



THE NEW DIGITAL MATHEMATICS OF THE MILLENNIUM

Constantine Konstantinos Kyritsis

► To cite this version:

Constantine Konstantinos Kyritsis. THE NEW DIGITAL MATHEMATICS OF THE MILLENNIUM. 2019. ⟨hal-02045153⟩

HAL Id: hal-02045153

<https://hal.science/hal-02045153v1>

Preprint submitted on 21 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

THE NEW DIGITAL MATHEMATICS OF THE MILLENNIUM

Konstantinos E. Kyritsis*

* Associate Prof. of University of Ioannina, Greece. ckiritsi@teiep.gr
C_kyrisis@yahoo.com, , Dept. Accounting-Finance Psathaki Preveza 48100

Prologue

This project is under the next philosophical principles

1) **Consciousness has the experience of the infinite.**

2) **But the physical material world is finite.**

3) **Therefore mathematical models in their ontology should contain only finite entities and should not involve the infinite at all which is a phenomenological abstraction. In digital mathematics we may have the concepts of seemingly infinite sets (or seemingly infinitesimal numbers) but they are essentially finite sets or numbers with finite decimal representation.**

These principles allow for an alternative and more realistic universe of mathematics, directly usable in **artificial intelligence of computers**. The development of digital mathematics is somehow more elaborate in the definitions but radically easier in the proofs compared to classical mathematics. It is certainly not equivalent to the classical mathematics. But here we give emphasis only to the theorems that are very similar and very familiar with corresponding to classical mathematics.

The previous principles require a rewriting of the classical axiomatic systems to those of

- 1) the natural digital numbers,
 - 2) the digital 1st and 2nd order formal logic,
 - 3) the digital set theory,
 - 4) the digital real numbers,
 - 5) the digital differential and integral calculus,
 - 6) the digital Euclidean axiomatic continuous geometry
- etc.

In the next two papers we present the digital real numbers with **the digital differential and integral calculus** and the **digital axiomatic continuous Euclidean geometry**, where all sets of points (visible or invisible) or numbers are finite sets of points and numbers with finite decimal representation.

Our perception and experience of the reality, depends on the system of beliefs that we have. In mathematics, the system of spiritual beliefs is nothing else than the axioms of the axiomatic systems that we accept. The rest is the work of reasoning and acting.

For each of the two papers we give at its end **a fictional philosophical or epistemological dialogue with the celebrated immortals of the classical mathematics** (like Euclid, Pythagoras, Hilbert, Cantor, Gödel, Newton, Leibniz etc) where the reader can grasp in non-technical terms the main differences and the main advantages (or disadvantages) of the digital mathematics where all are finite compared to the classical mathematics which are based on the phenomenological abstraction of the infinite

The 2nd of the two papers on the digital but continuous axiomatic Euclidean geometry has already been published in the *World Journal of Research and Review (WJRR)* ISSN:2455-3956, Volume-5, Issue-4, October 2017 Pages 31-43

OUTLINE OF THE INTRODUCTION TO THE DIGITAL DIFFERENTIAL AND INTEGRAL CALCULUS

Konstantinos E. Kyritsis*

** Associate Prof. of University of Ioannina, Greece. ckiritsi@teiep.gr
C_kyrisis@yahoo.com, , Dept. Accounting-Finance Psathaki Preveza 48100*

ABSTRACT

In this paper I go further from the digital continuous axiomatic Euclidean geometry ([8]) and introduce the basic definitions and derive the basic familiar properties of the differential and integral calculus without the use of the infinite, within finite sets only. No axioms are required in this only successfully chosen definitions. I call it the digital differential and integral calculus. Such mathematics is probably the old unfulfilled hitherto dream of the mathematicians since many centuries. Strictly speaking it is not equivalent to the classical differential and integral calculus which makes use of the infinite (countable and uncountable) and limits. Nevertheless for all practical reasons in the physical and social sciences it gives all the well known applications with a finite ontology which is directly realizable both in the physical ontology of atomic matter or digital ontology of operating systems of computers. Such a digital calculus has aspects simpler than the classical “analogue” calculus which often has a complexity irrelevant to the physical reality. It can become also more complicated than the classical calculus when more than 2 resolutions are utilized, but this complexity is directly relevant to the physical reality. The digital differential and integral calculus is of great value for the applied physical and social sciences as its ontology is directly corresponding to the ontology of computers. It is also a new method of teaching mathematics where there is integrity with what we say, write, see, and think. In this short outline of the basic digital differential and integral calculus, we include on purpose almost only the basic propositions that are almost identical with the corresponding of the classical calculus for reasons of familiarity with their proofs.

Key words: Digital mathematics, Calculus

MSC : 00A05

0. INTRODUCTION

Changing our concept of physical material, space and time continuum so as to utilize only finite points, numbers and sets, means that we change also our perception our usual mental images and beliefs about the reality. This project is under the next philosophical principles

- 1) **In the human consciousness we have the experience of the infinite.**
- 2) **But the ontology of the physical material world is finite.**
- 3) **Therefore mathematical models in their ontology should contain only finite entities and should not involve the infinite.**

4) Strange as it may seem, the digital mathematics are the really deep mathematics of the physical world, while the classical mathematics of the infinite ("analogue" mathematics) are a "distant" phenomenology, convenient in older centuries, but not the true ontology.

This paper is part of larger project which is creating again the basics of mathematics and its ontology with new definitions that do not involve the infinite at all.

Our perception and experience of the reality, depends on the system of beliefs that we have. In mathematics, the system of spiritual beliefs is nothing else than the axioms of the axiomatic systems that we accept. The rest is the work of reasoning and acting.

Quote: "It is not the world we experience but our perception of the world"

Nevertheless it is not wise to include in our perception of the material world and its ontology anything else than the finite, otherwise we will be lead in trying to prove basic facts with unsurpassed difficulties as the classical mathematics has already encounter , **The abstraction of the infinite is phenomenological** and it seems sweet at the beginning as it reduces some complexity, in the definitions, but later on it turns out to be bitter, as it traps the mathematical minds in to a vast complexity irrelevant to real life applications. Or to put it a more easy way, we already know the advantages of using the infinite but let us learn more about the advantages of using only the finite, for our perception, modelling and reasoning about empty space and physical reality. This is not only valuable for the applied sciences, through the computers but is also very valuable in creating a more perfect and realistic education of mathematics for the young people. H. Poincare used to say that mathematics and geometry is the art of correct reasoning over not-corresponding and incorrect figures. With the digital mathematics this is corrected. **The new digital continuums create a new integrity between what we see with our senses, what we think and write and what we act in scientific applications.**

The continuum with infinite many points creates an overwhelming complexity which is very often irrelevant to the complexity of physical matter. The emergence of the irrational numbers is an elementary example that all are familiar But there are less known difficult problems like the 3rd Hilbert problem (see [3] Boltianskii V. (1978)“). In the 3rd Hilbert problem it has been proved that two solid figures that are of equal volume are not always decomposable in to an in equal finite number of congruent sub-solids! Given that equal material solids consists essentially from the physical point of view from an equal number of sub-solids (atoms) that are congruent, this is highly non-intuitive! There are also more complications with the infinite like the Banach-Tarski paradox (see [2] Banach, Stefan; Tarski, Alfred (1924)) which is essentially pure magic or miracles making! In other words it has been proved that starting from a solid sphere S of radius r, we can decompose it to a finite number n of pieces, and then re-arrange some of them with isometric motions create an equal sphere S1 of radius again r and by rearranging the rest with isometric motions create a second solid Sphere S2 again of radius r! In other words like magician and with seemingly elementary operations we may produce from a ball two equal balls without tricks or “cheating”. Thus no conservation of mass or energy!. Obviously such a model of the physical 3-dimensional space of physical matter like the classical Euclidean geometry is far away from the usual physical material reality! I have nothing against miracles, but it is challenging to define a space , time and motion that behaves as we are used to know. In the model of the digital 3-dimesional space, where such balls have only finite many points such “miracles” are not possible!

The current digital version of the differential and integral calculus is based on the atomic structure of matter as hypothesized 2,000 years ago by the ancient Greek philosopher Democritus and which has developed in the modern the atomic physics. Also the role of computers and their digital world is important as it shows that space, time, motion, images,

sound etc can have finite digital ontology and still can create the continuum as a phenomenology of perception.

The famous physicist E. Schrodinger in his book ([12] *E. Schroedinger. Science and Humanism Cambridge University press 1961*) mentions that the continuum as we define it with the “analogue” mathematics involving the infinite is problematic and paradoxical, therefore needs re-creation and re-definition. It is exactly what we do here with the digital differential and integral calculus.

We enumerate some great advantages of the digital differential and integral calculus compared to the classical calculus with the infinite.

- 1) The digital continuity and smoothness (derivative) allows for a variable in scale of magnitude and resolution such concept and not absolute as in analogue classical mathematics. A curve may be smooth (differentiable) at the visible scale but non-smooth at finer scales and vice versa. This is not possible with classical definitions.
- 2) Corresponding to the concept of infinite of classical mathematics in digital mathematics there is the concept seemingly infinite and seemingly infinitesimal at its various orders, which is still finite. Thus many unprovable results in classical mathematics become provable in digital mathematics. This also resurrects the 17th and 18th century mathematical arguments in Calculus and mathematical physics that treated the “infinitesimals” as separate entities in the derivatives.
- 3) Many unsurpassed difficulties in proving desirable results in the infinite dimensional functional spaces of mathematical analysis disappear and allow for new powerful theorems because the seemingly infinite is still finite.
- 4) Integration is defined as finite (although seemingly infinite) weighted sum of the volumes of the points at some precision level, exactly as Archimedes was measuring and integrating volumes with water or sand. Contrary to classical mathematics any computably bounded function is integrable (see proposition 3.6)
- 5) Therefore, there are vast advantages compared to classical analogue calculus. The digital differential and integral calculus is a global revolution in the ontology of mathematics in teaching and applying them comparable with the revolution of digital technology of sounds, images . motion, etc compared to the classical analogue such technologies.
- 6) There are although “disadvantages” too , in the sense that if we do not restrict to a digital calculus relative 4 precision levels but include many more and grades of differentiability and integrability then the overall calculus will become much more complicated than the classical calculus.

In this short outline of the basic digital differential and integral calculus, we include on purpose only the basic propositions that are almost identical with the corresponding of the classical calculus for reasons of familiarity with their proofs. An exception is the proposition 3.6 which has an almost obvious proof.

1. THE DEFINITION OF THE DIGITAL REAL NUMBERS

THE MULTI-PRECISION DECIMAL DIGITAL REAL NUMBERS $R(m,n,p,q)$

Rules for phantasy and drawing of figures.

As initially we considered a system of digital real numbers $R(m,n,p,q)$ we consider the points of $P(m)$, $P(n)$ as visible in the figures while the points of $P(p)$ as invisible pixels, and those of $P(q)$ as invisible atoms. Therefore, even the points and seemingly infinitesimals that will be defined below, of $P(n)$ relative to $P(m)$ are considered as visible. This is in accordance with the habit in classical mathematics to make the points visible, although they claim that they have zero size.

- a) The **rational numbers** Q , as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** k/l of natural numbers k, l exist as numbers and are unique. The cost of course is that when we represent them with decimal representation they may have infinite many but with finite period of repetition decimal digits.
- b) The **classical real numbers** R , as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** of linear segments of Euclidean geometry, exist as numbers and are unique (**Eudoxus** theory of proportions). The cost of course finally is that when we represent them with decimal representation they may have infinite many arbitrary different decimal digits without any repetition.
- c) But in the physical or digital mathematical world, such costs are not acceptable. The infinite is not accepted in the ontology of digital mathematics (only in the subjective experience of the consciousness of the scientist). Therefore in the **multi-precision digital real numbers**, proportions are handled in different way, with priority in the Pythagorean-Democritus idea of the **creation of all numbers from an integral number of elementary units**, almost exactly as in the physical world matter is made from atoms (here the precision level of numbers in decimal representation) and the definitions are different and more economic in the ontological complexity.

We will choose for all practical applications of the digital real numbers to the digital Euclidean geometry and digital differential and integral calculus, the concept of a system of digital decimal real numbers with three precision levels, lower, low and a high.

Definition 1.1 The definition of a PRECISION LEVEL $P(n,m)$ where n, m are natural numbers, is that it is the set of all real numbers that in the decimal representation have not more than n decimal digits for the integer part and not more than m digits for the decimal part. Usually we take $m=n$. In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

THE DEFINITION 1.2 OF THE DIGITAL REAL NUMBERS $R(m,n,p,q)$

We assume at least four precision levels for an axiomatic decimal system of digital real numbers

Whenever we refer to a real number x of a (minimal in precision levels) system of real numbers $R(m,n,p,q)$, we will always mean that x belongs to the precision level $P(m)$ and that the system $R(m,n,p,q)$ has at least four precision levels with the current axioms.

Whenever we write an equality relation $=_m$ we must specify in what precision level it is considered. The default precision level that an equality of numbers is considered to hold, is the precision level $P(m)$.

Some of the Linearly ordered Field operations

The field operations in a precision level are defined in the usual way, from the decimal representation of the numbers. This would be an independent definition, not involving the infinite. Also equality of two numbers with finite decimal digits should be always specified to what precision level. E.g. if we are talking about equality in $P(m)$ we should symbolize it by $=_m$, while if talking about equality in $P(q)$ we should symbolize it by $=_q$. If we want to define these operations from those of the real numbers with infinite many decimal digits, then we will need the truncation function $[a]_x$ of a real number a , in the Precision level $P(x)$. Here for the rounding function we use the rounding to the left for positive numbers and to the right for negative numbers.

Then the operations e.g. in $P(m)$ with values in $P(n)$ $m < n$ would be

$$[a]_m + [b]_m =_n [a+b]_n \quad (\text{eq. 3})$$

$$[a]_m * [b]_m =_n [a*b]_n \quad (\text{eq. 4})$$

$$([a]_m)^{-1} =_n [a^{-1}]_n \quad (\text{eq. 5})$$

(Although, the latter definition of inverse seems to give a unique number in $P(n)$, there may not be any number in $P(n)$ or not only one number in $P(n)$, so that if multiplied with $[a]_m$ it will give 1. E.g. for $m=2$, and $n=5$, the inverse of 3, as $([3]_m)^{-1} =_n [1/3]_n = 0.33333$ is such that still $0.33333 * 3 \neq_n 1$).

Nevertheless here we will not involve the infinite and the classical real numbers, and we take the operation of digital real numbers from the standard operations of them as numbers with finite digital decimal representation and truncation by rounding.

Such a system of double or triple precision digital real numbers, has closure of the linearly ordered field operations only in a specific local way. That is If a, b belong to the Local Lower precision, then $a+b, a*b, -a, a^{-1}$ belong to the Low precision level, and the properties of the linearly ordered commutative field hold: (here the equality is always in $P(n)$, this it is mean the $=_n$).

1) if a, b, c belong to $P(m)$ then $(a+b), (b+c), (a+b)+c, a+(b+c)$ belong in $P(n)$ and $(a+b)+c = a+(b+c)$ for all a, b and c in $P(n)$.

2) There is a digital number 0 in $P(n)$ such that

2.1) $a+0=a$, for all a in $P(n)$.

2.2) For every a in $P(m)$ there is some b in $P(n)$ such that

$a+b=0$. Such a, b is symbolized also by $-a$, and it is unique in $P(n)$.

3) if a, b , belong to $P(m)$ then $(a+b), (b+a)$, belong in $P(n)$ and $a+b=b+a$

4) if a, b, c belong to $P(m)$ then $(a*b), (b*c), (a*b)*c, a*(b*c)$ belong in $P(n)$ and $(a*b)*c = a*(b*c)$.

5) There is a digital number 1 in $P(n)$ not equal to 0 in $P(n)$, such that

5.1) $a*1=a$, for all a in $P(n)$.

5.2) For every a in $P(m)$ not equal to 0, there may be one or none or not only one b in $P(n)$ such that $a*b=1$. Such b is symbolized also by $1/a$, and it may not exist or it may not be unique in $P(n)$.

6) if a, b , belong to $P(m)$ then $(a*b), (b*a)$, belong in $P(n)$ and $a*b=b*a$

7) if a, b, c belong to $P(n)$ then $(b+c), (a*b), (a*c), a*(b+c), a*b+a*c$, belong in $P(n)$ and $a*(b+c)=a*b+a*c$

Which numbers are positive and which negative and the linear order of digital numbers is precision levels $P(m), P(n)$, etc is something known from the definition of precision levels in the theory of classical real numbers in digital representation.

If we denote by $PP(m)$ the positive numbers of $P(m)$ and $PP(n)$ the positive numbers of $P(n)$ then

8) For all a in $PP(m)$, one and only one of the following 3 is true

8.1) $a=0$

8.2) a is in $PP(m)$

8.3) $-a$ is in $PP(m)$ ($-a$ is the element such that $a+(-a)=0$)

9) If a, b are in $PP(m)$, then $a+b$ is in $PP(n)$

10) If a, b are in $PP(m)$, then $a*b$ is in $PP(n)$

It holds for the inequality $a>b$ if and only if $a-b$ is in $PP(n)$

$a<b$ if $b>a$

$a\leq b$ if $a<b$ or $a=b$

$a\geq b$ if $a>b$ or $a=b$

and similar for $PP(n)$.

Similar properties as the ones from $P(m)$ to $P(n)$ hold if we substitute n with m , and m with p, q .

For the $R(m,n)$ the integers of $P(m)$ are also called computable finite or countable finite, while those of $P(n)$ are unaccountable finite or non-computable finite or also seemingly infinite relative to $P(m)$.

