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# Lovász-Schrijver PSD-operator on Claw-Free Graphs<sup>\*</sup>

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**Abstract.** The subject of this work is the study of the Lovász-Schrijver PSD-operator  $LS_+$  applied to the edge relaxation  $ESTAB(G)$  of the stable set polytope  $STAB(G)$  of a graph  $G$ . We are interested in the problem of characterizing the graphs  $G$  for which  $STAB(G)$  is achieved in one iteration of the  $LS_+$ -operator, called  $LS_+$ -perfect graphs, and to find a polyhedral relaxation of  $STAB(G)$  that coincides with  $LS_+(ESTAB(G))$  and  $STAB(G)$  if and only if  $G$  is  $LS_+$ -perfect. An according conjecture has been recently formulated ( $LS_+$ -Perfect Graph Conjecture); here we verify it for the well-studied class of claw-free graphs.

## 1 Introduction

The context of this work is the study of the stable set polytope, some of its linear and semi-definite relaxations, and graph classes for which certain relaxations are tight. Our focus lies on those graphs where a single application of the Lovász-Schrijver positive semi-definite operator introduced in [22] to the edge relaxation yields the stable set polytope.

The *stable set polytope*  $STAB(G)$  of a graph  $G = (V, E)$  is defined as the convex hull of the incidence vectors of all stable sets of  $G$  (in a stable set all nodes are mutually nonadjacent). Two canonical relaxations of  $STAB(G)$  are the *edge constraint stable set polytope*

$$ESTAB(G) = \{\mathbf{x} \in [0, 1]^V : x_i + x_j \leq 1, ij \in E\},$$

and the *clique constraint stable set polytope*

$$QSTAB(G) = \{\mathbf{x} \in [0, 1]^V : x(Q) = \sum_{i \in Q} x_i \leq 1, Q \subseteq V \text{ clique}\}$$

(in a clique all nodes are mutually adjacent, hence a clique and a stable set share at most one node). We have  $STAB(G) \subseteq QSTAB(G) \subseteq ESTAB(G)$  for any graph, where  $STAB(G)$  equals  $ESTAB(G)$  for bipartite graphs, and  $QSTAB(G)$  for perfect graphs only [5].

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According to a famous characterization achieved by Chudnovsky et al. [3], perfect graphs are precisely the graphs without chordless cycles  $C_{2k+1}$  with  $k \geq 2$ , termed *odd holes*, or their complements, the *odd antiholes*  $\overline{C}_{2k+1}$ . This shows that odd holes and odd antiholes are the only *minimally imperfect graphs*.

Perfect graphs turned out to be an interesting and important class with a rich structure and a nice algorithmic behavior [18]. However, solving the stable set problem for a perfect graph  $G$  by maximizing over  $\text{QSTAB}(G)$  does not work directly [17], but only via a detour involving a geometric representation of graphs [21] and a semi-definite relaxation  $\text{TH}(G)$  introduced in [18].

For some  $N \in \mathbf{Z}_+$ , an orthonormal representation of a graph  $G = (V, E)$  is a sequence  $(\mathbf{u}_i : i \in V)$  of  $|V|$  unit-length vectors  $\mathbf{u}_i \in \mathbf{R}^N$ , such that  $\mathbf{u}_i^T \mathbf{u}_j = 0$  for all  $ij \notin E$ . For any orthonormal representation of  $G$  and any additional unit-length vector  $\mathbf{c} \in \mathbf{R}^N$ , the orthonormal representation constraint is  $\sum_{i \in V} (\mathbf{c}^T \mathbf{u}_i)^2 x_i \leq 1$ .  $\text{TH}(G)$  denotes the convex set of all vectors  $\mathbf{x} \in \mathbf{R}_+^{|V|}$  satisfying all orthonormal representation constraints for  $G$ . For any graph  $G$ ,

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G)$$

holds and approximating a linear objective function over  $\text{TH}(G)$  can be done with arbitrary precision in polynomial time [18]. Most notably, the same authors proved a beautiful characterization of perfect graphs:

$$G \text{ is perfect} \Leftrightarrow \text{TH}(G) = \text{STAB}(G) \Leftrightarrow \text{TH}(G) = \text{QSTAB}(G). \quad (1)$$

For all imperfect graphs,  $\text{STAB}(G)$  does not coincide with any of the above relaxations. It is, thus, natural to study further relaxations and to combinatorially characterize those graphs where  $\text{STAB}(G)$  equals one of them.

