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Pseudo-stochastic simulation of turbulent channel flows with near-wall modelling

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Abstract

The pseudo-stochastic model recently proposed by Mémin (2014) is investigated and compared with the large-eddy simulation methodology. The theoretical analysis shows that this model is a generalisation of the eddy-viscosity model, which does not undergo the same restrictive physical assumptions and describes physical phenomena usually not considered (turbophoresis and turbulent compressibility). Numerical simulations of turbulent channel flows are performed. In order to better reproduce the turbulence anisotropy, a near-wall damping function is derived and successfully validated: the damping is imposed only on wall-normal direction (minimal constraint) and it requires to set a single parameter (reduced empirical content). Simulations show the accuracy of the new model, especially when the computational grid becomes coarse. A weak turbophoresis phenomenon is detected near the wall, while turbulent compressibility effects appear to be possibly related to the streaks structures.

Keywords: Stochastic model, Turbulence, Near-wall models, Numerical simulations, OpenFOAM.

1. Introduction

2 The use of stochastic calculus to describe fluid flows appears to be a suitable
3 strategy for turbulence modelling in computational fluid dynamics. The random
4 nature of turbulence cannot be completely represented by means of deterministic
5 variables, while it is the specific purpose of stochastic processes. Nevertheless,
6 the numerical solution of stochastic equations and the mathematical complexity
7 inherent to the use of stochastic calculus poses challenging issues. Turbulence
8 modelling with stochastic variables is of great interest in geophysical flow anal-
9 ysis, where the unresolved processes related to coarse spatial discretisation are
10 handled with probabilistic models. In the same spirit, stochastic models can be
11 applied to numerical simulations of environmental and engineering flows.

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12 In the last decades, several efforts have been made in this concern. In the
13 context of the Probability Density Function (PDF), the Langevin equation was
14 used to describe the velocity of a fluid particle subject to a turbulent flow,
15 modelled as a Brownian motion (see [Pope \(2000\)](#)). First applications focused
16 on homogeneous isotropic turbulence; later extended by [Pope \(1983\)](#) to inho-
17 mogeneous case and by [Durbin and Speziale \(1994\)](#) to anisotropic diffusion
18 case. In the Eddy-Damped Quasi-Normal Markovian (EDQNM) models, intro-
19 duced by [Orszag \(1970\)](#) and [Leslie \(1973\)](#), the large-scale governing equations
20 were closed in the spectral space by modelling the third/fourth-order moments
21 through a Gaussian closure. This strategy was found to be suitable in case of
22 strong non-linearity in the small-scale turbulence. [Chasnov \(1991\)](#) developed
23 a forced-dissipative model, where the large-eddy Navier-Stokes equations were
24 corrected by an eddy-viscosity and a stochastic force terms. Similarly, [Leith
25 \(1990\)](#) studied the case of plane shear mixing layer and improved the accu-
26 racy of LES with the Smagorinsky model by adding an empirical stochastic
27 backscatter. The work of [Kraichnan \(1961\)](#) exploited a different approach: the
28 Navier-Stokes equations were replaced by a set of equations having the same
29 mathematical properties, closed by a Gaussian stochastic model. This model
30 led to valuable results when applied to the study of mathematical properties
31 and physical effects, like turbulent diffusion and backscatter. [Frederiksen et al.
32 \(2013\)](#) showed that the same methodology can be used in the stochastic mod-
33 elling of barotropic flows or in quasi-geostrophic approximation, as well as for
34 the description of the interactions between topography and small-scale eddies.

35 Such attempts to include random functions in fluid dynamics modelling ex-
36 hibit some limitations: in POF and EDQNM models the solution is found in the
37 spectral space instead of the physical one; the explicit introduction of random
38 term relies mostly on empirical considerations and leads to a certain degree of
39 arbitrariness. For example, a question arises whether the random forcing term
40 should be multiplicative or additive.