Also, the Archimedean property holds only recursively in respect e.g. to the local lower precision level $P(m)$.

In other words, if $a, b, a<b$ belong to the precision level $P(m)$ then there is k integer in the precision level $P(n)$ such that $a*k>b$. And similarly for the precision levels $P(n)$ and $P(p), P(q)$.

The corresponding to the Eudoxus-Dedekind completeness in the digital real numbers also is relative to the three precision levels is simply that in the precision levels all possible combination of digits are included and not any decimal number of $P(m)$ or $P(n)$ is missing. Still this gives

THE SUPREMUM COMPLETENESS PROPERTY OF THE DIGITAL REAL NUMBERS.

From this completeness we deduce the supremum property of upper bounded sets (and infimum property of lower bounded sets) in the $P(m)$ (but also $P(n)$) precision levels. This is because in well ordered sets holds the supremum property of upper bounded sets. Here lower bounded sets have also the infimum property and this holds for any resolution of the digital real numbers

Mutual inequalities of the precision levels (AXIOMS OF SEEMINGLY (m,n) -INFINITE OR (m,n)-UNCOUNTABLE OR NON-COMPUTABLE FINITE AMONG RESOLUTIONS and seemingly finite or visibly finite or bounded or computable finite numbers.)

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, p, q .

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we feel safe to postulate such big differences.

There are definitions modeled after the definitions of **inaccessible cardinals** in classical mathematics. Here we give a weaker alternative definitions with **weaker concepts of seemingly infinite** that would correspond to that of inaccessible cardinals. In other words **we do not include the operation of power.**

We may conceive the **countable finite** as a finite computable by a computational power of some computer, and **unaccountable finite** as the **finite not computable** by a type of a computer

Transcendental Orders of (m,n) seemingly infinite, as in classical mathematics transcendental orders of ordinal numbers are also definable. E.g. if a, b are (m,n)=seemingly infinite then a is transcendental larger than b , in symbols $a \gg b$ iff $b/a =_m 0$ in $P(m)$.

And similarly transcendental orders of seemingly infinitesimals. E.g. if a, b are (m,n)=seemingly infinitesimals then a is transcendental smaller than b , in symbols $a \ll b$ iff $a/b =_m 0$ in $P(m)$.

We may compare them with the small $o()$ and big $O()$ definitions of the classical mathematics, but they are different as the latter involve the countable infinite, while former here involve only finite sets of numbers.

9) REQUIREMENTS OF THE SEEMINGLY INFINITE *If we repeat the operations of addition and multiplication of the linearly ordered commutative field starting from numbers of the precision level $P(m)$, so many times as the numbers of the local lower precision level $P(m)$, then the results are still inside the low precision level $P(n)$. In symbols if by $|P(m)|$ we denote the cardinality of $P(m)$, then $|P(m)| \cdot (10^m)$, and $(10^m)^{|P(m)|} \leq 10^n$. Similarly for the pair (m, q) . We may express it by saying that the 10^n is seemingly infinite or unaccountable finite compared to 10^m , or that the numbers less than 10^n are countable or computable finite. If we include besides the addition and multiplication the power operation too, then 10^m is inaccessible seemingly*

infinite compared to 10^m (a concept similar to inaccessible cardinal numbers in classical mathematics). Similarly for the precision levels $P(p)$, $P(q)$.

10) REQUIREMENTS OF THE SEEMINGLY INFINITESIMALS *The smallest magnitude in the low precision level $P(n)$ in other words the 10^{-n} , will appear as zero error in the low precision level $P(m)$, even after additive repetitions that are as large as the cardinal number of points of the lower precision level $P(m)$ and multiplied also by any large number of $P(m)$. In symbols*

$10^{-n} \cdot P(m) \cdot 10^m \leq 10^{-m}$. Similarly for the pairs (n,p) , (p,q) .

This may also be expressed by saying that the 10^{-n} is seemingly infinitesimal compared to the 10^{-m} . Other elements of $P(n)$ symbolized by dx with $|dx| < 10^{-n}$ with the same inequalities, that is $|dx| \cdot P(m) \cdot 10^m \leq 10^{-m}$ are also seemingly infinitesimals, provided the next requirements are also met:

*The seemingly infinitesimals dx of $P(p)$ relative to $P(m)$ (thus $|dx| \leq 10^{-m}$) are by definition required to have properties that resemble the **ideals in ring theory** (see e.g. [15] VAN DER WAERDEN ALGEBRA Vol I, chapter 3, Springer 1970). More precisely what it is required to hold is that*

If a, b are elements of $P(m)$, and dx, dy seemingly infinitesimals of $P(p)$ relative to $P(n)$ (thus $|dx|, |dy| \leq 10^{-n}$), thus relative to $P(m)$ too) then the linear combination and product are still seemingly infinitesimals. In symbols $adx + bdy$, are seemingly infinitesimals of $P(n)$ relative to $P(m)$ and $dx \cdot dy$ is seemingly infinitesimal of $P(q)$ relative to $P(p)$ and thus relative to $P(m)$ too.

*We call this **the ideal-like property of the seemingly infinitesimals**.*

.

One very important equation is of course that the digital real numbers is the union of the four precision levels.

$$R(m, n, p, q) = P(m) \cup P(n) \cup P(p) \cup P(q)$$

Two digital systems of Real numbers $R(m,n,p,q)$, $R(m',n',p',q')$ with $m=m'$, $n=n'$, $p=p'$, $q=q'$ and the above axioms are considered isomorphic.

2. THE DEFINITION OF THE DIGITAL FUNCTIONS, DIGITAL CONTINUITY AND DIGITAL DIFFERENTIABILITY.

A **digital real function** at 2 precision levels is a function in the ordinary set-theoretic sense, that sends elements of the digital real numbers to elements of the digital real numbers. It has to

be defined so that **it respects the precision levels**. This is defined so that a **parallelogram diagram**, of the two functions, the restriction function and the rounding function **commute** in the sense of the theory of **categories**. Usually the standard way is to **define it for the highest resolution** and then extend the definition for the lower resolutions by the **rounding function** (left for positive numbers and right for negative numbers). This process is called **natural rounding extension on lower resolutions**, and defines the rounded functions on the lower resolutions **so that the arrow diagrams commute** that $[f(a)]_n = f_n([a]_n)$ if $a, f(a)$ in $P(q)$ and we define f on $P(n)$ (The rounding of the image is the value of the rounded function on the rounded argument, so that rounding function and functions commute). **We only need to define the rounding for a pair of precision levels for differentiation and integration. Here for $P(m)/P(n)$. The f_m is the rounded function, and it is for all practical purposes the one only function observed. But it starts from a function f on $P(n)$. So for all digital function that we will consider, we will conceive them as double functions the finest**

of: $P(n) \rightarrow P(n)$ and the rounded , $f: P(m) \rightarrow P(m)$, and r is the restriction from $P(n)$ to $P(m)$ then a commutation of diagrams is the $[(of([x]_m)]_m = mf$

In some situations (e.g. definition of continuity) we will assume that the digital function is defined in 3 precision levels oof: $P(p) \rightarrow P(p)$ of: $P(n) \rightarrow P(n)$ and the rounded ,

$f: P(m) \rightarrow P(m)$, and by the restriction from $P(n)$ to $P(m)$ and from $P(n)$ to $P(p)$ a commutation of diagrams holds : $[(of([x]_m)]_m = mf$ and $[(oof([x]_n)]_n = n of$.

And in some cases we will need all 4-precision levels

For those that feel convenient to start with the classical mathematics with the infinite, and their functions, digital functions as above can be obtained by the rounding functions $[\]_m [\]_n$ in the precision levels $P(m), P(n)$. E.g. starting with the classical exponential function $g(x) = e^x$ to obtain a digital function in $P(m), P(n)$, we use the formulae $oof(x) = [e^{[x]_p}]_p$, $of(x) = [e^{[x]_n}]_n$ and $f(x) = [e^{[x]_m}]_m$

DEFINITION 2.1

A digital real function defined on a closed interval $f: [a, b]_m \rightarrow P(m)$, $of: [a, b]_n \rightarrow P(n)$, $oof: [a, b]_p \rightarrow P(p)$ is (digitally) $P(m)/P(n)/P(n)$ continuous at a point x of its domain of definition $[a, b]_m$ in $P(m)$, if and only if for every other point x' of the domain of definition $[a, b]_n$ in $P(n)$, such that x, x' are of seemingly infinitesimally distance $dx = x' - x$ (belongs to $P(n)$) , relative to $P(m)$, then also the $dy = of(x') - of(x)$ is seemingly infinitesimal of $P(n)$ relative to $P(m)$. It holds in particular:

$$dy = {}_ndof(x) = {}_mdx = {}_m0$$

Similar definitions hold for $P(m)/P(p)$, $P(n)/P(p)$ and $P(m)/P(q)$ continuity.

We concentrate on functions of $P(n)$ of $R(m,n,p,q)$ but we may we not leave unused the precision levels $P(p)$, $P(q)$. We mention also that the definitions can be also for the triples of precision levels $P(m)-P(n)-P(p)$, $P(n)-P(p)-P(q)$ as finer forms of continuity. If it is for all precision levels then it seems equivalent to the classical definitions.

If digital real function is digitally continuous at all points of its domain of definition it is called a **(digitally) $P(m)/P(n)$ continuous digital real function**.

DEFINITION 2.2

*A digital real function defined on a closed interval $f:[a,b]_m \rightarrow P(m)$, of: $[a,b]_n \rightarrow P(n)$, oof: $[a,b]_p \rightarrow P(p)$ is **(digitally) $P(m)/P(n)/P(p)$ continuous at a point x** of its domain of definition $[a,b]_m$ in $P(m)$, if and only if for every other point x' of the domain of definition $[a,b]_p$ in $P(p)$, such that x, x' are of seemingly infinitesimally distance $dx = x' - x$ (belongs to $P(p)$) , relative to $P(m)$, then also the $dy = of(x') - of(x)$ is seemingly infinitesimal of $P(n)$ relative to $P(m)$. It holds in particular:*

$$0 = {}_mdy = {}_ndof(x) = {}_mdx = {}_m0$$

Similar definitions hold for $P(n)/P(p)/P(q)$, $P(m)/P(p)/P(q)$ continuity etc.

It would be nice if it is possible to derive also the digital $P(m)/P(n)$ continuity as the standard continuity of topological space. The next definition gives the best idea for such a topological space. A topological space is defined by its open sets (see e.g. [9] J.Munkress) . But the open sets can also be definite by the limit points of sets too.

We consider the Cartesian product set $P(m) \times P(n) = P(m) \times P(n)$, where we define the disjoint union space $P(m) + P(n)$ and we do not consider that a coarse point of $P(m)$ contains fine points of $P(n)$ but we treat them separately. Our topological space will be the $Y = X + oX = P(m) + P(n)$. Subsets A of Y can be split to $A = oA + cA$, where oA are the fine points of A in $P(n)$ and cA are the coarse points of A in $P(m)$.

DEFINITION 2.3

A point x of $X = P(low) = P(m)$ or of $oX = P(high) = P(n)$ is a limit point of a subset A of $Y = P(low) + P(high)$ (and oA is a subset of oX) , iff there is a positive seemingly infinitesimal de of $P(high)$ such that for any positive seemingly infinitesimal da of $P(high)$ less than de , there is

a fine point y of ∂A such that $|x-y|=da$. We denote the set of fine points of P (high) limit points of A by $\partial L(A)$ and all coarse points of P (low) $=P(m)$ by $L(A)$. We define as closure $cl(A)$ of a subset A of Y , the $cl(A)=A$ union $Cl(A)$. A set is open if its complement in Y is the closure of a set.

Notice that with the closure we add only coarse visible points not fine (possibly invisible) points. For this reason the closure operator has the idempotent low $Cl(Cl(A))=Cl(A)$. For the relations of limit points, closure, boundary, open sets etc see [9] J. Munkress. In addition $Cl(A \text{ union } B) = Cl(A) \text{ union } Cl(B)$ and $Cl(A \text{ intersection } B) = Cl(A) \text{ intersection } Cl(B)$. We define that a x point of Y is seemingly in contact with the subset A of Y iff x belongs to A union $Cl(A)$. In other words either it belongs to the set or it is a limit point of it.

The concepts of boundary points and interior points are defined so as to have the usual properties as well as the concept of open set, base of open sets and base of neighbourhoods in Y . Similarly for connectedness. (See e.g. [9] J.Munkres)

The concept of topological **lowest visible or accountable or computable compactness** is defined in the usual way, where far the existence of finite sub-cover for any cover, we require, existence of **lowest visibly finite cardinality of a sub cover**. Similarly for the concept of **lower visible or computable or accountable compactness** or simply **visibly compactness** of a set of points. For a first outline of the Digital Calculus we will not proceed in these details.

The basic properties of continuity are:

- 1) Continuity is invariant by linear combinations
- 2) Continuity and product
- 3) Continuity and quotient
- 4) Composition of digital continuous functions are digital continuous
- 5) Bolzano theorem (after the supremum property of digital real numbers)
- 6) Mean value theorem.

PROPOSITION 2.1 (CONTINUOUS COMPOSITE)

Let two digital functions $f: [\alpha, \beta]_m \rightarrow R(m, n)$, with $oof: [\alpha, \beta]_p \rightarrow P(p)$, $of = [oof]_n$, $f = [of]_m$ and $h: [f(a), f(b)]_m \rightarrow R(m, n)$, with $ooh: [f(a), f(b)]_p \rightarrow P(p)$, $oh = [ooh]_n$, $h = [oh]$, that the first is (digitally) $P(m)/P(n)/P(p)$ continuous and the second $P(m)/P(p)/P(q)$ continuous such that their composition $f(h)(x): [\alpha, \beta]_m \rightarrow R(m, n)$ defined by $oor = [oof(ooh([x]_p))]$ (and or, r defined in the obvious way), is also a digital function with values in $P(m)$ (in other words its diagram commutes). Then this composition function is also a (digitally) $P(m)/P(n)/P(q)$ continuous function in $[a, b]_m$.

Hint for a proof: From the definition of the composite digital function oor on x of $[a, b]_p$

if dx is a seemingly infinitesimal at x of $P(p)$, then from the $P(m)/P(n)/P(p)$ continuity of ooh at x we get that the $dy = ooh(dx)$ is a seemingly infinitesimal of $P(n)$ relative to $P(m)$, and from the $P(m)/P(n)$ continuity of the of we get that the $of(dy)$ is a seemingly infinitesimal of $P(n)$, relative to $P(m)$. Thus the composite r is digitally $P(m)/P(p)$ continuous. QED

PROPOSITION 2.2 (CONTINUOUS LINEAR COMBINATIONS)

Let two digital functions $f: [a, b]_m \rightarrow R(m, n)$, with $of: [a, b]_n \rightarrow P(n)$, $f = [of]$

, and $h: [a, b]_m \rightarrow R(m, n)$, with $oh: [a, b]_n \rightarrow P(n)$, $h = [oh]$, that are (digitally) $P(m)/P(p)/P(q)$ continuous such that for any digital scalars a, b of $P(m)$, the functions $af + bh$, $f * h$, $1/f$ are also digital functions on $[a, b]_m$ with values in $P(m)$, then they are also (digitally) $P(m)/P(n)/P(q)$ continuous functions.

Hint for a proof: From the $P(m)/P(p)/P(q)$ continuity of the f and h we get that for dx seemingly infinitesimals of $P(q)$, the $df(x)$, $dh(x)$ are seemingly infinitesimals of $P(p)$ and from the ideal-like property of the $P(p)$ seemingly infinitesimals (see definition of digital real numbers 10)) the $adf(x) + bdh(x)$ is a seemingly infinitesimal of $P(n)$ relative to $P(m)$, thus the linear combination is $P(m)/P(n)/P(q)$ digital continuous. QED

PROPOSITION 2.3 (CONTINUOUS PRODUCT)

Let two digital functions $f: [a, b]_m \rightarrow R(m, n)$, with $of: [a, b]_n \rightarrow P(n)$, $f = [of]$

, and $h: [a, b]_m \rightarrow R(m, n)$, with $oh: [a, b]_n \rightarrow P(n)$, $h = [oh]$, that are (digitally) $P(m)/P(p)/P(q)$ continuous such that for any digital scalars a, b of $P(m)$, the functions $f * h$, is also digital functions on $[a, b]_m$ with values in $P(m)$, then they are also (digitally) $P(m)/P(n)/P(q)$ continuous functions.

Hint for a proof: From the $P(m)/P(p)/P(q)$ continuity of the f and h we get that for dx seemingly infinitesimals of $P(q)$, the $df(x)$, $dh(x)$ are seemingly infinitesimals of $P(p)$ and from the ideal-like property of the $P(p)$ seemingly infinitesimals (see definition of digital real numbers 10)) the $df(x) * dh(x)$ is a seemingly infinitesimal of $P(q)$ relative to $P(m)$. Then the $df(x)h(x) =_p f(x+dx)h(x+dx) - f(x)h(x) =_p (f(x) + df(x))(h(x) + dh(x)) - f(x)h(x) =_p$ by multiplying out we get a linear combination of seemingly infinitesimals of $P(p)$ and $P(q)$ that by the ideal-like property of the seemingly infinitesimals are also seemingly infinitesimals of $P(n)$ relative to $P(m)$. Thus the product is $P(m)/P(n)/P(q)$ digital continuous. QED

PROPOSITION 2.4 (CONTINUOUS INVERSE)

Let a digital functions $f: [a, b]_m \rightarrow R(m, n)$, with $of: [a, b]_n \rightarrow P(n)$, $f = [of]$

, that is (digitally) $P(m)/P(p)/P(q)$ continuous such that the functions $1/f$ is also definable digital functions on $[a,b]_m$ with values in $P(m)$, then it is also (digitally) $P(m)/P(n)/P(q)$ continuous function.

