hat{t}c \in Rep_{t,t'}^i(tc)$, whatever i , then for any route choice of t and t' at least a pair of them will be stringent. In this case, Theorem 2 allows to replace Constraints (13) and (14) with the newly defined ones and variable $y_{t,t',tc}$ can be eliminated. The so obtained reformulation, on the one side, may have significantly less binary variables than the standard model. On the other side, it may have more constraints, if several representative track-circuits are necessary to capture the precedence relation on a single tc depending on the route choices. Moreover, this new formulation may be slightly weaker than the original one, as Constraints (36) and (37) are a surrogate relaxation of Constraints (13) and (14) and Inequalities (34) and (35). However, remark that the presence of big M constants in both types of constraints make in any case the RECIFE-MILP model somehow weak. Then, from a computational perspective, the operation of reducing the number of binary variables appears promising if the consequent number of additional constraints is limited.

6.3 Selection of representative track-circuits

To limit as much as possible the number of variables and constraints of the reformulated RECIFE-MILP model, in this final part of the section, we present a possible approach to minimize the cardinality of the specific $Rep_{t,t'}^i(tc)$ used in the reformulation for each t, t' and tc . Specifically, we are interested in determining the minimal set $H_{t,t'}^*$ of representative track-circuits in $TC_t \cap TC_{t'}$ such that there exists a $Rep_{t,t'}^i(tc) \subseteq H_{t,t'}^*$ for all $tc \in TC_t \cap TC_{t'}$. To this end, for all $tc \in TC_t \cap TC_{t'}$, $S \in S_{t,t',tc}$, let

$$a_{tc,\hat{tc},S} = \begin{cases} 1 & \text{if } \hat{tc} \text{ is a representative of } tc \text{ and } \hat{tc} \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The optimal solution of the following binary programming problem, in the variables

$$z_{\hat{tc}} = \begin{cases} 1 & \text{if } \hat{tc} \in H_{t,t'}^* \\ 0 & \text{otherwise} \end{cases}$$

with $\hat{tc} \in TC_t \cap TC_{t'}$, identifies the elements of $H_{t,t'}^*$:

$$\begin{aligned} \min \quad & \sum_{\hat{tc} \in TC_t \cap TC_{t'}} z_{\hat{tc}} \\ & \sum_{\hat{tc} \in S} a_{tc,\hat{tc},S} z_{\hat{tc}} \geq 1, \quad \forall tc \in TC_t \cap TC_{t'}, S \in S_{t,t',tc} \\ & z_{\hat{tc}} \in \{0, 1\} \quad \forall \hat{tc} \in TC_t \cap TC_{t'} \end{aligned} \quad (38)$$

For each $tc \in TC_t \cap TC_{t'}$, we define the representative set for the reformulation of RECIFE-MILP, $Rep_{t,t'}^*(tc)$, as the minimum cardinality subset of $H_{t,t'}^*$ such that for all $S \in S_{t,t',tc}$, $S \cap Rep_{t,t'}^*(tc) \neq \emptyset$. In words, it is a minimal subset of $H_{t,t'}^*$ that includes a representative track-circuit of tc for each section S that includes tc itself.

Example 11 *In Figure 1, problem (38) has several possible solutions. As an example, one of them is $H_{t,t'}^* = \{tc5, tc10, tc15\}$.*

Problem (38) is NP-hard as it reduces to the hitting set problem (Karp, 1972). However, our experience suggests that its instances can be solved in a fraction of a second by any commercial solver even for rather large infrastructures. Indeed, all the operations involving the definition of set $H_{t,t'}^*$

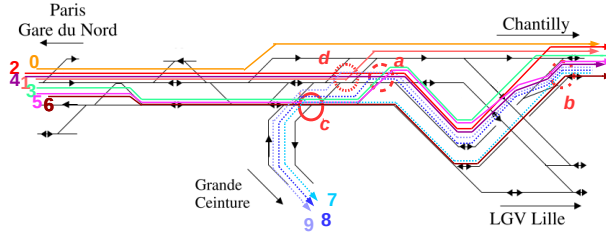


Figure 2: Infrastructure of the Pierrefitte-Gonesse junction, in France, with the indication of the routes available for two trains: 0, 1, 2, 3, 4, 5 and 6 for t , and 7, 8 and 9 for t' . The track-circuits in $H_{t,t'}^*$ are circled in red.

can be performed off-line, once the alternative routes for the trains which may travel along the infrastructure are established *a priori*. Moreover, if two trains t' and t'' share the same set of alternative routes, for any other train t we only need to compute $H_{t,t'}$ and set $H_{t,t'}^* = H_{t,t''}^*$. At maximum, we need to solve problem (38) $(|T| \times (|T| - 1))/2$ times.

Hereinafter, we refer as the *boosted RECIFE-MILP* the reformulation of RECIFE-MILP where only the $y_{t,t',\hat{t}c}$ -variables associated to track-circuits $\hat{t}c \in H_{t,t'}^*$ are defined for each pair of trains $t, t' \in T$, and where we replace Constraints (13) and (14) with Constraints (36) and (37) for all $\hat{t}c \in \text{Rep}_{t,t'}^*(tc), tc \in TC_t \cap TC_{t'}$.

7 Description of facets by valid inequalities

In this section, we present an instance of the rtRTMP representing traffic in a real-world control area, where inequalities of type (18) describe all the facets of the convex hull of the projection of the boosted RECIFE-MILP feasible solution sets in the space of the variables associated to route and precedence decisions.