41 An alternative approach was developed. It is based on the idea that the
42 velocity field itself is a random process, composed of a differentiable component
43 and a fast oscillating random term. Physically, the former describes the smooth
44 macroscopic velocity while the latter accounts for the stochastic turbulent mo-
45 tion. Under this assumption, the fluid dynamics equations are re-derived using
46 stochastic calculus, leading to a complete set of stochastic partial differential
47 equations. Pioneering work in this sense was made by [Brzeźniak et al. \(1991\)](#).
48 Subsequently, [Mikulevicius and Rozovskii \(2004\)](#) and [Flandoli \(2011\)](#) expanded
49 his formulation and studied the mathematical properties of the resulting stochas-
50 tic system. Such a model has been further developed by [Mémín \(2014\)](#) in view
51 of practical applications and takes the name of model under Location Uncer-
52 tainty (LU). Later, [Neves and Olivera \(2015\)](#) theoretically investigate a similar
53 system, while [Holm \(2015\)](#) derives an equivalent model using Lagrangian me-
54 chanics. This last model differs from LU because an extra term appears in the
55 momentum equation, which ensures helicity and circulation conservation but
56 may alter the kinetic energy budget.

57 The LU model was tested in several cases: [Resseguier et al. \(2017a,b,c\)](#)

58 successfully used such type of model to study geophysical flows, which was found
59 to be more accurate in the reproduction of extreme events and provided new
60 analysis tools. [Chapron et al. \(2017\)](#) investigated the Lorentz-63 case and state
61 that LU is more effective in exploring the regions of the deterministic attractor
62 than the classical models. Furthermore, it was used in conjunction with the
63 proper orthogonal decomposition technique by [Resseguier et al. \(2017d\)](#) for
64 studying a wake flow past a circular cylinder at $Re = 3900$. Recently, [Pinier
65 et al. \(2019\)](#) perform mathematical analysis of the turbulent boundary layer
66 through the LU equations. They propose a complete explicit profile for the
67 mean vertical velocity, that includes an expression for the velocity in the buffer
68 layer, for which a rigorous theoretical model is missing so far.

69 Despite these encouraging results, to perform stochastic numerical simula-
70 tions for practical applications poses some difficulties; e.g. the numerical reso-
71 lution techniques are not straightforward and they can possibly require a large
72 computational effort. In order to circumvent such difficulties, [Mémín \(2014\)](#)
73 introduced a hybrid model hereafter named *pseudo-stochastic model*: first, the
74 governing equations are decomposed into two coupled system of partial and
75 stochastic differential equations; second, the resolution of the latter is avoided
76 and the system is closed by modelling the effects of the random velocity term
77 through physical assumptions. Hence, the flow dynamics is described by a set
78 of classical partial differential equations, which includes terms that derive from
79 the stochastic representation of turbulence. [Harouna and Mémín \(2017\)](#) used
80 the pseudo-stochastic model to investigate the Green-Taylor vortex flow, testing
81 several closure models. [Chandramouli et al. \(2018\)](#) successfully employed the
82 model to simulate the transitional wake flow with coarse mesh resolution.

83 The present contribution aims to explore the potentiality of the pseudo-
84 stochastic model, making a direct comparison with the Large-Eddy Simulation
85 (LES) methodology. First, the model is described and discussed in details;
86 then numerical simulations on the turbulent channel flows are performed and
87 analysed. The main novelties here reported are: a detailed study of the pseudo-
88 stochastic equations with respect to the classical ones; the derivation of a (re-
89 solved) turbulent kinetic energy budget for LU; the development of a near-wall
90 model for pseudo-stochastic simulations from the study carried out by [Pinier
91 et al. \(2019\)](#).

92 The paper is organized as follows: section 2 describes the pseudo-stochastic
93 equations, along with the kinetic energy budget and the near-wall model; sec-
94 tion 3 reports a physical interpretation of the equation terms, as well as a com-
95 parison with the LES methodology; section 4 presents the simulation method-
96 ologies and settings; section 5 discusses the validation of the near-wall model
97 and the simulation results; section 6 reports some final remarks.

98 2. Pseudo-stochastic model

99 The pseudo-stochastic equations are described, together with the kinetic
100 energy budgets. We refer to [Mémín \(2014\)](#) and [Resseguier \(2017\)](#) for the formal
101 derivation.