*Linear relaxations and related graphs.* A natural generalization of the clique constraints are rank constraints  $\mathbf{x}(G') = \sum_{i \in G'} x_i \leq \alpha(G')$  associated with arbitrary induced subgraphs  $G' \subseteq G$ . By the choice of the right hand side  $\alpha(G')$ , denoting the size of a largest stable set in  $G'$ , rank constraints are valid for  $\text{STAB}(G)$ . A graph  $G$  is called *rank-perfect* by [32] if and only if  $\text{STAB}(G)$  is described by rank constraints only.

By definition, rank-perfect graphs include all perfect graphs. By restricting the facet set to rank constraints associated with certain subgraphs, several well-known graph classes are defined, e.g., *near-perfect graphs* [29] where only rank constraints associated with cliques and the whole graph are allowed, or *t-perfect* [5] and *h-perfect graphs* [18] where rank constraints associated with edges, triangles and odd holes resp. cliques of arbitrary size and odd holes suffice.

As common generalization of perfect, t-perfect, and h-perfect graphs, the class of *a-perfect graphs* was introduced in [33] as graphs  $G$  where  $\text{STAB}(G)$  is given by rank constraints associated with antiwebs. An *antiweb*  $A_n^k$  is a graph with  $n$  nodes  $0, \dots, n-1$  and edges  $ij$  if and only if  $k \leq |i-j| \leq n-k$  and  $i \neq j$ . Antiwebs include all complete graphs  $K_n = A_n^1$ , odd holes  $C_{2k+1} = A_{2k+1}^k$ , and their complements  $\overline{C}_{2k+1} = A_{2k+1}^{2k+1}$ . Antiwebs are *a-perfect* by [33].

A more general type of inequalities is obtained from complete joins of antiwebs, called *joined antiweb constraints*

$$\sum_{i \leq k} \frac{1}{\alpha(A_i)} x(A_i) + x(Q) \leq 1,$$

associated with the join of some antiwebs  $A_1, \dots, A_k$  and a clique  $Q$ . We denote the linear relaxation of  $\text{STAB}(G)$  obtained by all joined antiweb constraints by  $\text{ASTAB}^*(G)$ . By construction, we see that

$$\text{STAB}(G) \subseteq \text{ASTAB}^*(G) \subseteq \text{QSTAB}(G) \subseteq \text{ESTAB}(G).$$

In [6], a graph  $G$  is called *joined a-perfect* if and only if  $\text{STAB}(G)$  coincides with  $\text{ASTAB}^*(G)$ . Besides all a-perfect graphs, further examples of joined a-perfect graphs are *near-bipartite graphs* (where the non-neighbors of every node induce a bipartite graph) due to [30].

*A semi-definite relaxation and  $LS_+$ -perfect graphs.* In the early nineties, Lovász and Schrijver introduced the PSD-operator  $LS_+$  (called  $N_+$  in [22]) which, applied to  $\text{ESTAB}(G)$ , generates a positive semi-definite relaxation of  $\text{STAB}(G)$  stronger than  $\text{TH}(G)$  (see Section 2.1 for details). In order to simplify the notation we write  $LS_+(G) = LS_+(\text{ESTAB}(G))$ . In [22] it is shown that

$$\text{STAB}(G) \subseteq LS_+(G) \subseteq \text{ASTAB}^*(G).$$

As in the case of perfect graphs, the stable set problem can be solved in polynomial time for the class of graphs for which  $LS_+(G) = \text{STAB}(G)$ . These graphs are called  *$LS_+$ -perfect*, and all other graphs  *$LS_+$ -imperfect* (note that they are also called  $N_+$ -(im)perfect, see e.g. [1]). In [1], the authors look for a characterization of  $LS_+$ -perfect graphs similar to the characterization (1) for perfect graphs. More precisely, they intend to find an appropriate polyhedral relaxation  $P(G)$  of  $\text{STAB}(G)$  such that