Consider the infrastructure of the Pierrefitte-Gonesse junction, in France, shown in Figure 2. Let two trains traverse this infrastructure, t and t' . Train t goes from Paris to Chantilly along one of the following seven routes: 0, 1, 2, 3, 4, 5, 6. Train t' goes from Chantilly to Grande Ceinture along one of the following three routes: 7, 8, 9. Given these alternative routes, the minimal set of representative track-circuits $H_{t,t'}^*$ includes the four track-circuits

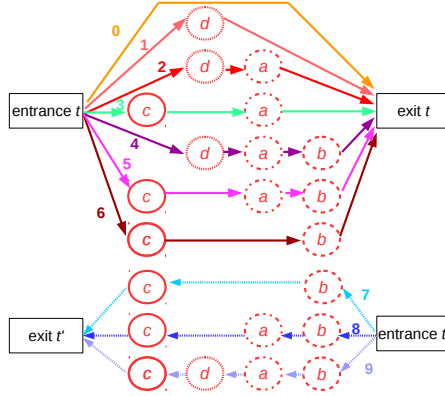


Figure 3: Schematic representation of the track-circuits traversed by trains t and t' traveling in the infrastructure shown in Figure 2. Only the track-circuits in $H_{t,t'}^*$ are considered.

circled in red in the figure, namely, a, b, c, d . Figure 3 depicts a schematic representation of the routes of the trains indicating their passing or not through the track-circuits in $H_{t,t'}^*$.

Next we introduce Inequalities (39) to (74). These inequalities, obtained through Polymake Gawrilow and Joswig (2000), are the facets of the convex hull that includes all the combinations of x and y -variables for track-circuits in $H_{t,t'}^*$ which satisfy Assumption 1. Clearly, these combinations satisfy also Constraints (7) of RECIFE-MILP, and hence Lemma 1, and the non-negativity constraints for variables $x_{t,r} \geq 0$. As we will show, Inequalities (39) to (74) are all instances of Inequality (26).

We preliminarily note that, having 4 representative track-circuits, we can have $3^4 - 1 = 80$ different choices for the sets $|Q^+|$ and $|Q^-|$. Not all of them induce some facets. In the following, we report the inequalities in ascending order of $|Q^+| + |Q^-|$ and of k .

Let us first consider the inequalities where $|Q^+| + |Q^-| = 1$.

Inequality (26) for $Q^+ = \{tc\}$, $tc \in \{a, b, c, d\}$ and $k = 0$ becomes

$$y_{t,t',tc} \leq 1 \quad tc \in \{a, b, c, d\}. \quad (39)$$

Differently, any choice $Q^- = \{tc\}$, $tc \in \{a, b, c, d\}$, and $k = 0$ would induce inequalities of type $y_{t,t',tc} \geq 0$, which do not turn out to be facet as they are dominated by the ones that follow.

In Inequalities (40)-(43), $Q^+ = \emptyset$, $Q^- = \{tc\}$, $tc \in \{a, b, c, d\}$, and $k = 1$.

$$y_{t,t',a} \geq x_{t,0} + x_{t,1} + x_{t,6} \quad (40)$$

$$y_{t,t',b} \geq x_{t,0} + x_{t,1} + x_{t,2} + x_{t,3} \quad (41)$$

$$y_{t,t',c} \geq x_{t,0} + x_{t,1} + x_{t,2} + x_{t,4} \quad (42)$$

$$y_{t,t',d} \geq x_{t,0} + x_{t,3} + x_{t,5} + x_{t,6} \quad (43)$$

Note that these inequalities are particular instances of (31). Consider as an example Inequality (40): it imposes $y_{t,t',a} = 1$ when t does not use a route that includes a .

In Inequalities (44) and (45), $Q^- = \emptyset$ and $Q^+ = \{tc\}$, $tc \in \{a, d\}$, and $k = 1$.

$$y_{t,t',a} \leq 2 - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t',7} \quad (44)$$

$$y_{t,t',d} \leq 2 - x_{t,1} - x_{t,2} - x_{t,4} - x_{t',7} - x_{t',8} \quad (45)$$

These inequalities are particular instances of (32). Consider as an example (45): $\Psi_t^1 = \{1, 2, 4\}$ and $\Psi_{t'}^1 = \{7, 8\}$. It imposes $y_{t,t',d} = 0$ when t uses a route that includes d whilst t' does not. Note that we do not have similar conditions on track-circuits b and c as any route of train t' includes both these track-circuits.

Hereafter, we present inequalities for $|Q^+| + |Q^-| \geq 2$. Then, the last argument of Section 5 guarantees that we will not encounter a facet defined by a k -double partition with $k = |Q^+| + |Q^-|$.

Consider the inequalities where $|Q^+| + |Q^-| = 2$.

In the following one, $Q^+ = \{a, d\}$, $Q^- = \emptyset$ and $k = 1$.

$$y_{t,t',a} + y_{t,t',d} \leq 3 - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t',7} \quad (46)$$

Inequality (46) imposes that at least one between $y_{t,t',a}$ and $y_{t,t',d}$ is equal to zero when t uses a route that includes at least one between a and d , whilst t' uses a route that includes none of them. Note that $Q^+ = \{a, d\}$ is the only set whose cardinality is greater than 1 such that $\{r' \in R_{t'} : TC^{r'} \cap Q^+ = \emptyset\}$ is not empty, as any route r' of t' includes both b and c .

In Inequalities (47)-(54), $|Q^+| = |Q^-| = 1$ and $k = 1$. From (47) to (51) the constant in the inequality is 1 since $\Psi_{t'}^k = R_{t'}$ for the particular sets Q^+

and Q^- considered.

$$y_{t,t',a} - y_{t,t',b} \leq 1 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} \quad (47)$$

$$y_{t,t',a} - y_{t,t',c} \leq 1 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} \quad (48)$$

$$y_{t,t',b} - y_{t,t',c} \leq 1 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,4} - x_{t,5} - x_{t,6} \quad (49)$$

$$y_{t,t',d} - y_{t,t',a} \leq 1 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,4} - x_{t,6} \quad (50)$$

$$y_{t,t',d} - y_{t,t',b} \leq 1 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} \quad (51)$$

$$y_{t,t',a} - y_{t,t',d} \leq 2 - x_{t,0} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',7} - x_{t',9} \quad (52)$$

$$y_{t,t',b} - y_{t,t',a} \leq 2 - x_{t,0} - x_{t,1} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',8} - x_{t',9} \quad (53)$$

$$y_{t,t',b} - y_{t,t',d} \leq 2 - x_{t,0} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',9} \quad (54)$$

Next, consider the inequalities where $|Q^+| + |Q^-| = 3$.