Hint for a proof: From the $P(m)/P(p)/P(q)$ continuity of the f we get that for dx seemingly infinitesimals of $P(q)$, the $df(x)$, is seemingly infinitesimals of $P(p)$. The $d(1/f(x)) =_p (f(x+dx) - f(x))/f(x) * (f(x+dx) - f(x)) =_p (df(x))/f(x) * (df(x) + f(x))$. The denominator is a computable finite number and non-seemingly infinitesimal of $P(m)$, while the numerator is a seemingly infinitesimals of $P(p)$. From the ideal-like properties of the seemingly infinitesimals we deduce that the ratio is a seemingly infinitesimal of $P(n)$. Thus the inverse is $P(m)/P(n)/P(q)$ digital continuous. QED

PROPOSITION 2.5 (BOLZANO)

Let a digital (digitally) continuous functions $f: [a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f = [of]$ $f: P(m) \rightarrow P(m)$, defined in a finite interval $[a,b]_m$ of $P(m)$ such that, $f(a), f(b)$ have opposite signs, that is $f(a)f(b) <_m 0$, (e.g. assume $f(a) <_m 0$) then there is at least one point c in the open interval $(a,b)_m$, such that for its next higher point c' in $[a,b]_m$ holds $f(c) <_m 0$ and $f(c') >_m 0$

Hint for a proof: We apply the supremum completeness property for upper bounded sets of the digital real numbers at the $P(m)$ precision level for the set $A = \{x / a \leq_m x \leq b \text{ that the } f \text{ is negative in the } [a,x] \}$. QED

PROPOSITION 2.6 (MAXIMUM)

Let a digital (digitally) continuous functions $f: [a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f = [of]$ $f: P(m) \rightarrow P(m)$, defined in a finite interval $[a,b]_m$ of $P(m)$, then it attains its maximum in $[a,b]_m$, in other words there is a number y in $[a,b]_m$ in $P(m)$, such that $f(x) <_m f(y)$ for all x in $[a,b]_m$ in $P(m)$.

Hint for a proof: We apply the supremum property of the digital real numbers at the $P(m)$ precision level for the set $A = f([a,b])$ in $P(m)$. As A is a finite set it has a maximum element.

DEFINITION 2.3

A digital real function defined on a closed interval $f: [a,b]_m \rightarrow P(m)$, of: $[a,b]_n \rightarrow P(n)$, is (digitally) is $P(m)/P(n)/P(n)$ differentiable at a point a of its domain of definition $[a,b]_m$ in $P(m)$, if for every other point x' of its domain of definition $[a,b]_n$ in $P(n)$, such that the distance of a and x' is seemingly infinitesimal belonging in $P(n)$ and relative to $P(m)$ with $dx =_n x' - a$, then $dy =_n f(x') - f(a)$ is a seemingly infinitesimal relative to $P(m)$, belonging to $P(n)$ and the ratio $dy/dx =_m (f(x') - f(a))/(x' - a)$ is always the same as number c of $P(m)$, independent from the choice of x' which is called the derivative of f at a , $c =_m df(x)/dx|_a$, while the $c - dy/dx =_n c - (f(x') - f(a))/(x' - a)$ is a seemingly infinitesimal relative to $P(m)$ and belonging to $P(n)$.

Notice that when change seemingly infinitesimals dx , the dy/dx may change as number of $P(n)$, but remains constant as number of $P(m)$.

Similarly we may define differentiation by the pairs of precision levels $P(m)$ - $P(p)$, and $P(m)$ - $P(q)$.

DEFINITION 2.4

A digital real function defined on a closed interval $f:[a,b]_m \rightarrow P(m)$, of: $[a,b]_n \rightarrow P(n)$, is (digitally) is $P(m)/P(n)/P(p)$ differentiable at a point a of its domain of definition $[a,b]_m$ in $P(m)$, if for every other point x' of its domain of definition $[a,b]_n$ in $P(n)$, such that the distance of a and x' is **seemingly infinitesimal** (that is in $P(p)$ and relative to $P(n)$ and $P(m)$) with $dx=_n x'-a$, then $dy=_n f(x')-f(a)$ is a seemingly infinitesimal relative to $P(m)$, belonging to $P(p)$ and the ratio $dy/dx=_m (f(x')-f(a))/(x'-a)$ is **always the same as number c of $P(m)$** , independent from the choice of x' which is called the derivative of f at a , $c=_m df(x)/dx|_a$, while the $c \cdot dy/dx=_p c \cdot (f(x')-f(a))/(x'-a)$ is a seemingly infinitesimal relative to $P(m)$ and belonging to $P(n)$.

Notice that when change seemingly infinitesimals dx , the dy/dx may change as number of $P(n)$, but remains constant as number of $P(m)$.

Similarly we may define differentiation by the pairs of precision levels $P(m)$ - $P(p)$, and $P(m)$ - $P(q)$.

The basic properties of differentiability are

- 1) Chain Rule
- 2) Linearity
- 3) Product or Leibniz rule
- 4) Quotient rule

PROPOSITION 2.7 (CHAIN RULE)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[f(a),f(b)]_m \rightarrow R(m,n)$, with of: $[f(a),f(b)]_n \rightarrow P(n)$, $h=[oh]$, that are the first (digitally) $P(m)/P(n)/P(p)$ differentiable at a (in $P(m)$) and the second $P(m)/P(p)/P(q)$ differentiable at $f(a)$ in $P(m)$ such that their composition $f(h)(x): [a,b]_m \rightarrow R(m,n)$ defined by $oor=[oof(ooh([x]_p))]$ (and or, r defined in the obvious way), is also a digital function with values in $P(m)$ (in other words its diagram commutes), and the product $(df/dx) \cdot (dh/dx)$ exists in $P(m)$ too. Then their composition function is also a (digitally) $P(m)/P(n)/P(q)$ differentiable function at a and

$$\frac{d(f(h(x)))}{dx} \Big|_a =_m \frac{df(y)}{dy} \Big|_{f(a)} * \frac{dh(x)}{dx} \Big|_a$$

Or in other symbols if $df(a)=db$, $df(h(a))=d\gamma$, $da=dx/a$

$$\frac{d\gamma}{da} =_m \frac{d\gamma}{db} * \frac{db}{da}$$

Hint for a proof: We start with a seemingly infinitesimal dx of $P(q)$ relative to $P(p)$, then from the $P(m)/P(n)/P(p)$ differentiability of h , the $dh(x)$ is a seemingly infinitesimal of $P(p)$, relative to $P(n)$ and the derivative $dh(x)/dx$ exists in $P(m)$. Taking this $dh(x)$ seemingly infinitesimal of $P(n)$ relative to $P(m)$, from the $P(m)/P(n)/P(p)$ differentiability of f , the $df(h(x))/dh(x)$ exists as element of $P(m)$ and thus by multiplying $(df(h(x))/dh(x)) * dh(x)/dx =_m df(h(x))/dx$, the quotient by the hypotheses exists in $P(m)$ therefore the composite is $P(m)/P(n)/P(q)$ differentiable and the chain rule holds. QED

PROPOSITION 2.8 (Linear combination)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with $of: [a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[a,b]_m \rightarrow R(m,n)$, with $oh: [a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) $P(m)/P(p)/P(q)$ differentiable at a point x such that their linear combination $adf(x)/dx + bdh(x)/dx$ for constants a, b of $P(m)$, is again inside $P(m)$. Then their linear combination $af(x) + bh(x)$ function at x is also a (digitally) $P(m)/P(n)/P(q)$ differentiable function and

$$\frac{d(af(x) + bh(x))}{dx} =_m a \frac{df(x)}{dx} + b \frac{dh(x)}{dx}$$

Hint for a proof: If dx is a seemingly infinitesimal of $P(q)$ relative to $P(p)$, then it holds that $The d(af(x)+bh(x)) =_n adf(x)+bdh(x)$ is seemingly infinitesimal of $P(p)$ relative to $P(n)$. Thus from the $P(m)/P(p)/P(q)$ differentiability of the f and h , the $d(af(x)+bh(x))/dx =_m adf(x)/dx + bdh(x)/dx$ is by hypotheses in $P(m)$ too, and the property holds. QED

PROPOSITION 2.9 (Leibniz product rule)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with $of: [a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[a,b]_m \rightarrow R(m,n)$, with $oh: [a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) $P(m)/P(p)/P(q)$ differentiable at a point x such that the expression $(df(x)/dx)*h(x) + f(x)*(dh(x)/dx)$ is again inside $P(m)$. Then the product $f(x)*h(x)$ function at x is also a (digitally) $P(m)/P(n)/P(p)$ differentiable function and

$$\frac{d(f * g)(x)}{dx} =_m \frac{df(x)}{dx} * h(x) + f(x) * \frac{dh(x)}{dx}$$

Hint for a proof: If dx is a seemingly infinitesimal of $P(q)$ relative to $P(p)$, then $d(f(x)*h(x)) =_p (f(x+dx)h(x+dx) - f(x)h(x)) =_p ((f(x)+df)(h(x)+dh) - f(x)h(x)) =_p (fdf+hdf+dfd h)$ and by the ideal-like property of the infinitesimals it is in $P(p)$. Thus $d(f(x)h(x))/dx =_m f(dh(x)/dx) + h(df(x)/dx) + df*(dh(x)/dx)$. The last terms is zero in $P(m)$ because

the df is seemingly infinitesimal relative to P(m), and the sum of the first two terms exists in P(m) by the hypotheses, thus the product is P(m)/P(n)/P(q) differentiable and the Leibniz product rule holds. QED

PROPOSITION 2.8 (Quotient)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with $of: [a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[a,b]_m \rightarrow R(m,n)$, with $oh: [a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) P(m)/P(p)/P(q) differentiable at a point x such that their quotient $f(x)/h(x)$ is defiable and in P(m) and the right hand of the formula below is computably finite, that is it belongs to P(m) when the terms of do. Then the quotient $f(x)/h(x)$ function at x is also a (digitally) P(m)/P(n)/P(p) differentiable function and

$$d \frac{f}{h}(x) =_m \frac{(\frac{df(x)}{dx} * h(x) - f(x) * \frac{dh(x)}{dx})}{[h(x)]^2}$$

Hint for a proof: Similar, as in the product rule. It is based on the ideal-like properties of the seemingly infinitesimals, and the hypotheses of the theorem. We start with a seemingly infinitesimal of P(q) relative to P(p), and calculate the $d(f/h)$. We substitute the $f(x+dx)$, $h(x+dx)$ with $f(x)+df$, $h(x)+dh$ in P(p), make the operations, we use the P(m)/P(p)/P(q) differentiability of the f and h , and that the right hand side of the formula in the theorem, also belongs to P(m) and we get the P(m)/P(n)/P(q) differentiability of the quotient. QED

PROPOSITION 2.10 (Continuity of differentiable function)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with $of: [a,b]_n \rightarrow P(n)$, $f=[of]$, which is (digitally) P(m)/P(n)/P(p) differentiable at a point a of P(m). Then it holds that it is also a (digitally) P(m)/P(n)/P(p) continuous function at a .

Hint for a proof: From $f(x)' =_m df(x)/dx$ in P(m) and a seemingly infinitesimal dx of P(p) we get that $df =_n f(x)' * dx$. And from the ideal-like properties of the seemingly infinitesimals, the right hand side is also in P(n) and seemingly infinitesimal. Thus the f by the definition of continuity is P(m)/P(n)/P(p) digitally continuous. QED

DEFINITION 2.4

(Higher dimension total derivative of a digital k-vector function.)

Let A_m closed rectangle subset of $P^k(m)$ and let a digital vector function $f:A_m \rightarrow P^s(m)$, $of: A_n \rightarrow P^s(n)$, $f=[of]_m$. We define that f is (digitally) **P(m)/P(n)/P(p) differentiable** at a point a in A_m iff there is a linear transformation $L: P^k(m) \rightarrow P^s(m)$, such that for any seemingly infinitesimal vector dh of $P^k(p)$ relative to $P^k(m)$, it holds that

$$\frac{\|f(z+dh) - f(a) - L(dh)\|}{\|dh\|} =_m 0 \text{ in } P(m)$$

The linear transformation L is denoted by $D(f(a))$ and is called total derivative of f at a . It can be proved that any such linear transformation L if it exists it is unique.

This is somehow equivalent to that

- 1) For every seemingly infinitesimal dh of $P(p)^k$ at a point a of $P(m)$, $d_h f(a) = {}_m L(dh)$
- 2) And also for this seemingly infinitesimal dh , the $d_h f(a) - L(dh)$ as seemingly infinitesimal of $P(n)^k$ is transcendently smaller than the seemingly infinitesimal dh of $P(p)^k$.

L is can be a function of $P(n)$ not only of $P(m)$ that is definable in seemingly infinitesimals too.

Properties of classical total derivative are:

- 1) Partial derivatives per coordinate exist and their Jacobean matrix is the matrix of the total derivative (differential)
- 2) Conversely if they exist and are continuous in a region then the total derivative exist, and the digital vector function is called continuously differentiable.

3. THE DEFINITION OF THE DIGITAL ARCHIMEDEAN MEASURE AND INTEGRAL.

At first we define the digital Archimedean Integral and then also the Archimedean measure, although it can be vice versa.

DEFINITION 3.1

Let a subset A of a closed interval $[a, b]_n$ of $P(n)$, with $[a, b]_m$ belonging to $P(m)$, of cardinal number of points $|A|$ which is a number of $P(n)$ and in general seemingly infinite relative to $P(m)$. We define as Archimedean measure of A , in symbols $m(A)$, and call A , $P(m)$ -countably measurable, or simply $P(m)$ -measurable, a possibly of seemingly infinite terms relative to $P(m)$ sum of $|A|$ times of the $P(n)$ -sizes of the points of A , such that the $P(m)$ rounding of the sum belongs to $P(m)$. In other words as each point of A in $P(n)$ has size 10^{-n} then $m(A) = [|A| * 10^{-n}]_m$ which is a number required to belonging to $P(m)$ for A to be $P(m)$ -measurable.

Similar definition exists for higher dimensions $R^k(m, n)$

DEFINITION 3.2

Let a digital functions $f: [a, b]_m \rightarrow P(m)$, with of: $[a, b]_n \rightarrow P(n)$, $f = [of]$.. Then we define as Archimedean integral of f on the closed interval $[a, b]_m$, and call the f Archimedean $P(m)$ -integrable, the possibly of seemingly infinite terms relative to $P(m)$ sum of $[a, b]_n$ times of the

$P(n)$ -sizes of the points dx of $[a,b]_n$ multiplied with the value of $f(x)$ at each point dx of $[a,b]_n$, such that the $P(m)$ rounding of this weighted sum belongs to $P(m)$. In symbols

$$I =_m \int_a^b f(x)dx \text{ in } P(m)$$

Notice that according to that definition the Archimedean measure of a subset A of $[a,b]_m$ is the Archimedean of the characteristic function X_A of A . In symbols

$$m(A) =_m \int_a^b X_A dx \text{ is in } P(m)$$

Similar definition exists for higher dimensions $\mathbb{R}^k(m,n)$

Similarly we may define measure and integration by the pairs of precision levels $P(m)$ - $P(p)$, and $P(m)$ - $P(q)$.

The basic properties of the classical Integral are:

- 1) Continuous= \Rightarrow Integrable
- 2) Linearity
- 3) Inequality
- 4) Additivity at the limits of integration
- 5) Upper, Lower bounds and the limits of integration
- 6) Absolute value inequality
- 7) Additive property of point measure

$$m(A \cup B) =_m m(A) + m(B) - m(A \cap B)$$

- 8) It holds also that functions that differ only at a set of measure zero have the integrals.

PROPOSITION 3.1(Measure zero)

Let two digital functions $f: [a,b]_m \rightarrow \mathbb{R}(m,n)$, with $of: [a,b]_n \rightarrow P(n)$, $f=[of]$, and $h: [a,b]_m \rightarrow \mathbb{R}(m,n)$, with $oh: [a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) integrable on $[a,b]_m$, such that they differ in values only on a subset of $[a,b]_m$ of (Archimedean) measure zero, then their (Archimedean) integrals are equal.

$$\int_a^b f(x)dx =_m \int_a^b h(x)dx$$

Hint for a proof:

PROPOSITION 3.2 (Continuity implies integrability)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, which is (digitally) continuous in the closed interval $[a,b]$. Then it holds that it is also a (digitally) integrable function at $[a,b]_m$ and

$$I =_m \int_a^b f(x)dx \text{ is in } P(m)$$

Hint for a proof:

PROPOSITION 3.3 (Additive decomposition of interval)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, which is (digitally) integrable on the closed interval $[a,b]_m$. Then for an c of $[a,b]_m$ in $P(m)$ it holds that f it is also a (digitally) integrable function on $[a,c]_m$ and $[c,b]_m$ and

$$\int_a^b f(x)dx =_m \int_a^c f(x)dx + \int_c^b f(x)dx$$

PROPOSITION 3.4 (Linear combination)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) integrable on $[a,b]_m$, such that their linear combination $kf(x)+lh(x)$ for constants k, l of $P(m)$, is again inside $P(m)$. Then their linear combination $kf(x)+lh(x)$ function is also (digitally) integrable digital function on $[a,b]_m$ and

$$\int_a^b (kf(x) + lh(x))dx =_m k \int_a^c f(x)dx + l \int_c^b h(x)dx$$

PROPOSITION 3.5 (Upper, Lower bounds inequalities)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, which is (digitally) integrable on the closed interval $[a,b]_m$, such that for constants m, M of $P(m)$, it holds that $m \leq_m f(x) \leq_m M$. Then

$$m^*(b-a) \leq_m \int_a^b f(x) dx \leq_m M(b-a).$$

Hint for a proof:

PROPOSITION 3.6 (Integrability)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, which is upper bounded by a number of $P(m)$: $f(x) \leq_m M$ and M and also $(b-a)M$ are in $P(m)$ for all x in $P(n)$. Then it is Archimedean integrable:

$$I =_m \int_a^b f(x) dx \text{ exists as a number of } P(m)$$

Hint for a proof:

Indication of Proof: In the definition of the Archimedean integral, in the finite (but seemingly infinite) sum of terms $f(x)dx$ in $P(n)$ we may substitute $f(x)$ with its bound M , and factor out the M , by the distributive law of finite sums, while the sum of dx 's give the length of the interval $[a,b] =_m b-a$. Therefore the integral is upper bounded by $(b-a)M$ in $P(m)$, which means that the rounded in $P(m)$ sum and Integral exists also in $P(m)$, thus the function is Archimedean integrable.

