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To cite this version:

Yirmeyahu J Kaminski, Jean Lévine, François Ollivier. On Singularities of Flat Affine Systems With n States and n − 1 Controls. International Journal of Robust and Nonlinear Control, 2020, 30 (9), pp.3547-3565. 10.1002/rnc.4943 . hal-02045321v3

HAL Id: hal-02045321

https://hal.science/hal-02045321v3

Submitted on 15 May 2020

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On Singularities of Flat Affine Systems With n States and $n-1$ Controls

Yirmeyahu J. Kaminski* Jean Lévine† François Ollivier‡

We study the set of intrinsic singularities of flat affine systems with $n-1$ controls and n states using the notion of Lie-Bäcklund atlas, previously introduced by the authors. For this purpose, we prove two easily computable sufficient conditions to construct flat outputs as a set of independent first integrals of distributions of vector fields, the first one in a generic case, namely in a neighborhood of a point where the $n-1$ control vector fields are independent, and the second one at a degenerate point where $p-1$ control vector fields are dependent of the $n-p$ others, with $p>1$. After introducing the Γ-accessibility rank condition, we show that the set of intrinsic singularities includes the set of points where the system does not satisfy this rank condition and is included in the set where a distribution of vector fields introduced in the generic case is singular. We conclude this analysis by three examples of apparent singularities of flat systems in generic and non generic degenerate cases.

1 Introduction

Differential flatness [4, 5, 15] is known to be a powerful notion in control theory. Roughly speaking, a system with m independent controls $u=(u_1,\ldots,u_m)\in\mathbb{R}^m$ and state $x=(x_1,\ldots,x_n)$ defined on a n-dimensional smooth manifold X, is said to be (differentially) flat at a given point $(x,u,\dot{u},\ddot{u},\ldots)$ of the infinite dimensional jet manifold

$$X\times\mathbb{R}^m_\infty \triangleq X\times\mathbb{R}^m\times\mathbb{R}^m\times\cdots$$

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if, and only if, its trajectories may be expressed, in a neighborhood of this point, as functions of the trajectories of m functionally independent smooth variables, called flat outputs, and a finite number of their time derivatives.

Although non generic from a mathematical standpoint, flatness is a property shared by many popular models in various branches of engineering and has been shown to be particularly useful to solve motion planning problems (see *e.g.* [15]).

In many cases, the flat outputs can only be defined in a dense open set, and one may need to use different parameterizations to cover the largest possible subset of the system configuration space, thus defining an atlas (see [2, 3, 12]). Therefore, obtaining local flatness criteria allowing to build atlases covering the *widest possible domain* is an important issue, in particular since the complementary of this domain, by definition, is equal to the *set of intrinsic singularities* [12]. It is remarkable that intrinsic flatness singularities may be interpreted as points in a neighbourhood of which the flatness-based control design is non robust since flat outputs stop existing there. On the contrary, an apparent singularity may allow a locally robust design by a suitable change of flat output.

In this paper, we continue our study of flatness singularities, initiated in [12], by restricting ourselves to control affine systems with n states and $n-1$ controls, of the form

$$\dot{x} = f_0(x) + \sum_{j=1}^{n-1} u_j f_j(x) \triangleq g(x,u).$$

After a brief recall, in this context, of concepts and notations related to flat systems and their singularities [12], we focus attention on the Γ-accessibility rank condition at a point, the restriction at a point of the criterion studied, *e.g.*, in [6, 10, 8], and the first order controllability around an integral curve generated by a constant control passing through this point, in the spirit of *e.g.* [22, 11], which are proven to be necessary conditions for flatness[1]. Therefore the points that do not satisfy them are naturally excluded from the above mentioned atlases and are thus contained in the so-called *intrinsic singularity set*. Then, we show in theorem 3 that the sufficient condition for Γ-accessibility

$$\dim(\mathsf{Span}\{f_1, \ldots, f_{n-1}, [g, f_k]\}) = n$$

for some k, is also a sufficient condition for flatness. Moreover, we prove that flat outputs can be obtained as *independent first integrals* of the above field f_k in each neighborhood where the condition holds: this is a consequence of the equivalence to the Brunovskỳ controllability canonical form where at least one of the controls, u_k, cannot appear in the first order derivatives of all components of the diffeomorphism but one, these $n-1$ components being therefore first integrals of the corresponding vector field f_k. This construction sheds a new light on a comparable result by Ph. Martin [18], obtained by input-output and structure at infinity considerations.

Since the points where the above condition holds are defined by the independence of n vectors, they are naturally said *generic*.

[1]Note that, to the authors knowledge, and in spite of a long standing study of nonlinear controllability by many authors, these particular results are not available in the literature.

Again using a result of Ph. Martin [17] clarifying the relationship between Lie-Bäcklund isomorphism and dynamic feedback, we then show that our result can be interpreted in terms of (extended state) feedback linearization [9, 7] and draw some consequences on the set of intrinsic singularities.

Let us also mention some complementary approaches, indirectly related to the present singularity study, on the existence of flat outputs in a neighborhood of a point, under assumptions on the differential weight or the prolongation length, *e.g.* , in [21, 19, 20].

We proceed in our singularity study with the following question: *Are there points of the state space where the dimension of the vector space generated by the control vector fields $f_1, \ldots, f_{n-1}$ drops down that are nevertheless apparent singularities?* Note that such points, if they exist, may be called *non generic* for obvious reasons. We prove a theorem giving a new construction of flat outputs at such points, thus providing a positive answer to the previous question.

The above mentioned theorems and their consequences on flatness singularities constitute the main results of this paper. Though the first theorem, that gives a sufficient condition for flatness at generic points, was already known in a different perspective, our approach is here completely renewed compared to our previous paper [12] since it deals with distributions of vector fields for systems represented by explicit differential equations and provides direct and computable flat output constructions by first integrals, as can be seen in the three examples at the end of this paper. It is also interesting to remark that the result at non generic points (theorem 5) is still valid, as is, with $m \leq n-1$ inputs, as illustrated by the third example. We may also stress that, in these results, the singularities are not given in terms of singularities of the parameterization as in [2, 3, 12] but rather as singularities of distributions of vector fields.

The paper is organized as follows: the basics of control affine systems with n states and $n-1$ controls as well as those on flatness singularities are recalled in sections 2 and 3. Section 4 is then devoted to the singularity study at generic and non generic points. Three academic examples are then presented in section 5, finally followed by concluding remarks in section 6.

2 Control Affine Systems with n States and $n-1$ Inputs

We consider a control affine system with drift given, in a local chart, by:

$$\dot{x} = f_0(x) + \sum_{j=1}^{n-1} u_j f_j(x) \tag{2.1}$$

where the state x evolves in a manifold X of dimension $n \geq 2$, with drift f_0, and with $m = n-1$ independent controls.

We also make the following classical assumption:

The vector fields[2] $f_1, \cdots, f_{n-1}$ are assumed to be C^∞ and linearly independent in a dense open set of X.

[2]The notations f_i, or g, etc. may be indifferently understood as usual vectors in prescribed local coordinates $(x_1, \ldots, x_n)$, *i.e.* $(f_{i,1}, \ldots, f_{i,n})^T$, or $(g_1, \ldots, g_n)^T$, etc., the superscript ^{T} standing

In other words, there is a dense open set where the matrix

$$G(x) \triangleq \begin{pmatrix} f_1 & \cdots & f_{n-1} \end{pmatrix}, \tag{2.2}$$

of size $n \times (n-1)$, has full rank.

In the sequel, for simplicity's sake, we denote by g the vector field in (2.1), pointwise defined by

$$g(x,u) \triangleq f_0(x) + \sum_{i=1}^{n-1} u_i f_i(x). \tag{2.3}$$

3 Recalls on the Infinite Order Jets Approach to Flat Systems with $n-1$ Inputs and Their Singularities

In this section, we briefly recall and adapt the main background and tools, introduced and defined in [12], to the present context of systems with $n-1$ inputs.

3.1 The Formalism of Infinite Order Jets

The definition of flatness introduced in [5] requires the use of infinite order jets. More precisely, we embed the manifold X and the associated system (2.1) in the manifold

$$\mathfrak{X} \triangleq X \times \mathbb{R}^{n-1}_{\infty} = X \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \cdots$$

with coordinates

$$(x, \overline{u}) \triangleq (x, u^{(0)}, u^{(1)}, u^{(2)}, \ldots, u^{(k)}, \ldots),$$

endowed with the product topology.

In this topology, a continuous (resp. differentiable) function from $X \times \mathbb{R}^{n-1}_{\infty}$ to $\mathbb{R}$, by construction, *only depends on a finite number of coordinates* and is continuous (resp. differentiable) with respect to these coordinates in the usual (finite dimensional) sense.

$\mathfrak{X} = X \times \mathbb{R}^{n-1}_{\infty}$ is also endowed with the Cartan vector field

$$C_g = \sum_{i=1}^{n} g_i(x,\overline{u}) \frac{\partial}{\partial x_i} + \sum_{i=1}^{n-1} \sum_{j \geq 0} u_i^{(j+1)} \frac{\partial}{\partial u_i^{(j)}} \triangleq g(x,\overline{u}) \frac{\partial}{\partial x} + \sum_{j \geq 0} u^{(j+1)} \frac{\partial}{\partial u^{(j)}} \tag{3.1}$$

with g defined by (2.3).

Considering C_g as a first order differential operator and $h : \mathfrak{X} \mapsto \mathbb{R}$ an arbitrary differentiable function, interpreting the expression $C_g h = \sum_{i=1}^{n} g_i(x,\overline{u}) \frac{\partial h}{\partial x_i} + \sum_{i=1}^{n-1} \sum_{j \geq 0} u_i^{(j+1)} \frac{\partial h}{\partial u_i^{(j)}}$, as the Lie derivative of h along the vector field g of $\mathrm{T}\mathfrak{X}$, the tangent bundle of $\mathfrak{X}$, this amounts to identify C_g with the vector $(g, \dot{u}, \ddot{u} \ldots)$ and equation (2.1) with the infinite number of equations

$$\dot{x} = g(x,\overline{u}), \quad \dot{u}^{(0)} = u^{(1)}, \quad \ldots, \quad \dot{u}^{(k)} = u^{(k+1)}, \quad \ldots$$

for transpose, or as the associated first order partial differential operators, *i.e.* $\sum_{j=1}^{n} f_{i,j} \frac{\partial}{\partial x_j}$, or $\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}$, etc.