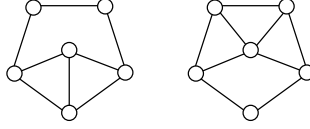
$$G \text{ is } LS_+\text{-perfect} \Leftrightarrow LS_+(G) = \text{STAB}(G) \Leftrightarrow LS_+(G) = P(G). \quad (2)$$

A conjecture has been recently proposed in [2], which can be equivalently reformulated as follows [12]:

*Conjecture 1 ( $LS_+$ -Perfect Graph Conjecture).* A graph  $G$  is  $LS_+$ -perfect if and only if  $LS_+(G) = \text{ASTAB}^*(G)$ .

The results of Lovász and Schrijver [22] prove that joined a-perfect graphs are  $LS_+$ -perfect, thus, the conjecture states that  $LS_+$ -perfect graphs coincide with joined a-perfect graphs and that  $\text{ASTAB}^*(G)$  is the studied polyhedral relaxation of  $\text{STAB}(G)$  playing the role of  $P(G)$  in (2).

In addition, every subgraph of an  $LS_+$ -perfect graph is also  $LS_+$ -perfect. This motivates the definition of *minimally  $LS_+$ -imperfect graphs* as these  $LS_+$ -imperfect graphs whose proper induced subgraphs are all  $LS_+$ -perfect. The two smallest such graphs, found by [10] and [20], are depicted in Figure 1.



**Fig. 1.** The graphs  $G_{LT}$  (on the left) and  $G_{EMN}$  (on the right).

Conjecture 1 has been already verified for near-perfect graphs by [1], for *fs-perfect graphs* (where the only facet-defining subgraphs are cliques and the graph itself) by [2], for *webs* (the complements  $W_n^k = \overline{A}_n^k$  of antiwebs) by [11] and for *line graphs* (obtained by turning adjacent edges of a root graph into adjacent nodes of the line graph) by [12], see Section 2.1 for details.

*The  $LS_+$ -Perfect Graph Conjecture for claw-free graphs.* The aim of this contribution is to verify Conjecture 1 for a well-studied graph class containing all webs, all line graphs and the complements of near-bipartite graphs: the class of *claw-free graphs* (i.e., the graphs not containing the complete join of a single node and a stable set of size three).

Claw-free graphs attracted much attention due to their seemingly asymmetric behavior w.r.t. the stable set problem. On the one hand, the first combinatorial algorithms to solve the problem in polynomial time for claw-free graphs [23, 28] date back to 1980. Therefore, the polynomial equivalence of optimization and separation due to [18] implies that it is possible to optimize over the stable set polytope of a claw-free graph in polynomial time. On the other hand, the problem of characterizing the stable set polytope of claw-free graphs in terms of an explicit description by means of a facet-defining system, originally posed in [18], was open for several decades. This motivated the study of claw-free graphs and its different subclasses, that finally answered this long-standing problem only recently (see Section 2.2 for details).

To verify the conjecture for claw-free graphs, we need not only to rely on structural results and complete facet-descriptions of their stable set polytope, but also to ensure that all facet-inducing subgraphs different from cliques, antiwebs or their complete joins are  $LS_+$ -imperfect. A graph  $G$  is said to be *facet-defining* if  $\text{STAB}(G)$  has a full-support facet.

The paper is organized as follows: In Section 2, we present the State-of-the-Art on  $LS_+$ -perfect graphs (including families of  $LS_+$ -imperfect graphs needed for the subsequent proofs) and on claw-free graphs, their relevant subclasses and the results concerning the facet-description of their stable set polytopes from the literature. In Section 4, we verify, relying on the previously presented results, Conjecture 1 for the studied subclasses of claw-free graphs. As a conclusion, we obtain as our main result:

**Theorem 1.** *The  $LS_+$ -Perfect Graph Conjecture is true for all claw-free graphs.*

We close with some further remarks and an outlook to future lines of research.

