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▶ **To cite this version:**

François Clautiaux, Boris Detienne, Gaël Guillot. An iterative dynamic programming approach for the temporal knapsack problem. European Journal of Operational Research, 2021, 293 (2), 10.1016/j.ejor.2020.12.036 . hal-02044832v3

HAL Id: hal-02044832
https://hal.science/hal-02044832v3

Submitted on 24 Jun 2021

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An iterative dynamic programming approach for the temporal knapsack problem

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Abstract

In this paper, we address the temporal knapsack problem (TKP), a generalization of the classical knapsack problem, where selected items enter and leave the knapsack at fixed dates. We model the TKP with a dynamic program of exponential size, which is solved using a method called *Successive Sublimation Dynamic Programming* (SSDP). This method starts by relaxing a set of constraints from the initial problem, and iteratively reintroduces them when needed. We show that a direct application of SSDP to the temporal knapsack problem does not lead to an effective method, and that several improvements are needed to compete with the best results from the literature.

Keywords: Temporal knapsack, Exact algorithm, Lagrangian Relaxation, Successive Sublimation Dynamic Programming method

1. Introduction

In this paper, we address the Temporal Knapsack Problem (TKP), a generalization of the well-known knapsack problem, where each item has a time window during which it can be added to the knapsack, and the capacity constraint is considered at each time period. The name Temporal Knapsack was introduced in [1], although the problem had already been studied in [2] as a bandwidth allocation problem. Formally, the TKP can be stated as follows.

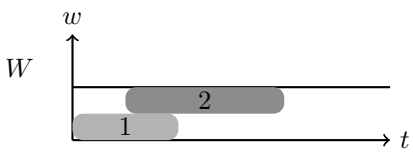
Problem 1 (Temporal Knapsack Problem). *Let $\mathcal{I} = \{1, \dots, n\}$ be a set of items. Each item $i \in \mathcal{I}$ has a profit $p_i \in \mathbb{R}_+$, a size $w_i \in \mathbb{N}$, and a time interval $[s_i, f_i)$, where $s_i, f_i \in \mathbb{N}$ and $s_i < f_i$. Moreover, let $W \in \mathbb{N}$ be the weight of the knapsack. A feasible solution comprises of a subset $\mathcal{J}$ of $\mathcal{I}$ such that for any value of $m \in \mathbb{N}$, the sum of the sizes of the items in $\mathcal{J}$ whose time interval contains m is less than or equal to W. The* Temporal Knapsack Problem *is the problem of finding a feasible subset $\mathcal{J}$ of $\mathcal{I}$ with maximum profit.*

Figure 1 represents an instance of TKP with three items, and its two maximal solutions. One can see that no solution can contain both items 2 and 3, since they are both active at time 3, and the sum of their sizes is larger than the capacity of the container. On the contrary 1 and 3 can be selected in the same solution, despite of their size, since their time intervals do not overlap.

$W = 8$

i	w_i	p_i	$[s_i, f_i)$
1	4	2	$[0, 2)$
2	4	20	$[1, 4)$
3	8	200	$[3, 5)$

(a) A TKP Instance



(b) A first solution of profit 22

(c) A second solution of profit 202

Figure 1: A TKP instance with three items and its two maximal solutions

In its general version, the TKP is NP-hard in the strong sense [3]. The first results proposed for TKP were focused on a theoretical characterization: a polynomially solvable case [4], and approximation

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results [2, 5]. Two dynamic programs were proposed by [2] and [6], which are described more precisely in the next section. A Dantzig-Wolfe reformulation was proposed by [6], where the idea is to partition the time horizon into consecutive time periods (blocks). For each block, the variables related to the items whose time intervals intersect the corresponding time period are duplicated. Each subproblem is a smaller TKP, while the master problem makes sure that the duplicated variables related to the same item have the same value. Based on this reformulation, [6] proposed a branch-and-price algorithm. These results were improved in [7] by using an innovative stabilization technique. This method relies on so-called dual-optimal inequalities, and uses dominance relations between (pairs of) items to add additional effective dual cuts that are satisfied by at least one optimal dual solution. Finally [8] proposed a method based on the previous Dantzig-Wolfe reformulation, where each subproblem is itself decomposed into a master problem and several smaller TKPs, and solved by branch-and-price.

In this paper, we propose a new exact algorithm for TKP. It is based on an exponential size dynamic program, where the size of the state-space depends exponentially on the number of items n. Several methods in the literature have been proposed to tackle such large dynamic programs [9, 10]. All of them are based on the concept of *state-space relaxation* [11]. Among them, we selected *Successive Sublimation Dynamic Programming* (SSDP) method, originally proposed by [10], which has been successfully adapted to several one-machine scheduling problems (see [12] for example).

SSDP consists in solving a relaxation of the original dynamic program, removing some transitions that cannot belong to an optimal solution, and reintroducing incrementally the relaxed constraints, until an optimality proof is reached. An originality of this method is that it does not use a label-setting algorithm, but builds explicitly the graph representation of each relaxed dynamic program. The effectiveness of the method is highly dependent on the capability to reuse information from the previous iterations (primal and dual bounds, variable fixing).

As is the case with many generic methods, obtaining an effective version of SSDP for a new problem is not straightforward. We show numerically that a basic application of this technique to TKP is not competitive compared to state-of-the-art solvers. We then propose several advanced algorithmic techniques, which allow a significant improvement on the computational results.

We implemented our algorithms and compared them empirically against a commercial MIP solver, using instances proposed in [6]. We also report results obtained by [7] on these instances. These experiments show that our algorithm is competitive compared to the state of the art.

The rest of the paper is organized as follows. In Section 2, we formally discuss integer programming formulations and dynamic programs for TKP. In Section 3, we describe an application of SSDP to TKP. Section 4 exposes the various refinements of the method that are necessary to obtain competitive results. We present our computational experiments in Section 5 before offering some brief concluding remarks and suggestions for future research.

2. Integer programming and dynamic programming models

In this section, we discuss compact MIP formulations and dynamic programs for TKP.

2.1. Integer programming formulations

We first recall the commonly used integer programming formulation (see *e.g.* [6, 8]) for TKP. In this model, each binary variable x_i is equal to one if item i is selected, and zero otherwise, similarly to the classical knapsack problem. As suggested in [6], it is sufficient to check the capacity constraints at starting time s_j of each item j.

$$\max \sum_{i \in \mathcal{I}} p_i x_i \tag{1}$$

$$\text{s.t.} \sum_{i \in \mathcal{I}: s_i \leq s_j < f_i} w_i x_i \leq W, \quad j \in \mathcal{I} \tag{2}$$

$$x_i \in \{0,1\}, \quad i \in \mathcal{I} \tag{3}$$

We now propose an alternative MIP formulation for TKP, which is not meant to be used directly to solve the problem, but simplifies the presentation of our dynamic program. In this model, we see the problem as a succession of *events* where decisions have to be taken (adding the item, or removing the item). This means that we split each original variable x_i into two distinct variables that have to take the same value.

Let $(1, \ldots, 2n+1)$ be an ordered list of *event* indices. There are two events in the list for each item, plus an additional dummy event $2n+1$. We distinguish the events related to the beginning of a time window (set $\mathcal{E}^{\text{in}}$) and those related to the end of a time window ($\mathcal{E}^{\text{out}}$). For each event in $1, \ldots, 2n$, we denote by $i(e) \in \mathcal{I}$, the item related to an index $1 \leq e \leq 2n$, and by $t(e)$ the time period when e occurs, *i.e.*, $t(e) = f_{i(e)}$ if $e \in \mathcal{E}^{\text{out}}$ and $t(e) = s_{i(e)}$ if $e \in \mathcal{E}^{\text{in}}$. Events related to items are ordered from 1 to $2n$ as follows: $e < e'$ if $t(e) < t(e')$ or $(t(e) = (e') \wedge e \in \mathcal{E}^{\text{out}} \wedge e' \in \mathcal{E}^{\text{in}})$ (ties are broken arbitrarily).

The decisions of the new MIP model are related to these events. For each event e, we define a binary variable y_e that indicates whether the action related to event e is performed or not. If $e \in \mathcal{E}^{\text{in}}$, this decision corresponds with adding $i(e)$ to the current solution. If $e \in \mathcal{E}^{\text{out}}$, the decision corresponds with removing $i(e)$. In a valid solution, an item leaves the knapsack if and only if it enters the knapsack in a previous event. Each variable ϕ_e $(e = 1, \ldots, 2n)$ is equal to the total size of the selected items at the end of event e.

$$\max \sum_{e=1,\ldots,2n} \frac{1}{2} p_{i(e)} y_e \tag{4}$$

$$\phi_1 = w_{i(1)} y_1 \tag{5}$$

$$\phi_e = \phi_{e-1} + w_{i(e)} y_e \quad e \in \mathcal{E}^{\text{in}} \setminus \{1\} \tag{6}$$

$$\phi_e = \phi_{e-1} - w_{i(e)} y_e \quad e \in \mathcal{E}^{\text{out}} \tag{7}$$

$$\phi_e \leq W \quad e = 1, \ldots, 2n \tag{8}$$

$$\phi_{2n} = 0 \tag{9}$$

$$y_e - y_{e'} = 0 \quad e \in \mathcal{E}^{\text{in}}, e' \in \mathcal{E}^{\text{out}}, i(e) = i(e') \tag{10}$$

$$y_e \in \{0, 1\}, \quad e = 1, \ldots, 2n \tag{11}$$

$$\phi_e \in \mathbb{R}_+, \quad e = 1, \ldots, 2n \tag{12}$$

The objective function is similar to that of model (1). The only difference is that the profit is split between the two events related to each item. Constraints (5)–(7) ensure that the capacity consumption at the end of each event is consistent with the contents of the knapsack. Constraints (8) and (9) guarantee that the capacity constraints are satisfied. Note that constraint (9) is redundant when no other constraint is relaxed. Constraints (10) state that if an item enters the knapsack, it has to leave it. We call constraints (10) *consistency constraints*. Using two variables for deciding if an item is selected in the solution is redundant. The idea behind this variable splitting is to apply Lagrangian relaxation to the consistency constraints. Our method is based on this relaxation, which is similar to the so-called Lagrangian decomposition technique [13].

2.2. Dynamic programs for TKP

To our knowledge, two dynamic programs for TKP were proposed in the literature. Both DP record in each state the subset of items that belong to a partial solution. In [2], a state $(i, \mathbf{d})$ is characterized by the current item $i \in \mathcal{I}$ and $\mathbf{d} \in \{0, 1\}^n$, the characteristic vector of the set of items currently in the knapsack. In the remainder of the paper, when two vectors are considered, symbols + and − stand for the component-wise addition and subtraction, respectively, and $\mathbf{0}$ stands for the vector $(0, \ldots, 0)$ of size n. The value of a state is computed with the following recursive formulas: $f(n+1, \mathbf{0}) = 0$, and $f(i, \mathbf{d}) = \max\{f(i+1, \mathbf{d}'), \quad p_i + f(i+1, \mathbf{d}' + \mathbf{i})\}$ where $\mathbf{d}'$ is obtained from $\mathbf{d}$ by removing all items whose departure date is before the starting time of i. The left-hand part of the alternative chooses not to select item i (and thus just removes the items whose interval do not overlap with item $i+1$), while the right-hand part corresponds with selecting item i (and collects the profit of i). In the latter case, i is recorded in the current vector, and allows to fathom configurations where the total size of the items is larger than the size of the bin. The optimal value is equal to $f(1, \mathbf{0})$. States where $\mathbf{d}$ represents an infeasible subset of items are discarded.

In [6], another dynamic program is proposed. The possible subsets of items that can be in the knapsack at the same instant are computed *a priori*, each of them is related to a state. More precisely, the approach is based on a reformulation of the problem as a maximum profit path problem in an exponentially large graph. The vertex set of this graph can be partitioned in so-called *layers*, one for each knapsack constraint (2). In each layer, a node is created for each feasible subset of items that can be in the knapsack. There is an arc between two nodes from two consecutive layers if and only if their contents are consistent. The cost of the arc is equal to the sum of the profits of the items that are added to obtain the new configuration. This method allows solving only the smallest instances of the literature,

since it cannot be applied when many items can be packed at the same time period, as the number of states in the dynamic program grows exponentially with this quantity.

There are some similarities between the two DP described above. Both state-spaces record the elements in the knapsack at a given step. The first DP builds the possible configurations item by item, and thus may consider non maximal configurations. On the other hand, it may allow to fathom some configurations using some dominance rules or cost considerations, while in the second DP, one has to build all possible configurations beforehand, which may not be possible when this number is large.

Our dynamic program is based on the concept of *events* used in (4)–(12). It is an adaptation of [2] where we add a redundant information to the states (the capacity consumption), and split the decision of adding an item into two decisions. Both modifications are necessary to compute our relaxations. The model works similarly to model (4)–(12) in the sense that it uses the same decisions, and the current capacity is updated event by event recursively, one has to ensure that the capacity constraint remains satisfied, and decisions related to items are consistent.