3.2 Lie-Bäcklund Equivalence

Consider two systems:

$$\dot{x} = g(x, u) \quad \text{and} \quad \dot{y} = \gamma(y, v) \tag{3.2}$$

and their prolongations on $X \times \mathbb{R}^{n-1}_\infty$ and $Y \times \mathbb{R}^{\mu}_\infty$ respectively with the associated Cartan fields, in condensed notations introduced in (3.1):

$$C_g = g(x, u)\frac{\partial}{\partial x} + \sum_{j\geq 0} u^{(j+1)}\frac{\partial}{\partial u^{(j)}}, \quad C_\gamma = \gamma(y, v)\frac{\partial}{\partial y} + \sum_{j\geq 0} v^{(j+1)}\frac{\partial}{\partial v^{(j)}} \tag{3.3}$$

We say that they are *Lie-Bäcklund equivalent* at a pair of points $(x_0, \overline{u}_0)$ and $(y_0, \overline{v}_0)$ if there exist neighborhoods of these points where every integral curve of one is mapped into an integral curve of the other and conversely.

In other words, the two systems are *Lie-Bäcklund equivalent* at the points $(x_0, \overline{u}_0)$ and $(y_0, \overline{v}_0)$ if there exists neighborhoods $\mathcal{N}_{x_0,\overline{u}_0} \subset X \times \mathbb{R}^{n-1}_\infty$ and $\mathcal{N}_{y_0,\overline{v}_0} \subset Y \times \mathbb{R}^{\mu}_\infty$ and a C^∞ isomorphism $\Phi : \mathcal{N}_{y_0,\overline{v}_0} \to \mathcal{N}_{x_0,\overline{u}_0}$ satisfying $\Phi(y_0, \overline{v}_0) = (x_0, \overline{u}_0)$, with C^∞ inverse Ψ, such that the respective Cartan fields are Φ and Ψ related, *i.e.* $\Phi_* C_\gamma = C_g$ in $\mathcal{N}_{x_0,\overline{u}_0}$ and $\Psi_* C_g = C_\gamma$ in $\mathcal{N}_{y_0,\overline{v}_0}$.

We recall, without proof, a most important result from [17] (see also [4, 5, 15]) giving an interpretation of the Lie-Bäcklund equivalence in terms of diffeomorphism and endogenous dynamic feedback, that will be useful in the next sections. We state it in the present context of systems with $n-1$ inputs for convenience, though the result is much more general.

Theorem 1 (Martin [17])**.** *If the systems* (3.2) *are Lie-Bäcklund equivalent at a given pair of points, then (i) and (ii) must be satisfied:*

(i) $n - 1 = \mu$, i.e. *they must have the same number of independent inputs;*

(ii) there exist

- *an endogenous dynamic feedback*[3]

$$u = \alpha(x, z, w), \quad \dot{z} = \beta(x, z, w), \tag{3.4}$$

- *a multi-integer*[4] $r \triangleq (r_1, \ldots, r_{n-1})$,
- *and a local diffeomorphism* χ,

all defined in a neighborhood of the considered points, such that the closed-loop system

$$\dot{x} = g(x, \alpha(x, z, w)), \quad \dot{z} = \beta(x, z, w) \tag{3.5}$$

[3]A dynamic feedback is said endogenous if, and only if, the closed-loop system and the original one are Lie-Bäcklund equivalent, *i.e.* if, and only if, the extended state z can be locally expressed as a smooth function of x, u and a finite number of time derivatives of u (see [17, 4, 5, 15]).

[4]Recall that we denote by $v^{(r)} \triangleq \left(v_1^{(r_1)}, \ldots, v_{n-1}^{(r_{n-1})}\right) = \left(\frac{\mathrm{d}^{r_1} v_1}{\mathrm{d}t^{r_1}}, \ldots, \frac{\mathrm{d}^{r_{n-1}} v_{n-1}}{\mathrm{d}t^{r_{n-1}}}\right)$.

is locally diffeomorphic to the extended one

$$\dot{y} = \gamma(y, v), \quad v^{(r)} = w \tag{3.6}$$

for all $w \in \mathbb{R}^{n-1}$, i.e.

$$(x, z) = \chi(y, v, \dot{v}, \ldots, v^{(r-1)}), \qquad (y, v, \dot{v}, \ldots, v^{(r-1)}) = \chi^{-1}(x, z) \tag{3.7}$$

and

$$\hat{g} = \chi_* \hat{\gamma}, \qquad \hat{\gamma} = \chi_*^{-1} \hat{g} \tag{3.8}$$

where we have denoted

$$\hat{g}(x, z, w) \triangleq g(x, \alpha(x, z, w)) \frac{\partial}{\partial x} + \beta(x, z, w) \frac{\partial}{\partial z}$$
$$\hat{\gamma}(y, v, \dot{v}, \ldots, v^{(r-1)}, w) \triangleq \gamma(y, v) \frac{\partial}{\partial y} + \sum_{j=0}^{r-2} v^{(j+1)} \frac{\partial}{\partial v^{(j)}} + w \frac{\partial}{\partial v^{(r-1)}}.$$

3.3 Flatness

We say that system (2.1) is *differentially flat* (or, more shortly, *flat*) at the pair of points $(x_0, \overline{u}_0)$ and $\overline{y}_0$ if and only if, it is *Lie-Bäcklund equivalent to the trivial system* $\mathbb{R}^{n-1}_\infty$ *endowed with the trivial Cartan field*

$$\tau = \sum_{j \geq 0} \sum_{i=1}^{n-1} u_i^{(j+1)} \frac{\partial}{\partial u_i^{(j)}}$$

at the considered points.

Otherwise stated, the locally defined flat output $y = \Psi(x, \overline{u})$ is such that $(x, \overline{u}) = \Phi(\overline{y}) = (\Phi_0(\overline{y}), \Phi_1(\overline{y}), \Phi_2(\overline{y}), \ldots)$ with $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(\overline{y}) = g(\Phi_0(\overline{y}), \Phi_1(\overline{y}))$ for all sufficiently differentiable y.

This definition immediately implies that a system is flat if there exists a generalized output $y = \Psi(x, \overline{u})$ of dimension $n-1$, thus depending at most on a finite number of derivatives of u, with independent derivatives of all orders, such that x and $\overline{u}$ can be expressed in terms of y and a finite number of successive derivatives, *i.e.* $(x, \overline{u}) = \Phi(\overline{y})$, and such that the system equation $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(\overline{y}) = g(\Phi(\overline{y}))$ is identically satisfied for all sufficiently differentiable y.

For a flat system, with the notations of subsection 3.2, the vector field γ, or $\widehat{\gamma}$ indifferently, corresponds to the linear system in Brunovský canonical form

$$y_i^{(r_i+1)} = w_i, \qquad i = 1, \ldots, n-1, \tag{3.9}$$

$C_\gamma = \tau$ (with global coordinates $\overline{y} \triangleq (y, \dot{y}, \ldots)$ in place of $\overline{u}$) and theorem 1 reads:

Corollary 1. *If system* (2.1)*, with notation* (2.3)*, is flat at a given point, it is dynamic feedback linearizable in a neighborhood of this point,* i.e. *there exists an endogenous dynamic feedback of the form* (3.4) *and a local diffeomorphism* χ *such that the closed-loop system* (3.5) *is transformed by* χ *into* (3.9) *for all* $w \in \mathbb{R}^{n-1}$.

3.4 Lie-Bäcklund Atlas

The notion of a Lie-Bäcklund atlas for flat systems was initially introduced in [12] in the context of implicit systems. Our presentation here adapts this definition to the case of systems in explicit form. It consists of a collection of charts on $\mathfrak{X}$, that we call *Lie-Bäcklund charts and atlas*, and that will allow us to define the notions of apparent and intrinsic singularities.

Definition 1. *(i) A Lie-Bäcklund chart on $\mathfrak{X}$ is the data of a pair $(\mathfrak{U}, \psi)$ where $\mathfrak{U}$ is an open set of $\mathfrak{X}$ and $\psi : \mathfrak{U} \to \mathbb{R}^{n-1}_\infty$ a local flat output, with local inverse $\varphi : \mathfrak{V} \to \mathfrak{U}$, $\mathfrak{V} \triangleq \psi(\mathfrak{U})$ being an open set of $\mathbb{R}^{n-1}_\infty$.*

(ii) Two charts $(\mathfrak{U}_1, \psi_1)$ and $(\mathfrak{U}_2, \psi_2)$ are said to be compatible if, and only if, the mapping

$$\psi_1 \circ \varphi_2 : \psi_2(\varphi_1(\mathfrak{V}_1) \cap \varphi_2(\mathfrak{V}_2)) \subset \mathbb{R}^{n-1}_\infty \to \psi_1(\varphi_1(\mathfrak{V}_1) \cap \varphi_2(\mathfrak{V}_2)) \subset \mathbb{R}^{n-1}_\infty$$

with $\mathfrak{V}_i = \psi_i(\mathfrak{U}_i)$, $i = 1, 2$, is a local Lie-Bäcklund isomorphism (with the same trivial Cartan field τ associated to both the source and the target) with local inverse $\psi_2 \circ \varphi_1$, as long as $\varphi_1(\mathfrak{V}_1) \cap \varphi_2(\mathfrak{V}_2) \neq \emptyset$.

Note that charts with nonempty intersection are always compatible since the composition of Lie-Bäcklund isomorphisms is a Lie-Bäcklund isomorphism, or, otherwise stated, in reason of the transitivity of the Lie-Bäcklund equivalence relation.

(iii) An atlas $\mathfrak{A}$ is a collection of compatible charts.

For a given atlas $\mathfrak{A} = (\mathfrak{U}_i, \psi_i)_{i \in I}$, let $\mathfrak{U}_{\mathfrak{A}}$ be the union $\mathfrak{U}_{\mathfrak{A}} = \bigcup_{i \in I} \mathfrak{U}_i$.

Remark 1. *In definition 1, we stress that Lie-Bäcklund isomorphisms play a similar role as the smooth diffeomorphisms appearing in the definition of a usual smooth manifold, at the exception that we do not require that $\mathfrak{U}_{\mathfrak{A}} = \mathfrak{X}$.*

Remark 2. *The charts are made of open sets that are homeomorphic to open sets of $(\mathbb{R}^{n-1})^N$ for some finite N, according to (i), and thus it is always possible to construct a larger atlas entirely made of topologically trivial (*i.e. *contractible) open sets.*

3.5 Apparent and Intrinsic Flatness Singularities

It is clear from what precedes that if we are given two Lie-Bäcklund atlases, their union is again a Lie-Bäcklund atlas. Therefore the union of all charts that form every atlas is well-defined as well as its complementary, which we call the set of intrinsic flatness singularities, as stated in the next definition.

Definition 2. *We say that a point in $\mathfrak{X}$ is an intrinsic flatness singularity if it is excluded from all charts of every Lie-Bäcklund atlas. Every other singular point, namely every point $\overline{x} \notin \mathfrak{U}_i$ for some chart $(\mathfrak{U}_i, \psi_i)$ but for which there exists another chart $(\mathfrak{U}_j, \psi_j)$, $j \neq i$, such that $\overline{x} \in \mathfrak{U}_j$, is called apparent.*

Clearly, this notion does not depend on the choice of atlas and charts. The concrete meaning of this notion is that at points that are intrinsic singularities there is no flat output, i.e. the system is not flat at these points.