In Inequalities (55) and (56), $|Q^+| = 2$, $|Q^-| = 1$ and $k = 1$. Observe that when $Q^+ = \{a, b\}$ and $Q^- = \{c\}$, we can have a k -double partition including $\Psi_t^k = R_t$ and $\Psi_{t'}^k = R_{t'}$, from which the constant value 1 at the r.h.s. of (55).

$$y_{t,t',a} + y_{t,t',b} - y_{t,t',c} \leq 1 \quad (55)$$

$$y_{t,t',b} + y_{t,t',d} - y_{t,t',a} \leq 3 - x_{t,0} - x_{t,1} - x_{t,2} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',8} - x_{t',9} \quad (56)$$

In Inequalities (57)-(61), $|Q^+| = 2$, $|Q^-| = 1$ and $k = 2$. In particular, $|\Psi_t^0| = |\Psi_{t'}^0| = 0$ in (57) and (58), whilst $|\Psi_t^0| = 1$ and $|\Psi_{t'}^0| = 0$ in (59)-(61).

$$y_{t,t',a} + y_{t,t',c} - y_{t,t',b} \leq 2 - x_{t,2} - x_{t,3} - x_{t',7} \quad (57)$$

$$y_{t,t',c} + y_{t,t',d} - y_{t,t',a} \leq 2 - x_{t,1} - x_{t',8} \quad (58)$$

$$y_{t,t',c} + y_{t,t',d} - y_{t,t',b} \leq 2 - x_{t,1} - x_{t,2} - x_{t',7} + x_{t,5} \quad (59)$$

$$y_{t,t',d} + y_{t,t',c} - y_{t,t',b} \leq 2 - x_{t,1} - x_{t,2} - x_{t',8} + x_{t,6} \quad (60)$$

$$y_{t,t',d} + y_{t,t',a} - y_{t,t',b} \leq 2 - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t',7} + x_{t,6} \quad (61)$$

Let us study Inequality (57) as it is the first one obtained for $k = 2$. Here, $Q^+ = \{a, c\}$ and $Q^- = \{b\}$. Looking at Figures 2 and 3, we can observe that Condition 1 of Lemma 1 is satisfied for track-circuit a for the pairs of routes (r, r') with $r \in \{2, 3, 4, 5\}$ and $r' = 7$: a belongs to r and not to r' . For track-circuit c , the same condition is never satisfied since c belongs to all $r' \in R_{t'}$. Condition 2 of Lemma 1 is satisfied for track-circuit b for all pairs (r, r') with $r \in \{0, 1, 2, 3\}$ and $r' \in R_{t'}$: it is only necessary that

Table 2: Validity of Lemma 1 for $Q^+ = \{a, c\}$ and $Q^- = \{b\}$, for Inequality (57).

track-circuits or pairs satisfying at least one condition of Lemma 1				number of times $\gamma_{r,r'}$ in which Lemma 1 holds			
$r \in R_t$	$r' \in R_{t'}$			$r \in R_t$	$r' \in R_{t'}$		
	7	8	9		7	8	9
0	b	b	b	0	1	1	1
1	b	b	b	1	1	1	1
2	a, b	b	b	2	2	1	1
3	a, b	b	b	3	2	1	1
4	a	(a, b)	(a, b)	4	1	1	1
5	a	$(a, b), (c, b)$	(a, b)	5	1	1	1
6	(c, b)	-	-	6	1	0	0

b does not belong to r . The pair (a, b) satisfies Condition 3 of Lemma 1 for (r, r') with $r \in \{4, 5\}$ and $r' \in \{8, 9\}$. The pair (c, b) satisfies the same conditions for the pairs of routes $(5, 8)$ and $(6, 7)$. Finally, Condition 4 of Lemma 1 is never satisfied because b never precedes either a or c along a route $r \in R_t$. Table 2 summarizes these observations on the track-circuits and pairs of track-circuits for which at least one condition of Lemma 1 holds given a pair of routes $r \in R_t$ and $r' \in R_{t'}$ (left-hand side of the table). The right-hand side of the table reports the consequent number of times the lemma holds for each pair of routes, which are needed for computing the k -double partitions. Remark that Lemma 1 holds only once for the pair of routes $(5, 8)$, since both pairs of track-circuits satisfying one of its conditions include b , and Corollary 1 states that they cannot both be counted. As a consequence, in the 2-double partition, $\Psi_t^2 = \{2, 3\}$ and $\Psi_{t'}^2 = \{7\}$: there are all the routes which, when paired, make Lemma 1 hold at least twice. Since all the pairs of routes make the lemma hold at least once, and exactly once in many cases, $\Psi_t^1 = \{0, 1, 4, 5, 6\}$ and $\Psi_{t'}^1 = \{8, 9\}$. Then the r.h.s. of

the inequality can be written as:

$$\begin{aligned}
|Q^+| + k - \left(\sum_{r \in \Psi_t^1} x_{t,r} + \sum_{r' \in \Psi_{t'}^1} x_{t',r'} \right) - 2 \left(\sum_{r \in \Psi_t^2} x_{t,r} + \sum_{r' \in \Psi_{t'}^2} x_{t',r'} \right) &= \\
= 2 + 2 - x_{t,0} - x_{t,1} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',8} - x_{t',9} - 2x_{t,2} - 2x_{t,3} - 2x_{t',7} &= \\
= 4 - \sum_{r \in R_t} x_{t,r} - \sum_{r' \in R_{t'}} x_{t',r'} - x_{t,2} - x_{t,3} - x_{t',7} = 2 - x_{t,2} - x_{t,3} - x_{t',7} &
\end{aligned}$$

For ease of visualization, we use a similar trick for expressing Inequalities (57)-(74), exploiting the fact that, in general, the r.h.s. of (26) can be rewritten as

$$|Q^+| - (k-2) - \sum_{r \in \Psi_t^k} x_{t,r} - \sum_{r' \in \Psi_{t'}^k} x_{t',r'} + \sum_{l=0}^{k-1} (k-1-l) \left(\sum_{r \in \Psi_t^l} x_{t,r} + \sum_{r' \in \Psi_{t'}^l} x_{t',r'} \right).$$