We now describe formally our dynamic program using *states* and *transitions*. We define a *state* as a tuple $(e, w, \mathbf{d})$ where $e \in \{1, \dots, 2n+1\}$ is the current event, $w \in \mathbb{Z}_+$ the current consumption of the knapsack capacity, and $\mathbf{d} \in \{0,1\}^n$ the characteristic vector of the set of items currently in the knapsack. Note that w is redundant, since it can be deduced from vector $\mathbf{d}$. We call a *transition* the possibility to pass from one state to another by taking a decision. A transition is defined by a tuple $(\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p)$ where $\Delta_e \in \mathbb{Z}_+$ describes the increase in the current event index, $\Delta_w \in \mathbb{Z}$ is the capacity consumed/released when the decision is taken, $\Delta_{\mathbf{d}} \in \{-1, 0, 1\}^n$ a vector that updates the content of the knapsack, and $p \in \mathbb{R}_+$ the profit obtained when the decision is taken.

The possible decisions that can be taken are defined by ψ, the function that associates to each state a set of feasible transitions. Let $\boldsymbol{\varepsilon}_k \in \{0,1\}^n$ be the characteristic vector of set $\{k\}$, for $k \in \mathcal{I}$. For any feasible state $(e, w, \mathbf{d})$, function $\psi((e, w, \mathbf{d}))$ is computed as follows.

$$\psi((e,w,\mathbf{d})) = \begin{cases} \left\{(1,0,\mathbf{0},0), (1, w_{i(e)}, \boldsymbol{\varepsilon}_{i(e)}, \frac{1}{2}p_{i(e)})\right\} & \text{if } e \in \mathcal{E}^{\text{in}} \wedge w + w_{i(e)} \leq W \\ \{(1,0,\mathbf{0},0)\} & \text{if } e \in \mathcal{E}^{\text{in}} \wedge w + w_{i(e)} > W \\ \left\{(1, -w_{i(e)}, -\boldsymbol{\varepsilon}_{i(e)}, \frac{1}{2}p_{i(e)})\right\} & \text{if } e \in \mathcal{E}^{\text{out}} \wedge \mathbf{d}_{i(e)} = 1 \\ \{(1,0,\mathbf{0},0)\} & \text{if } e \in \mathcal{E}^{\text{out}} \wedge \mathbf{d}_{i(e)} = 0 \end{cases} \tag{13}$$

When an event $e \in \mathcal{E}^{\text{in}}$ is considered, two transitions are possible: one corresponding with selecting $i(e)$, the other with not selecting $i(e)$ (the former exists only if the remaining capacity is large enough). When an event $e \in \mathcal{E}^{\text{out}}$ is considered, only one transition is possible, depending on the value $\mathbf{d}_{i(e)}$.

The cost function α from each state $(e, w, \mathbf{d})$ is then expressed in a backward recursive fashion.

$$\alpha((e,w,\mathbf{d})) = \begin{cases} \max\limits_{(\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p) \in \psi((e,w,\mathbf{d}))} \{p + \alpha((e + \Delta_e, w + \Delta_w, \mathbf{d} + \Delta_{\mathbf{d}}))\} & \text{if } e \in \{1, \dots, 2n\} \\ 0 & \text{if } e = 2n+1, w = 0, \boldsymbol{d} = \mathbf{0} \end{cases} \tag{14}$$

The optimal value of the TKP is $\alpha((1,0,\mathbf{0}))$.

3. Specializing Successive Sublimation Dynamic Programming to TKP

In this section, we explain how SSDP can be used to solve TKP. We first describe the generic algorithm, emphasizing the main points to be studied, namely choosing a relaxation, solving the relaxation, and updating the relaxation to obtain a refined model. We then address each point specifically.

3.1. Graph representation of the dynamic program

We first describe a graph representation of dynamic program (13), where states are represented by vertices, and transitions by arcs. The graph representation $G = (V, A)$ is obtained by creating a vertex for each possible reachable state of (13), and an arc for each possible transition. Each arc has the profit p of the corresponding transition. Starting from initial state $(1, 0, \mathbf{0})$, the nodes of the graph are created by computing recursively function $\psi(s)$ and creating the corresponding transitions to obtain the arcs and the vertices.

Figure 2 illustrates the graph representation of DP (13) applied to the instance of Figure 1a. The different paths from $(1, 0, (0, 0, 0)$ to $(7, 0, (0, 0, 0))$ are related to the different solutions for instance 1a.

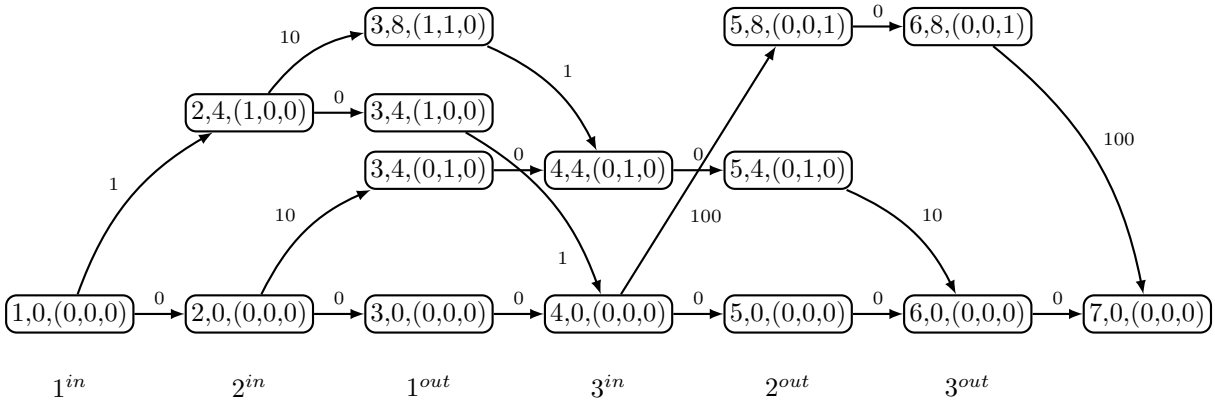

Figure 2: Graph representation of dynamic program (13) applied to the instance of Figure 1a.

We call a *layer* the set of vertices related to a given event. Note that in layers related to *out* events, vertices have exactly one outgoing arc, while in layers related to *in* events, there are at most two possible outgoing arcs.

Once the graph representation of the DP is built, the problem is solved by finding the maximum profit path between the vertex associated with $(1, 0, (0, 0, 0))$ and the vertex associated with the final state $(2n + 1, 0, (0, 0, 0))$. Since the graph has no directed cycles, we use Bellman's algorithm.

3.2. Iterative state-space relaxation

Building the graph representation of DP (13) and using Bellman's algorithm does not lead to a practical method. The size of the state-space in (13) is exponential in the size of $\mathbf{d}$: the size of the binary vector $\mathbf{d}$ is n, so the size of the state-space is in $O(n \times 2^n)$.

Several methods are used in the literature to solve dynamic programs with a large number of dimensions [9, 10]. All of them are based on the concept of state-space relaxation, introduced in [11]. The idea is to project the initial state-space onto a lower dimensional space in such a way that a dual bound is obtained. One then successively computes increasingly refined state-space relaxations until a stopping criterion is reached.

To our knowledge, the first method to use such a relaxation in an iterative algorithm is SSDP, which was proposed in [10]. At each step of the algorithm, SSDP builds explicitly the graph representation of the DP expressed in the current relaxed state-space (called extended graph in the remainder). This extended graph, which can be exponentially large, is used to fix the value of some variables based on an evaluation of the cost of any path passing through the corresponding arcs.

Another algorithm using iterative state-space relaxation was proposed in [9]. It is called *Decremental State Space Relaxation* (DSSR). This method differs from SSDP in the way each relaxation is solved. The known implementations of DSSR do not build explicitly the extended graph, but use labelling algorithms instead. Several labels are kept for each vertex of the graph representation of the initial relaxation. DSSR has been applied successfully to solve resource-constrained shortest path. A strength of this method is its ability to deal with elementary constraints effectively in routing problems, making use of strong dominance checks.

These two versions of iterative state-space relaxations have different advantages. On the one hand, DSSR allows more dominance checking than SSDP, since advanced label-setting/correcting techniques can be used. On the other hand, filtering techniques based on costs apply only to the initial graph in DSSR, whereas SSDP filters arcs from the extended graphs corresponding to all intermediate relaxations.

For TKP, we decided to use SSDP for several reasons. First, our preliminary experiments have shown that the gap between the dual and primal bounds was good enough to filter a good percentage of arcs on many instances, and thus the extended graph does not grow too fast. Second, the dominance relations between two labels are weak for TKP, since one has to take into account the residual capacity that is freed by the items leaving the knapsack over time.

Algorithm 1: SSDP

1 **Compute the graph related to the first relaxation**.
2 Build graph G^0, the graph representation of the initial relaxation ;
3 $\ell \leftarrow 0$;
4 **Solving the relaxation and filtering**.
5 Solve the relaxation corresponding with graph G^ℓ to obtain a solution `sol` ;
6 **if** `sol` *is feasible and has a cost equal to the current best dual bound* **then return** `sol`;
7 Remove non-optimal states and transitions, obtaining graph $\hat{G}^\ell$;
8 **Sublimation**.
9 Construct the new graph $G^{\ell+1}$ from $\hat{G}^\ell$ by reintroducing new constraints ;
10 $\ell \leftarrow \ell + 1$;
11 go back to *step 4* ;

3.3. Presentation of the generic algorithm

SSDP is a dual method that iteratively solves problems obtained by applying state-space relaxation to a dynamic program. An initial relaxation is obtained by relaxing constraints that cause the exponential size of the state-space. A first dual bound is obtained by solving the relaxation. This dual bound is improved by refining the relaxation (*i.e.* reintroducing constraints), until the duality gap to a known primal bound is closed. The bound obtained at each step is possibly reinforced using a Lagrangian relaxation of the constraints. At each step of the algorithm, some unnecessary states and transitions are identified and removed from subsequent relaxations.

An important feature of the algorithm is that it constructs explicitly at each step the graph representation of the current dynamic program. This representation is used to record variable fixing information from one relaxation to the next. The main steps of the method are summarized in Algorithm 1.

Similar to many generic frameworks, several ad-hoc key ingredients have to be designed for each new problem. The most important are the set of relaxed constraints, and the type of relaxation used. Another major ingredient is the algorithm used to solve each relaxed problem, and its capability to eliminate infeasible/non-optimal partial solutions. Finally, an effective method to update the relaxation at each step is necessary.

3.4. Relaxation used for TKP

Our relaxation consists in not considering consistency constraints for some items. This is equivalent to considering only a subset of the values in $\mathbf{d}$ in the states, which reduces the size of the state-space. In this case, an item can enter the knapsack and not leave it, or vice-versa. Our algorithm relies on the fact that there is a one-to-one correspondence between the consistency constraints and the dimensions of the DP related to vector $\mathbf{d}$ in (13).

Observation 1. *Projecting out vector* $\mathbf{d}$ *in* (13) *is equivalent to relaxing consistency constraints* (10) *in* (4)– (12).

We use a modified graph representation to compute the relaxation. At a given iteration of the algorithm, the relaxation is based on a set $\mathcal{J}$ of items that have to satisfy constraint (10). Let $G_\mathcal{J} = (V_\mathcal{J}, A_\mathcal{J})$ be the graph representation of the relaxed DP associated with $\mathcal{J}$. A vertex is now identified by a tuple $(e, w, \mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{J}$ is the subset of items from $\mathcal{J}$ that are in the knapsack. Each vertex v represents a set of states $\mathcal{S}_\mathcal{J}(v) = \mathcal{S}_\mathcal{J}((e, w, \mathcal{C})) = \{(e', w', \mathbf{d}) : e' = e, w' = w, \forall i \in \mathcal{J}, d_i = 1 \leftrightarrow i \in \mathcal{C}\}$. For a given state $s = (e, w, \mathbf{d})$, we denote by $\hat{v}_\mathcal{J}(s)$ the unique vertex v such that $s \in \mathcal{S}_\mathcal{J}(v)$.

Given the modified graph representation, the relaxed dynamic program is obtained by the following recursive functional equation that applies to the vertices of the graph: $\hat{\alpha}_\mathcal{J}((2n+1, 0, \emptyset)) = 0$ and

$$\hat{\alpha}_\mathcal{J}((e, w, \mathcal{C})) = \max_{(\Delta_e, \Delta_w, \Delta_\mathbf{d}, p) \in \cup_{s \in \mathcal{S}_\mathcal{J}((e,w,\mathcal{C}))} \psi(s)} \{p + \hat{\alpha}_\mathcal{J}(\hat{v}_\mathcal{J}(e + \Delta_e, w + \Delta_w, \mathbf{d} + \Delta_\mathbf{d}))\} \tag{15}$$

The optimal solution for the relaxation is obtained by computing $\hat{\alpha}_\mathcal{J}((1, 0, \emptyset))$.

For any arc a in $A_\mathcal{J}$, we denote by $\mu(a)$ the transition associated with arc a. For each arc $a \in A_\mathcal{J}$, let $\tau(a)$ be its tail and $h(a)$ be its head. For a vertex v, let $\Gamma^+(v)$ (resp. $\Gamma^-(v)$) be the set of outgoing arcs (resp. incoming arcs).