On the other hand, points that are apparent singularities are singular for a given set of flat outputs, but well defined points for another set of flat outputs defined in another chart containing these points.

4 Intrinsic Flatness Singularity Study of Control Affine Systems with $n-1$ Inputs

4.1 On Flat Output Computation and Lie-Bäcklund Atlas Construction at Generic Points

We start this section with the singularity study (and thus the flat output computation) at points $x \in X$ such that the vector space generated by the control vector fields $f_1(x), \ldots, f_{n-1}(x)$ has dimension equal to $n-1$ and remains constant in a suitable open neighborhood of x. We call these points *generic* for obvious reasons. We first prove that these generic points are such that the Γ-accessibility rank condition (see [6, 10]) is satisfied, or equivalently that the first approximation of the system around germs of integral curves passing through these points is controllable (see [22, 11]), and give a first sufficient condition for the existence of flat outputs at these points. We then construct the associated Lie-Bäcklund charts and atlas.

4.1.1 Γ-accessibility Rank Condition of Affine Systems with $n-1$ Inputs

Let us recall the classical Lie bracket notations: $[\eta, \gamma] \triangleq \frac{\partial \gamma}{\partial x}\eta - \frac{\partial \eta}{\partial x}\gamma$ denotes the Lie bracket of the vector fields η and γ, and, iteratively, $\mathrm{ad}_\eta \gamma \triangleq [\eta, \gamma]$, $\mathrm{ad}_\eta^k \gamma = [\eta, \mathrm{ad}_\eta^{k-1}\gamma]$, with $\mathrm{ad}_\eta^0 \gamma = \gamma$, for all $k \geq 0$. We also denote the $n \times (n-1)$ matrix $\mathrm{ad}_g^k G$ by

$$\mathrm{ad}_g^k G \triangleq \begin{pmatrix} \mathrm{ad}_g^k f_1 & \cdots & \mathrm{ad}_g^k f_{n-1} \end{pmatrix}, \tag{4.1}$$

with G defined by (2.2), and the $n \times (k+1)(n-1)$ matrix $\mathcal{G}_k$, for all $k \geq 1$, by

$$\mathcal{G}_k \triangleq \begin{pmatrix} G & -\mathrm{ad}_g G & \cdots & (-1)^k \mathrm{ad}_g^k G \end{pmatrix}, \tag{4.2}$$

that may be interpreted as the Wronskian matrix of G (see [22]), where the successive time derivative operators $\frac{\mathrm{d}^k}{\mathrm{d}t^k}$ are replaced by the iterated Lie bracket operators $(-1)^k \mathrm{ad}_g^k$. Note that, for linear systems, $\mathcal{G}_n$ is often called the Kalman controllability matrix (see (4.7) in the proof of lemma 1).

Definition 3. *Given the following sequence of distributions:*

$$\Gamma_0 \triangleq \mathsf{Span}\{f_1, \cdots, f_{n-1}\}, \qquad \Gamma_{k+1} \triangleq \Gamma_k + \mathrm{ad}_g \Gamma_k, \quad k \geq 0, \tag{4.3}$$

we denote by $\Gamma_k(x,u) = \{\gamma(x,u) : \gamma \in \Gamma_k\}$. *We say that the* Γ-accessibility rank condition *is satisfied at the point* (x_0,u_0) *if, and only if, there exists* $k^\star \in \mathbb{N}$ *such that* $\Gamma_{k^\star}(x_0,u_0) = \Gamma_\infty(x_0,u_0) = \mathrm{T}_{x_0,u_0}X$, i.e. *if* $\dim \Gamma_{k^\star}(x_0,u_0) = \dim \Gamma_\infty(x_0,u_0) = n$.

Remark 3. $\Gamma_{k^\star}$ *is thus the Lie ideal generated by* $f_1,\cdots,f_{n-1}$ *in the Lie algebra generated by* $g, f_1,\cdots,f_{n-1}$.

Definition 4. *Consider the other sequence of distributions*

$$\mathcal{D}_{k+1} = \overline{\mathcal{D}}_k + \mathrm{ad}_{f_0}\overline{\mathcal{D}}_k, \quad k \geq 0, \quad \mathcal{D}_0 = \Gamma_0 \tag{4.4}$$

where $\overline{\mathcal{D}}_k$ *is the involutive closure of* $\mathcal{D}_k$ *(see* e.g. *[8]). The condition* $\dim \mathcal{D}_\infty(x_0) = n$ *is called the* strong accessibility rank condition *[6].*

Proposition 1. *We have* $\Gamma_k \subset \mathcal{D}_k$ *for all* k *and* $u \in \mathbb{R}^m$ *and* $\Gamma_\infty(x_0,u_0) = \mathcal{D}_\infty(x_0)$ *for all* u_0 *in a dense subset of* $\mathbb{R}^m$. *Moreover,* $\dim \Gamma_\infty(x_0,u_0) = n$ *implies* $\dim \mathcal{D}_\infty(x_0) = n$. *In other words, the* Γ*-accessibility rank condition implies the strong accessibility rank condition.*

Proof. Since $g = f_0 + \sum_{i=1}^{n-1} u_i f_i$, we indeed have $\Gamma_k \subset \mathcal{D}_k$ for all k and $u \in \mathbb{R}^m$ and $\Gamma_\infty(x_0,u_0) = \mathcal{D}_\infty(x_0)$ for all u_0 in a dense subset of $\mathbb{R}^m$. Again, since $\Gamma_k \subset \mathcal{D}_k$ for all k and $u \in \mathbb{R}^m$, we immediately deduce that Γ-accessibility implies strong accessibility. □

Remark 4. *If the distributions* (4.3) *have constant rank in a neighborhood of* (x_0,u_0), *it can be readily verified that* $k^\star$, *defined by definition 3, satisfies* $k^\star \leq n-1$. *If, on the contrary, the rank of some of these distributions drops down at* (x_0,u_0), *then it is possible that* $k^\star > n$.

Theorem 2. *Every flat system at a point satisfies the* Γ*-accessibility rank condition (and thus the strong accessibility rank condition) at this point.*

Proof. Given a flat system at a point, we consider the associated equivalent linear system (3.9) which is indeed controllable. The Kalman controllability matrix of this system has constant rank $n' \triangleq \sum_{i=1}^{n-1} r_i$, with the notations of theorem 1. As already noted, to this matrix, there corresponds the increasing sequence of distributions generated by the control vector fields $\hat{\Gamma}_0 \triangleq \mathsf{Span}\{\chi_*\left(\frac{\partial}{\partial y_i^{(r_i)}}\right), i = 1,\ldots,n-1\}$ and their iterated Lie brackets with the vector field $\hat{g}$, defined by (3.8), *i.e.* $\hat{\Gamma}_{k+1} \triangleq \hat{\Gamma}_k + \mathrm{ad}_{\hat{g}}\hat{\Gamma}_k$, for $k \geq 0$, analogously to the construction (4.3). According to the properties of the image of the Lie bracket by diffeomorphism, it results that the distributions $\hat{\Gamma}_k$ must have locally a constant dimension and that the largest one, say $\hat{\Gamma}_\infty$, must have constant dimension equal to n'. We also note that since χ is a local diffeomorphism satisfying $(x,z) = \chi(y,\dot{y},\ldots,y^{(r-1)})$, we indeed have $n' \geq n$. We thus assume that the Γ-accessibility rank condition is not satisfied or, equivalently, that $\dim \Gamma_{k^\star}(x_0,u_0) < n$. Since, by construction, the projection on T_X of $\hat{\Gamma}_\infty$ is contained in $\Gamma_{k^\star}$, there must exist at least a non 0 combination of the x_j's, $j \in \{1,\ldots,n-1\}$, denoted by ξ, such that $d\xi \in \left(\hat{\Gamma}_\infty\right)^\perp$. But then an immediate computation shows that $d\chi^{-1}(\xi)$ must be independent of the inputs $y_i^{(r_i)}$ for all $i = 1,\ldots,n-1$, which contradicts the controllability of system (3.9), hence the result. □

A simple interpretation of the Γ-accessibility rank condition may be given in terms of controllability of the first order time-varying linear approximation of the system:

Definition 5. *System* (2.1) *is said controllable at the first order with constant input at* $(x_0, u_0) \in X \times \mathbb{R}^{n-1}$*, or first order controllable with constant input* $u_0 \in \mathbb{R}^{n-1}$*, at a point* $x_0 \in X$ *if, and only if, its tangent linear approximation along the integral curve* $(x(t), u_0)$ *passing through* (x_0, u_0) *at time* $t = 0$*, and with constant input* $u(t) = u_0$ *for all* t *in a small interval of time containing 0, namely* $\delta\dot{x} = \frac{\partial g}{\partial x}(x(t), u_0)\delta x + \sum_{i=1}^{n-1} f_i(x(t))\delta u_i$*, is controllable in the sense of linear time-varying systems (see e.g. [22, 11]).*

Lemma 1. *The system is controllable at the first order with constant input at* (x_0, u_0) *if, and only if, in an open neighborhood of* (x_0, u_0)*, there exists* $k^\star \in \mathbb{N}$ *such that:*

$$\mathsf{rank}\ \mathcal{G}_{k^\star}(x_0, u_0) = n \tag{4.5}$$

or, equivalently, if, and only if, the Γ*-accessibility rank condition is satisfied at* (x_0, u_0)*.*

Moreover, if in an open neighborhood of (x_0, u_0) *there exists* $k \in \{0, \ldots, n-1\}$ *such that*

$$\dim(\mathsf{Span}\{f_1, \ldots, f_{n-1}, [g, f_k]\}) = n \tag{4.6}$$

then condition (4.5) *is satisfied.*

Proof. The tangent linear system along the integral curve $t \mapsto (x(t), u_0)$, with constant $u_0 \triangleq (u_{1,0}, \ldots, u_{n-1,0})$ for t in a given open interval I containing 0, is given by: $\delta\dot{x} = A(t)\delta x + B(t)\delta u$, where

$$A(t) \triangleq \frac{\partial f_0}{\partial x}(x(t)) + \sum_{i=1}^{n-1} u_{i,0}\frac{\partial f_i}{\partial x}(x(t)), \qquad B(t) \triangleq G(x(t)).$$

According to [22, 11], this linear time-varying system is controllable at (x_0, u_0) if, and only if the controllability matrix

$$\mathcal{C}(0) \triangleq \left(B(t), (A(t) - \frac{\mathrm{d}}{\mathrm{d}t})B(t), \cdots, (A(t) - \frac{\mathrm{d}}{\mathrm{d}t})^{k^\star}B(t)\right)_{|t=0}$$

has rank n for some $k^\star$.