In Inequalities (62)-(66), $|Q^+| = 1$, $|Q^-| = 2$ and $k = 2$. In particular, $|\Psi_t^0| = |\Psi_{t'}^0| = 0$ in (62)-(64), whilst $|\Psi_t^0| = 1$ and $|\Psi_{t'}^0| = 0$ in (65) and (66). Moreover, $\Psi_{t'}^2 = R_{t'}$ in (65), which explains the absence of the coefficient 1 at the r.h.s.. As an example, Table 3 reports the track-circuits satisfying a condition of Lemma 1 and the value of $\gamma_{r,r'}$ for the construction of the 2-double partition used in Inequality (65): $\Psi_t^2 = 0, 1, 2, 3, 4$ and $\Psi_{t'}^2 = R_{t'}$; $\Psi_t^1 = 5$ and $\Psi_{t'}^1 = \emptyset$; $\Psi_t^0 = 6$ and $\Psi_{t'}^0 = \emptyset$.

$$y_{t,t',a} - y_{t,t',b} - y_{t,t',d} \leq 1 - x_{t,0} - x_{t,2} - x_{t,3} - x_{t,5} - x_{t',7} - x_{t',9} \quad (62)$$

$$y_{t,t',a} - y_{t,t',c} - y_{t,t',d} \leq 1 - x_{t,0} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t',7} - x_{t',9} \quad (63)$$

$$y_{t,t',b} - y_{t,t',c} - y_{t,t',d} \leq 1 - x_{t,0} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',9} \quad (64)$$

$$y_{t,t',a} - y_{t,t',b} - y_{t,t',c} \leq -x_{t,0} - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} + x_{t,6} \quad (65)$$

$$y_{t,t',b} - y_{t,t',a} - y_{t,t',c} \leq 1 - x_{t,0} - x_{t,1} - x_{t,4} - x_{t,6} - x_{t',8} - x_{t',9} + x_{t,3} \quad (66)$$

In Inequalities (67)-(69), $|Q^+| + |Q^-| = 4$. Specifically, $|Q^+| = 3$, $|Q^-| = 1$ and $k = 2$ in (67) and (68), and $|Q^+| = 2$, $|Q^-| = 2$ and $k = 2$ in (69). In this latter case, $|\Psi_t^0| = |\Psi_{t'}^0| = 0$.

Table 3: Validity of Lemma 1 for $Q^+ = \{a\}$ and $Q^- = \{b, c\}$, for Inequality (65).

track-circuits or pairs satisfying at least one condition of Lemma 1				number of times $\gamma_{r,r'}$ in which Lemma 1 holds			
$r \in R_t$	$r' \in R_{t'}$			$r \in R_t$	$r' \in R_{t'}$		
	7	8	9		7	8	9
0	b, c	b, c	b, c	0	2	2	2
1	b, c	b, c	b, c	1	2	2	2
2	a, b, c	b, c	b, c	2	3	2	2
3	a, b	$b, (a, c)$	$b, (a, c)$	3	2	2	2
4	a, c	$c, (a, b)$	$c, (a, b)$	4	2	2	2
5	a	$(a, b), (a, c)$	$(a, b), (a, c)$	5	1	1	1
6	-	-	-	6	0	0	0

$$y_{t,t',c} + y_{t,t',d} + y_{t,t',a} - y_{t,t',b} \leq 3 - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t',7} \quad (67)$$

$$y_{t,t',c} + y_{t,t',d} + y_{t,t',b} - y_{t,t',a} \leq 3 - x_{t,1} - x_{t,4} - x_{t',8} \quad (68)$$

$$y_{t,t',a} + y_{t,t',b} - y_{t,t',c} - y_{t,t',d} \leq 2 - x_{t,0} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',7} - x_{t',9} \quad (69)$$

In Inequality (70), $Q^- = \{b\}$, $k = 2$ and Q^+ is a multiset including two instances of track-circuit a and one of c :

$$2y_{t,t',a} + y_{t,t',c} - y_{t,t',b} \leq 3 - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t',7} \quad (70)$$

Let us build the k -double partition for this first occurrence of a multiset. Indeed, a belongs to $\Phi_1(r, r')$ if $r \in \{2, 3, 4, 5\}$ and $r'=7$. Since a appears twice in the multiset Q^+ , this implies that the Lemma holds twice for all these routes when paired. For no pairs of routes c belongs to $\Phi_1(r, r')$, since it is in all routes of t' . Finally, b belongs to $\Phi_2(r, r')$ if $r \in \{0, 1, 2, 3\}$ whatever $r' \in R_{t'}$. With $k = 2$, then, we can set $\Psi_t^2 = \{2, 3, 4, 5\}$ and $\Psi_{t'}^2 = \{7\}$. Indeed, for $r' = 7$ and $r \in \{0, 1\}$, Lemma 1 holds exactly once, and these routes of t are in Ψ_t^1 . Moreover, the pair (c, b) satisfies

Table 4: Validity of Lemma 1 for $Q^+ = \{c, d, d\}$ and $Q^- = \{a\}$, for Inequality (71).

track-circuits or pairs satisfying at least one condition of Lemma 1				number of times $\gamma_{r,r'}$ in which Lemma 1 holds			
$r \in R_t$	$r' \in R_{t'}$			$r \in R_t$	$r' \in R_{t'}$		
	7	8	9		7	8	9
0	a	a	a	0	1	1	1
1	a, d, d	a, d, d	a	1	3	3	1
2	d, d	d, d	(d, a)	2	2	2	1
3	-	(c, a)	-	3	0	1	0
4	d, d	d, d	(d, a)	4	2	2	1
5	-	(c, a)	-	5	0	1	0
6	a	a	a	6	1	1	1

Condition 3 of Lemma 1 for $r' = 7$ and $r = 6$. Hence, $\Psi_t^1 = \{0, 1, 6\}$ and, for the symmetric reasoning, $\Psi_{t'}^1 = \{8, 9\}$. As a consequence, $\Psi_t^0 = \Psi_{t'}^0 = \emptyset$.