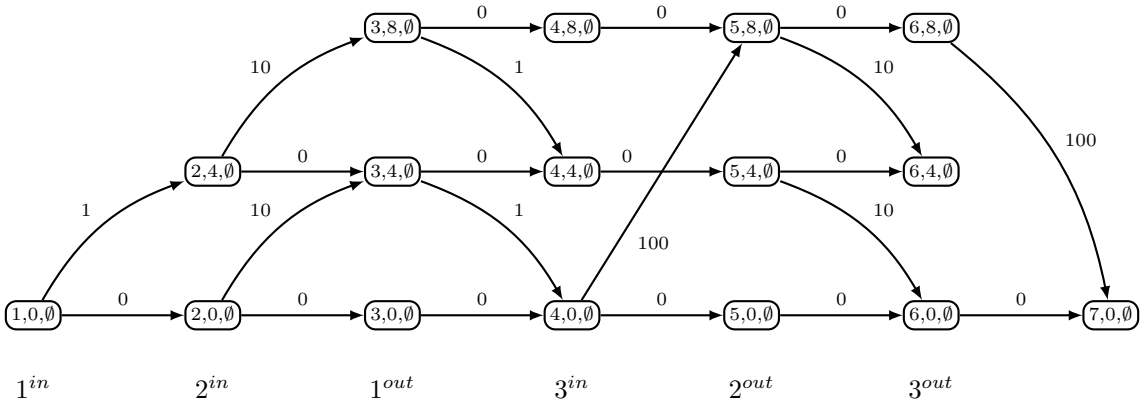

Figure 3: Graph representation of the relaxed DP with $\mathcal{J} = \emptyset$. In this relaxation, the presence of each item in the knapsack is not recorded. Note that state $(6, 4, \emptyset)$ does not belong to any path from the source to the sink of the graph, and will be deleted.

Figure 3 depicts the graph representation of the relaxed version of the dynamic program when $\mathcal{J} = \emptyset$. Note that in this relaxation, the vertices in the *out* layers can have up to two outgoing arcs, instead of one originally. For example, since vertex $(3, 4, (\_, \_, \_))$ represents states $(3, 4, (1, 0, 0))$ and $(3, 4, (0, 1, 0))$ of the original dynamic program, it has two outgoing arcs, related to these two original DP states. All feasible paths in the original graph representation remain feasible, but new paths that are not related to feasible solutions are now considered. The optimal solution for this relaxation is to add item 2, remove item 1, add item 3 and remove item 3, for a profit equal to 211.

3.5. Solving the relaxation and filtering

We use Lagrangian relaxation to produce stronger bounds than those of a combinatorial relaxation, while keeping the problem tractable. This method is used to compute a dual bound, and to remove some arcs that cannot belong to an optimal solution.

Let $\boldsymbol{\pi} \in \mathbb{R}^n$ be the vector of Lagrangian multipliers associated with Constraints (10) for indices $\mathcal{I} \setminus \mathcal{J}$. To simplify the notation, we assume that $\boldsymbol{\pi}$ and $\mathbf{d}$ are always of size n. Within this setting, for a given set $\mathcal{J}$, and a given vector of multipliers $\boldsymbol{\pi}$, the Lagrangian dual function can be written as:

$$L_{\mathcal{J}}(\boldsymbol{\pi}) = \max \sum_{e \in \mathcal{E}^{\text{in}}} \left(\frac{1}{2} p_{i(e)} + \boldsymbol{\pi}_{i(e)} \right) y_e + \sum_{e \in \mathcal{E}^{\text{out}}} \left(\frac{1}{2} p_{i(e)} - \boldsymbol{\pi}_{i(e)} \right) y_e \tag{16}$$

$$(6) - (9), (11), (12) \tag{17}$$

$$y_e - y_{e'} = 0 \quad e \in \mathcal{E}^{\text{in}}, e' \in \mathcal{E}^{\text{out}}, i(e) = i(e'), i(e) \in \mathcal{J} \tag{18}$$

In Figure 4, we report the graph obtained by adding Lagrangian costs to the initial state-space relaxation. There is one multiplier for each consistency constraint. Here, we choose $(0, -3, 0)$ to penalize the possibility of making item 2 entering the knapsack without leaving it. The new optimal solution is the same as in Figure 3, but its cost is now 208, leading to a better bound. Note that by choosing vector $(0, -10, 0)$ for the Lagrangian multipliers, the relaxation would have allowed to solve the problem optimally.

For fixed $\mathcal{J}$ and any vector $\boldsymbol{\pi}$, $L_{\mathcal{J}}(\boldsymbol{\pi})$ is an upper bound on the optimal value of (4)-(12). To compute a good bound using this relaxation, we need to solve approximately the Lagrangian dual problem $\min_{\boldsymbol{\pi} \in \mathbb{R}^n} \{L_{\mathcal{J}}(\boldsymbol{\pi})\}$. In the case of a maximization problem, function $L_{\mathcal{J}}(\boldsymbol{\pi})$ is known to be convex, which implies that minimizing this function can be done using a subgradient algorithm, or one of its refinements (see for example [14]).

We solve the Lagrangian dual problem using Volume algorithm proposed in [15]. This approximate method builds a sequence of solutions $\boldsymbol{\pi}$ that converges to an optimum. For each value of $\boldsymbol{\pi}$, $L_{\mathcal{J}}(\boldsymbol{\pi})$ is computed by applying Bellman's algorithm on graph $(V_{\mathcal{J}}, A_{\mathcal{J}})$, where the profits of the arcs are modified to take into account the penalization of the relaxed constraints. More precisely, for each arc a such that $\mu(a) = (\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p)$, the profit of a is now $p+ < \boldsymbol{\pi}, \Delta_{\mathbf{d}} >$. In what follows, we call $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ the graph

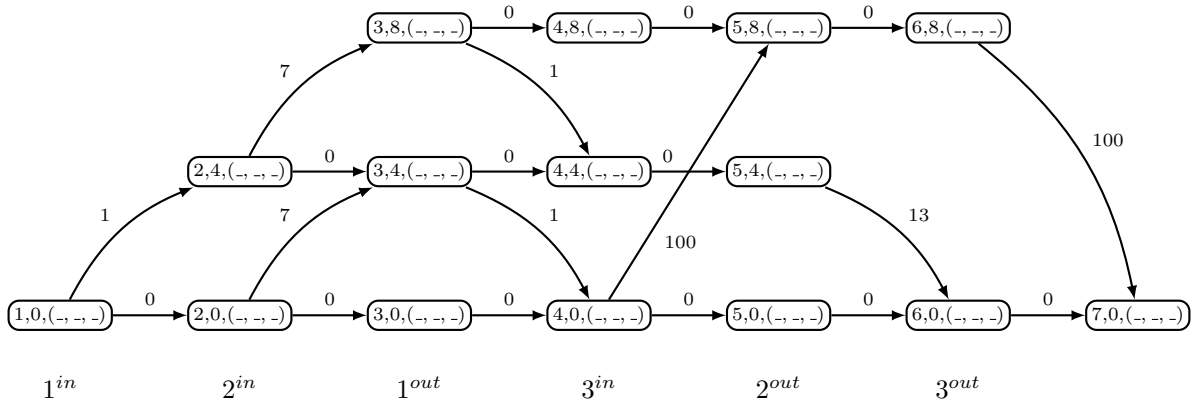

Figure 4: Graph representation of the relaxed DP for $\mathcal{J} = \emptyset$, with Lagrangian costs $(0, -3, 0)$. Adding item 2 has now a profit of 7 instead of 10, while removing item 2 has now an increased profit of 13 instead of 10.

$G_{\mathcal{J}}$ with the costs modified by the Lagrangian multipliers $\boldsymbol{\pi}$. We denote by $v^0 = (1, 0, \emptyset)$ the vertex representing the initial state, and by $v^{\Omega} = (2n + 1, 0, \emptyset)$ the vertex representing the terminal state. We denote by $\alpha^{\boldsymbol{\pi}}_{\mathcal{J}}((e, w, \mathcal{C}))$ the value of Bellman's function for vertex $(e, w, \mathcal{C})$. The value of $L_{\mathcal{J}}(\boldsymbol{\pi})$ is the maximum profit of a path from v^0 to v^{Ω}. Solving the Lagrangian subproblem has complexity $O(|V_{\mathcal{J}}| + |A_{\mathcal{J}}|)$ using Bellman's algorithm.

Figure 5 illustrates a case where a dimension has been added (namely the dimension related to item 2), and Lagrangian costs are used for the other dimensions. Note that all paths where 2 is added and not removed or vice-versa have been excluded.

Observation 2. *Problem* (4)-(12) *is equivalent to the problem defined by graph* $G^{\boldsymbol{\pi}}_{\mathcal{I}}$*, for all* $\boldsymbol{\pi} \in \mathbb{R}^n$*. Indeed, any path in* $G^{\boldsymbol{\pi}}_{\mathcal{I}}$ *defines a feasible solution of* (4)-(12) *with the same cost since the contributions of Lagrangian multipliers cancel out.*

Now, we recall a result used in [10, 16] to remove unnecessary vertices and arcs from $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ (and thus the corresponding states and transitions). For this purpose, let us remark that for any node $(e, w, \mathcal{C}) \in V_{\mathcal{J}}$, Bellman function value $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}((e, w, \mathcal{C}))$ is equal to the maximum cost of a path in $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ from $(e, w, \mathcal{C})$ to v^{Ω}. Likewise, we define $\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}}((e, w, \mathcal{C}))$ as the maximum cost of a path from v^0 to $(e, w, \mathcal{C})$.

Proposition 1 ([10]). *For* $\mathcal{J} \subseteq \mathcal{I}$*, let* $a \in A_{\mathcal{J}}$*, such that* $\mu(a) = (\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p)$*, between two vertices* $(e^1, w^1, \mathcal{C}^1)$ *and* $(e^2, w^2, \mathcal{C}^2)$*, and* $\boldsymbol{\pi} \in \mathbb{R}^n$*. The following value is an upper bound on the cost of any path in* $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ *that uses arc* a*:*

$$\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}}((e^1, w^1, \mathcal{C}^1)) + p + < \boldsymbol{\pi}, \Delta_{\mathbf{d}} > + \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}((e^2, w^2, \mathcal{C}^2))$$

This result allows us to remove unnecessary transitions from graph $G_{\mathcal{J}}$: if the upper bound for arc a is lower than a known lower bound for the problem, then arc a and the related transition cannot be in an optimal solution of the relaxation defined by $\mathcal{J}$. Moreover, the efficiency of SSDP lies in the fact that the corresponding arcs in subsequent stronger relaxations can be removed as well. However, to our knowledge, the validity of this permanent removal is only implicitly assumed in the literature. We provide, in Appendix (Proposition 9), a formal proof of this feature in our specific context. We also give in Figure .12 the graph obtained after applying the filtering phase to the graph of Figure 4.

Values $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}((e, w, \mathcal{C}))$ and $\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}}((e, w, \mathcal{C}))$ can be computed for all nodes $(e, w, \mathcal{C}) \in V_{\mathcal{J}}$ in two passes using Bellman's forward and backward dynamic programming algorithm, respectively.

If an arc is filtered from a graph $G^{\boldsymbol{\pi}}_{\mathcal{J}}$, it is filtered in the graph representation $(V_{\mathcal{J}}, A_{\mathcal{J}})$, and all vertices with no predecessors or no successors are removed from $V_{\mathcal{J}}$. The corresponding states and transitions will not be considered in subsequent iterations.