102 *2.1. Stochastic formalism*

103 The pathlines in a turbulent flow are modelled as a stochastic process, where
 104 a regular function is perturbed by a random (turbulent) process. Consequently,
 105 a Lagrangian fluid-particle displacement is described by a stochastic differential
 106 equation of the type:

$$dX_t^i(x_0) = w_i(X_t, t)dt + \int_{\Omega} \sigma_{ik}(X_t, y, t)dB_t^k(y) dy, \quad (1)$$

107 where the index $i = 1, 2, 3$ indicates respectively the x, y, z -component in the
 108 space domain Ω (they are placed at top or bottom indifferently) and the Ein-
 109 stein summation convention is adopted; X_t^i is the trajectory followed by a fluid-
 110 particle initially located in x_0 ; w_i is a differentiable function that corresponds
 111 to the drift velocity; $d\eta_t^i = \int_{\Omega} \sigma_{ik}dB_t^k dy$ is a stochastic process (accounting
 112 for turbulent effects) uncorrelated in time but correlated in space. This last is
 113 constructed as a combination of a cylindrical Wiener processes $B_t^k(x)$ not differ-
 114 entiable in time, and a time-differentiable symmetric *diffusion tensor* $\sigma_{ik}(x, y, t)$
 115 which acts as an integral kernel. Hence, they are fast oscillating stochastic com-
 116 ponents, possibly anisotropic and inhomogeneous in space.

117 The velocity field U_i in Eulerian coordinate x is derived from equation (1):

$$U_i(x, t) = w_i(x, t) + \dot{\eta}_t^i(x), \quad (2)$$

118 where the second term on the right-hand side expresses the stochastic velocity
 119 defined as the weak derivative of $\eta_t^i(x)$ in time. From a physical point of view,
 120 w_i is the velocity expected value and $\dot{\eta}_t^i(x)$ represents a noise: a generalised
 121 stochastic process that has to be defined in the space of temperate distribution,
 122 see [Øksendal \(2003\)](#).

123 In the derivation of the stochastic model, the quadratic variation of the
 124 diffusion tensor is of particular interest since it represents the time-variation
 125 of spatial variance of the stochastic increments along time. It is named as the
 126 *variance tensor* and it is defined as:

$$a_{ij}(x, t) = \int_{\Omega} \sigma_{ik}(x, y, t)\sigma_{kj}(x, y, t) dy, \quad (3)$$

127 it can be shown to be a symmetric and semi-positive definite matrix with di-
 128 mension [m^2/s].

129 *2.2. Pseudo-stochastic equations of motion*

130 The stochastic process (1) that described the flow is not time-differentiable
 131 in the framework of classical analysis. Thus, the Navier-Stokes equations need
 132 to be re-derived using the stochastic calculus, where the use of the Itô-Wentzell
 133 formula is crucial for computing the derivative in time, see [Kunita \(1997\)](#). The
 134 result is a complete system of stochastic partial differential equations that de-
 135 scribes the fluid flow. Assuming the drift velocity is of bounded variation (de-
 136 terministic) and using the unique decomposition of semi-martingale, the system

137 can be divided into a set of stochastic equations and a set of pure deterministic
 138 ones. The former allows finding an expression for the variance tensor a_{ij} , re-
 139 quired for the resolution of the latter. The pseudo-stochastic model is derived
 140 by neglecting the resolution of the stochastic equations and closing the system
 141 by giving an expression of the variance tensor, which is modelled through phys-
 142 ical hypothesis. This choice gives rise to a hybrid model where the terms that
 143 depend on a_{ij} accounts for the *Stochastic Unresolved Scales* (SUS) of motion.

144 The pseudo-stochastic equations for incompressible flows read:

$$\begin{cases} \frac{\partial w_i}{\partial t} + w_j^* \frac{\partial w_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 w_i}{\partial x_j \partial x_j} + \frac{1}{2} \frac{\partial}{\partial x_j} \left(a_{jk} \frac{\partial w_i}{\partial x_k} \right) \\ \frac{\partial w_i^*}{\partial x_i} = 0. \end{cases} \quad (4)$$

145 they represent the momentum and mass conservation, respectively, written in
 146 the non-conservative form proposed by Resseguier et al. (2017a). The *effective*
 147 *advection* velocity w^* is defined as:

$$w_i^* = w_i - \frac{1}{2} \frac{\partial a_{ik}}{\partial x_k}, \quad (5)$$

148 and the pressure is the sum of an hydrostatic pressure and an isotropic turbulent
 149 term:

$$p = p_h + \frac{\nu}{3} \frac{\partial w_\ell}{\partial x_\ell} = p_h + \frac{\nu}{6} \frac{\partial^2 a_{sk}}{\partial x_k \partial x_s}. \quad (6)$$

150 This last term does not contribute to the flow and it is included in the pressure
 151 gradient in the same manner as the isotropic residual stress in the Smagorinsky
 152 model, see Pope (2000).