On the other hand, we have, using (2.3), in matrix notation:

$$\mathrm{ad}_g f_k = \frac{\partial f_k}{\partial x}\left(f_0 + \sum_{i=1}^{n-1} u_{i,0} f_i\right) - \left(\frac{\partial f_0}{\partial x} + \sum_{i=1}^{n-1} u_{i,0}\frac{\partial f_i}{\partial x}\right) f_k$$

Therefore, an easy direct computation yields:

$$\mathrm{ad}_g G = \left(\frac{\mathrm{d}B}{\mathrm{d}t}(t) - A(t)B(t)\right)_{|t=0} = -\left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t}\right)B(t)_{|t=0}. \tag{4.7}$$

Thus, by induction, we get $\mathrm{ad}_g^k G = -\left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t}\right)\left((-1)^{k-1}\left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-1} B(t)\right)_{|t=0} = (-1)^k \left(\left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t}\right)^k B(t)\right)_{|t=0}$, since u_0 is constant, and the controllability matrix $\mathcal{C}(0)$ is proven to be equal to $\mathcal{G}_{k^\star}(x_0, u_0)$.

Moreover, assuming that, at (x_0, u_0), $\mathrm{ad}_g f_k \notin \mathsf{Span}\{f_1, \cdots, f_{n-1}\} = \mathrm{im}\, G$ for some $k \in \{1, \ldots, n-1\}$, then the matrix $\begin{pmatrix} G & -\mathrm{ad}_g G \end{pmatrix}$ has rank n, which immediately implies that the controllability matrix $\mathcal{G}_1(x_0, u_0) = \mathcal{C}(0)$ has full rank, hence the first order controllability at (x_0, u_0). □

4.1.2 Flat Outputs for Affine Systems with $n-1$ Inputs at Generic Points

Theorem 3. *Let $(x_0, u_0) \in X \times \mathbb{R}^{n-1}$ and k be such that assumption (4.6) holds in an open neighborhood of (x_0, u_0). Then, every $(n-1)$-tuple of first integrals of f_k, independent at (x_0, u_0), forms a vector of flat outputs in an open neighborhood of (x_0, u_0).*

Proof. We consider f_k satisfying assumption (4.6). Note that $f_k(x_0) \neq 0$, since otherwise, the rank of the distribution $\mathsf{Span}\{f_1, \ldots, f_{n-1}, [g, f_k]\}$ would be smaller than or equal to $n-1$ at this point. Thus, x_0 is a transient point of f_k, and, according to *e.g.* [1, 13], there exist $(n-1)$ differentially independent first integrals[5] of f_k, noted $z_1 = \psi_{0,1}(x), \ldots, z_{n-1} = \psi_{0,n-1}(x)$, *i.e.* satisfying:

$$L_{f_k}\psi_{0,i}(x) = 0, \quad i = 1, \cdots, n-1, \tag{4.8}$$

where we have denoted by $L_\gamma h$ the Lie derivative of an arbitrary differentiable function h along the vector field γ.

In order to show that, from these first integrals, one can deduce the diffeomorphism that puts the system in canonical form (3.9), whose Jacobian matrix is explicitly given by the matrix N below (see (4.15)), that we prove to be locally invertible, we first show that the Lie derivative of the vector function $\psi_0 \triangleq (\psi_{0,1}, \cdots, \psi_{0,n-1})$ of the first integrals along the system vector field g does not depend on the input u_k. Thanks to (4.8), we have:

$$\dot{z}_i = \sum_{j=1}^{n} \frac{\partial \psi_{0,i}}{\partial x_j}(x)\dot{x}_j = L_{f_0}\psi_{0,i}(x) + \sum_{j=1}^{n-1} u_j L_{f_j}\psi_{0,i}(x) = L_{f_0}\psi_{0,i}(x) + \sum_{j\neq k} u_j L_{f_j}\psi_{0,i}(x) \triangleq \psi_{1,i}(x, u). \tag{4.9}$$

Thus, $\frac{\partial \dot{z}_i}{\partial u_j} = \frac{\partial \psi_{1,i}}{\partial u_j} = L_{f_j}\psi_{0,i}$ for $j \neq k$ and $\frac{\partial \psi_{1,i}}{\partial u_k} = 0$ for all $i \in \{1, \cdots, n-1\}$, thus meaning that $\psi_{1,i}$ depends only on $(x, \hat{u})$ where $\hat{u}$ denotes the collection of inputs where u_k has been removed, *i.e.* $\hat{u} \triangleq (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{n-1})$.

Denoting by $z \triangleq (z_1, \cdots, z_{n-1}) = \psi_0(x)$ and $\dot{z} \triangleq (\psi_{1,1}(x, \hat{u}), \ldots, \psi_{1,n-1}(x, \hat{u})) \triangleq \psi_1(x, \hat{u})$, $D\psi_0(x) = \frac{\partial \psi_0}{\partial x}(x)$ the Jacobian matrix of ψ_0 at the point x, of rank $n-1$ according

[5] The existence of $n-1$ independent local first integrals is equivalent to the existence of local coordinates in which f_k is straightened out. Note that the practical computation of these coordinates is generally done by computing the associated first integrals (see again [1, 13]). Therefore, straightening out f_k would not provide significant simplifications in general.

to the independence of the first integrals, and $\hat{f} \triangleq (f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{n-1})$, we have that $\frac{\partial \dot{z}}{\partial \hat{u}} = \frac{\partial \psi_1}{\partial \hat{u}} = (D\psi_0)\,\hat{f}$ does not depend on u and $\mathsf{rank}\,\left(\frac{\partial \psi_1}{\partial \hat{u}}\right) = \mathsf{rank}\,\left((D\psi_0)\,\hat{f}\right) = n-2$ since, by (4.6), the $n-2$ columns of $\hat{f}$ are independent.

We next consider the following matrix:

$$M \triangleq \begin{pmatrix} \dfrac{\partial \psi_0}{\partial x} & 0 \\ \dfrac{\partial \psi_1}{\partial x} & \dfrac{\partial \psi_1}{\partial \hat{u}} \end{pmatrix}$$

Clearly, M is the Jacobian matrix of the mapping $\psi \triangleq (\psi_0, \psi_1) : (x, \hat{u}) \mapsto (\psi_0(x) = z, \psi_1(x, \hat{u}) = \dot{z})$. We prove that its rank is exactly $2n-2$ in an open neighborhood of (x_0, u_0).

Assume, on the contrary, that the columns of M are linearly dependent in a given neighborhood of (x_0, u_0). Thus, there exist smooth functions $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$, not all equal to 0, such that

$$\sum_{j=1}^{n} \lambda_j(x, \hat{u}) \frac{\partial \psi_0}{\partial x_j}(x) = 0. \tag{4.10}$$

$$\sum_{j=1}^{n} \lambda_j(x, \hat{u}) \frac{\partial \psi_1}{\partial x_j}(x, \hat{u}) + \sum_{j=1, j\neq k}^{n-1} \mu_j(x, \hat{u}) \frac{\partial \psi_1}{\partial u_j}(x, \hat{u}) = 0 \tag{4.11}$$

In view of equation (4.10), since $(\psi_{0,1}, \ldots, \psi_{0,n-1})$ are independent first integrals of f_k, we immediately deduce that there exists a smooth function λ such that

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \lambda f_k$$

and thus, using (4.9), equation (4.11) reads

$$\lambda(x, \hat{u}) \frac{\partial \psi_1}{\partial x}(x, \hat{u}) f_k(x) + \sum_{j=1, j\neq k}^{n-1} \mu_j(x, \hat{u}) \frac{\partial \psi_1}{\partial u_j}(x) = \lambda L_{f_k} \psi_1(x, \hat{u}) + \sum_{j=1, j\neq k}^{n-1} \mu_j L_{f_j} \psi_0(x, \hat{u}) = 0 \tag{4.12}$$

but, again using (4.9) and (4.8), we get $L_{f_k}\psi_1 = L_{f_k} L_g \psi_0 = -L_{[g, f_k]}\psi_0$ and (4.12) becomes

$$-\lambda L_{[g, f_k]}\psi_0 + \sum_{j=1, j\neq k}^{n-1} \mu_j L_{f_j} \psi_0 = L_{\left(-\lambda[g, f_k] + \sum_{j=1, j\neq k}^{n-1} \mu_j f_j\right)} \psi_0 = 0$$

which proves that the linear combination $-\lambda[g, f_k] + \sum_{j=1, j\neq k}^{n-1} \mu_j f_j$ of $f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{n-1}$ and $[g, f_k]$ must be colinear to f_k, hence contradicting the assumption (4.6). We have thus proven that $\mathsf{rank}\ M = 2n-2$ in a neighborhood of (x_0, u_0), as announced and thus that the mapping $\psi \triangleq (\psi_0, \psi_1) : (x, \hat{u}) \mapsto (\psi_0(x) = z, \psi_1(x, \hat{u}) = \dot{z})$ is locally invertible.

Therefore x and $\hat{u}$ can be obtained from z and $\dot{z}$ in a unique way

$$x = \varphi_0(z, \dot{z}), \quad \hat{u} = \varphi_1(z, \dot{z})$$

where (φ_0, φ_1) is the inverse mapping of $\psi = (\psi_0, \psi_1)$.

In order to obtain the last input u_k from the derivatives of z, and thus prove that z is a flat output, we compute $\ddot{z} = L_g\psi_1$:

$$\begin{aligned}\ddot{z} = L_g\psi_1 = L_{f_0}^2\psi_0 + \sum_{i\neq k}\hat{u}_i L_{f_0}L_{f_i}\psi_0 + \sum_{i\neq k}\hat{u}_i L_{f_i}L_{f_0}\psi_0 + u_k L_{f_k}L_{f_0}\psi_0 \\ + \sum_{i\neq k} u_k\hat{u}_i L_{f_k}L_{f_i}\psi_0 + \sum_{i,j\neq k}\hat{u}_i\hat{u}_j L_{f_i}L_{f_j}\psi_0 + \sum_{i\neq k}\dot{\hat{u}}_i L_{f_i}\psi_0 \triangleq \psi_2(x, \hat{u}, \dot{\hat{u}}, u_k)\end{aligned}$$

This immediately yields

$$\frac{\partial\psi_2}{\partial\dot{\hat{u}}_i} = L_{f_i}\psi_0 = \frac{\partial\psi_1}{\partial\hat{u}_i} \tag{4.13}$$

and

$$\frac{\partial\psi_2}{\partial u_k} = L_{f_k}L_{f_0}\psi_0 + \sum_{i\neq k}\hat{u}_i L_{f_k}L_{f_i}\psi_0 = L_{f_k}L_g\psi_0 = -L_{[g,f_k]}\psi_0 \neq 0 \tag{4.14}$$

as an immediate consequence of (4.6).