3.6. Sublimation and convergence

In SSDP, the sublimation phase consists in strengthening the current relaxation by enforcing some constraints that are violated in the current solution. In our application, the set $\mathcal{J}$ of consistency constraints taken into account in the DP is extended by adding new ones, defining $\mathcal{K} \supset \mathcal{J}$. Let $\rho_{\mathcal{J}}$ be

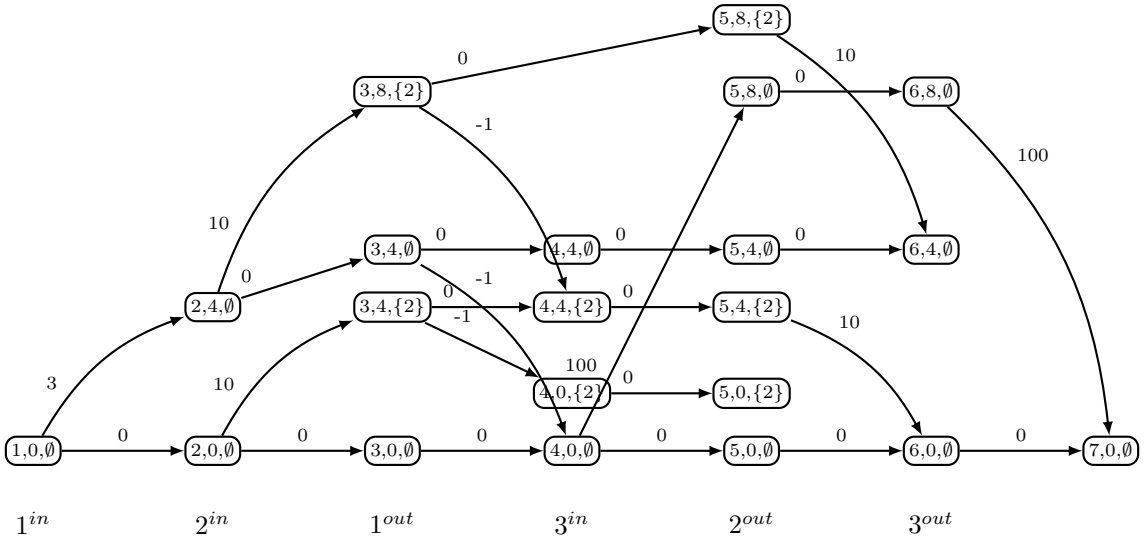

Figure 5: Graph representation of the relaxed DP for $\mathcal{J} = 2$, with Lagrangian costs $(2, 0, 0)$. Nodes $(4, 0, \{2\})$, $(5, 0, \{2\})$, $(4, 4, \emptyset)$, $(5, 4, \emptyset)$, $(5, 8\{2\})$ and $(6, 4, \emptyset)$ can be deleted since they are not contained in any path from the source to the sink.

the function that associates to state $s = (e, w, \mathbf{d})$ the set of transitions that were not filtered up to the iteration related to set $\mathcal{J}$.

$$\rho_{\mathcal{J}}((e, w, \mathbf{d})) = \begin{cases} \emptyset & \text{if } \hat{v}_{\mathcal{J}}((e, w, \mathbf{d})) \notin V_{\mathcal{J}} \\ \{\mu(a) : a \in \Gamma^+(\hat{v}_{\mathcal{J}}(e, w, \mathbf{d}))\} & \text{otherwise} \end{cases}$$

The sublimation phase builds a new graph $G_{\mathcal{K}}$ from filtered graph $G_{\mathcal{J}}$ using the following modified transition function $\hat{\psi}_{\mathcal{K}}$:

$$\hat{\psi}_{\mathcal{K}}((e, w, \mathbf{d})) = \psi((e, w, \mathbf{d})) \cap \rho_{\mathcal{J}}((e, w, \mathbf{d}))$$

This function defines the set of transitions going out of a given state $(e, w, \mathbf{d})$. These transitions should not have been discarded by filtering during previous iterations (*i.e.*, they should be in $\rho_{\mathcal{J}}((e, w, \mathbf{d}))$). Also, the latter set might contain infeasible transitions for $(e, w, \mathbf{d})$ specifically since it gathers all transitions from states which are not distinguishable from $(e, w, \mathbf{d})$ in the previous relaxation (*i.e.* from all the states associated to $\hat{v}_{\mathcal{J}}(e, w, \mathbf{d})$ in $V_{\mathcal{J}}$). Thus, only the transitions that are in $\psi((e, w, \mathbf{d}))$ as well are kept. The maximum number of iterations of the algorithm is n, since at least one item index is added to $\mathcal{J}$ at each sublimation step, and when $\mathcal{J} = \mathcal{I}$, the relaxation obtained is equivalent to (4)-(12) (Observation 2). However a feasible solution may be found at step 5 when $\mathcal{J} \neq \mathcal{I}$. In the latter case, the cost of this solution in model (16)-(18) is equal to its cost in (4)-(12), so that it provides dual and primal bounds with the same value and the algorithm terminates with this optimal solution.

4. Refinements of SSDP to solve TKP effectively

Preliminary computational experiments showed that a direct implementation of SSDP for TKP is not able to produce results that can compete with state-of-the-art TKP solvers. This can be explained by several issues: the method takes a large computation time to build the first relaxation, many states that are not useful are generated when the first relaxation is computed, and the gap is not reduced significantly when only one constraint at a time is reintroduced in the sublimation phase. We now propose several techniques to deal with these issues, and improve the performance of SSDP for solving TKP.

4.1. Attaching additional information to the states

Let $\mathcal{J}$ be the index set of constraints that are currently taken into account in the dynamic program. For a given vertex $v = (e, w, \mathcal{C})$ such that $e \in \mathcal{E}^{\text{out}}$, if $i(e) \notin \mathcal{J}$, two transitions are possible (removing item $i(e)$ or not). In some cases, all states from the original state-space represented by vertex v contain

$i(e)$. In some others, all such states do not contain $i(e)$. In both cases, only one transition should be created.

To detect these case, we attach to each vertex v an additional vector $\mathbf{d}^\eta(v) \in \{0,1,\emptyset\}^n$. If $\mathbf{d}^\eta(v)_i = 0$, v represents only states where i is not in the knapsack, while $\mathbf{d}^\eta(v)_i = 1$ means that v represents only states where i is in the knapsack. If $\mathbf{d}^\eta(v)_i = \emptyset$, v represents both types of states.

For $i \notin \mathcal{J}$, we set $\mathbf{d}^\eta(v^0)_i = 0$, and $\mathbf{d}^\eta(v)_i$ is computed recursively for each other vertex v as follows:

$$\mathbf{d}^\eta(v)_i = \begin{cases} 1 & \text{if } \forall a \in \Gamma^-(v), \quad \mathbf{d}^\eta(\tau(a))_i = 1 \text{ or } \mu(a) = (+1, \Delta_w, +\boldsymbol{\varepsilon}_i, p) \\ 0 & \text{if } \forall a \in \Gamma^-(v), \quad \mathbf{d}^\eta(\tau(a))_i = 0 \text{ or } \mu(a) = (+1, \Delta_w, -\boldsymbol{\varepsilon}_i, p) \\ \emptyset & \text{otherwise} \end{cases}$$

To illustrate the computation of this information, take the graph of Figure 3. Let v be vertex $(2, 4, (\_, \_, \_))$. This vertex can only be obtained by adding item 1 and the other items have not been considered yet. Therefore, for this vertex, $\mathbf{d}^\eta(v) = (0, 1, 0)$. On the contrary, let v' be vertex $(3, 4, (\_, \_, \_))$. This vertex can be obtained by either selecting item 2, or from vertex $(2, 4, (\_, \_, \_))$, so $\mathbf{d}^\eta(v') = (\emptyset, \emptyset, 0)$.

Consequently, vector $\mathbf{d}^\eta(v)$ is computed on the fly while $(V_\mathcal{J}, A_\mathcal{J})$ is created. We attach another information to each vertex v, which corresponds with redundant constraints. For each vertex $v = (e, w, \mathcal{C})$, we define $q^\eta_{\min}(v)$ (resp. $q^\eta_{\max}(v)$) as a lower (resp. upper) bound on the number of items that can be in the knapsack in the states of $\mathcal{S}_\mathcal{J}(v)$. These values can be computed recursively, in a similar way to vector $\mathbf{d}^\eta(v)$. We set $q^\eta_{\min}(v^0) = 0$ and for each other vertex v, $q^\eta_{\min}(v) = \min_{a=(\Delta_e, \Delta_w, \Delta_\mathbf{d}, p) \in \Delta^-(v)} \{q^\eta_{\min}(\tau(a)) + < \Delta_d.\mathbf{1} >\}$. Value $q^\eta_{\max}(v)$ can be obtained by replacing min by max in the expression. As an example, consider the graph of Figure 3. Although we are not able to know the configuration related to vertex $(3, 4, (\_, \_, \_))$, we know that all configurations represented by the vertex contain exactly one item.

Let $\mathcal{I}(e) = \{i \in \mathcal{I} : s_i \leq \hat{t}(e) < f_i\}$ be the set of items that may belong to the knapsack when event e occurs. For each event e, let $Q^{\max}(e) = \max\{|S| : S \subseteq \mathcal{I}(e), \sum_{i \in S} w_i \leq W\}$ be the maximum number of items that can belong to the knapsack when this event occurs (this value can be computed in time linear in $|\mathcal{I}(e)|$ for each event e when the elements of this set are sorted by non-decreasing order of their size). Obviously, for vertex $v = (e, w, \mathcal{C})$, the number of items in any valid state represented by v is in $[0, Q^{\max}(e)]$.

4.2. Feasibility tests and dominance

In the remainder, a *feasible state* is defined as a state that can be generated from s^0 by applying a feasible sequence of transitions following recurrence equations (13). We denote by $\mathcal{S}^+$ the set of feasible states. We recall that $V_\mathcal{J}$ is the set of vertices considered while solving the relaxation related to $\mathcal{J}$. Note that any state in $V_\mathcal{J}$ that is not related to a state in $\mathcal{S}^+$ can be removed from the graph without impairing the validity of the algorithm. We now describe several techniques used to detect infeasibilities of such states. The following results are stated without proof.

The first feasibility test checks that the number of items in the knapsack is consistent with the list of items that are either present or absent in all states represented by a vertex v.

Proposition 2. *Let $\mathcal{J} \subseteq \mathcal{I}$ and $v = (e, w, \mathcal{C}) \in V_\mathcal{J}$. If card$(\{i \in \mathcal{I} : \mathbf{d}^\eta(v)_i \neq 0\}) < q^\eta_{\min}(v)$ or card$(\{i \in \mathcal{I} : \mathbf{d}^\eta(v)_i = 1\}) > q^\eta_{\max}(v)$ then $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$.*

For example in Figure 3, vertex $(4, 8, (\_, \_, \_))$ would be eliminated, since the minimum number of items in the configuration represented by the vertex is 2, and for this layer, there cannot be more than one item.

Another feasibility test is based on the set of possible weights of subsets of items that can belong to the knapsack at a given event. For this purpose, we precompute for each event e the following set: $\mathcal{F}(e) = \{\sum_{i \in S} w_i : S \subseteq \mathcal{I}(e), \sum_{i \in S} w_i \leq W\}$, which corresponds with all reachable weights of a subset of items. Each of these sets can be computed in $\mathcal{O}(nW)$-time using a straightforward dynamic programming algorithm. Proposition (3) follows from the definition of $\mathcal{F}$.

Proposition 3. *Let $\mathcal{J} \subseteq \mathcal{I}$ and $v = (e, w, \mathcal{C}) \in V_\mathcal{J}$. If $w \notin \mathcal{F}(e)$ then $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$.*

This rule also allows to remove vertex $(4, 8, (\_, \_, \_))$ from the graph of Figure 3, since for this layer, there are no possible item combination whose size is 8 (the only possible values are 0 and 4).

This rule can be improved by considering additional information gathered from $\mathbf{d}^\eta(s)$. We precompute, for each event e and each item $i \in \mathcal{I}(e)$, $\mathcal{F}^+(e, i)$ and $\mathcal{F}^-(e, i)$, the possible weights that can be reached using item i, and without item i, respectively. We have $\mathcal{F}^+(e, i) = \{\sum_{j \in S \cup \{i\}} w_j : S \subseteq \mathcal{I}(e) \setminus \{i\}, \sum_{j \in S} w_j \leq W - w_i\}$ and $\mathcal{F}^-(e, i) = \{\sum_{j \in S} w_j : S \subseteq \mathcal{I}(e) \setminus \{i\}, \sum_{j \in S} w_j \leq W\}$.

Proposition 4. *Let $\mathcal{J} \subseteq \mathcal{I}$, $v = (e, w, \mathcal{C}) \in V_{\mathcal{J}}$ and $i \in \mathcal{I}(e)$. If $\mathbf{d}^\eta(v)_i = 1$ and $w \notin \mathcal{F}^+(e, i)$ then $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$. Likewise, if $\mathbf{d}^\eta(v)_i = 0$ and $w \notin \mathcal{F}^-(e, i)$ $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$.*

The following result allows detecting nodes related to states that can be generated only by removing an item from the knapsack without adding it first. In such cases, the weight recorded in the state might become less than the weight of the items whose presence in the knapsack is known for sure.

Proposition 5. *Let $\mathcal{J} \subseteq \mathcal{I}$ and $v = (e, w, \mathcal{C}) \in V_{\mathcal{J}}$. If $\sum_{i \in \mathcal{I}: \mathbf{d}^\eta(v)_i = 1} w_i > w$ then $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$.*

For a vertex $(e, w, \mathcal{C})$, the following feasibility test integrates the bounds on the number of items in the knapsack to ensure the consistency of set $\mathcal{C}$ with respect to value w.

Proposition 6. *Let $\mathcal{J} \subseteq \mathcal{I}$ and $v = (e, w, \mathcal{C}) \in V_{\mathcal{J}}$. Let $S^1 = \{i \in \mathcal{I}(e) : \mathbf{d}^\eta(v)_i \neq 0\}$ be the set of items potentially in the knapsack. If $\max\left\{\sum_{i \in S} w_i : S \subseteq S^1, |S| \leq q^\eta_{\max}(v)\right\} < w$ or $\min\left\{\sum_{i \in S} w_i : S \subseteq S^1, |S| \geq q^\eta_{\min}(v)\right\} > w$ then $\mathcal{S}(v) \cap \mathcal{S}^+ = \emptyset$.*

In the classical knapsack problem, an item i is dominated by item j if $p_i \leq p_j$, $w_i \geq w_j$ and one of the two inequalities is strict. In this case, from any feasible solution where item i is chosen but not item j, we can build another solution where j is chosen and whose profit is not smaller than that of the initial solution. Thus, among solutions that include i, only those including j as well need to be considered. For TKP, dominance relations must take into account the temporal aspect as well.