Hint for a proof:

PROPOSITION 3.7 (Inequality with absolute values)

Let a digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$ $f:[a,b]_n \rightarrow P(m)$, which is integrable on $[a,b]_m$. Then it holds that $|f|$ is also integrable on $[a,b]_m$ and

$$\left| \int_a^b f(x) dx \right| \leq_m \int_a^b |f(x)| dx$$

PROPOSITION 3.8 (Integration by parts)

Let two digital functions $f:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $f=[of]$, and $h:[a,b]_m \rightarrow R(m,n)$, with of: $[a,b]_n \rightarrow P(n)$, $h=[oh]$, that are (digitally) integrable on $[a,b]_m$, such that the next integrals on $[a,b]_m$ exist

$$\int_a^b f(x) \left(\frac{dh(x)}{dx} \right) dx, \int_a^b h(x) \left(\frac{df(x)}{dx} \right) dx$$

then

$$\int_a^b f(x) \left(\frac{dh(x)}{dx} \right) dx + \int_a^b h(x) \left(\frac{df(x)}{dx} \right) dx =_m f(b)h(b) - f(a)h(a)$$

Hint for a proof:

PROPOSITION 3.9 (Inequality 2)

Let two digital functions $f: [a, b]_m \rightarrow R(m, n)$, with $of: [a, b]_n \rightarrow P(n)$, $f = [of]$, and $h: [a, b]_m \rightarrow R(m, n)$, with $oh: [a, b]_n \rightarrow P(n)$, $h = [oh]$, that are (digitally) integrable on $[a, b]_m$ and $f(x) \leq_n h(x)$ in $[a, b]_n$ then it holds that

$$\int_a^b f(x) dx \leq_m \int_a^b h(x) dx$$

PROPOSITION 3.10 (Additivity of Archimedean measure)

Let a sets A, B , in $P(n)$ that are Archimedean measurable. Then also their union $A \cup B$ and their intersection $A \cap B$ are Archimedean measurable and it holds for their Archimedean measure symbolized by $m()$, that

$$m(A \cup B) =_m m(A) + m(B) - m(A \cap B).$$

Hint for a proof:

Fubini Theorem It can be deduced as in classical Calculus that we can get the value of the integral by iterative one dimensional integrals once the lower or upper one-dimensional integrals exist. It is the results Associative and commutative property of finite sums.

PROPOSITION 3. 12 (Fubini theorem iterated integrals)

Let A closed rectangle subset of $P^k(m)$ and B_m closed rectangle subset of $P^s(m)$ and let digital function $f: A \times B \rightarrow P(m)$, $of: A \times B \rightarrow P(n)$, $f = [of]$ (digitally) integrable.

For x in A let $h_x: B \rightarrow P(m)$ be defined by $h_x(y) =_m f(x, y)$, and we assume that it is also a digital function and let

$$I(x) =_m \int_B (y) dy$$

which is assumed also a digital function.

Then $I(x)$ is (digitally) integrable on A and it holds that

$$\iint_{A \times B} f(x, y) dy dx =_m \int_A I(x) dx =_m \int_A \left(\int_B h_x(y) dy \right) dx$$

Hint for a proof:

4. THE FUNDAMENTAL THEOREM OF THE DIGITAL CALCULUS AS THE RELATION OF THE DIGITAL DERIVATIVE WITH THE DIGITAL INTEGRAL.

It is simply the formal expression that a weighted sum that is the mass of segment when getting its derivative to length it will give the linear density of the segment, which is also a derivative.

PROPOSITION 4.1 (FUNDAMENTAL THEOREM OF CALCULUS)

Let a digital functions $f: [a, b]_m \rightarrow P(m)$, with $f = [of]$

$f: [a, b]_n \rightarrow P(n)$, which is (digitally) continuous thus integrable on the closed interval $[a, b]_m$ and also the next function on $[a, b]_n$ is a digital function.

$$h(x) =_n \int_a^x f(y) dy$$

Then it holds that the function at $[a, b]_m$

$$h(x) =_m \int_a^x f(y) dy$$

is (digitally) differentiable and at any c of $[a, b]_m$.

$$\left. \frac{dh(x)}{dx} \right|_c =_m f(c)$$

Hint for a proof:

5. CONCLUSIONS AND PERSPECTIVES.

For all practical reasons in the physical and social sciences the digital calculus gives all the well-known applications with a finite ontology which is directly realizable both in the physical

ontology of atomic matter or digital ontology of operating systems of computers. This has vast advantages in applications in, Engineering, Physics, Meteorology, Chemistry, Ecology , social sciences etc.

The digital Calculus is also an educational revolution in the Education of Mathematics. . It is a new method of teaching mathematics where there is higher integrity with what we say, write, see, and think.

After [8] that defines the axiomatic Euclidean geometry and the current outline of the digital Differential and Integral Calculus, one may define and solve the digital differential and partial differential equations (with easier applications in the physical sciences) , digital fluid dynamics (with easier applications in physics) , digital differential geometry, digital functional analysis (appropriate for easier applications in signal theory) etc. The road is open and the digital world of the computers is the direct tool for this.

REFERENCES

[1] APOSTOL T. 1974 “Mathematical Analysis” 2nd edition Addison Wesley Publishing Company. 1974

[2] Banach-Tarski paradox:

Banach, Stefan; Tarski, Alfred (1924). "Sur la décomposition des ensembles de points en parties respectivement congruentes" (PDF). *Fundamenta Mathematicae* (in French). **6**: 244–277.

[3] Boltianskii V. (1978)“Hilbert’s 3rd problem” J. Wesley & Sons 1978

[4] Pawlikowski, Janusz (1991). "The Hahn–Banach theorem implies the Banach–Tarski paradox" (PDF). *Fundamenta Mathematicae*. **138**: 21–22.

[5] Wilson, Trevor M. (September 2005). "A continuous movement version of the Banach–Tarski paradox: A solution to De Groot's problem". *Journal of Symbolic Logic*. **70** (3): 946–952. [JSTOR 27588401](#). [doi:10.2178/jsl/1122038921](#).

[6] EVANS L. C. (2010) Partial Differential Equations 2nd edition, Graduate Texts in Mathematics, American mathematic Society, Providence Rhode Island

[7] Hausdorf paradox:

Felix Hausdorff (1914). "Bemerkung über den Inhalt von Punktmengen". *Mathematische Annalen*. **75**: 428–434. [doi:10.1007/bf01563735](#)

[8] Kyritsis K. An Axiomatic System for a Physical or Digital but Continuous 3-Dimensional Euclidean Geometry, Without Infinite Many Points. *World Journal of Research and Review (WJRR)* ISSN:2455-3956, Volume-5, Issue-4, October 2017 Pages 31-43

[9] Munkres J. R. Topology (a first course) editions Prentice Hall

[10] Von Neumann paradox:

On p. 85 of: von Neumann, J. (1929), "Zur allgemeinen Theorie des Massen" (PDF), *Fundamenta Mathematicae*, 13: 73–116

[11] RUDIN, W. 1976 "Principles of Mathematical Analysis "3rd edition McGraw-Hill Inc. 1976

[12] E. Schroedinger. *Science and Humanism* Cambridge University press 1961

[13] SPIVAK, M. 1965 "Calculus on Manifolds" *The Benjamin/Cumming publishing company* 1965.

[14] SPIVAK, M. 1967 Calculus *The Benjamin/Cumming publishing company* 1967.

[15] VAN DER WAERDEN *ALGEBRA Vol I, II* Springer 1970

THE FICTIONAL DIALOGUE OF THE IMMORTAL MATHEMATICIANS ON THE OCCASION OF THE NEW DIGITAL DIFFERENTIAL AND INTEGRAL CALCULUS

ARXHINEOMEDES after presenting the immortals the basic introduction to the digital differential and integral calculus , invites them in a free discussion about it.

ARCHINEOMEDES, and **NEWCLID** are individuals representing the collective intelligence of the digital technology but also of mathematics of the 21st century.

The participants of the discussion are the next 20.

1. Pythagoras
2. Eudoxus
3. Euclid
4. Democritus
5. Archimedes
6. Newton
7. Leibnitz
8. Cartesius
9. Cauchy
10. Dedekind
11. Weierstrasse
12. Hilbert

13. Riemann
14. Cantor
15. von Neumann
16. Poincare
17. Gödel
18. Cavalieri
19. Lagrange
20. Helmholtz
21. Gauss
22. Jordan
23. Lobachevski
24. Zeno
25. Bolzano
26. Lebesgue

And 2 contemporary mortals:
20. Newclid, and Archineomedes.

ARCHINEOMEDES:

Welcome honourable friends that you have become immortals with your fame and contribution in the creation of the science and discipline of Mathematics among the centuries on the planet earth!

Now that you have watched my presentation of the digital differential and integral calculus , I would like to initiate a discussion that will involve your remarks , perspectives opinions about it. Who would like to start the conversation?

PYTHAGORAS:

Thank you Archineomedes for the honor in gathering us together. I must express that I like the new of the digital differential calculus as well as the approach of the Axiomatic Digital Euclidean Geometry of Newclid , that as you say is a resume of what already the beginning of the 21st century in the earthly Computer Science has realized through software in the computer operating systems and computer screens and monitors.

In fact, I was always teaching my students that the integer natural numbers are adequate for creating a mathematical theory of the geometric space! And this is so because matter has atomic structure as Democritus has taught and space and time are simply abstract properties of matter. One only has e.g. to take as unit of measurement of lengths, the length of a invisible points and all metric relations in the low precision level of the figures, including the Pythagorean theorem, become relations of positive integer numbers, or solutions of Diophantine equations! But at that time no such detailed and elaborate system of definitions was easy, neither a well accepted concept that matters consists from atoms, was available in the mathematicians of the ancient Greece, Egypt or Babylon.

EUCLID: I am impressed ARCHINEOMEDES with your skillful definitions of derivative and integral. To tell the truth I never was satisfied with the classical definitions through limits and the infinite which now I consider a phenomenological abstraction not so much appropriate for an ontology of mathematics with applications in the physical sciences. I am myself also indirectly responsible for it, as my axioms that for every two

points on line there is always a 3rd between them, was the beginning of the need for the infinite and I am glad now that we can do the mathematics without it.

ARCHIMEDES: I like your digital differential and integral calculus Archineomedes ! It is as my perceptions! Actually my heuristic experimental work with solids that I was filling with sand or water to make volume comparisons by mechanical balances, is just an experimental realization of your concept of measure and integral through those of the points and finite many points! That is how I discovered and proved the formula of the volumes of the sphere.

ARCHINEOMEDES: Thank you Archimedes, that is why for your honor I called them Archimedean point measures.

DEMOCRITUS: Bravo ARCHINEOMEDES! Exactly my ideas of atoms! Actually as in my theory of atoms, the water is made from finite many atoms, the volume experiments of Archimedes with water is rather the exact realization of your point measures for areas and volumes through that of the invisible points! Here the atoms of the water are invisible, while the granulation of the sand may resemble your concept of the visible points!

LEIBNITZ: I want to congratulate you ARCHINEOMEDES for your approach! In fact my symbols of infinitesimal dx in my differential calculus suggest what I had in mind: A difference $dx=x_2-x_1$ so that it is small enough to be zero in the Lowest phenomenological measurements precision level but still non-zero in the Highest ontological precision level! Certainly a finite number!

NEWTON: I must say here that the Leibnitz idea of infinitesimal as a finite number based on the concepts of Low and High precision is not what I had in my mind when I was writing about infinitesimals or fluxes. That is why I was calling them fluxes and symbolized them differently. The theory of null sequences of numbers (converging to zero) of Cauchy and Weierstrasse is I think the correct formulation of my fluxes. Nevertheless these null sequences need not be infinite, they can very well be finite ending on the finest bin of the highest precision level. I was believing in my time ,like Democritus, that matter consist from finite many atoms , but I never dared to make a public scientific claim of it, as no easy proof would convince the scientist of my time!

I want to ask an important question to ARCHINEOMEDES : Is your digital differential and integral calculus based on three levels of precision more difficult or simpler that the classical differential and integral calculus based on limits and infinite many real numbers? And could be formulated in an equivalent way with fluxes, that is finite sequences converging to a point?

ARCHINEOMEDES: Well Newton thank you for the good words! A differential and integral calculus based on three levels of precision is certainly less complicated than (and also not equivalent to) the classical calculus with infinite sequences or limits. But a differential and Integral calculus of 3 ,4 or more precision levels is by far more complicated than the classical analogue differential and Integral calculus. Only that this further complication is a complexity that does correspond to the complexity of the physical material reality, while the complexities of the infinite differential and integral calculus (in say Lebesgue integration theory or bounded variation functions etc) is a complexity rather irrelevant to the physical material complexity. Now I do believe , although I have not carried it out with proofs, that by using finite sequences converging

to a point of the highest precision level, as seemingly infinitesimals, would be an equivalent formulation for the digital derivative.

CARTESIUS: I want to congratulate you ARCHINEOMEDES for your practical , finite but comprehensive digital differential and integral calculus which is practically based on the digital analytic Cartesian geometry! And my arithmetization of the geometry is the prerequisite for a digitalization. This was my implicated intention too.

CAUCHY: I wish I had thought of such definitions of the real numbers and integral myself, including the concept of seemingly. I want nevertheless to ask a very important question ARCHINEOMEDES. You mentioned that the digital differential and integral calculus is not equivalent to the classical differential and integral calculus with the infinite. Is it possible in the context f your concepts to define a differential and integral calculus equivalent t the classical one?

ARCHINEOMEDES: Well CAUCHY I have thought about it, although I never carried out detailed proofs. It seems to me that if I take all possible, I mean all levels of fine sizes of precision levels and require that a function would be digital differentiable or digitally integrable in all of them , then this might be equivalent o the classical definitions. Nevertheless such a very strong requirement would be, absolute, and not corresponding to the situations of material ontology. It would be a very strong requirement tying strongly together the phenomenology and ontology of matter, and we do know, that they should differ.

EUDOXOS: Well in your digital real numbers ARCHINEOMEDES , my definition of the ratio of two linear segments which is the base of the complete continuity of the line seem not to be that critical in your system, although I thing that it still holds, no?

ARCHINEOMEDES: It still holds EUDOXOS, except it is restricted to rational numbers with finite decimal representation.

DEDEKIND: And as I reformulated the idea and definition of equality of ratio of linear segments of Eudoxus, to my concept of Dedekind cuts about the completeness of continuity of the real numbers, does this still holds in your digital real numbers?

ARCHINEOMEDES: It still holds DEDEKIND, except it is restricted to digital numbers with finite decimal representation. And the same with the supremum and infimum properties of bounded sets of real numbers.

BOLZANO: That is why my basic theorem of continuous curves of continuous functions holds in the digital calculus. As ARCHINEOMEDES presented we have a usual and classical topological space based in this continuity.

JORDAN: Which suggests also that my theorem of closed curves in the digital plane should hold too?

ARCHINEOMEDES: Certainly JORDAN although I have not carried out any detailed proof of it yet.

WEIESSTRASSE: What about my definition of continuity with the epsilon-delta inequalities, does it hold for the digital continuity ARCHINEOMEDES?

ARCHINEOMEDES: More or less yes, with slightly different details WEIESSTRASSE. The epsilon must be restricted to the lowest phenomenological precision level, while the delta in the highest ontological precision level. That is how I though initially to define the digital continuity, but later I preferred the concept of seemingly infinitesimal so as to resurrect as rigorous and correct the arguments of hundreds of mathematicians in the 7th, 18th and 19th century in the calculus that used infinitesimals with the Leibniz notation.

POINCARÉ: Yes indeed, my articles in mathematics are full of arguments using the infinitesimals in an isolated way. Thank you ARCHINEOMEDES that now they have a rigorous and exact, formulation within the finite. I used to mock those that made mathematics with transfinite numbers, but now with the digital real numbers I realize that the inverse of a seemingly infinitesimal is a seemingly infinite number. I used to say that mathematics and geometry is the art of correct reasoning over not-corresponding and incorrect figures. With the digital mathematics this is corrected.

CAVALIERI: Would the methods of digital calculus render my principle of indivisibles on the calculations of volumes of 3-dimensional bodies rigorous and exact too?