Therefore, we may complete the matrix M by the following $3(n-1)\times 3(n-1)$ square matrix

$$N \triangleq \begin{pmatrix} \frac{\partial\psi_0}{\partial x} & 0 & 0 & 0 \\ \frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_1}{\partial\hat{u}} & 0 & 0 \\ \frac{\partial\psi_2}{\partial x} & \frac{\partial\psi_2}{\partial\hat{u}} & \frac{\partial\psi_2}{\partial u_k} & \frac{\partial\psi_2}{\partial\dot{\hat{u}}} \end{pmatrix} \tag{4.15}$$

which, according to (4.13) and (4.14), is clearly invertible. We indeed recognize that N is the Jacobian matrix of the transformation $\overline{\psi} \triangleq (\psi_0, \psi_1, \psi_2)$, which proves that $\overline{\psi}$ is a local diffeomorphism and thus that x and $u = (\hat{u}, u_k)$ may be expressed as functions of $(z, \dot{z}, \ddot{z})$.

Putting these results together, we have proven that z is a vector of flat outputs □

Remark 5. *A comparable result has been proven by P. Martin in [18] using different ideas, related to system structure at infinity. More precisely, in [18], condition* (4.6) *is proven to be a flatness sufficient condition, but the role played by first integrals of one of the control vector fields in the construction of flat outputs has not been brought to light.*

Remark 6. *In the case of systems with $n-1$ inputs, thanks to theorem 3, the general approach to the computation of flat outputs presented in [15, 16], based on an implicit representation of system* (2.1)*, is not needed. Moreover, the present direct approach does not require an unbounded recursion as in the above mentioned references. These facts will yield important simplifications in the analysis of flatness singularities in the next sections.*

Remark 7. *The last part of the proof of theorem 3 could be slightly shortened by remarking that, since x has been proven to be a function of $(z, \dot{z})$, the last input u_k may be obtained by differentiating x with respect to time, but we have preferred a more explicit argument by constructing the Jacobian matrix of the diffeomorphism expressing $x, \hat{u}, u_k$ in function of $(z, \dot{z}, \ddot{z})$.*

4.1.3 Interpretation in terms of Feedback Linearization

Consider the $(2n-2)$-dimensional extended system (see also [18]):

$$\begin{aligned} \dot{x} &= \left(f_0(x) + \sum_{i \neq k} u_i f_i(x) \right) + u_k f_k(x) \\ \dot{u}_i &= v_i, \quad i \neq k \end{aligned} \tag{4.16}$$

with drift[6]

$$f \triangleq f_0 + \sum_{i \neq k} u_i f_i \tag{4.17}$$

and control vector fields defined by:

$$\tilde{f}_i \triangleq \begin{cases} \dfrac{\partial}{\partial u_i}, & i \in \{1, \ldots, n-1\}, \ i \neq k \\ f_k, & i = k. \end{cases}$$

Then we consider the distributions

$$G_0 = \mathsf{Span}\{\tilde{f}_1, \ldots, \tilde{f}_{n-1}\}, \quad G_1 = G_0 + \mathrm{ad}_f G_0$$

in a neighborhood of a generic point, where we have denoted $\mathrm{ad}_f G_0 \triangleq \{\mathrm{ad}_f \gamma \mid \gamma \in G_0\}$. A direct computation immediately shows that $[\tilde{f}_i, \tilde{f}_j] = 0$ for all i, j, that $\mathrm{ad}_f \tilde{f}_i = -f_i$ if $i \neq k$, and $\mathrm{ad}_f \tilde{f}_k = \mathrm{ad}_f f_k = [f, f_k] = [g, f_k]$ (where g is defined by (2.3)). Thus, according to assumption (4.6), G_0 is clearly involutive and

$$G_1 = \mathsf{Span}\left\{ \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{k-1}}, \frac{\partial}{\partial u_{k+1}}, \ldots, \frac{\partial}{\partial u_{n-1}}, f_1, \ldots, f_{n-1}, [g, f_k] \right\}$$

is involutive and has rank $2n-2$ in the neighborhood under consideration. Hence, according to [7, 9], system (4.16) is static feedback linearizable in a suitable neighborhood of the extended state manifold of dimension $(2n-2)$ and local coordinates $(x, \hat{u}) = (x_1, \ldots, x_n, u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{n-1})$, where $\hat{u} \triangleq (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{n-1})$ is the state extension of dimension $n-2$. In other words, there exists a regular extended-state feedback $v_i = a_i(x, \hat{u}, w)$, for $i = 1, \cdots, n-1$, $i \neq k$, and $u_k = a_k(x, \hat{u}, w)$, where $w \triangleq (w_1, \ldots, w_{n-1})$ is the new input, and a local diffeomorphism $(z, \dot{z}) = \psi(x, \hat{u})$, such that the closed-loop system of dimension $2n-2$, namely

$$\begin{aligned} \dot{x} &= f_0(x) + \sum_{i \neq k} \hat{u}_i f_i(x) + a_k(x, \hat{u}, w) f_k(x) \\ \dot{\hat{u}}_i &= a_i(x, \hat{u}, w), \quad i \neq k \end{aligned}$$

[6] hence we have $g = f + u_k f_k$.

where we have renamed $u_i \triangleq \hat{u}_i$, $i \neq k$, can be transformed by ψ into the $(2n-2)$-dimensional linear controllable one

$$\ddot{z}_i = w_i, \quad i = 1, \ldots, n-1.$$

4.2 A First Atlas Construction

Let $\Omega_0 \subset X \times \mathbb{R}^{n-1}$ be the set of points (x, u) that satisfy assumption (4.6)[7], *i.e.*

$$\Omega_0 \triangleq \{(x, u) \in X \times \mathbb{R}^{n-1} \mid \exists k : \mathsf{Span}\{f_1, \ldots, f_{n-1}, [f, f_k]\} = \mathrm{T}_x X\} \tag{4.18}$$

where $\mathrm{T}_x X$ denotes the tangent space of X at the point x and $f = f_0 + \sum_{i \neq k} u_i f_i$ (see (4.17)). We denote by $\tilde{\Omega}_0 \subset \mathfrak{X}$, the set of points $(x, \overline{u})$ whose projection (x, u) in $X \times \mathbb{R}^{n-1}$ belongs to Ω_0

We also consider the set

$$\Omega \triangleq \{(x, u) \in X \times \mathbb{R}^{n-1} \mid \exists k^\star \in \mathbb{N} : \mathsf{rank}\ \mathcal{D}_{k^\star}(x, u) = n\} \tag{4.19}$$

with $\mathcal{D}_k$ defined by (4.4). Recall indeed that $\Omega_0 \subset \Omega$ (see lemma 1 and proposition 1). We also denote by $\tilde{\Omega} \subset \mathfrak{X}$, the set of points $(x, \overline{u})$ whose projection (x, u) in $X \times \mathbb{R}^{n-1}$ belongs to Ω. We indeed also have $\tilde{\Omega}_0 \subset \tilde{\Omega}$.

Then the following assertion holds:

Theorem 4. *Under assumption* (4.6)*, for each point* $(x_0, \overline{u}_0) \in \tilde{\Omega}_0$*, there exists an open neighborhood* $U_{(x_0,u_0)} \subset \tilde{\Omega}_0$ *of* $(x_0, \overline{u}_0)$[8]*, and a well-defined flat output* $z = \Phi_{(x_0,u_0)}(x, u, \dot{u})$ *in* $U_{(x_0,u_0)}$*, constructed according to theorem 3. Moreover,* $(U_{(x_0,u_0)}, \Phi_{(x_0,u_0)})$ *constitutes a Lie-Bäcklund chart, the collection of which defines a Lie-Bäcklund atlas.*

Moreover, the set of intrinsic singularities is contained in $\tilde{\Omega}_0^{\mathsf{C}}$ *and contains* $\tilde{\Omega}^{\mathsf{C}}$*, where the superscript* ^{C} *stands for the complementary of the corresponding set.*

Proof. For each point $(x_0, \overline{u}_0) \in \tilde{\Omega}_0$, there exists some k and an open neighborhood $U_{(x_0,u_0)}$ of $(x_0, \overline{u}_0)$ such that condition (4.6) holds and such that there exists a flat output $z = \Phi_{(x_0,u_0)}(x, u, \dot{u})$ made of $n-1$ independent first integrals of f_k in $U_{(x_0,u_0)}$ according to theorem 3. The charts, therefore made of the pairs $(U_{(x_0,u_0)}, \Phi_{(x_0,u_0)})$ are indeed Lie-Bäcklund charts and naturally compatible (property (ii) of definition 1) thanks to the transitivity of the Lie-Bäcklund equivalence relation. Therefore they form a Lie-Bäcklund atlas.

Since $\tilde{\Omega}_0$ contains only regular points, and since, in $\tilde{\Omega}$, the Γ-accessibility rank condition $\mathsf{rank}\ \mathcal{G}_{k^\star}(x, u) = n$ is satisfied (see lemma 1) and is a necessary flatness condition (theorem 2), the last assertion of this theorem is proven. □

Remark 8. *When the fields are analytic, the complementary of* Ω_0*, made of the points where* $\dim(\mathsf{Span}\{f_1, \ldots, f_{n-1}, [f, f_k]\}) < n$ *for all* $k \in \{0, \ldots, n-1\}$ *is at least of codimension 1, hence the set of intrinsic singularities has zero Lebesgue measure. When the fields are* C^∞ *but not analytic, the Lebesgue measure of* Ω_0^{C} *may be non-zero.*

[7] Note that condition (4.6) depends on u by $f = f_0 + \sum_{i \neq k} u_i f_i$. Note furthermore that, according to (2.3) and (4.17), $[g, f_k] = [f, f_k]$.

[8] The neighborhood $U_{(x_0,u_0)}$ can always be chosen of the form $U^0_{(x_0,u_0)} \times \mathbb{R}^{n-1}_\infty$ where $U^0_{(x_0,u_0)}$ is a neighborhood of (x_0, u_0) in Ω_0 and thus only depends on (x_0, u_0).

4.3 More on the Set of Intrinsic Singularities, Non Generic Points

In the previous section, we have proven that the set of intrinsic singularities is contained in $\tilde{\Omega}_0^{\mathsf{C}}$ and contains $\tilde{\Omega}^{\mathsf{C}}$. In this section, we investigate more in depth the structure of $\tilde{\Omega}_0^{\mathsf{C}}$ and show that there might exist points of $\tilde{\Omega}_0^{\mathsf{C}}$ where the system is still flat, or otherwise stated, $\tilde{\Omega}_0^{\mathsf{C}}$ might contain some apparent singularities.

The next result studies degenerated situations, compared to (4.6), namely when some of the control vector fields become linearly dependent of the others. Therefore, it is intended to be applied to points $(x, u) \notin \Omega_0$.

We thus consider a point x_0 and the distribution $\Gamma_0 \triangleq \mathsf{Span}\{f_1, \cdots, f_{n-1}\}$ and denote by $\Gamma_0(x_0)$ the vector space generated by the vectors $\{f_1(x_0), \cdots, f_{n-1}(x_0)\}$. We assume that $\dim \Gamma_0(x_0) = n-p$, with $p > 1$[9]. Without loss of generality and up to a renumbering of the f_i's, $i = 1, \ldots, n-1$, we note

$$\Gamma_0^a \triangleq \mathsf{Span}\{f_1, \ldots, f_{n-p}\}, \quad \Gamma_0^b \triangleq \mathsf{Span}\{f_{n-p+1}, \ldots, f_{n-1}\}, \tag{4.20}$$

and assume that $\Gamma_0(x_0) = \Gamma_0^a(x_0)$, thus meaning that the dimension of Γ_0 drops down from $n-1$ to $n-p$ at x_0 and that $\Gamma_0^b(x_0) \subset \Gamma_0^a(x_0)$.