Proposition 7. *Item i is dominated by item j if $p_i \leq p_j$, $w_i \geq w_j$, $s_i \leq s_j$, $f_i \geq f_j$ and one of the four inequalities is strict.*

In the course of SSDP, we can take advantage from it by modifying the recurrence equations as follows. Let us consider vertex $v = (e, w, \mathcal{C}) \in A_{\mathcal{J}}$ such that $e \in \mathcal{E}^{\text{in}}$, $i(e) = j$ and $\mathbf{d}^\eta(v)_i = 1$ with i an item which is dominated by item j. Then any transition $(\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p) \in \cup_{s \in \mathcal{S}_{\mathcal{J}}(v)} \psi(s)$ with $(\Delta_{\mathbf{d}})_j = 0$ can be discarded from equation (15). Indeed, selecting this transition would mean choosing item i but not item j.

4.3. Partial enumeration of transitions

Combining several consecutive transitions into single compound transitions allows to enforce some consistency constraints locally even if the corresponding dimensions are not included in the current state space.

Our implementation of this idea uses an input parameter k^{enum}, which controls the depth of the enumeration of consecutive transitions. More precisely, from a given state, instead of computing the two possible transitions, we compute the $O(2^{k^{\text{enum}}})$ possible sequences of k^{enum} successive transitions. Sequences that violate consistency constraints are not generated. Figure 6 illustrates a case where three consecutive events are considered.

In some cases this technique might lead to a larger network. However, when the enumeration of some transitions spans the two events of the same item, the associated consistency constraint is always satisfied, which results in a stronger relaxation. Moreover, the filtering procedure may filter a compound transition because the related sequence of transitions has a poor cost, while in the initial graph, each intermediate arc of this sequence may individually belong to a path with a better cost, preventing the removal of any transition. This contributes to limiting the growth of the network, and improves the quality of the relaxation at the same time.

4.4. Criteria for selecting constraints to reintroduce

At each sublimation step, one has to select the dimensions related to violated constraints that are integrated into the state-space. To make this selection, we record some information that will be used in our method. At the iteration determined by set $\mathcal{J}$, we record a list $(\boldsymbol{\pi}^1, \ldots, \boldsymbol{\pi}^{k^{\text{nbsol}}})$ of Lagrangian multipliers that led to the best upper bounds in Volume algorithm. This list is sorted by decreasing value of $L_{\mathcal{J}}(\boldsymbol{\pi}^q)$. For each recorded vector $\boldsymbol{\pi}^q$, let $\boldsymbol{y}^q$ be the corresponding solution expressed in terms of the variables of (16)-(18).

First, we do not consider all dimensions for inclusion in the state-space. Only those that are related to violated constraints are considered. Let $\mathcal{J}^{\neq} = \left\{i \in \mathcal{I} \setminus \mathcal{J} : \exists q \in \{1, \ldots, k^{\text{nbsol}}\}, y^q_{e^{in}(i)} \neq y^q_{e^{out}(i)}\right\}$ be the set of items whose consistency constraint is violated in at least one of the solutions of $(\boldsymbol{y}^1, \ldots, \boldsymbol{y}^{k^{\text{nbsol}}})$. The corresponding dimensions are natural candidates to be considered for the sublimation phase.

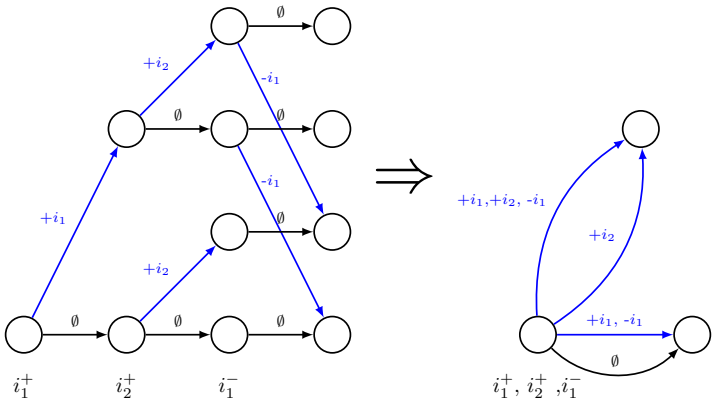

Figure 6: Partial enumeration of three events. Four sequences out of eight are feasible.

We now describe three criteria for estimating the computational attractiveness of adding a specific dimension to set $\mathcal{J}$.

The first criterion (**Lagrangian Multipliers**) is to use the best Lagrangian multipliers $\boldsymbol{\pi}^1$ found to determine the attractiveness of each consistency constraint: $\psi_i^1 = |\boldsymbol{\pi}_i^1|$. A Lagrangian cost of large magnitude tends to indicate that many solutions violate the corresponding constraint, which is penalized by Volume.

The second criterion (**Network Size**) aims to control the number of states in the network after the sublimation step. For each constraint, we compute an estimation of the growth of the vertex set if the corresponding constraint – and only that one – is included into the state-space. When adding a single constraint related to item i, only states s such that $\mathbf{d}^\eta(s)_i = \emptyset$ can yield two different states after sublimation. Hence, the second criterion we define is the opposite (smaller is better) of an upper bound on the number of additional states due to i: $\psi_i^2 = -|v \in V_\mathcal{J} : \mathbf{d}^\eta(v)_i = \emptyset|$.

The third criterion (**Number of violations**) favors the constraints that are violated in many solutions recorded in the list $(\boldsymbol{y}^1, \ldots, \boldsymbol{y}^{k^{\text{nbsol}}})$. This can be computed as follows: $\psi_i^3 = \frac{1}{k^{\text{nbsol}}} \sum_{q=1}^{k^{\text{nbsol}}} \left| y^q_{e^{in}(i)} - y^q_{e^{out}(i)} \right|$.

4.5. Reintroducing batches of constraints

Adding several violated constraints at once is generally a good strategy for TKP. The sublimation step is a time-consuming procedure, and preliminary experiments have shown that in many cases, adding only one constraint is not sufficient to ensure a significant decrease in the dual bound. However, adding too many constraints increases dramatically the size of the network. Therefore, we have to find a good trade-off between the quality of the bound and the size of the network.

A first issue is to compute a reliable estimation of the size of the network when a new constraint is introduced. The following proposition provides an upper bound on the number of labels in the network given a set of consistency constraints enforced in the state space.

Proposition 8. *Let $\mathcal{J}$ and $\mathcal{K}$ such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$. It holds that*

$$|V_\mathcal{K}| \leq \sum_{e=1,\ldots,2n+1} \left(2^{|(\mathcal{K}\setminus\mathcal{J})\cap\mathcal{I}(e)|} \, card(\{(e', w, \mathcal{C}) \in V_\mathcal{J} : e' = e\})\right)$$

.

Proof. The set of vertices created in $V_\mathcal{K}$ from a given vertex $(e, w, \mathcal{C}) \in V_\mathcal{J}$ is included in the set

$$\{(e, w, \mathbf{d}') : \mathbf{d}'_i = \mathbf{d}_i \ \forall i \in \mathcal{J}, \mathbf{d}'_i \in \{0, 1\} \ \forall i \in (\mathcal{K} \setminus \mathcal{J}) \cap \mathcal{I}(e), \mathbf{d}'_i = 0 \ \forall i \in \mathcal{I} \setminus (\mathcal{J} \cup (\mathcal{K} \cap \mathcal{I}(e)))\}$$

whose cardinality is $2^{|(\mathcal{K}\setminus\mathcal{J})\cap\mathcal{I}(e)|}$. Summing up over all states in $V_\mathcal{J}$ yields the result. □

This indicates that adding constraints related to items with pairwise disjoint time windows is particularly attractive: in such cases, for all events $e \in \{1, \ldots, 2n+1\}$ we have $\text{card}((\mathcal{K} \setminus \mathcal{J}) \cap \mathcal{I}(e)) \leq 1$. It follows that the network grows by a constant factor only, as stated formally in the next corollary.

Let $G^{\text{int}} = (\mathcal{I}, E^{\text{int}})$ be the interval graph related to intervals $[s_i, f_i), i \in \mathcal{I}$. An arc in this graph represents a pair of dimensions that should not be added together (if one wants to avoid a too rapid growth of the network size). We also define the subgraph of G^{int} induced by $\mathcal{J}^{\neq}$: $G^{\text{int}}_{\neq} = (\mathcal{J}^{\neq}, E^{\text{int}}_{\neq})$, where $E^{\text{int}}_{\neq} = E^{\text{int}} \cap (\mathcal{J}^{\neq} \times \mathcal{J}^{\neq})$.

Corollary 1. *Let $\mathcal{J}$ and $\mathcal{K}$ be such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$, and $\mathcal{K} \setminus \mathcal{J}$ is a stable set in graph G^{int}. Then $|V_{\mathcal{K}}| \leq 2|V_{\mathcal{J}}|$.*

This corollary guided our strategies to select the set of dimensions that are added at each sublimation step. We propose four strategies that aim at finding a good tradeoff between the approximate growth of the network and the quality of the relaxation.

The first strategy, that we call **Weighted stable set**, limits the expected network growth during the sublimation step. To this end, we use the upper bound on the number of additional states in the new DP provided by Corollary 1: when set $\mathcal{J} \subseteq \mathcal{K}$ forms a stable set in G^{int}, $\sum_{i \in \mathcal{J} \subseteq \mathcal{K}} \psi_i^2$ is an upper bound on the overall number of the new states. This upper bound allows to account for the effect of including item i in set $\mathcal{J}$. Note that, in [17], the authors try to heuristically restrain the growth of the DP by choosing the constraints impacting the smallest number of arcs. Here, we rely on properties of our specific problem, allowing for a more strict control of the number of additional states. Based on criteria ψ_i^1, ψ_i^2 and ψ_i^3, we define a weight ψ_i for each item. We then seek for a stable set in G^{int} of maximum weight, such that the estimated growth of the number of states is less than parameter MAXG.

This is done by solving the model below, where a binary variable x_i is created for all $i \in \mathcal{J}^{\neq}$, indicating whether i is selected or not.

$$\max \left\{ \sum_{i \in \mathcal{J}^{\neq}} \psi_i x_i : \sum_{i \in \mathcal{J}^{\neq}} -\psi_i^2 x_i \leq \text{MAXG}, x_i + x_j \leq 1 \ \forall (i,j) \in E^{\text{int}}_{\neq}, x_i \in \{0,1\} \ \forall i \in \mathcal{J}^{\neq} \right\}$$

This problem is a knapsack problem with conflicts, which is NP-complete, but can be solved in pseudo-polynomial time through dynamic programming for interval graphs (see [18]). For the instances we considered, the time needed to solve this subproblem is negligible compared to the overall time of the algorithm.

Our second strategy, called **Cardinality constrained stable set**, aims at circumventing a major drawback of the Weighted stable set strategy, which sometimes adds too few new constraints, leading to a slow convergence of the overall algorithm. Our strategy consists in solving first the model above with unit profits to find a stable set of maximum cardinality (which we denote by $C_{\max}$). We then seek a stable set of cardinality larger than $C_{\max} * k^{\text{rstable}}$ where k^{rstable} is a parameter in $(0, 1]$, by solving the following cardinality constrained maximum weight stable set problem. Using the same variables as the model above, one obtains the following model.

$$\max\{ \sum_{i \in \mathcal{J}^{\neq}} \psi_i x_i : \sum_{i \in \mathcal{J}^{\neq}} x_i \geq C_{\max} * k^{\text{rstable}}, x_i + x_j \leq 1 \ \forall (i,j) \in E^{\text{int}}_{\neq}, x_i \in \{0,1\} \ \forall i \in \mathcal{J}^{\neq}\}$$

This problem is also NP-complete, but is solved effectively by modern MILP solvers (*i.e.*, the time needed to solve it is also negligible compared to the overall time of the algorithm).

The third strategy, called **k-coloring**, not only considers stable sets in $G^{\text{int}}_{\neq}$, but in a larger graph representing also constraints that were added in the previous iterations. To this end, we denote this extended set $\mathcal{J}^{+} = \mathcal{J} \cup \mathcal{J}^{\neq}$, we define $E^{\text{int}}_{+} = E^{\text{int}} \cap (\mathcal{J}^{+} \times \mathcal{J}^{+})$, and we consider the subgraph $G^{\text{int}}_{+} = (\mathcal{J}^{\neq} \cup \mathcal{J}, E^{\text{int}}_{+})$ of G^{int} induced by $\mathcal{J}^{+}$. Let k^{color} be equal to the number of colors that we allow at the current step. At each iteration, we seek a k^{color}-colorable subgraph of G^{int}_{+} of maximum weight. If the solution is different from $\mathcal{J}$, then we add the new constraints selected. If the solution only contains $\mathcal{J}$, then we increase the value of k^{color} by one unit, and solve the model again. Initially, k^{color} is set to one. We solve repeatedly the following model, where z_{ij} are binary variables indicating that item i is

assigned to color j.

$$\max \sum_{i \in \mathcal{J}^{\neq}} \sum_{j=1,\ldots,k^{\text{color}}} \psi_i z_{ij}$$

$$z_{ij} + z_{\ell j} \leq 1, \forall (i,\ell) \in E_+^{\text{int}}, j \in \{1,\ldots,k^{\text{color}}\}$$

$$\sum_{j=1,\ldots,k^{\text{color}}} z_{ij} \leq 1, \forall i \in \mathcal{J}^{\neq}$$

$$\sum_{j=1,\ldots,k^{\text{color}}} z_{ij} = 1, \forall i \in \mathcal{J}^{+}$$

$$z_{ij} \in \{0,1\}, \forall i \in \mathcal{J}^{\neq} \cup \mathcal{J}^{+}, j \in \{1,\ldots,k^{\text{color}}\}$$

We also solve this subproblem with a general purpose MILP solver. Again, the solution time is negligible compared to the time needed to build the new graph at each iteration.