ARCHINEOMEDES: With the right new details I believe yes CAVALIERI. A slice of a 3-D body by a digital plane, would consist from finite many highest precision level invisible points as tiny cubes that still have finite thickness (although zero in the phenomenological lowest precision level) therefore they make a kind of indivisibles and indivisible slices.

LEBESQUE: So in your digital calculus ARCHINEOMEDES, the definition of the digital integral is somehow a Lebesgue integral or a Riemann integral?

ARCHINEOMEDES: I did not define the integral with partitions to answer it precisely. If I would do so, then in your integral could start with seemingly infinite partitions, while a finite Riemann integral with only computable finite partitions. I defined it directly with seemingly infinite many, seemingly infinitesimal rectangles. So it is closer to your definition, and its relation with digital functions with points of discontinuity of measure seemingly zero rather confirms it.

HILBERT: I like your brave and perfect approach ARCHINEOMEDES! No infinite in your calculus till very realistic and useful, so as to have easy physical applications, as nothing in the physical material reality is infinite. Congratulations!

Von NEUMANN: I like too your digital differential and integral calculus ARCHINEOMEDES! I believe that I could easily make it myself, except at that time I

was busy in designing a whole generation of computers! I believe your work is a direct descendant of my work on computers. As you said your ideas came from software developers in the operating system of a computer!

ARCHINEOMEDES: Indeed, von Neumann! Thank you!

CANTOR: Pretty interesting your digital differential and integral calculus
ARCHINEOMEDES! But what is wrong with the infinite? Why you do not allow it in your mathematic? I believe that the infinite is a legitimate creation of the human mind! Your Digital differential and integral calculus lacks the charm and magic of the infinite!

PYTHAGORAS: Let me, **ARCHINEOMEDES**, answer this question of **CANTOR**! Indeed **CANTOR** the human mind may formulate with a consistent axiomatic way what it wants! E.g. an axiomatic theory of the sets where infinite sets exist! And no doubt that the infinite is a valuable and sweet experience of the human consciousness! But as in the physical material reality there is nowhere infinite many atoms, mathematical models that in their ontology do not involve the infinite, will be more successful for physical applications! In addition, there will not be any irrelevant to the physical reality complexity as in the mathematical models of e.g. of physical fluids that use infinite many points with zero dimensions in the place of the finite many only physical atoms with finite dimensions. The infinite may have its charm. Actually I believe that the human consciousness gives the feeling of the infinite. Consciousness is not a property of matter like energy, and it has to remain outside the ontology of matter and of mathematics. The Digital Differential and Integral Calculus has its own and different magic too!

RIEMANN: Very impressive **ARCHINEOMEDES** ,your phenomenological-ontological and logical approach to the Differential and Integral! But what about my Riemannian geometric spaces? Could they be formulated also with Local, Low and High precision levels and finite many visible and invisible points?

ARCHINEOMEDES: Thank you Riemann! Well my friend any digital system of your Riemannian Geometric spaces, with finite many points might require more than 2 probably 3 or 4 precision levels! That is why I start with a system of digital numbers of 4 precision levels. The reason might be that at any A point of a Riemannian Space, the tangent or infinitesimal space at A is Euclidean! And here the interior of the point A will be a whole flat Euclidean space which already requires two or 3 precision levels and both the visible and invisible points of the tangent Euclidean space will have to be invisible, while the point A visible point! But let us have patience! In the future I will study and answer your question with details and clarity! Originally me and Newclid had defined the digital real numbers only with two precision levels. But for the sake of differential manifolds and your Riemannian geometry I decided to put in the definition 4 precision levels. In modern software technology e.g. in scalable software maps, there are many map scales or precision levels that might be involved.

LOBACHEVSKI: I assume that the digitalization of the Riemannian geometry will derive automatically a digital version of my Hyperbolic non-Euclidean 3-dimensional geometry too!

ARCHINEOMEDES: Certainly **LOBACHEVSKI**!

HELMHOLTZ: I think that the idea of digital space and calculus is closer to the physical reality. Even my theory and study of sound, when stepping down to the molecules and atoms of air and matter becomes a digital ontology.

LAGRANGE: Of course **HELMHOLTZ!** As Cartesius arithmetization of geometry by coordinates is a prerequisite for the digitalization of geometry, so my arithmetization of the physical magnitudes of motion like velocity , force acceleration etc is a prerequisite for the realistic digitalization of such physical magnitudes of motion through the digital real numbers and digital derivative and integral.

ZENO: So if the magnitudes of motion after Lagrange arithmetization, are now digital in your calculus **ARCHINEOMEDES**, would this mean that my paradox with Achilles and the turtle resolve differently?

ARCHINEOMEDES: Certainty Zeno. In the classical “analogue” mathematics of the infinite, your paradox is resolved , as the sum of infinite series which is nevertheless a finite number. In the digital calculus, the corresponding series is already a finite series (as the space and magnitudes motion are themselves digital and finite) and it has also a computable finite number as its sum.

GAUSS: I agree that the digital differential and integral calculus is more transparent lucid and practical. What about my different proofs of the fundamental theorem of algebra that any polynomial has at least one root in the complex numbers. Do you think **ARCHINEOMEDES** that they could be transferable to proofs in the digital complex numbers?

ARCHINEOMEDES: Although I have not carried out in detail any such transfer of your alternative proofs, I believe it can be done. The digital ontology as Newclid had mentioned in previous discussions allows also for a new type of proofs which is that of finite induction on the (finite many) points. Maybe still an new alternative proof can be obtained in this way.

GOEDEL: You and **NEWCLID** , **ARCHINEOMEDES** mentioned somewhere that all of your arguments take place in the digital logic. What is the difference of digital logic say compared to a 1st order formal logic of classical mathematics?

ARCHINEOMEDES: The main difference **GOEDEL** is that in the digital 1st order formal Logic there do not exist countably infinite many proofs, or countable many formulae. Only finite many up to some size, as the digital natural numbers are used and not the classical natural numbers of Peano axioms.

GOEDEL: This mean that my theorem that for every axiomatic theory that contains the natural numbers it exist at least one proposition A than neither A, neither the negation of A can be proved, does not exist in your digital meta-mathematics.

ARCHINEOMEDES: That is correct **GOEDEL**.

GOEDEL: So in your digital mathematics, there might exist at least one axiomatic theory T, that contains the digital natural numbers N , and within a digital logic L, so that for

every proposition A, there is an appropriate size digital logic $L(A)$, such that there is a proof either of A or of the negation of A?

ARCHINEOMEDES: It is rather a more optimist theorem for the powers of rational thinking this theorem, compared to your celebrated theorem GOEDEL is it not? Well I have not laid down the details of its statement and the details of any proof of it, but I certainly hope that it might hold true.

NEWCLID: I want also to ask you **ARCHINEOMEDES** to make it clear if your concepts of digital line and digital plane and visible geometric points on them are different compared to those in my axiomatic system of Digital (but continuous) Euclidean geometry.

ARCHINEOMEDES: There are certainly **NEWCLID** considerable difference. Actually I did not have to define, what a digital line or digital plane is. But if I would have to, I would use linear equations of analytic geometry with rounding in the precision levels. And in my case the visible points are clearly tiny cubes. In your axiomatic system you have as initial concept the linear segment and plane and visible and invisible points, and you impose axiomatically mutual properties of them. As I understand your axiomatic system it is open at each instance how the visible points are chosen say for a linear segment. Maybe there are more than one possible choices each time of visible points that satisfies the axioms. The very inability to have simultaneously that the coordinates are in 1-1 correspondence with finite many points and that congruence is also a 1-1 correspondence also of the finite many points was the source of incommensurable magnitudes and irrational numbers in the ancient Greece. And although you impose axiomatically that each point has coordinated and each coordinate is coordinate of some point, nowhere in your axioms there as the requirement of an 1-1 such correspondence but only of a many to many relation. In my case I prefer to have a 1-1 correspondence of coordinates with points and avoid any claims of a congruence relation of figures that I never define for the needs of differentiation and integration. In addition you use a 3-precision levels system of digital real numbers with one resolution of visible and 2 resolutions of invisible points and I prefer to use a 4-precision levels system of digital real numbers with 2 resolutions of visible points and 2 resolutions of invisible points.

ARCHINEOMEDES: If there no more questions or remarks, let us end here our discussion, and let us take a nice and energizing walk under the trees in the park close to Plato's academy.

AT THIS POINT THE DISCUSSION ENDS.

The next paper on the digital but continuous axiomatic Euclidean geometry is preprint version of finally published version in the *World Journal of Research and Review (WJRR)* ISSN:2455-3956, Volume-5, Issue-4, October 2017 Pages 31-43

AN AXIOMATIC SYSTEM FOR A PHYSICAL OR DIGITAL BUT CONTINUOUS 3-DIMENSIONAL EUCLIDEAN GEOMETRY , WITHOUT INFINITE MANY POINTS.

Konstantinos E. Kyritsis*

** Associate Prof. of TEI of Epirus Greece. ckiritsi@teiep.gr*

C_kyrisis@yahoo.com, TEI of Epirus, Dept Accounting-Finance Psathaki Preveza 48100

Abstract

This paper is concerned with finding an axiomatic system, so as to define the 3-dimensional Euclidean space, without utilizing the infinite ,that can imply all the known geometry for practical applied sciences and engineering applications through computers , and for more natural and perfect education of young people in the Euclidean geometric thinking. In other words by utilizing only finite many visible and invisible points and only finite sets, and only real numbers with finite many digits, in the decimal representation. The inspiration comes from the physical matter , rigid, liquid and gaseous, which consists of only finite many particles in the physical reality. Or from the way that continuity is produced in a computer screen from only finite many invisible pixels . We present such a system of axioms and explain why it is chosen in such a way. The result is obviously not equivalent, in all the details, with the classical Euclidean geometry. Our main concern is consistency and adequacy but not independence of the axioms between them. It is obvious that within the space of a single paper, we do not attempt to produce all the main theorems of the Euclidean geometry, but present only the axioms.

Key words: Axiomatic systems of Euclidean geometry, Digital Mathematics, Digital space, Constructive mathematics, Non-standard mathematics.

Mathematical Subject Classification: 03F99, 03H99, 93C62, 51M05

§1. INTRODUCTION.

Changing our axiomatic system of the Euclidean geometry so as as to utilize only finite points, numbers and sets, means that we change also our perception our usual mental images and beliefs about the reality. This project is under the next philosophical principles

- 1) Consciousness has the experience of the infinite.**
- 2) But the physical material world is finite.**
- 3) Therefore mathematical models in their ontology should contain only finite entities and should not involve the infinite.**

This paper is part of larger project which is creating again the basic of mathematics and its ontology with new axioms that do not involve the infinite at all.

Our perception and experience of the reality, depends on the system of beliefs that we have. In mathematics, the system of spiritual beliefs is nothing else than the axioms of the axiomatic systems that we accept. The rest is the work of reasoning and acting.

Quote: "It is not the world we experience but our perception of the world"

The abstraction of the infinite seems sweet at the beginning as it reduces some complexity, in the definitions, but later on it turns out to be bitter, as it traps the mathematical minds in to a vast complexity irrelevant to real life applications. Or to put it a more easy way, we already know the advantages of using the infinite but let us learn more about the advantages of using only the finite, for our perception, modelling and reasoning about empty space. This is not only valuable for the applied sciences, through the computers but is also very valuable in creating a more perfect and realistic education of mathematics for the young people. The new axioms of the Euclidean geometry create a new integrity between what we see with our senses, what we think and write and what we act in scientific applications.

The Euclidean geometry with infinite many points creates an overwhelming complexity which is very often irrelevant to the complexity of physical matter. The emergence of the irrational numbers is an elementary example that all are familiar. But there are less known difficult problems like the 3rd Hilbert problem (see [Boltianskii V. (1978)“]). In the 3rd Hilbert problem it has been proved that two solid figures that are of equal volume are not always decomposable in to an in equal finite number of congruent sub-solids! Given that equal material solids consists essentially from the physical point of view from an equal number of sub-solids (atoms) that are congruent, this is highly non-intuitive! There are also more complications with the infinite like the Banach-Tarski paradox (see [[Banach, Stefan; Tarski, Alfred \(1924\)](#)]) which is essentially pure magic or miracles making! In other words it has been proved that starting from a solid sphere S of radius r, we can decompose it to a finite number n of pieces, and then re-arrange some of them with isometric motions create an equal sphere S1 of radius again r and by rearranging the rest with isometric motions create a second solid Sphere S2 again of radius r! In other words like magician and with seemingly elementary operations we may produce from a ball two equal balls without tricks or “cheating”. Thus no conservation of mass or energy!. Obviously such a model of the physical 3-dimensional space of physical matter like the classical Euclidean geometry is far away from the usual physical material reality! I have nothing against miracles, but it is challenging to define a space that behaves as we are used to know. In the model of the 3-dimesional space, as new axiomatic system where such balls have only finite many points such “miracles” are not possible!

The continuous 3-dimensional space, defined axiomatically, is closer to what we know from the continuity of matter and fluids in physical reality, and strictly logically different from the traditional Euclidean space of infinite many points. It is not only the Hilbert’s 3rd problem, and the Banach-Tarski paradox which do not hold anymore for the physical or digital 3-dimensional Euclidean space, but also elementary topics like the constructability with ruler and compass.

We know e.g. that the squaring of the circle is not constructible with ruler and compass in the classical Euclidean Geometry with infinite many points. Because it involves the solution of the equation $\pi R^2 = x^2$,

(eq. 1)

and the number π is a transcendental irrational number. But in the digital Euclidean space $E_3(n,m,q)$ the (eq. 1) becomes the equation of rational numbers

$$[\pi]_m R^2 = [x]_m^2,$$

(eq.2)

Where by $[\]_m$ we denote the truncation of a real number of infinite decimal points to m only decimal points in the precision level P(m) (right for positive numbers left for negative numbers), and by $=_m$ the equality within the precision level P(m).

And so the constructability with ruler and compass of squaring of the circle must be put together with the next two facts

- 1) Equation (eq.2) is an equation of rational numbers
- 2) Rational numbers, that is of the form k/l (k, l positive integers), are constructible with ruler and compass (as linear segments in a line with a unit length)

Still we should not jump in to conclusions. General rational numbers of the form k/l as above may not necessarily belong to the precision level $P(m)$. So it might be necessary to resort to a higher precision level digital geometric space $E_3(n', m', q')$, $n' \gg n$, $m' \gg m$, $q' \gg q$, make the construction with ruler and compass, and then return back to the lower precision level space $E_3(n, m, q)$, to construct a square with equal area with the initial circle.

We shall not only describe a new axiomatic system of the Euclidean geometry but also new axiomatic system of the natural numbers and real numbers, where only finite many numbers with finite many decimal digits are involved. Actually we could start in the meta-mathematics with new axioms and definitions of 1st order and 2nd order formal Logic where only finite many symbols, finite many natural numbers and proofs with finite only steps are involved. But we have not sufficient space for this in this paper, so we shall start only from the natural numbers. We present such a new system of axioms and explain why it is chosen in such a way. The result is obviously not equivalent, in all the details, with the classical Euclidean geometry. Our main concern is consistency and adequacy but not independence of the axioms between them. It is obvious that within the space of a single paper, we do not attempt to produce all the main theorems of the Euclidean geometry, but present only the axioms. The next step is obviously to define a digital differential and Integral calculus over such a digital Euclidean geometry and digital real numbers without convergence of infinite sequences or limits. But again this is not for the space of the current paper but probably of a future such paper.

The next presentation of such an axiomatic system is a design of logically organized realistic thinking in the area of numbers and space. It is also a realistic ontology of an operating system for numbers and space, for all practical scientific and engineering applications.

§2. The new axioms

As I am a computer programmer too, besides being a mathematician, it became easier for me to, find out the necessary changes of the axioms of traditional mathematics, so as to derive axioms for the digital mathematics.

The axiomatic system adopted here, is that of Hilbert axiomatic system for the Euclidean Geometry, with modifications. (See e.g. http://en.wikipedia.org/wiki/Hilbert's_axioms)

Surfing among the euclidean figures of this geometry, is like turning pages in a e-book of a touch-screen mobile or i-pad.

We make here some small modifications of the Hilbert axioms of synthetic visual Euclidean Geometry. Some of the axioms of Hilbert will not hold, (like that which claims that between two points there is always a third), and some new initial concepts will be added, like that of two types of points visible and invisible, plus some relevant axioms.

I do not claim here that the axioms of the Digital Euclidean Geometry, below, are independent, in other words none of them can be proved from the others. As the elements are finite, there may be such a case. But I am strongly interested a) at first that are non-contradictory, and b) second that are adequate many, so as to describe the intended structure. later simplified and improved in elegance versions of the axioms may be given.

Before we proceed we remind the properties of the axiomatic digital natural numbers and axiomatic decimal digital real numbers, where again no infinite exists.

§2.1 SIMILAR TO PEANO, AXIOMS

We define the natural numbers in two scales (and later precision levels) that are two unequal initial segments of the natural numbers $N(\omega) < N(\Omega)$. The number ω is called the Ordinal size ω of the local system of natural numbers $N(\omega)$ while the Ω is the cardinal size of the global system of natural numbers. $\omega < \Omega$. If we start with integers n_1, n_2, n_3 from $N(\omega)$, then their addition and multiplication, have the commutative semiring properties but without closure in $N(\omega)$, but with values in $N(\Omega)$. We call the $N(\omega)$, the local segment while the $N(\Omega)$ the global segment.

We have here an initial relation among the natural numbers which is called successor or next of a natural number x and it is denoted by $S(x)$.