In Inequality (71), $k = 2$, $Q^+ = \{c, d, d\}$ and $Q^- = \{a\}$:

$$y_{t,t',c} + 2y_{t,t',d} - y_{t,t',a} \leq 3 - x_{t,1} - x_{t,2} - x_{t,4} - x_{t',8} \quad (71)$$

Indeed, we can observe that: c belongs to no $\Phi_1(r, r')$ set because it is in all routes of t' ; $d \in \Phi_1(r, r')$ for $r \in \{1, 2, 4\}$ and $r' \in \{7, 8\}$; $a \in \Phi_2(r, r')$ for $r \in \{0, 1, 6\}$ and $r' \in R_{t'}$; (c, a) satisfies Condition 3 of Lemma 1 for (3,8) and (5,8); (d, a) satisfies the same condition for (2,9) and (4,9). Table 4 summarizes these observations and helps defining the following 2-double partition: $\Psi_t^2 = \{1, 2, 4\}$ and $\Psi_{t'}^2 = \{8\}$; $\Psi_t^1 = \{0, 3, 5, 6\}$ and $\Psi_{t'}^1 = \{7, 9\}$; $\Psi_t^0 = \emptyset$ and $\Psi_{t'}^0 = \emptyset$.

For $|Q^+| + |Q^-| = 5$, we have Inequality (72) for $k = 2$, $Q^- = \{b\}$ and Q^+ multiset including two instances of track-circuit a in addition to c and d :

$$2y_{t,t',a} + y_{t,t',c} + y_{t,t',d} - y_{t,t',b} \leq 4 - x_{t,1} - x_{t,2} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t',7} \quad (72)$$

Here, as summarized in Table 5: c never belongs to a $\Phi_1(r, r')$ because it is in all routes of t' ; $d \in \Phi_1(r, r')$ for $r \in \{1, 2, 4\}$ and $r' \in \{7, 8\}$; $a \in \Phi_1(r, r')$

for $r \in \{2, 3, 4, 5\}$ and $r' \in \{7\}$; $b \in \Phi_2(r, r')$ for $r \in \{0, 1, 2, 3\}$ and $r' \in R_{t'}$; (c, b) satisfies Condition 3 of Lemma 1 for (6,7); (a, b) satisfies the same condition for (4,8), (4,9), (5,8) and (5,9); (d, b) does so for (4,9). Hence a 2-double partition is: $\Psi_t^2 = \{1, 2, 3, 4, 5\}$ and $\Psi_{t'}^2 = \{7\}$ $\Psi_t^1 = \{0, 6\}$ and $\Psi_{t'}^1 = \{8, 9\}$ $\Psi_t^0 = \emptyset$ and $\Psi_{t'}^0 = \emptyset$.

In Inequality (73), $k = 3$, $Q^- = \{b, d\}$ and Q^+ is the multiset $\{a, a, c\}$.

$$2y_{t,t',a} + y_{t,t',c} - y_{t,t',b} - y_{t,t',d} \leq 2 - x_{t,2} - x_{t,3} - x_{t,5} - x_{t',7} + x_{t,1} + x_{t',8} \quad (73)$$

This inequality is defined according to the following 3-double partition: $\Psi_t^3 = \{2, 3, 5\}$ and $\Psi_{t'}^3 = \{7\}$; $\Psi_t^2 = \{0, 4, 6\}$ and $\Psi_{t'}^2 = \{9\}$; $\Psi_t^1 = \{1\}$ and $\Psi_{t'}^1 = \{8\}$; $\Psi_t^0 = \emptyset$ and $\Psi_{t'}^0 = \emptyset$. In fact (Table 6): $a \in \Phi_1(r, r')$ for $r \in \{2, 3, 4, 5\}$ and $r' \in \{7\}$; $c \in \Phi_1(r, r')$ never since it is in all $r' \in R_{t'}$; $b \in \Phi_2(r, r')$ for $r \in \{0, 1, 2, 3\}$ and $r' \in R_{t'}$; $d \in \Phi_2(r, r')$ for $r \in \{0, 3, 5, 6\}$ and $r' \in R_{t'}$; Condition 3 of Lemma 1 is satisfied by (a, b) for (4,8), (4,9), (5,8), (5,9), by (a, d) for (2,9) and (4,9), and (c, b) for (6,7).

Finally, in Inequality (74), $k = 3$, $Q^+ = \{a, b\}$ and Q^- is the multiset $\{c, d, d\}$.

$$y_{t,t',a} + y_{t,t',b} - y_{t,t',c} - 2y_{t,t',d} \leq 1 - x_{t,0} - x_{t,3} - x_{t,4} - x_{t,5} - x_{t,6} - x_{t',9} + x_{t,1} + x_{t',8} \quad (74)$$

To define it, observe that: $a \in \Phi_1(r, r')$ for $r \in \{2, 3, 4, 5\}$ and $r' \in \{7\}$; b is in no set $\Phi_1(r, r')$ because it is in all $r' \in R_{t'}$; $c \in \Phi_2(r, r')$ for $r \in \{0, 1, 2, 4\}$ and $r' \in R_{t'}$; $d \in \Phi_2(r, r')$ for $r \in \{0, 3, 5, 6\}$ and $r' \in R_{t'}$; Condition 3 of Lemma 1 is satisfied by (a, c) for (3,8) and (5,8), by (a, d) for (2,9) and (4,9), by (b, c) for (6,7) and (5,8), and by (b, d) for (4,9), Condition 4 of Lemma 1 is satisfied by (a, c) for (3,9) and (5,9), and by (b, c) for (6,8), (6,9) and (5,9). Table 7 summarizes these observations and allows identifying the 3-double partition considered in the inequality: $\Psi_t^3 = \{0, 3, 4, 5, 6\}$ and $\Psi_{t'}^3 = \{9\}$; $\Psi_t^2 = \{2\}$ and $\Psi_{t'}^2 = \{7\}$; $\Psi_t^1 = \{1\}$ and $\Psi_{t'}^1 = \{8\}$; $\Psi_t^0 = \emptyset$ and $\Psi_{t'}^0 = \emptyset$.