153 It is worthwhile to notice that system (4) reduces to the classical Navier-
 154 Stokes equations when the variance tensor tends to the zero matrix, i.e. when
 155 the stochastic contributions disappear.

156 In the framework of computational fluid dynamics, the drift velocity w_i can
 157 be interpreted as the (numerically) resolved velocity field, while the random
 158 field η_t^i assembles the (turbulent) unresolved motions. Therefore, giving an
 159 expression on variance tensor is equivalent to specifying a turbulence model.

160 2.3. Resolved kinetic energy budget

161 Equations for mean and turbulent kinetic energy budget of the resolved
 162 scales of motion are here derived. The resolved velocity is decomposed in a
 163 mean and a fluctuating part, respectively:

$$w_i = W_i + w'_i, \quad (7)$$

164 where the capital letter indicates the averaged field, $W_i = \langle w_i \rangle$. Variance tensor
 165 and pressure are decomposed in a similar way: $a_{ij} = A_{ij} + a'_{ij}$ and $p = P + p'$.
 166 The variance tensor accounts for the SUS effects on the mean flow.

The budget of resolved kinetic energy $K = (W_i W_i)/2$ is obtained multiplying momentum equation (4-first) by W_i and averaging. Applying the conservation of mass (4-second) and rearranging the terms, one gets:

$$\frac{\partial K}{\partial t} + \left(W_j - \frac{\partial}{\partial x_k} \frac{A_{jk}}{2} \right) \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j} \left[-PW_j + \left(\nu \delta_{jk} + \frac{A_{jk}}{2} \right) \frac{\partial K}{\partial x_k} \right] \quad (8)$$

$$- \left\langle \left(w'_j - \frac{\partial}{\partial x_k} \frac{a'_{jk}}{2} \right) w'_i \right\rangle W_i + \left\langle a'_{jk} \frac{\partial w'_i}{\partial x_k} \right\rangle W_i \quad (9)$$

$$+ \frac{p'}{2} \frac{\partial^2 A_{jk}}{\partial x_j \partial x_k} - \left(\nu \delta_{jk} + \frac{A_{jk}}{2} \right) \frac{\partial W_i}{\partial x_j} \frac{\partial W_i}{\partial x_k} \quad (10)$$

$$+ \left\langle \left(w'_j - \frac{\partial}{\partial x_k} \frac{a'_{jk}}{2} \right) w'_i \right\rangle \frac{\partial W_i}{\partial x_j} - \left\langle \frac{a'_{jk}}{2} \frac{\partial w'_i}{\partial x_j} \right\rangle \frac{\partial W_i}{\partial x_k} \quad (11)$$

167 The second term on the left-hand side represents the rate of change by means of
 168 the effective (mean) advection. The first four terms on the right-hand side ex-
 169 press the energy transport by pressure, molecular and turbulent viscous stresses,
 170 resolved turbulence, turbulent SUS motion (respectively). The fifth term is due
 171 to the non-solenoidal velocity field and is related to the compression-expansion
 172 work made by the SUS; it can be a production or dissipation term. The sixth
 173 term is a viscous and turbulent dissipation (it can be proven that A_{ij} is positive
 174 defined), while the seventh term is a loss due to resolved turbulence; the same
 175 term but with opposite sign is present in the turbulent kinetic energy budget
 176 presented later in this section. The last term indicates dissipation/production
 177 due to SUS.

The (resolved) turbulent kinetic energy $\kappa = w'_i w'_i/2$ budget is obtained following the procedure described in Kundu and Cohen (2004): the equation for resolved fluctuations is obtained subtracting expression (8) from (4-first), then multiplying by w_i and averaging. Using the continuity equation (4-second) to simplify the terms and rearranging them, one obtains the following expression for stochastic Turbulent Kinetic Energy (TKE):