- 1) The number 1 is a natural number and belongs both to $N(\omega)$, and $N(\Omega)$.
- 2) There is no natural number whose successor is 1.
- 3) If x is a natural number of $N(\omega)$, its successor $S(x)$, is also a natural number belonging in $N(\Omega)$.
- 4) If two different numbers of $N(\Omega)$, have the same successor, then they are equal, Formally if $S(x)=S(y)$ then $x=y$.
- 5) (Peano axiom of induction) If a property or formal proposition $P()$ holds for 1 (that is $P(1)=\text{true}$) and if when holding for x in $N(\omega)$ holds also for $P(S(x))$ with $S(x)$ in $N(\Omega)$, then it holds for all natural numbers of $N(\omega)$.

6) Axiom of sufficient large size or SEEMINGLY INFINITE many numbers. If we repeat the operations of the commutative semiring starting from elements of the local version $N(\omega)$, ω -times, the results are still inside the larger set $N(\Omega)$.

This last Peano axiom of induction is useful only if the natural numbers are formulated within a formal logic (the axiom itself as a formal proposition is in 2nd order formal logic) that its size $\Omega(1)$ is less than the size of the objective system of natural numbers Ω . Otherwise for sufficient large

$\Omega(L) \gg \Omega$, we may simply construct a lengthy proof of this axiom starting from $P(1)$ then $P(2)$...and finally $P(\Omega)$, which then it is a theorem.

Any two models M_1 M_2 of the digital natural numbers $N(\Omega), N(\omega)$ of equal size ω, Ω are isomorphic.

§ 2.2 THE AXIOMATIC MULTI-PRECISION DECIMAL DIGITAL REAL NUMBERS $R(n,m,q)$.

- a) The **rational numbers** Q , as we known them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** k/l of natural numbers k, l exist as numbers and are unique. The cost of course is that when we represent them with decimal representation they may have infinite many but with finite period of repetition decimal digits.
- b) The **classical real numbers** R , as we know them, do involve the infinite, as they are infinite many, and are created with the goal in mind that **proportions** of linear segments of Euclidean geometry, exist as numbers and are unique (**Eudoxus** theory of proportions). The cost of course finally is that when we represent them with decimal representation they may have infinite many arbitrary different decimal digits without any repetition.

- c) But in the physical or digital mathematical world, such costs are not acceptable. The infinite is not accepted in the ontology of mathematics (only in the subjective experience of the consciousness of the scientist). Therefore in the **multi-precision digital real numbers**, proportions are handled in different way, with priority in the Pythagorean idea of the **creation of all numbers from an integral number of elementary units**, almost exactly as in the physical world matter is made from atoms (here the precision level of numbers in decimal representation) and the definitions are different and more economic in the ontological complexity.

We will choose for all practical applications of the digital real numbers to the digital Euclidean geometry and digital differential and integral calculus, the concept of a system of digital decimal real numbers with three precision levels, lower, low and a high.

Definition 2.2 The definition of a PRECISION LEVEL $P(n,m)$ where n, m are natural numbers, is that it is the set of all real numbers that in the decimal representation have not more than n decimal digits for the integer part and not more than m digits for the decimal part. Usually we take $m=n$. In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

THE AXIOMS OF THE DIGITAL REAL NUMBERS $R(n,m,q)$

We assume at least three precision levels for an axiomatic decimal system of digital real numbers $R(n,m,q)$: **THE LOCAL LOWER PRECISION LEVEL $P(n)$** , **THE LOW PRECISION LEVEL $P(m)$** , **AND THE HIGH PRECISION LEVEL $P(q)$** .

Whenever we refer to a real number x of a (minimal in precision levels) system of real numbers $R(n,m,q)$, we will always mean that x belongs to the local lower precision level $P(n)$ and that the system $R(n,m,q)$ has at least three precision levels with the current axioms.

Whenever we write an equality relation $=_m$ we must specify in what precision level it is considered. The default precision level that a equality of numbers is considered to hold, is the low or standard precision level $P(n)$.

Some of the Linearly ordered Field operations

The field operations in a precision level are defined in the usual way, from the decimal representation of the numbers. This would be an independent definition, not involving the infinite. Also equality of two numbers with finite decimal digits should be always specified to what precision level. E.g. if we are talking about equality in $P(m)$ we should symbolize it by $=_m$, while if talking about equality in $P(q)$ we should symbolize it by $=_q$.If we want to define these operation from those of the real numbers with infinite many decimal digits, then we will need the truncation function $[a]_x$ of a real number a , in the Precision level $P(x)$.

Then the operations e.g. in $P(n)$ with values in $P(m)$ $n < m$ would be

$$[a]_n + [b]_n =_m [a+b]_m \quad (\text{eq. 3})$$

$$[a]_n * [b]_n =_m [a*b]_m \quad (\text{eq. 4})$$

$$([a]_n)^{(-1)} =_m [a^{(-1)}]_m \quad (\text{eq. 5})$$

(Although, the latter definition of inverse seems to give a unique number in $P(m)$, there may not be any number in $P(m)$ or not only one number in $P(m)$, so that if multiplied with $[a]_n$ it will give 1. E.g. for $n=2$, and $m=5$, the inverse of 3, as $([3]_n)^{-1} = [1/3]_m = 0.33333$ is such that still $0.33333 \cdot 3 \neq 1$).

Nevertheless here we will not use at all the infinite or infinite many digits, but only finite many.

Such a system of double or triple precision digital real numbers, has closure of the linearly ordered field operations only in a specific local way. That is If a, b belong to the Local Lower precision, then $a+b, a*b, -a, a^{-1}$ belong to the Low precision level, and the properties of the linearly ordered commutative field hold: (here the equality is always in $P(m)$, this it is mean the $=_m$).

- 1) if a, b, c belong to $P(n)$ then $(a+b), (b+c), (a+b)+c, a+(b+c)$ belong in $P(m)$ and $(a+b)+c=a+(b+c)$ for all a, b and c in $P(m)$.
- 2) There is a digital number 0 in $P(m)$ such that
 - 2.1) $a+0=a$, for all a in $P(m)$.
 - 2.2) For every a in $P(n)$ there is some b in $P(m)$ such that $a+b=0$. Such a, b is symbolized also by $-a$, and it is unique in $P(m)$.
- 3) if a, b , belong to $P(n)$ then $(a+b), (b+a)$, belong in $P(m)$ and $a+b=b+a$
- 4) if a, b, c belong to $P(n)$ then $(a*b), (b*c), (a*b)*c, a*(b*c)$ belong in $P(m)$ and $(a*b)*c=a*(b*c)$.
- 5) There is a digital number 1 in $P(m)$ not equal to 0 in $P(m)$, such that
 - 5.1) $a*1=a$, for all a in $P(m)$.
 - 5.2) For every a in $P(n)$ not equal to 0, there may be one or none or not only one b in $P(m)$ such that $a*b=1$. Such b is symbolized also by $1/a$, and it may not exist or it may not be unique in $P(m)$.
- 6) if a, b , belong to $P(n)$ then $(a*b), (b*a)$, belong in $P(m)$ and $a*b=b*a$
- 7) if a, b, c belong to $P(n)$ then $(b+c), (a*b), (a*c), a*(b+c), a*b+a*c$, belong in $P(m)$ and $a*(b+c)=a*b+a*c$

Which numbers are positive and which negative and the linear order of digital numbers is precision levels $P(n), P(m), P(q)$ is something known from the definition of precision levels in the theory of classical real numbers in digital representation.

If we denote by $PP(n)$ the positive numbers of $P(n)$ and $PP(m)$ the positive numbers of $P(m)$ then

8) For all a in $PP(n)$, one and only one of the following 3 is true

- 8.1) $a=0$
- 8.2) a is in $PP(n)$
- 8.3) $-a$ is in $PP(n)$ ($-a$ is the element such that $a+(-a)=0$)

9) If a, b are in $PP(n)$, then $a+b$ is in $PP(m)$

10) If a, b are in $PP(n)$, then $a*b$ is in $PP(m)$

It holds for the inequality $a>b$ if and only if $a-b$ is in $PP(m)$

$a < b$ if $b > a$
 $a \leq b$ if $a < b$ or $a = b$
 $a \geq b$ if $a > b$ or $a = b$

and similar for $PP(m)$.

Similar properties as the ones from $P(n)$ to $P(m)$ hold if we substitute n with m , and m with q .

Also, the Archimedean property holds only recursively in respect e.g. to the local lower precision level $P(n)$.

In other words, if $a, b, a < b$ belong to the Local lower precision level $P(n)$ then there is n integer in the Low precision level $P(m)$ such that $a \cdot n > b$. And similarly for the precision levels $P(m)$ and $P(q)$.

The corresponding to the Eudoxus-Dedekind completeness in the digital real numbers also is relative to the three precision levels.

We define that two visible points A, B , are in contact or of zero distance $\text{distance}(A, B) = 0$, if and only if in their Cartesian coordinates they are at a face, at an edge or at a vertex successive. If this is so then there are invisible points A' belonging to A (see axioms of incidence) and B' belonging to B , so that $\text{distance}(A', B') \leq 1/(10^{2q})$. Two visible points in contact do not have in general the same Cartesian measures distance. The distance of the invisible points is defined from the coordinates of the invisible points in the precision level $P(q)$ of $R(n, m, q)$ from the standard formula of Euclidean distance, that is a Cartesian measure as in Definition 2.3.I.2 or with the Archimedean measures but the values are identical in the standard or low precision level $P(n)$.

In other words for every visible point A in the Low precision level, there are exactly two other points B_1, B_2 again in the Low precision level with $B_1 < A < B_2$, such that the distance between A and B_1 , and A, B_2 is zero in the Low precision level, and there is no other visible point C strictly between A and B_1 and A and B_2 . This can be derived also from the requirement that all possible combinations of decimal digits in the local lower, low and high precision levels are being used as numbers of the system of digital real numbers.

Sufficient Mutual inequalities of the precision levels (AXIOMS OF SEEMINGLY INFINITE AMONG RESOLUTIONS)

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, q .

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we feel safe to postulate such big differences.

1) If we repeat the operations of addition and multiplication of the linearly ordered commutative field starting from numbers of the local lower precision level $P(n)$, so many times as the numbers of the local lower precision level $P(n)$, then the results are still inside the low precision level $P(m)$. (This in particular gives that $(10^n)^{(10^n)} \leq (10^m)$). This

may also be expressed by saying that the 10^m is seemingly infinite compared to the 10^n

2) The largest error in the high precision level $P(q)$, which we may also identify as the smallest magnitude in the low precision level $P(m)$ in other words the 10^{-m} , will appear as zero error in the low precision level $P(n)$, even after additive repetitions that are as large as the cardinal number of points of the lower precision level $P(n)$. This is e.g. is guaranteed if $5n+2\log_2 m < m$ or rounded $6n < m$ (Where by \log we denote the logarithm with base 10) . The points in 1-dimensional geometry are $10^{4n+2\log_2 m}$ and if an error of order 10^{-m} is repeated so many times and still be less than 10^{-n} , then $10^{(4n+2\log_2 m)*10^{-m}} < 10^{-n}$, thus $5n+2\log_2 m < m$. For the Euclidean geometry cube, this requires that $10^{(12n+6\log_2 m)*10^{-m}} < 10^{-n}$ thus $13n+6\log_2 m < m$ or rounded $14n < m$. This may also be expressed by saying that the 10^{-m} is seemingly infinitesimal compared to the 10^{-n}

3) The smallest magnitude in the high precision level $P(q)$ in other words the 10^{-q} , will appear as zero error in the low precision level $P(m)$, even after additive repetitions as large as the cardinal number of points of the low precision level $P(m)$. This is e.g. guaranteed if $5m+2\log_2 q < q$ or rounded $6m < q$, and for Euclidean geometry applications $13m+6\log_2 q < q$ or rounded $14m < q$. This may also be expressed by saying that the 10^{-q} is seemingly infinitesimal compared to the 10^{-m}

If instead of three precision levels $P(n)$, $P(m)$, $P(q)$, we would introduce four precision levels (still another $P(r)$), with the same mechanism of recursive axioms, then we would denote it by $R(n,m,q,r)$ and we would call it a 4-precisions levels system of digital real numbers.

Two digital systems of Real numbers $R(n,m,q)$, $R(n',m',q')$ with $n=n'$, $m=m'$, $q=q'$ and the above axioms are considered isomorphic.

Two digital systems of Real numbers $R(m,n,p,q)$, $R(m',n',p',q')$ with $m=m'$, $n=n'$, $p=p'$, $q=q'$ and the above axioms are considered isomorphic.

§2.3 AN AXIOMATIC SYSTEM OF THE PHYSICAL OR DIGITAL BUT CONTINUOUS 3-DIMENSIONAL EUCLIDEAN GEOMETRY $E_3(n,m,q)$.

We have as initial concepts of objects

- a) The High resolution or precision points, or invisible points or atoms
- b) The Low resolution or precision points, or visible points or pixels.
- c) The Lower or standard precision level of measurements.

Remark 2.3.1

We introduce in the digital Euclidean geometry the next two types of points:

1) All visible points (or low precision level points) are finite in number. And of **non zero but minimum possible dimension in the single or Low precision**, but not in the High precision level. Between two visible points there is not always another visible point. The case of non-existence on intermediate points will be used in the concept of completeness up to some density or resolution and continuity of the space. Visible points are called visible in our usual material realizations of geometric figures because if we put our eyes close enough to the paper surface or screen where a line or a circle is drawn, we can see the point, while at a normal distance we cannot see the points but only the linear segment or circle arc. E.g. **pixels** of lines on the computer screen. Nevertheless the smallest magnitude of the standard precision level is by far larger than the visible points.

2) All invisible points or pixels or atoms (or high precision level points) are finite in number and of minimum dimension in the high precision level. Invisible points are called invisible in our usual material realizations of geometric figures because no matter how close we may put our eyes to the paper surface or screen where a line or a circle is drawn, we cannot see these points. E.g. **atoms** of a metallic material surface. The main reason of introducing here the invisible points is so as to have at least two alternative systems of measures (lengths, areas, volumes), that of Archimedes and that of Cartesius. The full significance of the invisible points will become apparent only when introducing digital curved space like digital Riemannian space or manifolds, which is not in the scope of the current paper.

For the AXIOMATIC DIGITAL OR PHYSICAL EUCLIDEAN GEOMETRY we do not intent to use the system of Real numbers as it is defined as the minimal complete linearly ordered commutative field (in the order to topology), but instead all measurements of linear geometric segments lengths, areas, volumes etc will be done with a Low Precision level and a high precision level of real numbers. The definition of a PRECISION LEVEL $P(n,m)$ where n, m are natural numbers , is that it is the set of all real numbers that in the decimal representation have not more than n decimal digits for the integer part and not more than m digits for the decimal part. In other words as sets of real numbers it is a nested system of lattices each one based on units of power of 10, and as union a lattice of rational numbers with finite many decimal digits. We could utilize other bases than 10 e.g. 2 or 3 etc, but for the sake of familiarity with the base 10 and the 10 fingers of our hands we leave it as it is.

Whenever we refer to a real number x of as (minimal in precision levels) system of real numbers r we will always mean that x belongs to the local lower precision level and that the system has at least three precision levels with the current axioms.

Whenever we write an equality relation we must specify in what precision level it is considered. The default precision level that a equality of numbers and geometric elements of geometric figures like length, area and volume, is considered to hold, is the standard or lower precision level

c) The visible lines

d) The visible planes

c) We may apply finite sets only on the points of the digital Euclidean geometry

- d) And of course we may apply digital formal logic to make arguments and proofs.
- e) Besides the *congruence* as equivalence relations we have the next initial relations among visible or invisible elements.

An invisible point A *belongs* to a visible point B, denoted by $A \in B$

A visible point A *belongs* to a line L or , denoted by $A \in L$

A visible point A *belongs* to a plane P or , denoted by $A \in P$

A line L *belongs* to a Plane P, denoted by $L \in P$

A visible point A is *between* two visible points B, C.

We design 7 groups of axioms

- 1) Of finite decimal coordinates
- 2) Of lengths, areas and volumes
- 3) Of Incidence
- 4) Of Order
- 5) Of Congruence
- 6) Of Continuity
- 7) Of Resolution

In the next axioms the term point if we do not specify that it is invisible, refers to visible or low precision point. It has the minimum no-zero size (length) in the Low resolution real numbers that can be constructed on a geometric line, by the Cartesian coordinates as tiny cube, as we shall see. We use the axioms of Hilbert, but we modify them and add more axioms.

1 Axioms of finite decimal coordinates of points

- 1) Every invisible point P has 3 numerical coordinates $P(x_1)$, $P(x_2)$, $P(x_3)$ that are rational numbers that in decimal notation have finite many digits so many as the definition of the High measurement precision.
- 2) Every visible point P has 3 numerical coordinates $P(x_1)$, $P(x_2)$, $P(x_3)$ that are rational numbers that in decimal notation have finite many digits so many as the definition of the Low measurement precision.
- 3) The density of the visible is uniform through-out the spherical space. For every triad of decimal rational numbers of the low precision level $P(m)$, there is an invisible point with these coordinates.
- 4) The density of the invisible is uniform through-out the spherical space. For every triad of decimal rational numbers of the high precision, there is an invisible point with these coordinates.