8 Experimental analysis

In this section we describe the experimental analysis we performed to assess the performance of three boosted versions of RECIFE-MILP taking the standard RECIFE-MILP as a benchmark. Specifically, we consider boosted

Table 5: Validity of Lemma 1 for $Q^+ = \{a, a, c, d\}$ and $Q^- = \{b\}$, for Inequality (72).

track-circuits or pairs satisfying
at least one condition of Lemma 1

$r \in R_t$	$r' \in R_{t'}$		
	7	8	9
0	b	b	b
1	d, b	d, b	b
2	a, a, d, b	d, b	b
3	a, a, b	b	b
4	a, a, d	$d, (a, b),$ (a, b)	$(a, b), (a, b),$ (d, b)
5	a, a	$(a, b), (a, b)$	$(a, b), (a, b)$
6	(c, b)	-	-

number of times $\gamma_{r,r'}$ in
which Lemma 1 holds

$r \in R_t$	$r' \in R_{t'}$		
	7	8	9
0	1	1	1
1	2	2	1
2	4	2	1
3	3	1	1
4	3	2	1
5	2	1	1
6	1	0	0

Table 6: Validity of Lemma 1 for $Q^+ = \{a, a, c\}$ and $Q^- = \{b, d\}$, for Inequality (73).

track-circuits or pairs satisfying at least one condition of Lemma 1				number of times $\gamma_{r,r'}$ in which Lemma 1 holds			
$r \in R_t$	$r' \in R_{t'}$			$r \in R_t$	$r' \in R_{t'}$		
	7	8	9		7	8	9
0	b, d	b, d	b, d	0	2	2	2
1	b	b	b	1	1	1	1
2	a, a, b	b	$b, (a, d),$ (a, d)	2	3	1	2
3	$a, a, b,$ d	b, d	b, d	3	4	2	2
4	a, a	$(a, b),$ (a, b)	$(a, b), (a, b),$ $(a, d), (a, d)$	4	2	2	1
5	a, a, d	$d, (a, b),$ (a, b)	$d, (a, b),$ (a, b)	5	3	2	2
6	$b, (c, b)$	d	d	6	1	1	1

Table 7: Validity of Lemma 1 for $Q^+ = \{a, b\}$ and $Q^- = \{c, d, d\}$, for Inequality (74).

track-circuits or pairs satisfying at least one condition of Lemma 1				number of times $\gamma_{r,r'}$ in which Lemma 1 holds			
$r \in R_t$	$r' \in R_{t'}$			$r \in R_t$	$r' \in R_{t'}$		
	7	8	9		7	8	9
0	c, d, d	c, d, d	c, d, d	0	3	3	3
1	c	c	c	1	1	1	1
2	a, c	c	$c, (a, d)$	2	2	1	2
3	a, d, d	$d, d,$ (a, c)	$d, d,$ (a, c)	3	3	3	3
4	a, c	c	$c, (a, d),$ (b, d)	4	2	1	3
5	a, d, d	$d, d,$ $(a, c), (b, c)$	$d, d,$ $(a, c), (b, c)$	5	3	3	3
6	$d, d,$ (b, c)	$d, d,$ (b, c)	$d, d,$ (b, c)	6	3	3	3

RECIFE-MILP, where we reduce the number of binary variables and reformulate the constraints are explained in Section 6. Then, we add to this formulation all the inequalities of type (26) which are obtainable with sets Q^+ and Q^- of total cardinality of at most one, and we obtain *all-ineq. RECIFE-MILP*. Here, for each considered pair of sets Q^+ and Q^- we define one inequality, based on the following procedure for the identification of a k -double partition. We consider the routes of R_t and $R_{t'}$ in increasing order of their index, and we take the first pair $(r, r') \in R_t \times R_{t'}$ for which Lemma 1 holds at least k times: we put r in Ψ_t^k and r' in $\Psi_{t'}^k$. Then, we take the routes $r \in R_t$ one by one, a part from the one already in Ψ_t^k , and we include them in the sets $\Psi_t^{l_1}$ where l_1 is the maximal value less than or equal to k such that Lemma (1) holds at least l_1 times for (r, r') , r' being the route previously inserted in $\Psi_{t'}^k$. When no more $r \in R_t$ remain, we take the routes $r' \in R_{t'}$ one by one, excluding the one already in $\Psi_{t'}^k$. We include each of them in the set $\Psi_{t'}^{l_2}$, with l_2 maximal value not greater than k such that Lemma (1) holds at least $\max\{0, l_1 + l_2 - k\}$ times for (r, r') , for all routes r in $\Psi_t^{l_1}$ and for all $0 \leq l_1 \leq k$. It can be proven that the so obtained k -double partition is maximal. After building these k -double partitions, we eliminate the ones which are dominated by others according to Lemma 2. Moreover, to account for the fact that adding many inequalities may strongly increase the computational burden of the solution process, we consider *ineq. RECIFE-MILP*. Here, we select the inequalities to be added to the model after the solution of the root node in the branch and bound: we add all the inequalities obtained as in *all-ineq. RECIFE-MILP* including at least one y -variable whose value in the relaxed solution is between 0.01 and 0.99.

The algorithms are implemented in C++, exploiting the IBM CPLEX MILP solver v 12.6 through the CPLEX Concert technology. We run the experiments on a computer with eight Intel Xeon 3.5 Ghz processors and 128 GB RAM, and we allow three wall clock minutes of computational time to each run.

We assess the performance of the algorithms on instances representing traffic in four French control areas. For each of them, we consider a whole-day timetable and we create traffic perturbations by delaying the entrance in the infrastructure of 20% of trains, between 5 and 15 minutes. The selec-

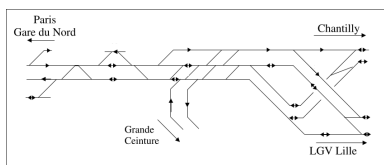


Figure 4: Representation of the Gonesse control area.

tion of the trains and the specific delays is random, considering a uniform probability distribution. By repeating the random selection, we obtain 100 whole-day perturbed scenarios per control area. From each of them, we generate one instance by considering all the trains entering the control area between 7:00 and 8:00 am, which is the morning peak-hour.