$$\begin{aligned} & \frac{\partial \langle \kappa \rangle}{\partial t} + \underbrace{\left(W_j - \frac{\partial}{\partial x_k} \frac{A_{jk}}{2} \right) \frac{\partial \langle \kappa \rangle}{\partial x_j} + \left\langle \left(w'_j - \frac{\partial}{\partial x_k} \frac{a'_{jk}}{2} \right) \frac{\partial \kappa}{\partial x_j} \right\rangle}_{\text{advection}} = \\ & = \underbrace{\frac{\partial}{\partial x_j} \left[-\langle p' w'_j \rangle + \left(\nu \delta_{jk} + \frac{A_{jk}}{2} \right) \frac{\partial \langle \kappa \rangle}{\partial x_j} + \left\langle \frac{a'_{jk}}{2} \frac{\partial \kappa}{\partial x_j} \right\rangle + \left\langle \frac{a'_{jk} w'_i}{2} \right\rangle \frac{\partial W_i}{\partial x_k} \right]}_{\text{transport}} \\ & + \underbrace{\left\langle \frac{p'}{2} \frac{\partial^2 a'_{jk}}{\partial x_j \partial x_k} \right\rangle}_{\text{turb. compress.}} - \underbrace{\left(\nu \delta_{jk} + \frac{A_{jk}}{2} \right) \left\langle \frac{\partial w'_i}{\partial x_j} \frac{\partial w'_i}{\partial x_k} \right\rangle - \left\langle \frac{a'_{jk}}{2} \frac{\partial w'_i}{\partial x_j} \frac{\partial w'_i}{\partial x_k} \right\rangle}_{\text{dissipation}} \\ & - \underbrace{\left\langle \left(w'_j - \frac{\partial}{\partial x_k} \frac{a'_{jk}}{2} \right) w'_i \right\rangle \frac{\partial W_i}{\partial x_j}}_{\text{production}} - \underbrace{\left\langle \frac{a'_{jk}}{2} \frac{\partial w'_i}{\partial x_j} \right\rangle \frac{\partial W_i}{\partial x_k}}_{\text{loss to SUS}} \quad (12) \end{aligned}$$

178 On the left-hand side, the second and third terms represent the TKE advection
 179 by mean and SUS effective advection velocity, respectively. On the right-hand
 180 side:

- 181 • the first four terms express spatial transport;
- 182 • the fifth term is a turbulent compression/expansion term due to SUS;
- 183 • the sixth and seventh terms account for dissipation by molecular viscosity,
 184 resolved turbulence and SUS motions;
- 185 • the eighth term represents the shear production, including the contribution
 186 by the fluctuations of turbulent advection velocity;
- 187 • the last term indicates a loss due to SUS; this term is also present in the
 188 resolved kinetic energy budget.

189 Both the kinetic energy and TKE expressions reduce to the classical ones if the
 190 stochastic contribution is negligible $a_{ij} \simeq 0$.

191 2.4. Isotropic constant model for variance tensor

192 Several strategies can be adopted to model the variance tensor. The isotropic
 193 model is developed by analogy with the Smagorinsky model, e.g. see [Deardorff](#)
 194 (1970), and was first proposed by [M emin](#) (2014). The variance tensor is given
 195 by:

$$a_{ij} = c_m \Delta^2 |S| \delta_{ij}, \quad (13)$$

196 where c_m is a model parameter, $|S|$ is the strain-rate tensor norm, and Δ is
 197 the computational cell width. The variance tensor reduces to a diagonal matrix
 198 with equal elements because turbulence is assumed isotropic and homogeneous
 199 in all directions.

200 2.5. Near-wall modelling of variance tensor

201 In a very recent work, [Pinier et al.](#) (2019) studied the mean velocity profile
 202 of the turbulent boundary layer through the LU equations. They proposed
 203 a modification of the classical velocity expression for wall-bounded flow and
 204 provided an analytical formula for the buffer layer, not available till now. Notice
 205 that the modified advection velocity plays a crucial role in the mathematical
 206 derivation of this formula; therefore, such a profile cannot be deduced using
 207 the classical formulation of the Navier-Stokes equations, where the modified
 208 advection is not explicitly taken into account. In the viscous sublayer ($y^+ < y_0^+$)
 209 and in the logarithm region ($y_L^+ < y^+ < y_1^+$) the linear and log-law velocity
 210 profiles (respectively) are retrieved, while in the buffer layer ($y_0^+ < y^+ < y_L^+$) a
 211 hyperbolic profile is specified:

$$u^+(y^+) = \begin{cases} y^+ & y^+ \in [0, y_0^+] \\ u^+(y_0^+) + \frac{2}{\tilde{\kappa}} - \frac{4}{\tilde{\kappa} [(\tilde{\kappa} y^+ - y_0^+) + 2]} & y^+ \in [y_0^+, y_L^+] \\ u^+(y_L^+) + \frac{4y_L^+}{[\tilde{\kappa}(y_L^+ - y_0^+) + 2]^2} \ln\left(\frac{y^+}{y_L^+}\right) & y^+ \in [y_L^+, y_1^+] \end{cases} \quad (14)$$