Remark 2.3.I.1

The visible and invisible points due to their orthogonal and rectangular coordinates may be considered **tiny little balls and cubes respectively**. If we would like to make a model of the present axioms of digital Euclidean geometry from the idea of digital decimal coordinates, so that it would be proved a consistent system of axioms, then we should start to define lines and planes as finite sets of points satisfying the standard linear equations of lines or planes of analytic cartesian geometry both in the high and low resolution and **up-to the corresponding precision level**. That is they consist only from finite many points (although seemingly infinite as we shall see) *We notice that we need not claim 1-1 and onto correspondence of the digital coordinates with the visible points.* Actually either we could assume an 1-1- onto correspondence of points and coordinates but then the congruence of figures would not be an 1-1- onto correspondence of points, or assume that the congruence would be a 1-1 onto correspondence of points but then the coordinates of (visible) points would not be 1-1- and onto. **Here the axioms leave the two possibilities (analytic or synthetic geometry) open.** If we would do assume 1-1 onto correspondence of visible points then a diagonally positioned linear segment at 45 degrees, might have less points than a horizontal positioned linear segment of equal length. For the case of cubic points of course we may define their cross sectional length, area and volumes as $10^{(-m)}$, $10^{(-2m)}$, $10^{(-3m)}$ for the visible and $10^{(-q)}$, $10^{(-2q)}$, $10^{(-3q)}$ for the invisible points.

Definition 2.3.I.1 of the local lower LLS and low resolution LS finite sphere or space.

There is a central visible point O of the space with coordinates (0,0,0) such that all the visible and invisible points of the space that have distance at most ω of it, where ω belongs in the $P(n,n)$, and $\omega=10^n$ is called the local lower resolution space or in short LLS. If we take the corresponding sphere from the center with coordinates (0,0,0) with all visible and invisible points with radius $\Omega=10^m$, is called the Low resolution space or LS.

Definition 2.3.I.2 Cartesian measures of length, areas and volumes

From the elementary Cartesian analytic geometry, we may define the distance of two points $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$ through the Pythagorean or Euclidean formula of distance (norm with rule of parallelogram). We may similarly define the area of three points not lying in a line, through the well known formula that is involving the determinant and their coordinates, and similarly for the 3-dimensional simplex or tetrahedron. Then we may define the area of finite sets of points that are in contact (see Definition xyz below) by triangulation with non-overlapping triangles. Similarly define the volume of finite sets of points that are in contact or connected (see **Definition 2.3.VI.1** below) by simplicialization with non-overlapping tetrahedral (simplexes). Such measures of area, and volumes of finite sets of points that are in contact (connected) we call in the next the **Cartesian measures of areas and volumes**.

Remark 2.3.I.2

Notice that in the synthetic axioms that we introduce here we do not impose geometric structure to the invisible points, but only to the visible points. In other words we do not define invisible lines and invisible planes. But we could as well do so, from the coordinates of the invisible points and the standard equations of lines and planes in the analytic geometry.

II Axioms of Archimedes measures of length, area, and volumes and compatibility with the coordinates

- 1) Every invisible point P , as belonging to a line L , has a non-zero length $l(P)$ which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.
- 2) Every invisible point P , as belonging to a plane E , has a non-zero area $a(P)$ which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.
- 3) Every invisible point P , has a non-zero volume $v(P)$ which is a rational number that in decimal notation has finite many digits so many as the definition of the High measurement precision while it is zero in the low measurement precision.
- 4) For every visible point P , there are a sets $VIn(P)$ of invisible points of it, so that volume of the visible point is defined as the sum of the lengths, of the volumes of the invisible points of the above sets correspondingly. These sets $VIn(P)$ for the volume are not unique for the point P , but all the alternative such sets give the same values volume of the point, and the same for all visible points.
- 5) For every visible point P , there are a sets $LIn(P)$, $AIn(P)$ of invisible points of it, so that the length and area of the visible point is defined as the sum of the lengths, of the lengths and areas of the invisible points of the above sets correspondingly. These sets $LIn(P)$, $AIn(P)$ and also their values for the lengths and areas are not unique for the point P , but depend and their values depend also, on the linear segment or plane correspondingly that the point P is considered that it belongs.
- 6) The length of linear segment is defined as the sum of the lengths of its visible points that in their turn define a partition of the invisible points of the segment. The length of the unit segment OA , with coordinates of O , $(0,0,0)$ and $(0,0,1)$ is equal to 1. (similarly by cyclic permutation of the coordinates and the other unit lengths from O).
- 7) The area of figure (set of visible point) is defined as the sum of the areas of all of its visible points that in their turn define a partition of the invisible points of the figure. The length of the unit square $OA-OB$, with coordinates of O , $(0,0,0)$ and $(0,0,1)$, $(1,1,1)$, $(1,0,0)$ is equal to 1 (similarly by cyclic permutation of the coordinates and the other unit squares)
- 8) The volume of a figure is defined as the sum of the volumes of its visible points that in their turn define a partition of the invisible points of the figure. The volume of the unit cube, with coordinates $(0,0,0)$ and $(0,0,1)$, $(1,1,1)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,1,1)$, $(1,0,1)$ is equal to 1
- 9) Congruent sets of points (of the LLS) have length, area, and volumes either in the Cartesian measures or the Archimedean measures, correspondingly that differ only by errors that are zero in the standard or low precision level. Furthermore they remain zero error, even if are repeated additively as many times as the cardinal number of elements of the low precision level $P(n)$.
- 10) For a finite connected set of visible points (of the LLS) the difference of its measure in the Cartesian measure and the Archimedean measure, correspondingly differ only by error that is zero in the standard or low precision level. Furthermore it remains zero error, even if it is repeated additively as many times as the cardinal number of elements of the low precision level $P(n)$.

Remark 2.3.II.1

Both types of measures Cartesian measures and Archimedes measures of, areas and volumes have the additive property of disjoint unions of finite sets of points in contact (connected sets of points see **Definition 2.3.VI.1**).

After the above axioms and definitions of such measures , it can be shown that the lengths, areas and volumes, are set functions l, a, v of sets of visible points, (but also of invisible points) ,with values in the positive Low precision level of decimal numbers, with the additive property of disjoint unions: (By \cap we denote the intersection and by \cup the union of sets)

$$l(A \cup B) =_n l(A) + l(B) - l(A \cap B)$$

$$a(A \cup B) =_n a(A) + a(B) - a(A \cap B)$$

$$v(A \cup B) =_n v(A) + v(B) - v(A \cap B)$$

$$AB \text{ congruent to } A'B' \text{ then } d(AB) =_n d(A'B')$$

Furthermore more properties for linear segments AB, BC, AD hold like

$$l(AB) =_n l(BA)$$

$$l(AC) \leq_n l(AB) + l(BC)$$

Also angular measures $\text{ang}()$ again with values in positive Low precision level are defined , through areas of circular sectors of unit circular discs or of the length of the corresponding circular segment of unit circular discs.

Axioms of Sufficient many points (visible and invisible) and mutual inequalities of the precision levels for length, areas and volumes. (Axioms of the seemingly infinite of the points of the geometry)

We impose also axioms for the sufficiently large size of the high precision level relative to the other two, and the sufficient large size of the low precision level relative to the local lower precision level. That is for the mutual relations of the integers m, n, q .

It may seem that these differences of the resolution or the precision levels are very severe and of large in between distance, and not really necessary. It may be so, as the future may show. But for the time being we feel safe to postulate such big differences.

The axioms are essential those of the digital real numbers $R^3(n,m,q)$ with numbers 1) 2) 3) as coordinates of the visible and invisible points of the digital Euclidean space $E_3(n,m,q)$.

In all the next axioms we start from visible points and geometric elements of the Local Lower resolution space LLS (which belongs in the coordinates cube $P(n,m)$), and we result in to the Lower resolution space LS (which belongs in the coordinates cube $P(m,m)$), because of the recursive and not absolute closeness in the digital real numbers $R(n,m,q)$. Angles in LLS are essentially circular sectors of length of radius equal to one unit.

III. Incidence

Terminology convention

- 1) In all the axioms of incidence, order, and congruence below, when we say and write the term “point” without specifying it to be an invisible point we will mean a visible point.
- 2) when we say and write the term “angle”, we will mean, a circular sector of unit radius.
- 3) when we say and write the term “line”, we will mean, a linear segment starting and ending at points of the Local Lower Sphere (LLS). The ending visible points of the line do not count as (interior) visible points of the line
- 4) when we say and write the term “plane”, we will mean, a circular disc , with boundary circle at surface points of the Local Lower Sphere (LLS). The boundary visible points of a plane do not count as (interior) points of the plane.

1. For every two points A and B (in LLS) there exists a line a (in LLS) that contains them both. We write $AB = a$ or $BA = a$. Instead of “contains,” we may also employ other forms of expression; for example, we may say “ A lies upon a ”, “ A is a point of a ”, “ a goes through A and through B ”, “ a joins A to B ”, etc. If A lies upon a and at the same time upon another line b , we make use also of the expression: “The lines a and b have the point A in common,” etc.
2. For every two points (in LLS) there exists no more than one line (in LLS) that contains them both; consequently, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.
3. There exist at least two points on a line (in LLS) . There exist at least three points that do not lie on a line.
4. For every three points A, B, C (in LLS) not situated on the same line there exists a plane α (in LLS) that contains all of them. For every plane (in LLS) there exists a point which lies on it. We write $ABC = \alpha$. We employ also the expressions: “ A, B, C , lie in α ”; “ A, B, C are points of α ”, etc.
5. For every three points A, B, C (in LLS) which do not lie in the same line, there exists no more than one plane (in LLS) that contains them all.
6. If two points A, B of a line a (in LLS) lie in a plane α (in LLS) , then every point of a lies in α . In this case we say: “The line a lies in the plane α ,” etc.
7. If two planes α, β (in LLS) have an (interior) point A in common, then they have at least a second (interior) point B in common.
8. There exist at least four points (in LLS) not lying in a plane.
9. For every invisible point A , there is a visible point B , so that A belongs to B .
10. Two invisible points A, B belong to the same visible point C is an equivalence relation among the invisible points.

IV. Order

1. If a point B (of LLS) lies between points A and C (of LLS) , B is also between C and A , and there exists a line containing the distinct points A, B, C .
2. Of any three points situated on a line (of LLS) , there is no more than one which lies between the other two.
3. **Pasch's Axiom**: Let A, B, C be three points (of LLS) not lying in the same line and let a be a line (of LLS) lying in the plane ABC and not passing through any of the

points A, B, C . Then, if the line a passes through a point of the segment AB , it will also pass through either a point of the segment BC or a point of the segment AC .

V. Congruence

- d) If A, B are two points on a line a , (of LLS), and if A' is a point upon the same or another line a' (of LLS), then, upon a given side of A' on the straight line a' , we can always find a point B' (of LS) so that the segment AB is congruent to the segment $A'B'$. We indicate this relation by writing $AB \cong A'B'$. Every segment is congruent to itself; that is, we always have $AB \cong AB$.

We can state the above axiom briefly by saying that every segment can be *laid off* upon a given side of a given point of a given straight line in at least one way (Always starting from the sphere LLS and resulting in the larger sphere LS).

2. If a segment AB (of LLS) is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \cong A'B'$ and $AB \cong A''B''$, then $A'B' \cong A''B''$.

Remark 2.3.V.1 Limited transitivity?

As we noticed in the axioms II.9 -II.10, congruent linear segments, and angles have equal measures with zero error in the standard precision level $P(n)$, but non-zero in the low precision level $P(m)$. Therefore in the transitivity of the congruence in the previous axiom, the error may be added and propagated. Still by the same axiom II.9, the repetition may be many times as the elements of the standard precision level $P(n)$ and still be zero. Therefore we know that the transitivity of the congruence will still hold up to as many times as the number of the elements of the standard precision level $P(n)$. Now if in the meta-mathematics of the formal logic we utilize the digital natural numbers $N(\omega)$ with $\omega=n$, then certainly even the largest allowable number of formal propositions and therefore repetitions of the transitivity of congruence will not lead to a non-zero error in the standard precision level. Therefore we may accept that the transitivity of the congruence is valid for all practical applications, although theoretically it is limited. Another way to keep the transitivity of the congruence is the next: We may define when modelling this axiomatic system to test the consistency, the congruence with a standard types transformation of the coordinates (e.g. isometric transformations). Then as the composition of two isometries is an isometry, and the error of an isometry can be uniformly bounded for all isometries so as to be zero in the low precision level, the transitivity of the congruence is valid from the point of view of the standard precision level.

3. Let AB and BC be two segments of a line a (of LLS) which have no points in common aside from the point B , and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another line a' (of LS) having, likewise, no point other than B' in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, we have $AC \cong A'C'$.
4. Let an angle $\text{ang}(h, k)$ be given in the plane α (of LLS) and let a line a' be given in a plane α' (of LLS). Suppose also that, in the plane α' , a definite side of the straight line a' be assigned. Denote by h' a ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one ray k' (of LS) such that the angle $\text{ang}(h, k)$, or $\text{ang}(k, h)$, is congruent to the angle $\text{ang}(h', k')$ and at the same time all interior points of the angle $\text{ang}(h', k')$ lie upon the given side of a' . We express this relation by means of the notation $\text{ang}(h, k) \cong \text{ang}(h', k')$.
5. If the angle $\text{ang}(h, k)$ (of LLS) is congruent to the angle $\text{ang}(h', k')$ and to the angle $\text{ang}(h'', k'')$, then the angle $\text{ang}(h', k')$ is congruent to the

angle $\text{ang}(h'', k'')$; that is to say, if $\text{ang}(h, k) \cong \text{ang}(h', k')$ and $\text{ang}(h, k) \cong \text{ang}(h'', k'')$, then $\text{ang}(h', k') \cong \text{ang}(h'', k'')$.

6. If, in the two triangles ABC and A'B'C' (of LLS) the congruencies $AB \cong A'B'$, $AC \cong A'C'$, $\text{ang}(BAC) \cong \text{ang}(B'A'C')$ hold, then the congruence $\text{ang}(ABC) \cong \text{ang}(A'B'C')$ holds (and, by a change of notation, it follows that $\text{ang}(ACB) \cong \text{ang}(A'C'B')$ also holds).

VI. Continuity and Completeness up to some density or resolution, relative to the digital real numbers $R(n,m,q)$.

The corresponding to the *Eudoxus-Cartesius-Dedekind, completeness* also is relative to the three precision levels of $R(n,m,q)$.

Definition 2.3.VI.1

We define that two visible points A, B, are in contact or of zero distance $\text{distance}(A,B)=0$ in $P(n)$, if and only if in their Cartesian coordinates they are at a face, at an edge or at a vertice successive. If this is so then there are invisible points A' belonging to A (see axioms of incidence) and B' belonging to B, so that $\text{distance}(A'.B') \leq 1/(10^{2q})$. Two visible points in contact do not have in general the same Cartesian measures distance. The distance of the invisible points is defined from the coordinates of the invisible points in the precision level $P(q)$ of $R(n,m,q)$ from the standard formula of Euclidean distance, that is a Cartesian measure as in Definition 2.3.I.2 or with the Archimedean measures but the values are identical in the standard or low precision level $P(n)$.

1. **Axiom of Digital Continuity and Completeness:** For every non-ending visible point A of LLS, of a linear segment a, there are exactly two other divisible points B1, B2 on a in LS, and with $B1 < A < B2$, such that the distance between A and B1, and A, B2 is zero, and there is no other visible point C strictly between A and B1 and a and B2. This can be derived also from the requirement that all possible combinations of decimal digits in the low and high precision levels are being used as numbers of the system of digital real numbers and correspond to visible and invisible points.

Remark 2.3.VI.1 : *An alternative way that we could formulate the completeness of points is the next.* An extension of a set of visible points on any line, plane and the space, with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-VIII based on the given density of Coordinates in $R(n,m,q)$, is impossible. In short we cannot add more visible points relative to the Low precision level of measurements and coordinates, and the same for the visible points and low precision level of coordinates. This comes also from the axiom of the density-completeness of all possible but finite many coordinates of points in $R(n,m,q)$.

VII. Axioms of Resolution or of Density

These axioms are of the same nature as the corresponding axioms of the digital natural numbers, and multi-precision digital real numbers, and the axioms 1,2,3 of the digital real numbers.

1. **Axiom of sufficient high resolution or density** . Let a line a passing from the center of the space O and the units of measurements OA on it, and let $\omega(a)$, $\Omega(a)$, denote the finite cardinal number which is the cardinal number of visible and invisible points that belong to a . Estimates of them are $\omega(a)=10^{(4m+2\log 2)}$ and $\Omega(a)=10^{(4q+2\log 2)}$ where by \log we denote the logarithm with base 10. And let $\omega(n)$ be the size of the model of the natural numbers constructed on the line a through congruence and the above axioms. An estimate of it is $\omega(n)=10^{(2n+\log 2)}$ Then it holds that

$$\omega(n) \leq \omega(a) \text{ or } 10^{(2n+\log 2)} \leq 10^{(4m+2\log 2)}$$

(Strong version of the axiom $2^{\omega(n)} \leq \omega(a)$)

$$\omega(a) \leq \Omega(a) \text{ or } 10^{(4m+2\log 2)} \leq 10^{(4q+2\log 2)}$$

(Strong version of the axiom $2^{\omega(a)} \leq \Omega(a)$)

2. . Let $\omega(S)$, $\Omega(S)$, denote the finite natural numbers which are the cardinal numbers of visible and invisible points that can belong to the spherical S 3-dimensional space. And let also $\omega(P)$ be the cardinal number of invisible points that any visible point may contain . (From the group II of axioms about lengths, areas and volumes and axioms 8,9,10, we have an estimate that $\omega(S) \leq 10^{(12m+6\log 2)}$ and $\Omega(S) \leq 10^{(12q+6\log 2)}$ where by \log we denote the logarithm with base 10, and m and q are the orders of the precision level of the visible and invisible points respectively and $\omega(P) \leq 10^{(q-m)}$)

3. Then it holds that

- a) Any number of visible points of the total spherical space is less than any number of invisible points that a visible point may contain . In particular

$$(\omega(S)) \leq \omega(P) \text{ or } 10^{(12m+6\log 2)} \leq 10^{(q-m)}$$

Strong version of the axiom $2^{(\omega(S))} \leq \omega(P)$).