## 2 State-of-the-Art

### 2.1 About $LS_+$ -perfect graphs

We denote by  $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$  the vectors of the canonical basis of  $\mathbf{R}^{n+1}$  (where the first coordinate is indexed zero),  $\mathbf{1}$  the vector with all components equal to 1 and  $S_+^n$  the convex cone of  $(n \times n)$  symmetric and positive semi-definite matrices with real entries. Given a convex set  $K$  in  $[0, 1]^n$ , let

$$\text{cone}(K) = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \in \mathbf{R}^{n+1} : \mathbf{x} = x_0 \mathbf{y}; \mathbf{y} \in K \right\}.$$

Then, we define the polyhedral set

$$\begin{aligned} M_+(K) = \{ Y \in S_+^{n+1} : & Y\mathbf{e}_0 = \text{diag}(Y), \\ & Y\mathbf{e}_i \in \text{cone}(K), \\ & Y(\mathbf{e}_0 - \mathbf{e}_i) \in \text{cone}(K), i = 1, \dots, n \}, \end{aligned}$$

where  $\text{diag}(Y)$  denotes the vector whose  $i$ -th entry is  $Y_{ii}$ , for every  $i = 0, \dots, n$ . Projecting this lifting back to the space  $\mathbf{R}^n$  results in

$$LS_+(K) = \left\{ \mathbf{x} \in [0, 1]^n : \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} = Y\mathbf{e}_0, \text{ for some } Y \in M_+(K) \right\}.$$

In [22], Lovász and Schrijver proved that  $LS_+(K)$  is a relaxation of the convex hull of integer solutions in  $K$  and that  $LS_+^n(K) = \text{conv}(K \cap \{0, 1\}^n)$ , where  $LS_+^0(K) = K$  and  $LS_+^k(K) = LS_+(LS_+^{k-1}(K))$  for every  $k \geq 1$ .

In this work we focus on the behavior of a single application of the  $LS_+$ -operator to the edge relaxation of the stable set polytope of a graph.

Recall that  $G_{LT}$  and  $G_{EMN}$  are the smallest  $LS_+$ -imperfect graphs. Further  $LS_+$ -imperfect graphs can be obtained by applying operations preserving  $LS_+$ -imperfection.

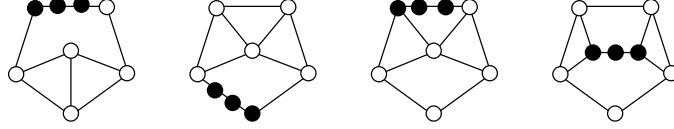
In [20], the *stretching* of a node  $v$  is introduced as follows: Partition its neighborhood  $N(v)$  into two nonempty, disjoint sets  $A_1$  and  $A_2$  (so  $A_1 \cup A_2 = N(v)$ , and  $A_1 \cap A_2 = \emptyset$ ). A stretching of  $v$  is obtained by replacing  $v$  by two adjacent nodes  $v_1$  and  $v_2$ , joining  $v_i$  with every node in  $A_i$  for  $i \in \{1, 2\}$ , and subdividing the edge  $v_1v_2$  by one node  $w$ . In [20] it is shown:

**Theorem 2 ([20]).** *The stretching of a node preserves  $LS_+$ -imperfection.*

Hence, all stretchings of  $G_{LT}$  and  $G_{EMN}$  are  $LS_+$ -imperfect, see Figure 2 for some examples.

In [12], the authors characterized  $LS_+$ -perfect line graphs by showing that the only minimally  $LS_+$ -imperfect line graphs are stretchings of  $G_{LT}$  and  $G_{EMN}$  and occur as subgraphs in all facet-defining line graphs different from cliques and odd holes:

**Theorem 3 ([12]).** *A facet-defining line graph  $G$  is  $N_+$ -perfect if and only if  $G$  is a clique or an odd hole.*