For simplicity's sake, we note

$$\begin{aligned} u_a &\triangleq (u_1, \ldots, u_{n-p}), & f_a &\triangleq (f_1, \ldots, f_{n-p}), & u_a f_a &\triangleq \sum_{i=1}^{n-p} u_i f_i, \\ u_b &\triangleq (u_{n-p+1}, \ldots, u_{n-1}), & f_b &\triangleq (f_{n-p+1}, \ldots, f_{n-1}), & u_b f_b &\triangleq \sum_{i=n-p+1}^{n-1} u_i f_i. \end{aligned} \tag{4.21}$$

Thus, the system equation (2.1) reads

$$\dot{x} = f_0(x) + u_a f_a(x) + u_b f_b(x)$$

and $u_b f_b$ may now be considered as part of the drift, the independent controls being restricted to u_a. Therefore, we embed the drift vector field $f_0 + u_b f_b$ in a vector field of $X \times \mathbb{R}_\infty^{p-1}$, given by

$$F_0^b(x, \overline{u}_b) \triangleq f_0(x) + u_b f_b(x) + \tau_b, \tag{4.22}$$

where τ_b is the trivial Cartan field of $\mathbb{R}_\infty^{p-1}$ with global coordinates $\overline{u}_b \triangleq (u_b, \dot{u}_b, \ldots)$:

$$\tau_b \triangleq \sum_{k \geq 0} \sum_{j=n-p+1}^{n-1} u_j^{(k+1)} \frac{\partial}{\partial u_j^{(k)}} \triangleq \sum_{k \geq 0} u_b^{(k+1)} \frac{\partial}{\partial u_b^{(k)}}. \tag{4.23}$$

We also introduce the following sequence of distributions of the tangent bundle $\mathrm{T}X \times \mathrm{T}\mathbb{R}_\infty^{n-1}$:

$$\Gamma_{k+1}^a \triangleq \Gamma_k^a + \mathrm{ad}_{F_0^b} \Gamma_k^a, \quad \forall k \geq 0. \tag{4.24}$$

The following lemma shows that, in fact, the Γ_k^a's are all contained in $\mathrm{T}X$ and thus have dimension less than or equal to n.

[9] Recall that Γ_0 is assumed to have dimension $n-1$ in an open dense subset $\mathcal{O} \subset X$. Thus, we indeed have $x_0 \notin \mathcal{O}$

Lemma 2. *We have $\Gamma_k^a(x,\overline{u}_b) \subset \mathrm{T}_x X$ for every k, every $x \in X$ and every $\overline{u}_b \in \mathbb{R}_\infty^{p-1}$, and there exists $k^\star \in \mathbb{N}$ such that $\dim \Gamma_k^a(x,\overline{u}_b) \leq \dim \Gamma_{k^\star}^a(x,\overline{u}_b) \leq n$ for every $k \in \mathbb{N}$ and $(x,\overline{u}_b) \in X \times \mathbb{R}_\infty^{p-1}$. Moreover, if $(x,\overline{u}_b)$ is such that the $\Gamma_k^a(x,\overline{u}_b)$'s have locally constant dimensions, then $k^\star \leq p$.*

Proof. Consider the sequence of distributions of $\mathrm{T}X$:

$$\Delta_{k+1}^a \triangleq \Delta_k^a + \mathrm{ad}_{f_0}\Delta_k^a + [\Gamma_0^b, \Delta_k^a], \quad \forall k \geq 0, \qquad \Delta_0^a = \Gamma_0^a. \tag{4.25}$$

Every Δ_k^a is a subdistribution of $\mathrm{T}X$, hence $\dim \Delta_k^a(x) \leq n$ for all $k \in \mathbb{N}$ and all $x \in X$, with $\Delta_k^a \subset \Delta_{k+1}^a$. Thus, there exists an integer $k_\Delta^\star$, possibly depending on x, this dependence being omitted for simplicity's sake, such that $\dim \Delta_k^a(x) \leq \dim \Delta_{k_\Delta^\star}^a(x)$ for all $k \leq k_\Delta^\star$ and $x \in X$, and $\dim \Delta_k^a(x) = \dim \Delta_{k_\Delta^\star}^a(x)$ for all $k \geq k_\Delta^\star$ and $x \in X$.

Next, considering the vector field F_0^b given by (4.22), we show by induction that the distributions (4.24) satisfy $\Gamma_k^a(x,\overline{u}_b) \subset \Delta_k^a(x)$ for all $k \geq 0$, all $x \in X$ and all $\overline{u}_b$ in $\mathbb{R}_\infty^{p-1}$.

This relation is indeed valid for $k = 0$.

Assuming that it holds up to j, a vector field $\gamma \in \Gamma_j^a$ has the form

$$\gamma(x,\overline{u}_b) \triangleq \sum_{r=1}^{r_j} \alpha_r(x,\overline{u}_b)\gamma_r(x),$$

where $\{\gamma_r, r = 1,\ldots,r_j\}$ are chosen in a basis of Δ_j^a, hence independent of $\overline{u}_b$ and commuting with τ_b, *i.e.* $[\tau_b,\gamma_r] = 0$, and where r_j stands for the dimension of $\Gamma_j^a(x,\overline{u}_b)$.

Let us compute $\mathrm{ad}_{F_0^b}\gamma$. We have

$$\begin{aligned}\mathrm{ad}_{F_0^b}\gamma &= [f_0 + u_b f_b + \tau_b, \gamma] = \mathrm{ad}_{f_0}\gamma + u_b[f_b,\gamma] + \sum_{r=1}^{r_j} (L_{\tau_b}\alpha_r)\,\gamma_r \\ &\in \mathrm{ad}_{f_0}\Delta_j + [\Gamma_0^b, \Delta_j] + \Delta_j = \Delta_{j+1}\end{aligned}$$

which proves that $\Gamma_{j+1}^a(x,\overline{u}_b) \subset \Delta_{j+1}^a(x) \subset \mathrm{T}_x X$ for all $x \in X$ and all $\overline{u}_b$ in $\mathbb{R}_\infty^{p-1}$.

Since $\Gamma_j^a(x,\overline{u}_b) \subset \Gamma_{j+1}^a(x,\overline{u}_b) \subset \mathrm{T}_x X$ for all $(x,\overline{u}_b) \in X \times \mathbb{R}_\infty^{p-1}$, we immediately conclude that $\dim \Gamma_j^a(x,\overline{u}_b)$ is bounded above by n for all $(x,\overline{u}_b) \in X \times \mathbb{R}_\infty^{p-1}$. If, moreover, $(x,\overline{u}_b)$ is such that the $\Gamma_k^a(x,\overline{u}_b)$'s have locally constant dimensions, this bound is reached at some integer $k^\star \leq p$ since $\dim \Gamma_{k+1}^a - \dim \Gamma_k^a \geq 1$ for all $k < k^\star$, with $\dim \Gamma_0^a = n - p$, hence the lemma. □

From now on, we shall use the notation $\overline{u}^{(\alpha)} \triangleq \left(u, \dot{u}, \ldots, u^{(\alpha)}\right)$ for every finite $\alpha \in \mathbb{N}$ and $\overline{u} \in \mathbb{R}_\infty^m$, *i.e.* the vector of successive derivatives of u from the order 0, with $u^{(0)} \triangleq u$, up to the order α.

Theorem 5. *Assume that, in a given neighborhood $V(x_0) \subset X$ of x_0 and for all $\overline{u}_b$ in an open dense subset of $\mathbb{R}_\infty^{p-1}$:*

(i) Γ_k^a has constant dimension and is involutive for every $k \geq 0$,

(ii) $\Gamma_p^a = \mathrm{T}\mathbb{R}^n$.

Then, system (2.1) *is flat with flat output* $(z_1, \ldots, z_{n-p}, u_{n-p+1}, \ldots, u_{n-1})$*, where* $z_i \triangleq \varphi_{i,0}(x, \overline{u}_b^{(k_i-3)})$*,* $i = 1, \ldots, n-p$*, is defined in a neighborhood of every point of a dense open set of* $V(x_0) \times \mathbb{R}^{(p-1)(2k_1-5)}$*, the integers* $k_1 \geq \ldots \geq k_{n-p} \geq 0$ *being the list of Brunovský controllability indices (see proposition 2 below), and* x_0 *is an apparent singularity.*

Before stating the proof, let us sketch the ideas, that display strong similarities with the construction of the diffeomorphism that transforms the nonlinear dynamics into the Brunovský controllability canonical form of the static feedback linearization problem (see [9, Corollary 1], [8, Chap.1, Sec. 3 and Chap. 5, Sec. 6] or [14, Sec. 4.1.3]), the main difference being that the distributions (4.24) may depend on u_b and a finite number of its derivatives. More precisely, transforming x and u_b and successive derivatives into a flat output z implies that the Brunovský controllability indices of the corresponding linear system are obtained from (4.24). Moreover, the successive derivatives of z with respect to the drift (4.22), up to a certain order, cannot depend on u_a (see (4.37) and (4.38) below). The corresponding conditions are expressed as conditions on the iterated Lie brackets (4.36) that generate the distributions (4.24) and, thanks to properties (i) and (ii) of theorem 5, they imply the existence of $n-p$ local first integrals of the Γ_k^a's, by Frobenius theorem (see proposition 2 below), first integrals that constitute the $n-p$ first components of a flat output, the $p-1$ remaining ones being the components of u_b.