Remark 2.3.VII.1 The axiom must guarantees that lengths, areas and volumes that are defined by summing the corresponding values of the invisible points, will have in general total errors zero in the low precision, and that the failure if the transitivity of relations congruence due to their limited character can be avoided to occur, with sufficient high resolution of invisible points relative to the size of the total space and our repetitive construction in it.

- b) The diameter of the total spherical 3--dimensional space in integer number of units of length denoted by $\omega(n)$,is less that any maximum number of visible points that the spherical space may contain. An estimate of $\omega(n)$ is $\omega(n)=10^{(2n+\log 2)}$. In particular

$$\omega(n) \leq \omega(S) \text{ or } 10^{(2n+\log 2)} \leq 10^{(12m+6\log 2)}$$

(Strong version of the axiom $2^{\omega(n)} \leq \omega(S)$).

Remark 2.3.VII.2 The axiom guarantees that a lattice of points with say integer coordinates will always be by far less dense than the lattice of visible points.

Remark 2.3.VII.3 Notice that the inequalities of the current axioms of resolution are stronger than those of axioms 1,2,3 of the digital real numbers.

Remark 2.3.VII.4.

Notice that we did not postulate anything similar to the **axiom of parallel lines** of Euclid! One reason is that the digital lines are eventually linear segments and do not extend to infinite, as

this in the current setting would make them have infinite many points. Furthermore for this reason there are more than one linear segment passing from a point outside another linear segment which do not have any point in common. But this of course does not make the digital geometry a **Lobachevskian or hyperbolic** geometry. Still in the low precision level there would be only one linear segment that passing from a point outside a linear segment so that the angles from a third crossing both linear segment have sum exactly equal to 180 degrees. The main reason that we did not postulate the axiom of parallels is that in the digital Euclidean geometry the properties of such parallels in the low precision level are deduced from the axioms of the coordinates. Furthermore they are already non-unique in the low precision level $P(m)$, but only in the lower or standard precision level.

Remark 2.3.VII.4

Any digital space $E_3(n,m,q)$ is determined essentially from the integer parameters n,m,q of the corresponding digital system of real numbers $R(n,m,q)$ which is used as coordinate system. To have that any two finite models of $E_3(n,m,q)$ are isomorphic, one has to define and model appropriately the incidence, order and congruence at first before defining the appropriate form of isomorphism of the models of $E_3(n,m,q)$.

§ 3 Conclusions

The axiomatic system of the digital Euclidean geometry $E_3(n,m,q)$ may seem complicated and elaborate compared to the Hilbert's axioms of the classical Euclidean geometry. But as we remarked from the beginning the cost of the simple Hilbert's axioms with infinite many points is paid later with overwhelming complexity, and many non-intuitive paradoxes like Hilbert's 3rd problem, and Banach-Tarski paradox. In addition the theory of the measures of areas and volume that require integration and limits is very complex. In contrast in the digital Euclidean geometry we may start with elaborate axioms, but later the theory of measures of areas and volumes is very simple and intuitive!

E.g. The areas of circular discs are calculated in the Cartesian measure of area, by triangulation with non-overlapping triangles of all the (finite many) points of the circular disc! And in the Archimedean measure of area the calculation is even simpler as the sum of the areas of the (finite many) points of the circular discs that are tiny rectangles. Integration is simple finite sums.

Similarly other elementary or non-elementary theorems can have easier proofs in the digital Euclidean geometry! We even have a new type of proofs not possible in the classical Euclidean Geometry: **Proof by induction on the number of (visible) points!** It is quite interesting to reformulate classical non-solved so far problems in the context of digital Euclidean geometry e.g. Riemann hypothesis of the roots of the zeta function, and try to prove it by induction on the number of visible points!

References

Banach-Tarski paradox:

[*Banach, Stefan; Tarski, Alfred \(1924\). "Sur la décomposition des ensembles de points en parties respectivement congruentes" \(PDF\). *Fundamenta Mathematicae* \(in French\). 6: 244–277.*](#)

Boltianskii V. (1978) "Hilbert's 3rd problem" J. Wesley & Sons 1978

Euclid The 13 books of the Elements Dover 1956

Hilbert D. (1977) "Grundlagen der Geometrie" Taubner Studienbucher 1977

Hausdorf paradox:

Hausdorff Felix (1914). "[Bemerkung über den Inhalt von Punktmengen](#)". *Mathematische Annalen*. **75**: 428–434. [doi:10.1007/bf01563735](#)

Moise E. E. (1963) "Elementary geometry from an advanced standpoint" Addison –Wesley 1963

Von Neumann paradox:

On p. 85 of: [von Neumann, J. \(1929\), "Zur allgemeinen Theorie des Masses" \(PDF\), *Fundamenta Mathematicae*, **13**: 73–116](#)

Pawlikowski, Janusz (1991). "[The Hahn–Banach theorem implies the Banach–Tarski paradox](#)" (PDF). *Fundamenta Mathematicae*. **138**: 21–22.

Wilson, Trevor M. (September 2005). "A continuous movement version of the Banach–Tarski paradox: A solution to De Groot's problem". *Journal of Symbolic Logic*. **70** (3): 946–952. [JSTOR 27588401](#). [doi:10.2178/jsl/1122038921](#).

APPENDIX

A DIALOGUE OF THE IMMORTALS OF GEOMETRY

(This is a fictional dialogue of the immortally famous mathematicians of the past that have significantly contribute to the mathematics of the Euclidean Geometry and comment on the new axiomatic system of the **Axiomatic Digital Euclidean Geometry**. The list is only indicative, not exhaustive.

THE DIALOGUE OF THE IMMORTALS MATHEMATICIANS ON THE OCCASION OF THE NEW AXIOMS OF THE AXIOMATIC DIGITAL EUCLIDEAN GEOMETRY BY NEWCLID

NEWCLID after presenting the immortals the new axioms of the Axiomatic Digital Euclidean Geometry, invites them in a free discussion about it.

NEWCLID, is an individual representing the collective intelligence of the digital technology but also of mathematics of the 21st century.

The participants of the discussion are the next 20.

1. Pythagoras
2. Aristarchus from Samos
3. Eudoxus
4. Euclid
5. Democritus
6. Archimedes
7. Apollonius

8. Copernicus
9. Galileo
10. Newton
11. Leibnitz
12. Cartesius
13. Cauchy
14. Dedekind
15. Weierstrass
16. Hilbert
17. Riemann
18. Cantor
19. von Neumann
- And a mortal:
20. Newclid

NEWCLID:

Welcome honourable friends that you have become immortals with your fame and contribution in the creation of the science and discipline of Mathematics among the centuries on the planet earth!

Now that you have watched my presentation of the axiomatic system of the new Axiomatic Digital Euclidean Geometry, I would like to initiate a discussion that will involve your remarks and opinions about it. Who would like to start the conversation?

PYTHAGORAS:

Thank you Newclid for the honour in gathering us together. I must express that I like the new approach of the Axiomatic Digital Euclidean Geometry, that as you say is a resume of what already the beginning of the 21st century in the earthly Computer Science has realized through software in the computer operating systems and computer screens and monitors.

I must say that I like the approach! In fact, I was always teaching my students that the integer natural numbers are adequate for creating a mathematical theory of the geometric space! One only has e.g. to take as unit of measurement of lengths, the length of a visible points and all metric relations in the low precision level of the figures , including the Pythagorean theorem, become relations of positive integer numbers, or solutions of Diophantine equations! But at that time no such detailed and elaborate axiomatic system, neither a well accepted concept that matters consists from atoms, was available in the mathematicians of the ancient Greece, Egypt or Babylon.

EUCLID: I am impressed Newclid for your elaborate axiomatic system. The axioms that I had gathered in my books with title “Elements” for the Euclidean geometry in my time were much less! I would like to ask you a question that puzzles me since I watched your presentation: How do we know that the more than 20 axioms of Hilbert about my Euclidean geometry, or your axioms of the Digital Euclidean Geometry are enough to prove all that we want to prove?

NEUCLID: This is a very good question, Euclid! Maybe our friend here Hilbert might like to answer it!

HILBERT: Well my friends, this is a question that I posed also to myself when writing my more than 20 axioms of the classical Euclidean Geometry! I have not read any such proof! It is by the rule of the thumb as they say! I collected them , through my experience and according to the theorems of Euclidean geometry till my time but also according to the standards of proofs in my time!

NEWCLID: What do you mean Hilbert? That maybe in the future we might discover that we need more axioms?

HILBERT: Exactly! That is what the History teaches us!

CARTESIUS: If I may enter the discussion here, I propose that a proof that the axioms of Hilbert are enough could be proving from the Hilbert axioms, the basic numerical axioms of my Analytic Geometry with coordinates! This, in my opinion, would be a proof!

NEWCLID: Very good idea Cartesius! This in my opinion suggests also that my axioms of the Digital Euclidean Geometry, that involve coordinates too, most probably are enough. But I am almost sure that they are not independent and some of them can be proved from the rest. Still I cannot claim that I have any proof, more than just experience and a rule of the thumb, that my axioms are adequate! Maybe in the future I may discover that I need a couple more!

ARISTARCHUS: May I ask Newclid if your concept of digital Euclidean space which is in the shape of a spherical ball is intended to be large enough so as to allow e.g. astronomical calculations like my calculations of the size of earth, moon , sun and their mutual distances?

GALILEO: I have the same question Newclid! Good that ARISTARCHUS asked it!

COPERNICUS: Me too Euclid!

NEWCLID: Certainly ARISTARCHUS! The spherical digital Euclidean space can be so large so as to include all the observable galaxies of the astronomical world as we know it! But it can be also small as a planet to accommodate for planetary calculations only too! The axioms do not specify how large or small it should be!

ARCHIMEDES: I like your axiomatic system and concept of space Newclid! It is as my perceptions! Actually my experimental work with solids that I was filling with sand or water to make volume comparisons is just an experimental realisation of your axioms of volumes through those of the points and finite many points!

DEMOCRITUS: Bravo Newclid! Exactly my ideas of atoms! Actually as in my theory of atoms, the water is made from finite many atoms, the volume experiments of Archimedes with water is rather the exact realisation of your axioms of volume through that of the invisible points! Here the atoms of the water are invisible, while the granulation of the sand may resemble your axioms of the visible points!

NEWCLID: Thank you, my friends! I agree!

EUDOXOS: Well in your digital Geometry Newclid, my definition of the ratio of two linear segments which is the base of the complete continuity of the line is not that critical in your axiomatic system, although I think that it still holds!

DEDEKIND: As I reformulated the idea and definition of equality of ratio of linear segments of Eudoxus, as my concept of Dedekind cuts about the completeness of continuity of the real numbers, I must say the same thing as Eudoxus!

WEIESSTRASSE: The same with my definitions of convergent sequences though the epsilon-and-delta formulation! They still hold in your approach!

APOLLONIUS: I would like to know Newclid, if my theory of circles in mutual contact would be provable as I know it in the classical Geometry of Euclid. E.g. if two circles are in contact externally, are they in contact in one only point, as I know it, or in more than one point in your geometry?

NEWCLID: I think APOLLONIUS that in my geometry what you observe in the real world is also more or less what is provable with the visible points. For sure two circles in contact even if they have only one common visible point they will have many common invisible points, all those inside the common visible point! But I am afraid that they may even have more than one common visible point , depending on their size and the definition of circle intersecting circle or line. The reason is that it may happen that

two different visible points have an error of distance from the centers of the circles which is zero in the Low precision although not zero in the High precision. Still one may gibe an appropriate definition where one of them has a maximal property thus a unique point of contact.

GALILEO: I would like to ask Newclid if your concept of invisible and visible points could be large enough and both of them visible, so as to account for the real planet earth (which is not a perfect sphere) as if a perfect sphere!

NEWCLID: Well **GALILEO**, the initial intention is the invisible points are indeed small enough to be invisible. But as you understand what is visible and invisible is not absolute and depends at least on the closeness of our eye. Theoretically one could conceive a model of my axioms where both visible and invisible points are visible and even large!

LEIBNITZ: I want to congratulate you Newclid for your approach! In fact my symbols of infinitesimal dx in my differential calculus suggest what I had in mind: A difference $dx=x_2-x_1$ so that it is small enough to be zero in the Low precision but still non-zero in the High precision! Certainly a finite number!

NEWTON: I must say here that the Leibnitz idea of infinitesimal as a finite number based on the concepts of Low and High precision is not what I had in my mind when I was writing about infinitesimals. That is why I was calling them fluxes and symbolized them differently. The theory of null sequences of numbers (converging to zero) of Cauchy and Weierstrasse is I think the correct formulation of my fluxes. So that such fluxes fit to a Geometry as Euclid and Hilbert was thinking it and not as Newclid formulated here. Still for physical applications I thing that Newclid's concept of space with finite many points only is better and closer to the physical reality! I was believing in my time that matter consist from finite many atoms , but I never dared to make a public scientific claim of it, as no easy proof would convince the scientist of my time!

I want to ask an important question to Newclid: Is your differential and integral calculus based on three levels of precision more difficult or simpler that the classical differential and integral calculus based on limits and infinite many real numbers?

NEWCLID: Well Newton thank you for the good words! Actually I have not yet developed all of a differential and integral calculus based on digital real numbers and digital Euclidean geometry, therefore the question runs ahead of our presentation. But I have thought myself about it, and I can remark the next: A differential and integral calculus based on three levels of precision is certainly less complicated than (and also not equivalent to) the classical calculus with infinite sequences or limits. But a differential and Integral calculus of 3 ,4 or more precision levels is by far more complicated than the classical differential and Integral calculus. Only that this further complication is a complexity that does correspond to the complexity of the physical material reality, while the complexities of the infinite differential and integral calculus (in say Lebesgue integration theory or bounded variation functions etc) is a complexity rather irrelevant to the physical material complexity.

CARTESIUS: I want to congratulate you Newclid for your practical , finite but axiomatic too approach for the physical space, and the introduction of my idea of rectangular coordinates right from the beginning of the axioms! I have a question though! You correspond points to coordinates, but they also have volume. If we think of a cubic lattice with its points and coordinates, which of the 8 cubes that surround the point you assume as voluminous point in your geometry?

NEWCLID: If I understand your question well **CARTESIUS**, it is the cube that its left upper corner is the point. Thanks for your praise!

CAUCHY: I wish I had thought of such an axiomatic system of space with finite many only points, and the concept of infinitesimals as Leibnitz mentioned with your Local, Low

and High precision levels! But there is a reason for this! Your axioms are much more elaborate and complicated than the Hilbert axioms of Euclidean Geometry!

NEWCLID: Indeed CAUCHY! But later the proofs of many other theorems, on areas, volumes and even derivatives, will become much simpler!

HILBERT: I like your brave and perfect approach Newclid! No infinite in your axioms so as to have easy physical applications, as nothing in the physical material reality is infinite. Congratulations! I am glad that my axioms of the classical Euclidean Geometry were of a good use to your work.

Von NEUMANN: I like too your axiomatic system Newclid! I believe that I could easily make it myself, except at that time I was busy in designing a whole generation of computers! I believe your work is a direct descendant of my work on computers. As you said your ideas came from software developers in the operating system of a computer!

NEWCLID: Indeed von Neumann! Thank you!

CANTOR: Pretty interesting your axiomatic system Newclid! But what is wrong with the infinite? Why you do not allow it in your axiomatic system? I believe that the infinite is a legitimate creation of the human mind! Your Digital Euclidean Geometry lacks the magic of the infinite!

PYTHAGORAS: Let me, Newclid, answer this question of CANTOR! Indeed CANTOR the human mind may formulate with a consistent axiomatic way what it wants! E.g. an axiomatic theory of the sets where infinite sets exist! And no doubt that the infinite is a valuable and sweet experience of the human consciousness! But as in the physical material reality there is nowhere infinite many atoms, mathematical models that in their ontology do not involve the infinite, will be more successful for physical applications! In addition there will not be any irrelevant to the physical reality complexity as in the mathematical models of e.g. of physical fluids that use infinite many points with zero dimensions in the place of the finite many only physical atoms with finite dimensions. The infinite may have its magic, but the axioms of the Digital Euclidean Geometry have their own and different magic!

RIEMANN: Very impressive Newclid your logical approach to the Euclidean space! But what about my Riemannian geometric spaces? Could they be formulated also with Local, Low and High precision levels and finite many visible and invisible points?

NEWCLID: Thank you Riemann! Well my friend any axiomatic system of your Riemannian Geometric spaces, with finite many points would require at least 3 or 4 precision levels! The reason is that at any A point of a Riemannian Space, the tangent or infinitesimal space at A is Euclidean! And here the interior of the point A will be a whole spherical Euclidean space which already requires two precision levels and both the visible and invisible points of the tangent Euclidean space will have to be invisible, while the point A visible point! But let us have patience! When I will be able to develop fully the digital differential calculus on a digital Euclidean Geometry we will reach and answer your question with clarity!

NEWCLID: If there no more questions or remarks, let us end here our discussion, and let us take a nice and energizing walk under the trees in the park close to our building.

AT THIS POINT THE DISCUSSION ENDS.