**Fig. 2.** Some node stretchings ( $v_1, w, v_2$  in black) of  $G_{LT}$  and  $G_{EMN}$ .

The proof relies on a result of Edmonds & Pulleyblank [8] who showed that a line graph  $L(H)$  is facet-defining if and only if  $H$  is a 2-connected hypomatchable graph (that is, for all nodes  $v$  of  $H$ ,  $H-v$  admits a perfect matching). Such graphs  $H$  have an ear decomposition  $H_0, H_1, \dots, H_k = H$  where  $H_0$  is an odd hole and  $H_i$  is obtained from  $H_{i-1}$  by adding an odd path (ear) between distinct nodes of  $H_{i-1}$ . In [12], it is shown that the line graph  $L(H_1)$  of any ear decomposition is a node stretching of  $G_{LT}$  and  $G_{EMN}$  and, thus,  $LS_+$ -imperfect by [20].

Again, using stretchings of  $G_{LT}$  and  $G_{EMN}$  and exhibiting one more minimally  $LS_+$ -imperfect graph, namely the web  $W_{10}^2$ ,  $LS_+$ -perfect webs are characterized in [11] as follows:

**Theorem 4 ([11]).** *A web is  $LS_+$ -perfect if and only if it is perfect or minimally imperfect.*

The proof shows that all imperfect not minimally imperfect webs with stability number 2 contain  $G_{EMN}$  and all webs  $W_n^2$  different from  $W_7^2, W_{10}^2$ , some stretching of  $G_{LT}$ . Furthermore, all other webs contain some  $LS_+$ -imperfect  $W_{n'}^2$  and are, thus, also  $LS_+$ -imperfect.

Finally, in [1], there is another family of  $LS_+$ -imperfect graphs presented that will play a central role in some subsequent proofs:

**Theorem 5 ([1]).** *Let  $G$  be a graph with  $\alpha(G) = 2$  such that  $G - v$  is an odd antihole for some node  $v$ .  $G$  is  $LS_+$ -perfect if and only if  $v$  is completely joined to  $G - v$ .*

## 2.2 About claw-free graphs

In several respects, claw-free graphs are generalizations of line graphs. An intermediate class between line graphs and claw-free graphs form *quasi-line graphs*, where the neighborhood of any node can be partitioned into two cliques (i.e., quasi-line graphs are the complements of near-bipartite graphs).

For this class, it turned out that so-called clique family inequalities suffice to describe the stable set polytope. Given a graph  $G$ , a family  $\mathcal{F}$  of cliques and an integer  $p < n = |\mathcal{F}|$ , the clique family inequality  $(\mathcal{F}, p)$  is the following valid inequality for  $\text{STAB}(G)$

$$(p-r) \sum_{i \in W} x_i + (p-r-1) \sum_{i \in W_o} x_i \leq (p-r) \left\lfloor \frac{n}{p} \right\rfloor \quad (3)$$

where  $r = n \bmod p$  and  $W$  (resp.  $W_o$ ) is the set of nodes contained in at least  $p$  (resp. exactly  $p - 1$ ) cliques of  $\mathcal{F}$ .

This generalizes the results of Edmonds [7] and Edmonds & Pulleyblank [8] that  $\text{STAB}(L(H))$  is described by clique constraints and rank constraints

$$x(L(H')) \leq \frac{1}{2}(|V(H')| - 1) \quad (4)$$

associated with the line graphs of 2-connected hypomatchable induced subgraphs  $H' \subseteq H$ . Note that the rank constraints of type (4) are special clique family inequalities. Chudnovsky and Seymour [4] extended this result to a superclass of line graphs. They showed that a connected quasi-line graph  $G$  is either a fuzzy circular interval graph or  $\text{STAB}(G)$  is given by clique constraints and rank constraints of type (4).

Let  $\mathcal{C}$  be a circle,  $\mathcal{I}$  a collection of intervals in  $\mathcal{C}$  without proper containments and common endpoints, and  $V$  a multiset of points in  $\mathcal{C}$ . The *fuzzy circular interval graph*  $G(V, \mathcal{I})$  has node set  $V$  and two nodes are adjacent if both belong to one interval  $I \in \mathcal{I}$ , where edges between different endpoints of the same interval may be omitted.