The first control area considered is the Pierrefitte-Gonesse junction (from now on *Gonesse*) shown in Figure 4. It is a complex junction about 18 km long where a freight, a conventional and a high speed line cross. The second control area is a portion of the line around the station of Rouen-Rive-Droite (from now on *Rouen*), approximately 27 km long. It is schematized in Figure 5. Also in this case, the traffic is mix including freight, passenger conventional and high speed trains. Six stations with two to six platforms are included in this control area, in addition to a junction. The third control area studied is the one including the Lille Flandres station (from now on *Lille*), measuring in total 12 km. This is a terminal station linked to national and international lines, with 17 platforms used by local, intercity and high speed trains. It is shown in Figure 6. Finally, the fourth control area includes the St. Lazare station in Paris (from now on *St. Lazare*). It is a terminal station area of slightly more than 4 km, with 27 platforms. It is depicted in Figure 7. Some characteristics of the instances representing traffic in each of these control areas are reported in Table 8. Table 9 summarizes the size of the formulation of standard, boosted, ineq. and all-ineq. RECIFE-MILP. As it can be observed, on the one hand, the number of binary variables obtained through the reformulation reported in Section 6 is remarkable: it goes from 35% to 64%, Rouen being impacted the least and St. Lazare the most. On the other hand, the increase of the number of constraints is almost negligible: it is null for Gonesse and Rouen and it is extremely small for Lille and St. Lazare.

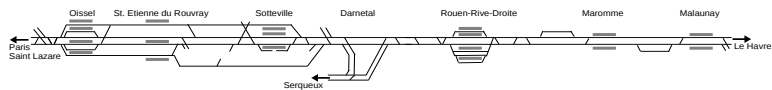


Figure 5: Representation of the Rouen control area.

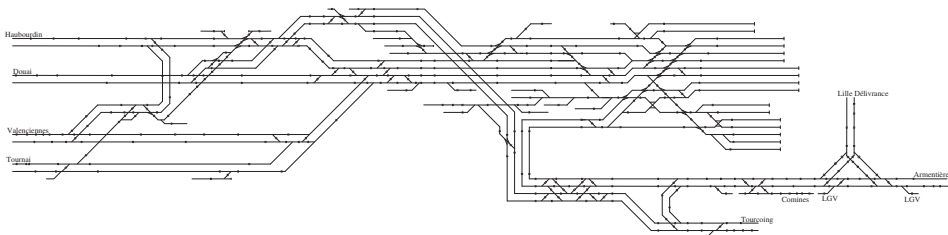


Figure 6: Representation of the Lille control area.

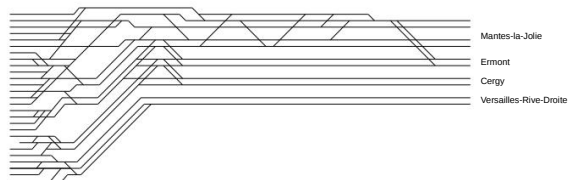


Figure 7: Representation of the St. Lazare control area.

Table 8: Characteristics of the instances for the four control areas considered

	Gonesse	Rouen	Lille	St. Lazare
track-circuits	89	190	299	212
block sections	79	176	829	197
average # trains per instance	26	12	48	101
average # routes per train	7	53	14	5

Table 9: Average model size for the four control areas considered

	Gonesse	Rouen	Lille	St. Lazare
continuous variables	15 021	86 819	37 467	24 301
standard RECIFE-MILP				
binary variables	3 299	1 282	8 004	12 158
constraints	72 217	261 427	137 875	114 606
boosted, ineq. and all-ineq. RECIFE-MILP				
binary variables	1 305	838	4 121	4 381
constraints	72 217	261 427	137 927	114 695
ineq. RECIFE-MILP				
inequalities	0 to 2	0 to 14	0 to 6	46 to 131
all-ineq. RECIFE-MILP				
inequalities	1103 to 1944	138 to 314	4660 to 6704	4925 to 6825

Table 10: Average results over 100 instances at each control area: final objective function value and computational time (seconds)

RECIFE-MILP	Gonesse		Rouen		Lille		St. Lazare	
	obj	time	obj	time	obj	time	obj	time
standard	334	90	100	82	3178	180	9633	180
boosted	323	66	85	80	3254	180	9406	180
ineq.	321	67	94	134	3198	180	9505	180
all-ineq.	327	67	93	131	3329	180	9502	180

Table 11: Statistical significance of the difference between algorithms behavior according to the Wilcoxon rank-sum test (confidence level of 0.95): an X indicates when the algorithm corresponding to the line is significantly better than the one corresponding to the column

RECIFE-MILP	Gonesse				Rouen				Lille				St. Lazare			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
standard (1)	-				-				-	X		X	-			
boosted (2)	X	-		X	X	-						X	X	-		
ineq. (3)	X		-	X			-		X	-	X		X		-	
all-ineq. (4)				-				-				-	X			-

Table 10 summarizes the results obtained for each control area by the four RECIFE-MILP algorithms. In particular, it reports the average objective function value and computational time. Recall that when the optimality of a solution is proven, the algorithm stops even if it has not consumed the whole available computational time. When this happens for some instances the average computational time is less than 180 seconds. Table 11 shows whether the difference in the behavior of pairs of algorithms is statistical significant according to the Wilcoxon rank-sum test with a confidence level of 0.95. Here, for each control area, we report an X if the algorithm indexing the line achieves results which are significantly better than the algorithm indexing the column. Recall that if the difference between two algorithms is significant, for example in favor of the first one, it means that if we draw a novel sample from the same distribution of results, i.e., if we solve of a further instance with similar characteristics, we will in principle have the first algorithm performing better than the second. If no significant difference is observed, we have no reason to expect one to be better than the other on a further sample.