Finally, we consider a strategy called **Hybrid**, which favours a strong improvement in the gap in the first iterations, and then favours a network of manageable size when the number of vertices reaches a threshold. This is based on the following observation: in the first iterations, one would like to close the gap between the primal and dual bounds as fast as possible to allow a better performance of filtering procedure, but when the number of vertices in the network becomes large, the most important criterion becomes its size, since a too large network may lead to intractable Lagrangian subproblems. Therefore, the hybrid strategy uses one of the strategies above in the first iterations, and when the number of nodes is larger than a given threshold, the choice is only based on the expected size of the network.

4.6. Implementation issues

The effectiveness of the filtering step heavily depends on the fact that a good primal solution is known. In general, during the optimization of the Lagrangian multipliers, it may happen that a primal solution is computed as a side product of the method. However, one cannot rely on this for TKP, since many constraints are often violated in a solution of a relaxation. To produce a lower bound for our problem, we heuristically solve model (1)–(3) by giving a small amount of time to a general purpose integer linear programming solver.

A good dual solution is also useful to warmstart Volume algorithm, which may take a large amount of time to converge when the DAG are large. To find a good set of multipliers, we solve the LP relaxation of (4)–(10), and use the optimal dual values of constraints (10). Solving the LP relaxation is fast and provides a good starting point for Volume algorithm.

We implemented a parallel version of Bellman's algorithm. We first compute the longest path (in term of number of arcs) from s^0 to all vertices. All vertices at the same distance are stored in a common bucket. The treatment of vertices in the same bucket can be done in parallel.

5. Computational experiments

In this section, we provide experimental results for our methods. For each refinement proposed, we evaluate its impact on the performance of the general algorithm. Finally, we compare our results to those of [7] and to the results obtained using an all-purpose commercial Integer Linear Programming solver. In this section, we consider an instance as solved if the algorithm finds an optimal solution and proves its optimality.

All our experiments are conducted using 2 Dodeca-core Haswell Intel Xeon E5-2680 v3 2,5 GHz with 128Go RAM. For each instance, our code was run on 6 threads and a 32 Go RAM limit. All models considered in subroutines are solved with IBM ILOG Cplex 12.7.

We use instances proposed in [6], composed of two groups. For instances in the first group (I), the number of items ranges from 2071 to 13025, the size of the knapsack is 100, and each item has a profit and a weight between 10 and 100. Since the method described in [7] and our methods can solve all those instances to optimality in a small amount of time, we do not report the corresponding results. For the 100 instances in the second group (called U), the number of items is 1000, the size of the knapsack ranges from 500 to 520. Each item has a profit and a weight between 1 and 100.

The goal of our experiments is twofold. We first want to determine the best parameters for our algorithm, and evaluate the impact of the different improvements that we have proposed. Secondly, we want to assess their effectiveness against the state-of-the-art methods from the literature. In the following we present aggregated results. Detailed tables can be found in appendix (online supplement).

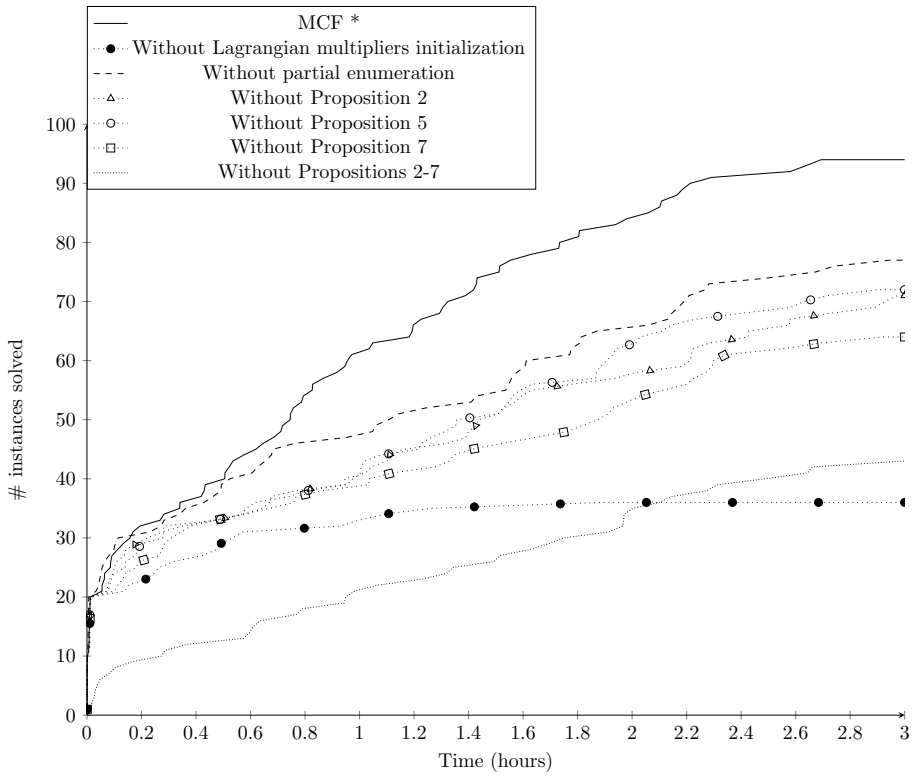

Figure 7: Number of instances of TKP from data set U ([8]) solved along time, for the best version of our algorithm, and six versions obtained by deactivating some techniques.

Table 1: Average size of first network for different value of k^{enum}, after the filtering step. "k" stands for thousands.

k^{enum}	1	2	3	4	5	6
Average nodes	703 k	384 k	264 k	202 k	162 k	135 k
Average arcs	1,392 k	1,344 k	1,639 k	2,269 k	3,155 k	4,658 k

5.1. Parameters of the method

We first evaluate the impact of the improvements proposed. For this purpose, we report the results obtained by the best combination of techniques (called SSDP* below), and methods obtained from SSDP* by deactivating some features. We deactivated the partial enumeration technique (subsection 4.3) and dominance and feasibility tests (subsection 4.2). For each method, we report in Figure 7 the number of instances solved along the time, with a limit of three hours.

The number of instances solved optimally within 3 hours increases by about 20% when partial enumeration is used. This means that enumerating sequences of transitions allows to include useful information that is used by the filtering algorithm. To illustrate the effect of partial enumeration, we report in Table 1 the size of the first network constructed, for different values of k^{enum}. As one can expect, increasing the number of consecutive transitions considered reduces the number of nodes in the network but increases the number of arcs (since combinations of arcs are replaced by single arcs). Although it yields a higher memory consumption and a longer solution time of the relaxations, it can be advantageous to some extent: selecting one arc means deciding more arcs simultaneously and more impact on the Lagrangian cost of the solution. Thus, such long sequences of decisions are more easily discarded by Lagrangian filtering. This also explains, along with the removal of short infeasible sequences, why the network with $k^{\text{enum}} = 2$ has fewer arcs than that with $k^{\text{enum}} = 1$.

The feasibility tests proposed in Propositions 2 to 6 have a crucial impact on the performance of our algorithm, since they allow solving 40 more instances in one hour-time limit, and 51 more instances in three hour-time limit. These tests remove about 20% of the nodes and 32% of the arcs included in the first network when $k^{\text{enum}} = 4$. One can also remark that each of them improves the overall procedure significantly.

Table 2: Parameters of the tested methods.

Configuration	Strategy	Criterion ψ_i
SSDP*	Cardinality constrained	$\psi_i^2\left(1-\psi_i^3\right)$
Stable	Weighted stable set	$\frac{-\psi_i^2+\max_{j\in\mathcal{J}^{\neq}}\psi_j^2}{\max_{j\in\mathcal{J}^{\neq}}\psi_j^2}\psi_i^3$
NbLabels	Cardinality constrained	ψ_i^2
Hybrid	Cardinality constrained	First ψ_i^3, then ψ_i^2
LagMult	Cardinality constrained	ψ_i^1
KColor	K-Color	$\frac{-\psi_i^2+\max_{j\in\mathcal{J}^{\neq}}\psi_j^2}{\max_{j\in\mathcal{J}^{\neq}}\psi_j^2}\psi_i^3$

5.2. Strategies for reintroducing constraints

As stated in most papers working on iterative state-space relaxations, the selection of the constraints to reintroduce is the most critical component in the method. In what follows, we empirically compare our different strategies to determine the most effective. In Table 2, we report how each configuration of our algorithm is parameterized. Preliminary experiments led us to set parameter $k^{\text{nbsol}} = 20$ for the computation of criterion ψ^3. The performance of the overall method was impaired by values of k^{nbsol} less than 10, but does not seem to be affected by values larger than 20. In method based on Cardinality constrained strategy, parameter $k^{rstable}$ is empirically fixed to 0.7. For method SSDP*, the weight associated with each constraint tries to balance the upper bound on the number of additional labels and the frequency of violation of this constraint in the best relaxed solutions considered. When using Weighted stable set strategy, minimizing the expected number of added labels comes to selecting no new constraint. That is why we maximize the complement to the maximum number of expected additional labels. This value is weighted by the frequency of violation of the constraint. Configuration Hybrid aims at improving the dual bound as fast as possible by enforcing the most frequently violated constraints. Once the network is too large (we empirically fixed a limit at 4,000k nodes), adding fewer labels is preferred.

Figure 8 numerically compares our different methods for selecting the constraints to add during the sublimation phase. Similarly to Figure 7, we report the total number of instances solved along the time, with a limit of three hours. We observe that $k-$coloring strategy is clearly not competitive compared to strategies based on stable sets. The fact that this strategy performs poorly shows that our method to evaluate the size of the network in the stable-set based strategies is useful, and that one cannot just rely on the interval structure of the constraints. All strategies based on stable sets have similar behaviour. Configuration Stable performs reasonably well within medium time limits. However, it does not seem to be more effective when more time is allocated. That can be explained by the fact that, the larger the network is, the less the number of new constraints added is controlled. Indeed, we empirically observed that only a few constraints are added in general by this method once a critical size is reached. The configurations based on Cardinality constrained stable set strategy do not suffer from that drawback. For group U of instances which are composed of 1000 items, both SSDP* and Stable configurations reintroduce between 5 and 20 constraints in more than 75% of the iterations, while the maximum number of constraints added by the algorithm at each iteration is respectively 29 and 41. The total number of constraints added is less than or equal to 15 for 20% of the instances, more than 274 (resp. 267) for 20% of the instances for configuration SSDP* (resp. Stable). The maximum number of constraints added for a single instance is 328 (resp. 348).

The performance of Hybrid configuration is disappointing. This might be due to the difficulty of finding a good rule for switching between the two criteria. Overall, an important conclusion is that taking into account the increase of the size of the network appears to be crucial for the method.

5.3. Comparison with the branch-and-price of [7]

We now compare our method with the algorithm of [7]. The authors have kindly provided us with the results obtained with their algorithm within a three-hour time limit. They implemented a pure branch-and-price without any primal heuristics and using best-first as node selection strategy. Their experiments were performed on a standard PC with an Intel(R) Core(TM) i7-2600 at 3.4 GHz with 16.0 GB main memory using a single thread. Figure 9 reports the performance of our best algorithm (SSDP*) and those obtained by [7](GI) using a similar computer (same processor, same amount of RAM), using a single thread only. The processor speeds in this setting and in the one described at the beginning of Section 5 are roughly comparable. However, the limited amount of memory on this machine is not

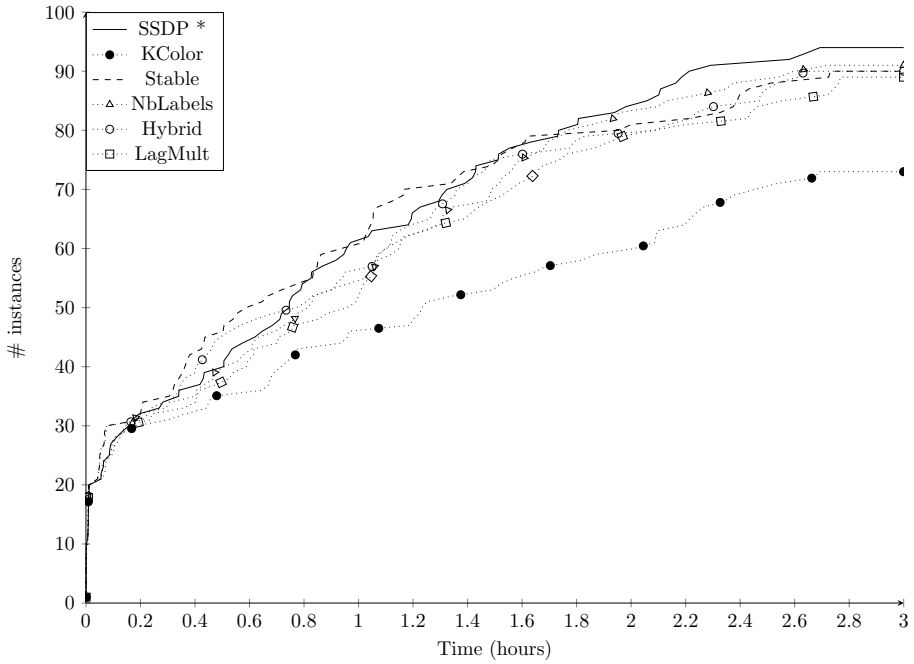

Figure 8: Number of instances solved along time for different methods used to determine the constraints to add at each iteration.

always enough for our method (the algorithm ran out of memory for eight instances that are solved on the other machine).