*Semi-line graphs* are either line graphs or quasi-line graphs without a representation as a fuzzy circular interval graph. Due to [4], semi-line graphs are rank-perfect with line graphs as only facet-defining subgraphs.

Eisenbrand et al. [9] proved that clique family inequalities suffice to describe the stable set polytope of fuzzy circular interval graphs. Stauffer [31] verified a conjecture of [25] that every facet-defining clique family inequality of a fuzzy circular interval graph  $G$  is associated with a web in  $G$ .

All these results together complete the picture for quasi-line graphs. However, there are claw-free graphs which are not quasi-line; the 5-wheel is the smallest such graph and has stability number 2. Due to Cook (see [30]), all facets for graphs  $G$  with  $\alpha(G) = 2$  are *clique-neighborhood constraints*

$$2x(Q) + x(N'(Q)) \leq 2 \quad (5)$$

where  $Q \subseteq G$  is a clique and  $N'(Q) = \{v \in V(G) : Q \subseteq N(v)\}$ . Therefore all non-trivial facets in this case are 1, 2-valued.

This is not the case for graphs  $G$  with  $\alpha(G) = 3$ . In fact, all the known difficult facets of claw-free graphs occur in this class. Some non-rank facets with up to five different non-zero coefficients are presented in [16, 19]. All of these facets turned out to be so-called *co-spanning 1-forest constraints* due to [26], where it is also shown that it is possible to build a claw-free graph with stability number three inducing a co-spanning 1-forest facet with  $b$  different left hand side coefficients, for every positive integer  $b$ .

The problem of characterizing  $\text{STAB}(G)$  when  $G$  is a connected claw-free but not quasi-line graph with  $\alpha(G) \geq 4$  was studied by Galluccio et al.: In a series of results [13–15], it is shown that if such a graph  $G$  does not contain a clique cutset, then 1, 2-valued constraints suffice to describe  $\text{STAB}(G)$ . Here, besides 5-wheels, different rank and non-rank facet-defining inequalities of the geared graph  $G$  shown in Fig. 3 play a central role.



In addition, graphs of this type can be decomposed into strips. A *strip*  $(G, a, b)$  is a (not necessarily connected) graph with two designated simplicial nodes  $a$  and  $b$  (a node is *simplicial* if its neighborhood is a clique). A claw-free strip containing a 5-wheel as induced subgraph is a *5-wheel strip*. Given two node-disjoint strips  $(G_1, a_1, b_1)$  and  $(G_2, a_2, b_2)$ , their *composition* is the union of  $G_1 \setminus \{a_1, b_1\}$  and  $G_2 \setminus \{a_2, b_2\}$  together with all edges between  $N_{G_1}(a_1)$  and  $N_{G_2}(a_2)$ , and between  $N_{G_1}(b_1)$  and  $N_{G_2}(b_2)$  [4].

As shown in [24], this composition operation can be generalized to more than two strips: Every claw-free but not quasi-line graph  $G$  with  $\alpha(G) \geq 4$  admits a decomposition into strips, where at most one strip is quasi-line and all the remaining ones are 5-wheel strips having stability number at most 3. There are only three “basic” types of 5-wheel strips (see Fig. 3) which can be extended by adding nodes belonging to the neighborhood of the 5-wheels (see [24] for details).

Note that a claw-free but not quasi-line graph  $G$  with  $\alpha(G) \geq 4$  containing a clique cutset may have a facet-inducing subgraph  $G'$  with  $\alpha(G') = 3$  (inside a 5-wheel strip of type 3), see [27] for examples.

Taking all these results together into account gives the complete list of facets needed to describe the stable set polytope of claw-free graphs.

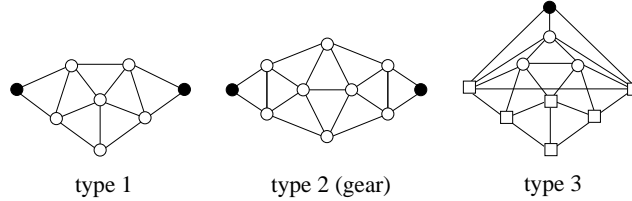


Fig. 3. The three types of basic 5-wheel strips.