We start with the following proposition:

Proposition 2. *Assume that the assumptions (i)-(ii) of theorem 5 are valid. Then there exist integers* $k_1, \ldots, k_{n-p}$*, with* $p+1 \geq k_1 \geq \ldots, \geq k_{n-p} \geq 0$ *and* $\sum_{i=1}^{n-p} k_i = n$*, (the so-called* Brunovský controllability indices of the Γ_k^a's*) and* n *smooth independent functions* $(\varphi_{1,0}, \ldots, \varphi_{1,k_1-1}, \ldots, \varphi_{n-p,0}, \ldots, \varphi_{n-p,k_{n-p}-1})$ *defined in a neighborhood* $W(\xi, \overline{\nu}_b)$ *of every* $(\xi, \overline{\nu}_b)$ *in a dense subset* W *of* $V(x_0) \times \mathbb{R}_\infty^{(p-1)}$*, with*

$$\begin{aligned} &\varphi_{i,j} : (x, \overline{u}_b^{(k_i+j-3)}) \in W(\xi, \overline{\nu}_b) \to \mathbb{R}, \quad j = 0, \ldots, k_i - 1, \\ &\varphi_{i,j+1}(x, \overline{u}_b^{(k_i+j-2)}) = L_{F_0^b} \varphi_{i,j}(x, \overline{u}_b^{(k_i+j-2)}), \quad j = 0, \ldots, k_i - 2 \end{aligned} \tag{4.26}$$

satisfying, in $W(\xi, \overline{\nu}_b)$*, and for every* $(\xi, \overline{\nu}_b) \in W$*:*

$$< d\varphi_{i,j}, \Gamma_k^a >= 0, \quad k = 0, \ldots, k_i - j - 2, \quad j = 0, \ldots, k_i - 2, \quad i = 1, \ldots, n-p, \tag{4.27}$$

the $(n-p) \times (n-p)$ *matrix* Δ *whose entries are*

$$\Delta_{i,j} \triangleq L_{ad_{F_0^b}^{k_j-1} f_i} \varphi_{j,0}, \quad i, j = 1, \ldots, n-p \tag{4.28}$$

being invertible in $W(\xi, \overline{\nu}_b)$*.*

Moreover, the mapping $x \in \mathrm{pr}_X W(\xi, \overline{\nu}_b) \mapsto \Phi(x, \overline{u}_b^{(2k_1-4)}) \in \mathbb{R}^n$, *with* Φ *defined by*

$$\begin{aligned}\Phi(x, \overline{u}_b^{(2k_1-4)}) &\triangleq \Big(\varphi_{1,0}(x, \overline{u}_b^{(k_1-3)}), \dots, \varphi_{1,k_1-1}(x, \overline{u}_b^{(2k_1-4)}), \dots \\ &\qquad \dots, \varphi_{n-p,0}(x, \overline{u}_b^{(k_{n-p}-3)}), \dots, \varphi_{n-p,k_{n-p}-1}(x, \overline{u}_b^{(2k_{n-p}-4)})\Big) \\ &= \Big(\varphi_{1,0}(x, \overline{u}_b^{(k_1-3)}), \dots, L_{F_0^b}^{k_1-1}\varphi_{1,0}(x, \overline{u}_b^{(2k_1-4)}), \dots \\ &\qquad \dots, \varphi_{n-p,0}(x, \overline{u}_b^{(k_{n-p}-3)}), \dots, L_{F_0^b}^{k_{n-p}-1}\varphi_{n-p,0}(x, \overline{u}_b^{(2k_{n-p}-4)})\Big)\end{aligned} \tag{4.29}$$

is a local diffeomorphism from $\mathrm{pr}_X W(\xi, \overline{\nu}_b)$ *to an open subset of* $\mathbb{R}^n$, *for all* $\overline{\nu}_b$ *in an open dense subset of* $\mathbb{R}^{p-1}_\infty$.

Proof. As already announced, the proposition results from [9, Corollary 1]. See also [8, Chapter1, Section 3 and Chapter 5, Section 6] or [14, Section 4.1.3].

We define the *Brunovský controllability indices* $k_1, \dots, k_{n-p}$ associated to the Γ_k^a's by first introducing the integers r_j by

$$r_j \triangleq \dim \Gamma_j^a - \dim \Gamma_{j-1}^a \quad \forall j \geq 1, \qquad r_0 \triangleq \dim \Gamma_0^a = n - p.$$

Indeed,

$$\dim \Gamma_j^a = \sum_{k=0}^{j} r_k \triangleq \rho_j, \quad \forall j \geq 0.$$

Let us prove that $r_{j+1} \leq r_j$ for all $j \geq 0$.

We denote by $\{\gamma_1^j, \dots, \gamma_{\rho_j}^j\}$ a basis of the vector space Γ_j^a for all $j \geq 0$ and $\{\eta_1^j, \dots, \eta_{r_{j+1}}^j\}$ a basis of the supplementary subspace of Γ_j^a in Γ_{j+1}^a.

We thus have

$$\begin{aligned}\Gamma_{j+1}^a &= \mathsf{Span}\{\gamma_1^{j+1}, \dots, \gamma_{\rho_{j+1}}^{j+1}\} \\ &= \mathsf{Span}\{\gamma_1^j, \dots, \gamma_{\rho_j}^j\} \oplus \mathsf{Span}\{\eta_1^j, \dots, \eta_{r_{j+1}}^j\} \\ &= \mathsf{Span}\{\gamma_1^{j-1}, \dots, \gamma_{\rho_{j-1}}^{j-1}\} \oplus \mathsf{Span}\{\eta_1^{j-1}, \dots, \eta_{r_j}^{j-1}\} \oplus \mathsf{Span}\{\eta_1^j, \dots, \eta_{r_{j+1}}^j\}\end{aligned}$$

Therefore we have: $\mathsf{Span}\{\eta_1^j, \dots, \eta_{r_{j+1}}^j\} \subset \mathsf{Span}\{\mathrm{ad}_{F_0^b}\eta_1^{j-1}, \dots, \mathrm{ad}_{F_0^b}\eta_{r_j}^{j-1}\}$ which immediately yields: $r_{j+1} \leq r_j$.

The sequence $r_j, j \geq 0$ being non increasing with $r_j \geq 0$ for all $j \geq 0$, there exists an ultimate $j^\star$ such that $r_{j^\star} > 0$ and $r_j = 0$ for all $j > j^\star$. We indeed have $j^\star \leq p$ and $\sum_{j=0}^{j^\star} r_j = \dim \Gamma_{j^\star}^a = \Gamma_p^a = n$ by assumption (ii) of theorem 5.

Then, the *Brunovský controllability indices* $k_1, \dots, k_{n-p}$ are given by

$$k_i \triangleq \#\{j \mid r_j \geq i\}, \quad i \geq 1$$

where $\#A$ denotes the number of elements of an arbitrary set A.

Again following the same lines as in [9, 8, 14], we immediately get that $k_i \leq k_1 = j^\star + 1 \leq p + 1$ for all i, or

$$\max\{k_i \mid i \geq 0\} = k_1 \leq p + 1.$$

Moreover, for all $i \geq 1$, the dimension jumps r_j, sorted by jumps of equal dimension i, being $k_i - k_{i+1}$, we get $\dim \Gamma_p^a = \sum_{i=1}^{n-p} i(k_i - k_{i+1}) = k_1 + \ldots + k_{n-p}$ and, since $\Gamma_p^a = \Gamma_{k_1}^a = \mathrm{T}\mathbb{R}^n$ by assumption (ii), that $k_1 + \ldots + k_{n-p} = n$.

Moreover, possibly up to a new renumbering of the f_i's and for all $\overline{u}_b$ in a dense subset of $\mathbb{R}_\infty^{p-1}$, they satisfy:

- if $0 \leq i \leq k_{n-p} - 1$:

$$\Gamma_i^a = \mathsf{Span}\{f_1, \ldots, \mathrm{ad}_{F_0^b}^i f_1, \ldots, f_{n-p}, \ldots, \mathrm{ad}_{F_0^b}^i f_{n-p}\} \tag{4.30}$$

- if $k_{j+1} \leq i \leq k_j - 1$, $1 \leq j \leq n-p-1$:

$$\begin{aligned}\Gamma_i^a = \mathsf{Span}\{&f_1, \ldots, \mathrm{ad}_{F_0^b}^i f_1, \ldots, f_j, \ldots, \mathrm{ad}_{F_0^b}^i f_j, \ldots \\ &\ldots, f_{j+1}, \ldots, \mathrm{ad}_{F_0^b}^{k_{j+1}-1} f_{j+1}, \ldots, f_{n-p}, \ldots, \mathrm{ad}_{F_0^b}^{k_{n-p}-1} f_{n-p}\}\end{aligned} \tag{4.31}$$

- and for $i \geq k_1$:

$$\Gamma_i^a = \Gamma_{k_1}^a = \mathsf{Span}\{f_1, \ldots, \mathrm{ad}_{F_0^b}^{k_1-1} f_1, \ldots, f_{n-p}, \ldots, \mathrm{ad}_{F_0^b}^{k_{n-p}-1} f_{n-p}\}. \tag{4.32}$$

Note, moreover, that $\Gamma_{k_1}^a$ so defined depends at most on the

$$\max\{k_{n-p} - 2,\, k_{n-p-1} - 2, \ldots, k_1 - 2\} = k_1 - 2$$

first successive derivatives of $\overline{u}_b$, *i.e.* on $\overline{u}^{(k_1-2)}$.

Hence, since all the Γ_i^a's are involutive by assumption (i), by Frobenius theorem, there exist $n-p$ independent first integrals $\varphi_{1,0}(x, \overline{u}_b^{(k_1-3)}) = z_1, \ldots, \varphi_{n-p,0}(x, \overline{u}_b^{(k_{n-p}-3)}) = z_{n-p}$ defined in a neighborhood $W(\xi, \overline{\nu}_b)$ of every $(\xi, \overline{\nu}_b)$ in a dense subset W of $V(x_0) \times \mathbb{R}_\infty^{(p-1)}$ satisfying

$$L_{ad_{F_0^b}^j f_k} \varphi_{i,0} = 0, \quad j = 0, \ldots, k_i - 2, \quad i, k = 1, \ldots, n-p, \tag{4.33}$$

or equivalently (4.27) and for which the matrix Δ given by (4.28), is invertible for all $\overline{u}_b$ in a dense subset of $\mathbb{R}_\infty^{p-1}$.

Next, considering the functions defined by (4.26):

$$\varphi_{i,j}(x, \overline{u}_b) \triangleq L_{F_0^b}^j \varphi_{i,0}(x), \quad j = 1, \ldots, k_i - 2, \quad i = 1, \ldots, n-p,$$

by (4.30)-(4.31)-(4.32), they are such that

$$L_{ad_{F_0^b}^{j-l} f_k} \varphi_{i,l} = 0, \quad 0 \leq l \leq j \leq k_i - 2, \quad i, k = 1, \ldots, n-p. \tag{4.34}$$

We can prove again, as in [9, 8, 14], that the mapping

$$\begin{aligned} x \mapsto &\left(\varphi_{1,0}, \ldots, \varphi_{1,k_1-1}, \ldots, \varphi_{n-p,0}, \ldots, \varphi_{n-p,k_{n-p}-1}\right) \\ &= (z_1, \ldots, z_1^{(k_1-1)}, \ldots, z_{n-p}, \ldots, z_{n-p}^{(k_{n-p}-1)}) \end{aligned} \tag{4.35}$$

where $z_i^{(j)} = \varphi_{i,j}(x, \overline{u}_b^{(k_i+j-3)}) = L^j_{F_0^b}\varphi_{i,0}(x, \overline{u}_b^{(k_i+j-3)})$, is a local diffeomorphism of X for every $\overline{u}_b$ in a dense open subset of $\mathbb{R}^{p-1}_\infty$. Moreover, as a consequence of (4.33), it is easy to prove by induction that, for all $j = 0, \ldots, k_i - 2$, $i, k = 1, \ldots, n-p$ and $r = 0, \ldots, j$,

$$L_{ad^j_{F_0^b} f_k}\varphi_{i,0} = 0 = (-1)^j L_{f_k} L^j_{F_0^b}\varphi_{i,0} = (-1)^j L_{f_k} L^{j-r}_{F_0^b}\varphi_{i,r}. \tag{4.36}$$

□

Proof of theorem 5. Assume that (i) and (ii) hold true in a neighborhood $V(x_0)$ of x_0. Then, according to Propositon 2, there exist n smooth independent functions $(\varphi_{1,0}, \ldots, \varphi_{1,k_1-1}, \ldots, \varphi_{n-p,0}, \ldots, \varphi_{n-p,k_{n-p}-1})$ that satisfy (4.27) with (4.28) and such that the mapping Φ defined by (4.29) is a local diffeomorphism.