The approach of [7] is more efficient within short computing time: it solves 41 instances in 30 minutes, whereas the best version of our algorithm, when restricted to a single thread on the same machine, solves only 35 instances. Both algorithms perform similarly within a one-hour time limit (47 instances solved versus 49). However, only a few more instances are solved by the branch-and-price approach within 3 hours of computing time (55 instances in total), while our approach solves 50 percent of the still unsolved instances between 1 and 3 hours (75 instances in total).

5.4. Comparison with a general-purpose MILP solver

Figure 10 reports the performance of our best algorithm (SSDP*), and those obtained by ILP solver IBM Ilog Cplex when solving model (1)-(3) (CPLEX) . For the sake of completeness, we also report the results obtained with model (4)-(12) (CPLEX-Event-based). We limited all methods to a single thread.

First, the straightforward use of the event-based model (4)-(12) is not a competitive approach. All instances but five were solved by at least one of the two other methods. In terms of number of instances solved after three hours, SSDP* shows the best performance. After three hours, SSDP* solves optimally 83 instances, which is 14 more than CPLEX. CPLEX is better than SSDP* for several instances, mostly instances numbered from 1 to 55. The solver is able to solve most of them in a few seconds, while SSDP* may need minutes or hours. That can be explained by the powerful procedures embedded in such solvers to deal with knapsack constraints (for example to derive cuts), as well as good generic heuristics. From instance 55 to 99 however, CPLEX is only able to solve 16 instances. This can be explained by the structure of the instances: each batch of ten consecutive instances has a similar structure, most notably the maximum number of items in a clique. This number increases with the index of the instances. It transpires from these experiments that linear programming based methods are highly sensitive to this parameter, which is not the case for SSDP*.

We now report the performances of CPLEX and our method in their multiple thread setting. Figure 11 reports the results with six threads for SSDP*, CPLEX and CPLEX-Event-based.

Using more cores is beneficial for all methods, although CPLEX-Event-based only solves two more instances within the time limit. After those three hours, SSDP* with six threads solved optimally 94 instances, which is 12 more than CPLEX. Our method is not six times faster, since only the Lagrangian problem solver is parallelized. The time needed to construct and update the graph representation of the dynamic program represents a large percentage of the total running time, and this part of the algorithm does not benefit from a multi-core architecture. Only two instances are solved by CPLEX and not by

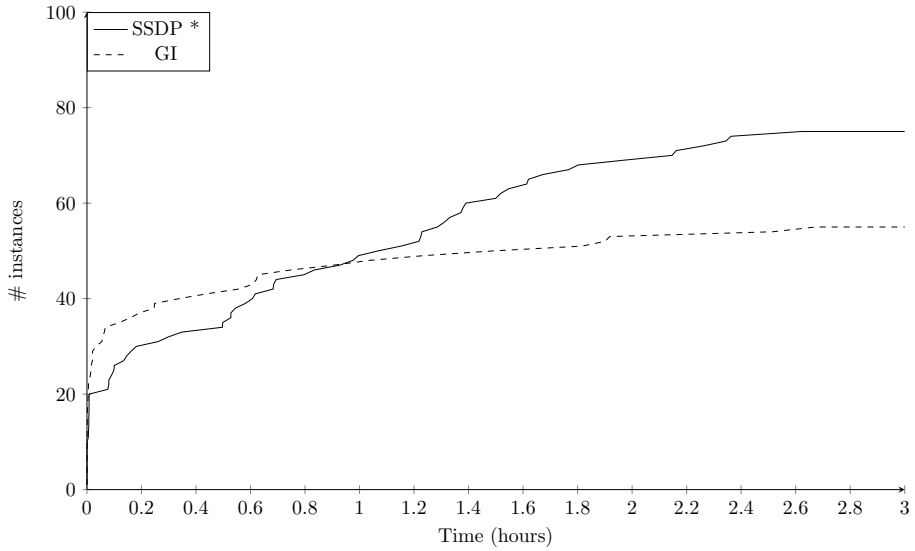

Figure 9: Number of instances solved along time for our best algorithm (SSDP*) and the algorithm of [7] on Intel(R) Core(TM) i7-2600 at 3.4 GHz with 16.0 GB main memory using a single thread only.

Table 3: Sensitivity of our method to the knapsack capacity. In these experiments, Proposition 4 is not used within the SSDP algorithm.

Instance type	Capacity	#Instance per status (over 20)			
		CPLEX only	SSDP only	solved by both	unsolved
I	1000	0	12	8	0
U	1000	5	0	7	8
I	10000	0	11	9	0
U	10000	14	0	0	6

SSDP* (U69 and U78). On the contrary, SSDP* is able to find 14 solutions when CPLEX does not reach convergence.

5.5. Sensitivity of our method to the knapsack capacity

The size of our dynamic program depends on the knapsack capacity W. When this value increases, so does the time and memory needed to create the graph representation of the problem. On the contrary, MIP solvers are generally less sensitive to the value of this parameter.

We run additional experiments on instances with larger knapsack capacities. We implemented the data generators of [6], the same that were used to create the instances we use in the previous experiments. As mentioned above, the authors generated 20 different classes of instances ($I1$ to $I10$ and $U1$ to $U10$). We generated four new instances for each class: two with $W = 1000$ and two with $W = 10000$. Therefore, the instances we generated have the same structure as the instances generated by [6], but with a larger knapsack capacity. For these experiments, Proposition 4 is not used within the SSDP algorithm, because the memory required for the related additional data is too large.

Our results are reported in Table 3. For each instance type (I or U), and each knapsack capacity (1000 and 10000), we report the number of instances (over the 20 generated) that are solved by CPLEX only, SSDP only, and both methods, within a three-hour time limit. We also report the number of unsolved instances.

According to these results, SSDP clearly suffers from a larger knapsack capacity and the deactivation of Proposition 4, as expected. On average, SSDP remains competitive even with larger capacities (47 instances solved against 43), in particular for instances of type I. Although these instances were solved in a handful of seconds by all methods for $W = 100$, CPLEX was only able to solve 8 instances over the 20 generated for $W = 1000$, and 9 instances when $W = 10000$. It is able to find a good solution quickly, but is not able to close the gap after three hours of computation time. SSDP is able to solve all large type I instances, although it comes with a larger computational cost (respectively 284s and 596s on average for $W = 1000$ and $W = 10000$). On the contrary, CPLEX is more effective for instances of

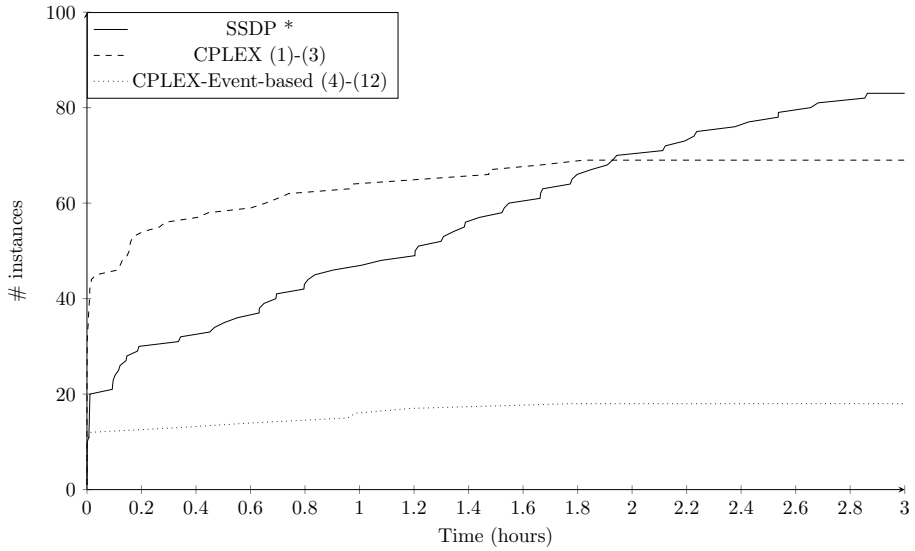

Figure 10: Number of instances solved along time for our best algorithm (SSDP*) and an ILP solver solving model (1)-(3) (CPLEX) and model (4)-(12) (CPLEX-Event-based), using a single thread.

type U. For $W = 10000$, it is able to solve 14 instances, while SSDP is not able to solve any instance of this set, because the memory needed to store the graph representation of the first dynamic program is too large.

6. Conclusion

In this paper we have proposed a new algorithm for solving the temporal knapsack problem. It is based on an exponentially large dynamic program, which is solved effectively using SSDP. With the help of several refinements that we describe, our method is able to obtain results that are competitive with those of the literature. The strategies that we propose are subject to many parameters, and the numerical experiments suggest that better parameterization could yield better results (notably the Hybrid strategy). The most crucial ingredient is the choice of the constraints to add during each sublimation phase. Machine learning algorithms could be an option both for fine-tuning proposed approaches and guiding the selection of constraints in a more generalized way. Several techniques used in this manuscript could be easily adapted to other applications of SSDP. We plan to develop a generic library providing these features for the community.

7. Acknowledgements.

We would like to thank Fabio Furini and Enrico Malaguti for sending us the detailed results of their experiments. We also would like to thank Timo Gschwind for running additional experiments for us.

This work was funded by Investments for the future Program IdEx Bordeaux, Cluster of Excellence SySNum.

Experiments presented in this paper were carried out using the PLAFRIM experimental testbed, being developed under the Inria PlaFRIM development action with support from Bordeaux INP, LABRI and IMB and other entities: Conseil Régional d'Aquitaine, Université de Bordeaux, CNRS and ANR in accordance to the programme d'investissements d'Avenir (see `https://www.plafrim.fr/`).

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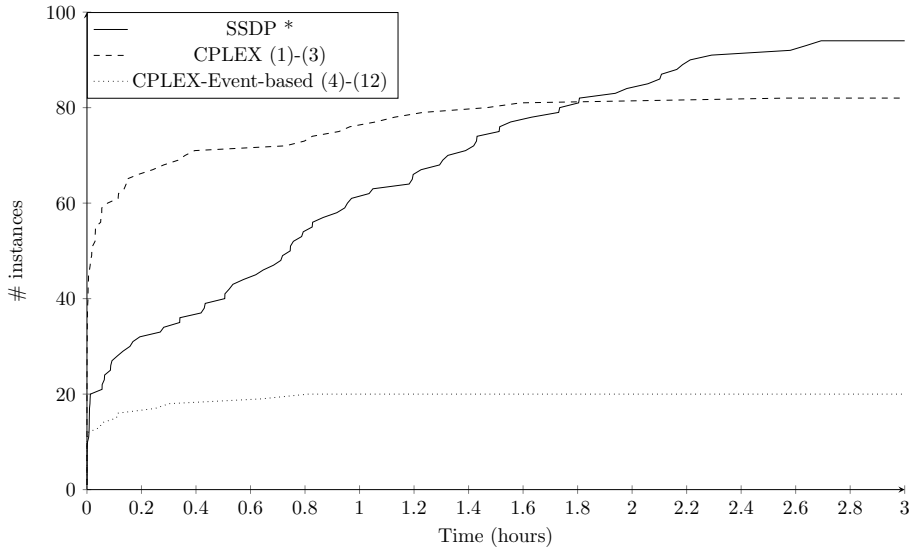

Figure 11: Number of instances solved along time for our best algorithm (SSDP*) using 6 threads and an ILP solver on model (1)-(3) (CPLEX) and model (4)-(12) (CPLEX-Event-based), using 6 threads.

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Appendix

Proofs for the validity of the filtering process

The following lemma shows that the set of arcs going out of each vertex of a refined relaxation related to $\mathcal{J}$' is a subset of the arcs going out of the vertex, in the relaxation related to $\mathcal{J}$, it comes from through sublimation.