The results show that no RECIFE-MILP variant is always better than another. In particular, for Gonesse and St. Lazare both boosted and ineq. RECIFE-MILP are significantly better than the standard and all-ineq. ones, but they are not different from each other. For Rouen the only significant difference is between boosted and standard RECIFE-MILP, with ineq. and

all-ineq. being between the two and not really distinguishable from any of them. Differently, for Lille standard RECIFE-MILP is, together with the ineq. one, the best algorithm; the worst is all-ineq. RECIFE-MILP. Finally, for St. Lazare the standard RECIFE-MILP is outperformed by all the other algorithms, which are not distinguishable from one another in statistical terms. In all the cases, the objective function values of boosted and standard RECIFE-MILP are however quite similar, with an average difference of 2 to 3% (in absolute value), but for Rouen where it gets to 16%. The corresponding figures for ineq. and all-ineq. RECIFE-MILP are 1 to 6% and 1 to 7%, respectively, for all control areas. Unfortunately, we could not identify any characteristic of the different sets of instances which would allow the explanation of the relative performance of the algorithms: the reason why standard RECIFE-MILP obtains the best performance for Lille is still object of investigation. In general, however, we think we can conclude that boosted RECIFE-MILP is the best, since it is significantly better than standard RECIFE-MILP in three over four control areas, and it is outperformed by the ineq. one only in this “unlucky” case.

9 Conclusions

In this paper we have defined valid inequalities for a MILP model for the rtRTMP. Specifically, we have focused on the model at the basis of RECIFE-MILP, which is among the state-of-the-art algorithms for this problem. Exploiting these valid inequalities, we also proposed a reformulation which allows a remarkable reduction of the number of binary variables. Although the valid inequalities proposed can be, at least in some cases, able to represent all the facets of the projection of the convex hull of the problem in the subspace of the binary variables, the inclusion of all of them in the model does not seem a viable option. In a thorough experimental analysis, we tested the possibility of adding some of them, according to different criteria. Unfortunately, we could not find out how to exploit their strength without suffering from the increased computational burden. Instead, the reduction of the number of binary variables allows a remarkable performance improvement with respect to the standard RECIFE-MILP, in almost all the cases considered.

In future works we will focus on the design of an oracle capable of iden-

tifying the suitable inequalities to be added during the solution process to further improve the performance of RECIFE-MILP.

References

- Cacchiani, V., Huisman, D., Kidd, M., Kroon, L., Toth, P., Veelenturf, L., and Wagenaar, J. (2014). An overview of recovery models and algorithms for real-time railway rescheduling. *Transportation Research Part B: Methodological*, 63:15–37.
- Caimi, G., Chudak, F., Fuchsberger, M., Laumanns, M., and Zenklusen, R. (2011). A new resource-constrained multicommodity flow model for conflict-free train routing and scheduling. *Transportation Science*, 45(2):212–227.
- Castro, P. M. (2015). Tightening piecewise McCormick relaxations for bilinear problems. *Computers & Chemical Engineering*, 72:300 – 311. A Tribute to Ignacio E. Grossmann.
- Corman, F., D’Ariano, A., Pacciarelli, D., and Pranzo, M. (2012). Optimal inter-area coordination of train rescheduling decisions. *Transportation Research Part E*, 48:71–88.
- Corman, F. and Meng, L. (2015). A review of online dynamic models and algorithms for railway traffic management. *Intelligent Transportation Systems, IEEE Transactions on*, 16(3):1274–1284.
- D’Ariano, A., Pacciarelli, D., and Pranzo, M. (2007). A branch and bound algorithm for scheduling trains in a railway network. *European Journal of Operational Research*, 183:643–657.
- Fischetti, M. and Monaci, M. (2017). Using a general-purpose mixed-integer linear programming solver for the practical solution of real-time train rescheduling. *European Journal Of Operational Research*, 263:258–264.
- Gawrilow, E. and Joswig, M. (2000). polymake: a framework for analyzing convex polytopes. In Kalai, G. and Ziegler, G. M., editors, *Polytopes — Combinatorics and Computation*, pages 43–74. Birkhäuser.

- Karp, R. M. (1972). Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer.
- Lamorgese, L. and Mannino, C. (2015). An exact decomposition approach for the real-time train dispatching problem. *Operations Research*, 63(1):48–64.
- McCormick, G. P. (1976). Computability of global solutions to factorable nonconvex programs: Part I — Convex underestimating problems. *Mathematical Programming*, 10(1):147–175.
- Meng, L. and Zhou, X. (2014). Simultaneous train rerouting and rescheduling on an n-track network: A model reformulation with network-based cumulative flow variables. *Transportation Research Part B: Methodological*, 67:208–234.
- Pachl, J. (2008). Timetable design principles. In Hansen, I. and Pachl, J., editors, *Railway Timetable & Traffic*, chapter 2, pages 9–42. Eurailpress — DVV Rail Media, Hambourg, Germany.
- Pellegrini, P., Marliere, G., Pesenti, R., and Rodriguez, J. (2015). RECIFE-MILP: An Effective MILP-Based Heuristic for the Real-Time Railway Traffic Management Problem. *IEEE Transactions on Intelligent Transportation Systems*, 16(5):2609–2619.
- Pellegrini, P., Marlière, G., and Rodriguez, J. (2014). Optimal train routing and scheduling for managing traffic perturbations in complex junctions. *Transportation Research Part B*, 59:58–80.
- Pellegrini, P., Marliere, G., and Rodriguez, J. (2016). A detailed analysis of the actual impact of real-time railway traffic management optimization. *Journal of Rail Transport Planning & Management*, 6(1):13–31.
- Quaglietta, E., Pellegrini, P., Goverde, R., Albrecht, T., Jaekel, B., Marlière, G., Rodriguez, J., Dollevoet, T., Ambrogio, B., Carcasole, D., Giaroli, M., and Nicholson, G. (2016). The ON-TIME real-time railway traffic management framework: A proof-of-concept using a scalable standardised data communication architecture. *Transportation Research Part C: Emerging Technologies*, 63:23–50.

- Samà, M., Pellegrini, P., D'Ariano, A., Rodriguez, J., and Pacciarelli, D. (2016). Ant colony optimization for the real-time train routing selection problem. *Transportation Research Part B: Methodological*, 85:89–108.
- Törnquist, J. and Persson, J. (2007). N-tracked railway traffic re-scheduling during disturbances. *Transportation Research Part B*, 41:342–362.