### 3 $LS_+$ -Perfect Graph Conjecture for claw-free graphs

In this section, we verify the  $LS_+$ -Perfect Graph Conjecture for all relevant subclasses of claw-free graphs.

Graphs with  $\alpha(G) = 2$  play a crucial role. Relying on the behavior of the stable set polytope under taking complete joins [5] and the result on  $LS_+$ -(im)perfect graphs  $G$  with  $\alpha(G) = 2$  (Theorem 5), we can prove:

**Theorem 6.** *All facet-inducing  $LS_+$ -perfect graphs  $G$  with  $\alpha(G) = 2$  are odd antiholes or complete joins of odd antihole(s) and a (possibly empty) clique.*

This shows that all facet-inducing  $LS_+$ -perfect graphs  $G$  with  $\alpha(G) = 2$  are joined  $a$ -perfect, and we conclude:

**Corollary 1.** *The  $LS_+$ -Perfect Graph Conjecture is true for graphs with stability number 2.*

*Quasi-line graphs* partition into the two subclasses of semi-line graphs and fuzzy circular interval graphs.

Chudnovsky and Seymour [4] proved that the stable set polytope of a semi-line graph is given by rank constraints associated with cliques and the line graphs of 2-connected hypomatchable graphs. Together with the result from [12] (presented in Theorem 3), we directly conclude that the  $LS_+$ -Perfect Graph Conjecture holds for semi-line graphs.

Based on the results of Eisenbrand et al. [9] and Stauffer [31], combined with the characterization of  $LS_+$ -imperfect webs from [11] (Theorem 4), we are able to show:

**Theorem 7.** *All facet-inducing  $LS_+$ -perfect fuzzy circular interval graphs are cliques, odd holes or odd antiholes.*

As a consequence, every  $LS_+$ -perfect fuzzy circular interval graph is a-perfect. This verifies the  $LS_+$ -Perfect Graph Conjecture for fuzzy circular interval graphs.

Since the class of quasi-line graphs partitions into semi-line graphs and fuzzy circular interval graphs, we obtain as direct consequence:

**Corollary 2.** *The  $LS_+$ -Perfect Graph Conjecture is true for quasi-line graphs.*

*Claw-free graphs that are not quasi-line* are distinguished according to their stability number.

Relying on the behavior of the stable set polytope under clique identification [5] and the result on  $LS_+$ -(im)perfect graphs from Theorem 5, we can prove:

**Theorem 8.** *Every facet-defining claw-free not quasi-line graph  $G$  with  $\alpha(G) = 3$  is  $LS_+$ -imperfect.*

Hence, the only facet-defining subgraphs  $G'$  of  $LS_+$ -perfect claw-free not quasi-line graphs  $G$  with  $\alpha(G) = 3$  have  $\alpha(G') = 2$  and are, by Theorem 6, cliques, odd antiholes or their complete joins. We conclude that  $LS_+$ -perfect facet-defining claw-free not quasi-line graphs  $G$  with  $\alpha(G) = 3$  are joined a-perfect and, thus, the  $LS_+$ -Perfect Graph Conjecture is true for this class.

To treat the case of claw-free not quasi-line graphs  $G$  with  $\alpha(G) \geq 4$ , we rely on the decomposition of such graphs into strips, where at most one strip is quasi-line and all the remaining ones are 5-wheel strips [24]. By noting that 5-wheel strips of type 3 contain  $G_{LT}$  and exhibiting  $LS_+$ -imperfect line graphs in the other two cases, we are able to show:

**Theorem 9.** *Every facet-defining claw-free not quasi-line graph  $G$  with  $\alpha(G) \geq 4$  is  $LS_+$ -imperfect.*

This together with Theorem 8 shows that the only facet-defining subgraphs  $G'$  of  $LS_+$ -perfect claw-free not quasi-line graphs  $G$  with  $\alpha(G) \geq 4$  have  $\alpha(G') = 2$  and are, by Theorem 6, cliques, odd antiholes or their complete joins. Thus, every  $LS_+$ -perfect claw-free not quasi-line graph  $G$  with  $\alpha(G) \geq 4$  is joined a-perfect and, thus, the  $LS_+$ -Perfect Graph Conjecture holds true for this class.