Lemma 1. *Let us consider $e \in \{1,\dots,2n+1\}$, $w \in \{0,\dots,W\}$, $\mathcal{J}$ and $\mathcal{J}$' such that $\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{I}$, $v = (e,w,\mathcal{C}) \in V_{\mathcal{J}}$ and $v' = (e,w,\mathcal{C}') \in V_{\mathcal{J}'}$ such that $\mathcal{C} \cap \mathcal{J} = \mathcal{C}' \cap \mathcal{J}$ (i.e. vertex v' comes from the sublimation of vertex v). Then $\cup_{s \in \mathcal{S}_{\mathcal{J}}((e,w,\mathcal{C}))} \psi(s) \supseteq \cup_{s \in \mathcal{S}_{\mathcal{J}'}((e,w,\mathcal{C}'))} \psi(s)$.*

Proof. Let $(e,w,\mathbf{d}) \in S_{\mathcal{J}'}(e,w,\mathcal{C}')$. Then by definition of $S_{\mathcal{J}}$, for all $i \in \mathcal{J}'$, $d_i = 1 \leftrightarrow i \in \mathcal{C}'$. Since $\mathcal{J} \subset \mathcal{J}'$, for all $i \in \mathcal{J}$, $d_i = 1 \leftrightarrow i \in \mathcal{C}'$, and so $d_i = 1 \leftrightarrow i \in \mathcal{C}$ because $\mathcal{C} \cap \mathcal{J} = \mathcal{C}' \cap \mathcal{J}$. Thus $(e,w,\mathbf{d}) \in S_{\mathcal{J}'}(e,w,\mathcal{C})$, from which the result follows. □

The next lemma formally shows that for a given vector of multipliers $\boldsymbol{\pi}$ the Lagrangian cost of taking a decision from a specific state cannot increase when the relaxation is refined.

Lemma 2. *Let us consider $e \in \{1,\dots,2n+1\}$, $w \in \{0,\dots,W\}$, $\mathcal{J}$ and $\mathcal{J}$' such that $\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{I}$, $v = (e,w,\mathcal{C}) \in V_{\mathcal{J}}$ and $v' = (e,w,\mathcal{C}') \in V_{\mathcal{J}'}$ such that $\mathcal{C} \cap \mathcal{J} = \mathcal{C}' \cap \mathcal{J}$ (i.e. vertex v' comes from the sublimation of vertex v) and $\boldsymbol{\pi} \in \mathbb{R}^n$ a vector of Lagrangian multipliers. Then $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e,w,\mathcal{C}) \geq \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e,w,\mathcal{C}')$ and $\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}}(e,w,\mathcal{C}) \geq \hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e,w,\mathcal{C}')$.*

Proof. We proceed by induction on e to prove the part of the proposition involving $\hat{\alpha}$. A straightforward adaptation of the proof yields the result for $\hat{\gamma}$. At rank $e = 2n+1$, the property is satisfied since we have $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e,0,\mathcal{C}) = \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e,0,\mathcal{C}') = 0$ and $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e,w,\mathcal{C}) = \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e,w,\mathcal{C}') = -\infty$ if $w \neq 0$, $\mathcal{C} \neq \emptyset$ or $\mathcal{C}' \neq \emptyset$.

At rank $e \in \{1,\dots,2n\}$, assume that $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e+1,\bar{w},\bar{\mathcal{C}}) \geq \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e+1,\bar{w},\bar{\mathcal{C}}')$ for all $(e+1,\bar{w},\bar{\mathcal{C}}) \in V_{\mathcal{J}}$ and $(e+1,\bar{w},\bar{\mathcal{C}}') \in V_{\mathcal{J}'}$ such that $\bar{w} \in \{0,\dots,W\}$ and $\bar{\mathcal{C}} \cap \mathcal{J} = \bar{\mathcal{C}}' \cap \mathcal{J}$. Then for all $(e,w,\mathcal{C}) \in V_{\mathcal{J}}$ and $(e,w,\mathcal{C}') \in V_{\mathcal{J}'}$ such that $w \in \{0,\dots,W\}$ and $\mathcal{C} \cap \mathcal{J} = \mathcal{C}' \cap \mathcal{J}$, and for all $(\Delta_e,\Delta_w,\Delta_{\mathbf{d}},p) \in \cup_{s \in \mathcal{S}_{\mathcal{J}}((e,w,\mathcal{C}))} \psi(s)$, we have

$$\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e+\Delta_e, w+\Delta_w, \mathcal{C}_+) \geq \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e+\Delta_e, w+\Delta_w, \mathcal{C}'_+)$$

with $\mathcal{C}_+ = \mathcal{C} \cup \{i \in \mathcal{I} : (\Delta_d)_i = 1\} \setminus \{i \in \mathcal{I} : (\Delta_d)_i = -1\}$ and $\mathcal{C}'_+ = \mathcal{C}' \cup \{i \in \mathcal{I} : (\Delta_d)_i = 1\} \setminus \{i \in \mathcal{I} : (\Delta_d)_i = -1\}$. Indeed, $\mathcal{C}_+ \cap \mathcal{J} = \mathcal{C}'_+ \cap \mathcal{J}$. From (15) and Lemma 1, we have that $\hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e,w,\mathcal{C}) \geq \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e,w,\mathcal{C}')$. □

The proposition below is crucial for the efficiency of SSDP algorithm: it shows that once an arc is eliminated from a graph at a given iteration, all corresponding arcs can be removed from the graphs built during subsequent iterations.

Proposition 9. *Let LB be a valid lower bound for the problem, $\mathcal{J} \subseteq \mathcal{I}$, $a \in A_{\mathcal{J}}$ such that $\tau(a) = (e^1, w^1, \mathcal{C}^1)$, $h(a) = (e^2, w^2, \mathcal{C}^2)$, $\mu(a) = (\Delta_e, \Delta_w, \Delta_{\mathbf{d}}, p)$, and $\boldsymbol{\pi} \in \mathbb{R}^n$. If $\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}}(e^1, w^1, \mathcal{C}^1) + < \Delta_{\mathbf{d}}, \boldsymbol{\pi} > + \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}}(e^2, w^2, \mathcal{C}^2) < LB$, then the shortest path problem in $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ without arc a is a relaxation of (4)-(12). Moreover, for any $\mathcal{J}' \supseteq \mathcal{J}$ and $\boldsymbol{\pi}' \in \mathbb{R}^n$, the shortest path problem $G^{\boldsymbol{\pi}'}_{\mathcal{J}'}$ without any arc related to transition $\mu(a)$ from states $(e^1, w^1, \mathcal{C}^{1'})$ such that $\mathcal{C}^{1'} \cap \mathcal{J} = \mathcal{C}^1 \cap \mathcal{J}$ is a relaxation of (4)-(12) as well.*

Proof. First, we prove the validity of the proposition for the same vector $\boldsymbol{\pi}$ and a relaxation refined by enforcing a larger set of consistency constraints $\mathcal{J}'$. Let us consider arc $b \in \mathcal{A}_{\mathcal{J}'}$, such that $\mu(a) = \mu(b)$, $\tau(b) = (e^1, w^1, \mathcal{C}^{1'})$ such that $\mathcal{C}^{1'} \cap \mathcal{J} = \mathcal{C}^1 \cap \mathcal{J}$. Then $h(b) = (e^2, w^2, \mathcal{C}^{2'})$, such that $\mathcal{C}^{2'} \cap \mathcal{J} = \mathcal{C}^2 \cap \mathcal{J}$. Indeed, $\mathcal{C}^{2'} \cap \mathcal{J} = (\mathcal{C}^{1'} \cap \mathcal{J}) \cup (\{i \in \mathcal{I} : (\Delta_d)_i = 1\} \cap \mathcal{J}) \setminus \{i \in \mathcal{I} : (\Delta_d)_i = -1\} = \mathcal{C}^1 \cup (\{i \in \mathcal{I} : (\Delta_d)_i = 1\} \cap \mathcal{J}) \setminus \{i \in \mathcal{I} : (\Delta_d)_i = -1\} = \mathcal{C}^2 \cap \mathcal{J}$. Hence Lemma 2 implies that $\hat{\gamma}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e^1, w^1, \mathcal{C}^{1'}) + < \Delta_{\mathbf{d}}, \boldsymbol{\pi} > + \hat{\alpha}^{\boldsymbol{\pi}}_{\mathcal{J}'}(e^2, w^2, \mathcal{C}^{2'}) < LB$. Arc b being part of a feasible solution of the relaxation defined by $G^{\boldsymbol{\pi}}_{\mathcal{J}'}$ that is optimal for the problem would contradict LB being a lower bound. It follows that b can be removed from $G^{\boldsymbol{\pi}}_{\mathcal{J}'}$ that shall still define a relaxation of (4)-(12).

Second, we prove the validity of the proposition for a fixed set of consistency constraints $\mathcal{J}$ and a different vector of multipliers $\boldsymbol{\pi}'$. Lemma 1 shows that arc u cannot be part of a feasible solution of the relaxation associated with $G^{\boldsymbol{\pi}}_{\mathcal{J}}$ that would be optimal (and feasible) for the problem, since that would imply that LB is not a lower bound. Hence, no solution using a in $G^{\boldsymbol{\pi}'}_{\mathcal{J}}$, $\boldsymbol{\pi}' \in \mathbb{R}^n$ can be optimal for the problem, and removing a does not remove optimal solutions of the problem from those relaxations. □

An example of sublimation using information from the filtered graph

We now report in Figure .12 the graph obtained by eliminating in the graph of Figure 3 all arcs that only belong to paths whose profit is less than an upper bound equal to 202. The filtering also works if a value smaller than the optimal, although less arcs may be removed. Ke weep the same Lagrangian multipliers for the filtering.

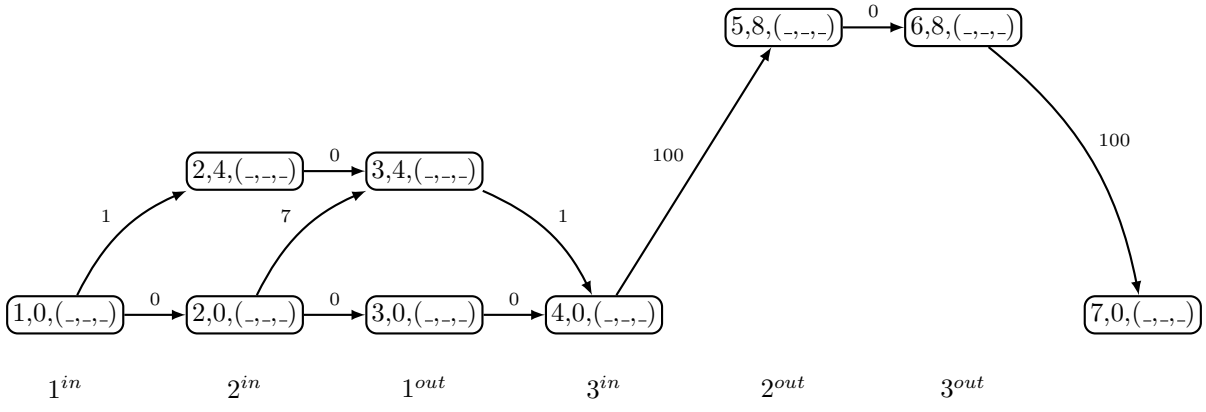

Figure .12: Graph representation of the relaxed DP for $\mathcal{J} = \emptyset$, with Lagrangian multipliers $(0, -3, 0)$ after filtering arcs whose lagrangian profit is not large enough (using a lower bound equal to 202).

In Figure .13, we report the graph obtained after the sublimation step that adds dimesion 2 to the state space. Note that only arcs that are represented by an arc in the graph of Figure .12 are created. Vertex $(3, 4, (\_, 1, \_)$ is eliminated, since the only possible arc from vertex $(3, 4, (\_, \_, \_)$ removes item 1, which would lead to an infeasible state (total weight of 0 and item 2 included in the knapsack).

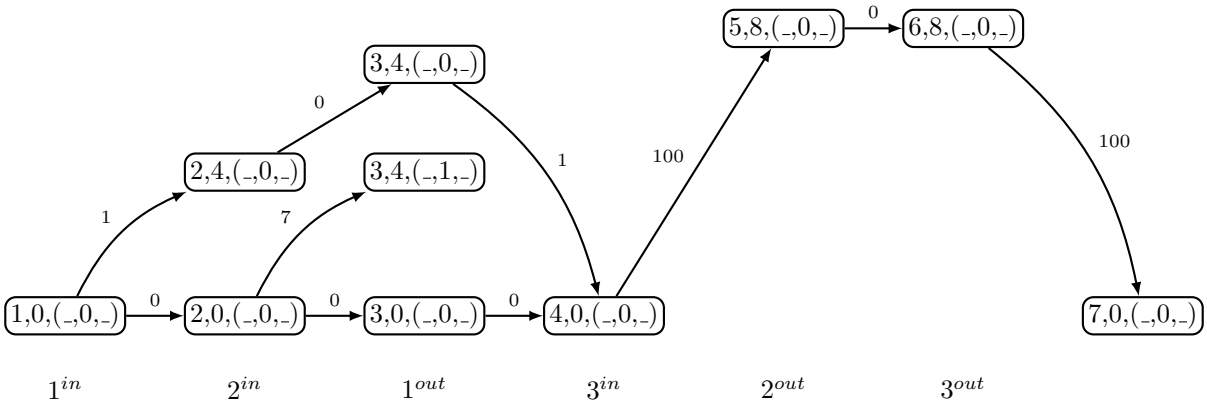

Figure .13: Graph representation built from the filtered graph of Figure .12 by adding dimension 2 to the state space.