Differentiating k times $z_i(t) \triangleq \varphi_{i,0}(x(t))$, for $i = 1, \ldots, n-p$, with respect to time, where $x(t)$ is an integral curve of system (2.1), we prove by induction, thanks to (4.33)-(4.34), that, for all $k = 0, \ldots, k_i - 2$,

$$z_i^{(k)}(t) = L^k_{F_0^b}\varphi_{i,0} + u_a L_{f_a} L^{k-1}_{F_0^b}\varphi_{i,0} = L^k_{F_0^b}\varphi_{i,0} \tag{4.37}$$

and

$$z_i^{(k_i-1)}(t) = L^{k_i-1}_{F_0^b}\varphi_{i,0} + u_a L_{f_a} L^{k_i-2}_{F_0^b}\varphi_{i,0}. \tag{4.38}$$

Thanks to (4.28), the latter relation allows to obtain u_a as a function of $z = (z_1, \ldots, z_{n-p})$ and u_b and derivatives up to $k_1 - 2$. Moreover, according to (4.35), this proves that x and u_a can be expressed as functions of the pair (z, u_b) and successive derivatives in finite number, hence the flatness property. □

Remark 9. *As for theorem 3, the reader may easily check* a posteriori *that theorem 5 may be interpreted in terms of feedback linearization by extending the state as* $(x, u_b, \ldots, u_b^{(2k_1-5)})$ *and inputs* $(u_a, u_b^{(2k_1-4)})$*. However, if we try to directly apply the feedback linearization theorem by finding the extension length by essay-error, not only the number* $2k_1 - 5$ *is not intuitive, and may be large, but its relation with the largest dimension jump,* k_1*, of the sequence of distributions* (4.24) *would be ignored.*

Remark 10. *It must be noted that theorem 5 remains valid in the case of* $m \leq n-1$ *control inputs, with* $n - m \leq p$*. This trivial verification is left to the reader.*

Remark 11. *To the authors knowledge, theorem 5 is the first one giving a practical condition for which the degenerating directions at a given point, producing a rank drop at this point, are replaced by those generated by another tower of constant rank and involutive subdistributions, namely the* Γ^a_k*'s. In contrast, the apparent singularities dealt with in [2, 3, 12] only concern singularities of the parameterization and are not related to a singularity of the distribution of control vector fields.*

Remark 12. *The application to the global or semi-global motion planning, as presented in [12] for the non-holonomic car, may be indeed easily adapted to the present context. However, though important in applications, we do not shed new light on this topic. Therefore, this aspect is not developed again here.*

5 Examples

We now give three simple academic examples, with computational aspects as simple as possible, to illustrate our previous results and emphasize the role played by suitably chosen local first integrals in the analysis of flatness singularities. This is why, apart from the first part of example 2, the flat outputs could also be easily determined by inspection. However, we do not want to make the reader believe that the computational complexity of our conditions is low. It is in fact comparable to the one of the already cited static feedback linearization conditions [7, 9].

5.1 Example 1

Consider the following system:

$$\begin{cases} \dot{x}_1 &= x_1 u_1 + x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u_2. \end{cases}$$

We have: $f_0 = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$, $f_1 = x_1 \frac{\partial}{\partial x_1}$ and $f_2 = \frac{\partial}{\partial x_3}$. When $x_1 \neq 0$, the fields f_1 and f_2 are linearly independent and condition (4.6) holds since $[g, f_2] = [f_0, f_2] = -\frac{\partial}{\partial x_2}$. Therefore by theorem 3, the system is flat in the dense open set $\mathbb{R}^3 \setminus \{x_1 = 0\}$. A flat output is given by two independent first integrals of f_2, e.g. $y_1 \triangleq x_1$, $y_2 \triangleq x_2$), which is easily confirmed by the formulas $x_1 = y_1$, $x_2 = y_2$, $x_3 = \dot{y}_2$, $u_1 = \frac{\dot{y}_1 - y_2}{y_1}$ and $u_2 = \ddot{y}_2$. One can also easily verify that, after extending the system by adding an integrator to u_1, *i.e.* $\dot{u}_1 = v_1$, the extended system is feedback linearizable.

At a point (x_1, x_2, u_1), where $x_1 = 0$, the system is degenerated but still flat. Since $f_1 = 0$ at this point, we have $\Gamma_0 = \mathsf{Span}\{f_2\}$, with $\dim \Gamma_0 = 1$. We thus apply theorem 5 with $p = 2$, $\Gamma_0^a = \mathsf{Span}\{f_2\}$ and $\Gamma_0^b = \mathsf{Span}\{f_1\}$. Here $u_b = u_1$ and $F_0^b(x, \overline{u}_b) = f_0 + u_1 f_1 + \tau_1 = (x_2 + u_1 x_1)\frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + \sum_{k \geq 0} u_1^{(k+1)} \frac{\partial}{\partial u_1^{(k)}}$. The reader may easily check that $\Gamma_0^a = \mathsf{Span}\{\frac{\partial}{\partial x_3}\} = \overline{\Gamma_0^a}$, $\Gamma_1^a = \mathsf{Span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}\} = \overline{\Gamma_1^a}$ and $\Gamma_2^a = \mathsf{Span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\} = \mathrm{T}\mathbb{R}^3$. Hence we conclude that the system is flat with $\tilde{y}_1 \triangleq x_1$ and $\tilde{y}_2 \triangleq u_1$ as flat output, which indeed implies that $x_1 = 0$ is an apparent singularity.

5.2 Example 2

We now consider the following 4-dimensional driftless system with 3 control inputs:

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 u_1 \\ \dot{x}_3 &= x_4 u_1 + x_1 u_3 \\ \dot{x}_4 &= u_2. \end{cases} \tag{5.1}$$

The drift is thus $f_0 = 0$ and the control fields are respectively given by

$$f_1 = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}, \quad f_2 = \frac{\partial}{\partial x_4}, \quad f_3 = x_1 \frac{\partial}{\partial x_3}.$$

At points where $x_1 \neq 0$ and $u_1 \neq 0$, these three vector fields are linearly independent and $[g, f_3] = u_1[f_1, f_3] = u_1\left(\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_2}\right) \notin \mathsf{Span}(f_1, f_2, f_3)$. Therefore assumption (4.6) is satisfied and, by theorem 3, the system is flat with flat outputs given by three independent first integrals of f_1. An easy computation yields the following flat output:

$$z_1 = 2x_2x_4 - x_3^2, \quad z_2 = x_1x_4 - x_3, \quad z_3 = x_4.$$

One could instead take three independent first integrals of f_3 as well to get

$$z_1' = x_1, \quad z_2' = x_2, \quad z_3' = x_4.$$

Note that at points $x_1 = 0$, the distribution $\mathsf{Span}\{f_1, f_2, f_3\}$ has dimension 2. The reader can verify that $\Gamma_0^a = \mathsf{Span}\{f_1, f_2\}$ is not involutive, and therefore that theorem 5 does not apply. Nevertheless, x_1, x_2, and u_3, is a flat output whenever $u_1 \neq 0$.

5.3 Example 3

We now consider an extension of system (5.1) by adding two integrators to u_1, which makes the system a 6 dimensional one with 3 inputs:

$$\begin{cases} \dot{x}_1 &= x_5 \\ \dot{x}_2 &= x_3x_5 \\ \dot{x}_3 &= x_4x_5 + x_1u_3 \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_1. \end{cases} \tag{5.2}$$

System (5.2) is indeed Lie-Bäcklund equivalent to (5.1) and therefore flat at points where $x_1 \neq 0$ and $u_1 \neq 0$.

We have

$$f_0 = x_5\frac{\partial}{\partial x_1} + x_3x_5\frac{\partial}{\partial x_2} + x_4x_5\frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_5}, \quad f_1 = \frac{\partial}{\partial x_6}, \quad f_2 = \frac{\partial}{\partial x_4}, \quad f_3 = x_1\frac{\partial}{\partial x_3}.$$

We now show that theorem 5, for $m = 3$ and $n = 6$ (see remark 10), applies in the open set defined by $x_5 \neq 0$, without restriction on x_1, even if the distribution $\mathsf{Span}\{f_1, f_2, f_3\} = \mathsf{Span}\{\frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_4}, x_1\frac{\partial}{\partial x_3}\}$ degenerates at points such that $x_1 = 0$. We have:

$$\Gamma_0^a = \mathsf{Span}\{f_1, f_2\}, \quad \Gamma_0^b = \mathsf{Span}\{f_3\}, \quad p = 4 \geq n - m = 3,$$

and

$$F_0^b = f_0 + u_3f_3 + \tau_3 = x_5\frac{\partial}{\partial x_1} + (x_4x_5 + u_3x_1)\frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_5} + \sum_{k\geq 0} u_3^{(k+1)}\frac{\partial}{\partial u_3^{(k)}}.$$

In the open set defined by $x_5 \neq 0$, we have $\Gamma_0^a = \mathsf{Span}\{\frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_4}\}$, $\Gamma_1^a = \mathsf{Span}\{\frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_4}, x_5\frac{\partial}{\partial x_3}\}$, and $\Gamma_2^a = \mathrm{T}\mathbb{R}^6$, which proves that the assumptions of theorem 5 are satisfied. We immediately get the flat output $z_1 = x_1$, $z_2 = x_2$ and $z_3 = u_3$.

6 Conclusions

In this paper, we have studied the set of intrinsic singularities of control affine flat systems with one input less than the number of states. We have proven two theorems in this context, showing how to construct flat outputs and their Lie-Bäcklund atlases, thus allowing to deduce some inclusions of their associated set of intrinsic singularities. We also give three examples which may be interpreted in a potentially interesting way for applications: if the system degenerates at a point but is still flat there, and if the degeneracy point corresponds to some damaged state, *e.g.* loss of a motor or of a wing of an aircraft, then the present analysis may be helpful for input reconfiguration of the damaged system and emergency motion planning. This idea will be more thoroughly studied in a forthcoming work of the authors.

7 Acknowledgement

The authors wish to express their warmest thanks to one of the anonymous reviewers for his/her careful reading and most useful remarks.

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