Combining Corollary 1 with the above results shows that all  $LS_+$ -perfect claw-free but not quasi-line graphs are joined a-perfect and we obtain:

**Corollary 3.** *The  $LS_+$ -Perfect Graph Conjecture is true for all claw-free graphs that are not quasi-line.*

Finally, we obtain our main result as direct consequence of Corollary 2 and Corollary 3: The  $LS_+$ -Perfect Graph Conjecture is true for all claw-free graphs.

## 4 Conclusion and future research

The context of this work was the study of  $LS_+$ -perfect graphs, i.e., graphs where a single application of the Lovász-Schrijver PSD-operator  $LS_+$  to the edge relaxation yields the stable set polytope. Hereby, we are particularly interested in finding an appropriate polyhedral relaxation  $P(G)$  of  $STAB(G)$  that coincides with  $LS_+(G)$  and  $STAB(G)$  if and only if  $G$  is  $LS_+$ -perfect. An according conjecture has been recently formulated ( $LS_+$ -Perfect Graph Conjecture); here we verified it for the well-studied class of claw-free graphs (Theorem 1).

Note further that, besides verifying the  $LS_+$ -Perfect Graph Conjecture for claw-free graphs, we obtained the complete list of all minimally  $LS_+$ -imperfect claw-free graphs. In fact, the results in [1, 11, 12] imply that the following graphs are minimally  $LS_+$ -imperfect:

- graphs  $G$  with  $\alpha(G) = 2$  such that  $G - v$  is an odd antihole for some node  $v$ , not completely joined to  $G - v$ ,
- the web  $W_{10}^2$ ,
- $LS_+$ -imperfect line graphs (which are all node stretchings of  $G_{LT}$  or  $G_{EMN}$ ).

Our results from Section 3 on facet-defining  $LS_+$ -perfect claw-free graphs imply that they are the only minimally  $LS_+$ -imperfect claw-free graphs.

Finally, the subject of the present work has parallels to the well-developed research area of perfect graph theory also in terms of polynomial time computability. In fact, it has the potential of reaching even stronger results due the following reasons. Recall that calculating the value

$$\eta_+(G) = \max \mathbf{1}^T x, x \in LS_+(G)$$

can be done in polynomial time for every graph  $G$  by [22]. Thus, the stable set problem can be solved in polynomial time for a strict superset of perfect graphs, the  $LS_+$ -perfect graphs, by  $\alpha(G) = \eta_+(G)$ . Hence, our future lines of research include to find

- new families of graphs where the conjecture holds (e.g., by characterizing the minimally  $LS_+$ -imperfect graphs within the class),
- new subclasses of  $LS_+$ -perfect or joined a-perfect graphs,
- classes of graphs  $G$  where  $STAB(G)$  and  $LS_+(G)$  are “close enough” to have  $\alpha(G) = \lfloor \eta_+(G) \rfloor$ .

In particular, the class of graphs  $G$  with  $\alpha(G) = \lfloor \eta_+(G) \rfloor$  can be expected to be large since  $LS_+(G)$  is a much stronger relaxation of  $STAB(G)$  than  $TH(G)$ .

In all cases, the stable set problem could be solved in polynomial time in these graph classes by optimizing over  $LS_+(G)$ . Finally, note that  $LS_+(P(G))$  with

$$\text{STAB}(G) \subseteq P(G) \subseteq \text{ESTAB}(G)$$

clearly gives an even stronger relaxation of  $\text{STAB}(G)$  than  $LS_+(G)$ . However, already optimizing over  $LS_+(\text{QSTAB}(G))$  cannot be done in polynomial time anymore for all graphs  $G$  by [22]. Hence, in view of the polynomial time solvability of the stable set problem,  $LS_+$ -perfect graphs or their generalizations satisfying  $\alpha(G) = \lfloor \eta_+(G) \rfloor$  are the most promising cases in this context.

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