212 where $u(y)$ is the streamwise velocity as a function of the wall-normal coordinate.
 213 Quantities are made non-dimensional by means of the friction velocity u_τ and
 214 molecular viscosity ν , as usual: $y^+ = yu_\tau/\nu$ and $u^+ = u/u_\tau$. The $\tilde{\kappa}$ is a model
 215 constant (to not be confused with the von Kármán constant); for a plain channel
 216 flow it has been estimated to be $\tilde{\kappa} = 0.158$ from direct numerical simulations.
 217 The boundaries of the three regions are: $y_0^+ \simeq 5$, $y_L^+ \simeq 50$, and $y_1^+ \simeq 150$ even if
 218 the profile is often extended till the half of the channel. Let us stress that these
 219 profiles are rigorously derived from the LU models. See [Pinier et al. \(2019\)](#) for
 220 an extensive validation on the pipe flow, turbulent boundary layer, and channel
 221 flows.

222 An additional result concerns the expression of the variance tensor. In the
 223 viscous sublayer, a_{ij} is almost zero, while in the buffer layer the wall-normal
 224 component depends only from the distance from the wall and exhibits a linear
 225 profile:

$$a_{yy}^+(y^+) = \tilde{\kappa} (y^+ - y_0^+), \quad (15)$$

226 where $a_{ij}^+ = a_{ij}/\nu$. In the log-law region, it scales as the square-root of the wall
 227 distance:

$$a_{yy}^+(y^+) = \tilde{\kappa} (y_L^+ - y_0^+) \sqrt{y^+/y_L^+}. \quad (16)$$

228 No estimations are provided for the other components.

229 Preliminary pseudo-stochastic simulations with the isotropic constant model
 230 (13) have shown an excessive energy dissipation near the solid boundaries, given
 231 by high values of a_{ij} in the buffer and viscous layer. This is not unexpected since
 232 the LES Smagorinsky model (that is the classical counterpart of the isotropic
 233 model) exhibit the same behaviour (see discussion in following section 3.2).

234 To correct this behaviour, a damping function for variance tensor is here
 235 formulated, exploiting the above-described characterisation of wall-normal compo-
 236 nent. Away from the wall, a_{yy} is given by the isotropic model; at a point y_B^+
 237 placed in the buffer layer, a linear decrease is imposed in such a way to reach
 238 the zero value at y_0^+ ; in the viscous sublayer, it is set to be zero. Hence, the LU
 239 near-wall model reads:

$$a_{yy}^+(y^+) = \begin{cases} 0 & y^+ \in [0, y_0^+] \\ a_{yy}^+(y_B) \frac{y^+ - y_0^+}{y_B^+ - y_0^+} & y^+ \in [y_0^+, y_B^+] \\ \frac{c_m \Delta^2}{\nu} |S| \delta_{ij} & y^+ \in [y_B^+, y_1^+] \end{cases} \quad (17)$$

240 The coordinate y_B^+ is a model parameter that have to be set after theoretical
 241 or numerical estimation. No constraints are imposed on the other components;
 242 they are computed according to the isotropic model (13).

243 3. Physical interpretation and comparison with LES models

244 The pseudo-stochastic equations (4) are analysed from a physical point of
 245 view, and a comparison with the eddy-viscosity model used in LES is reported.

246 *3.1. Physical interpretation*

Recalling the decomposition of the velocity gradient as the sum of the symmetric and the antisymmetric part, respectively the strain-rate tensor $S_{ij} = \frac{1}{2} (\partial w_i / \partial x_j + \partial w_j / \partial x_i)$ and the rotation-rate tensor $\Omega_{ij} = \frac{1}{2} (\partial w_i / \partial x_j - \partial w_j / \partial x_i)$, the pseudo-stochastic equations (4) are rearranged as:

$$\frac{\partial w_i}{\partial t} + \left(w_j - \frac{1}{2} \frac{\partial a_{jk}}{\partial x_k} \right) \frac{\partial w_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\left(2\nu \delta_{jk} + \frac{a_{jk}}{2} \right) S_{ki} \right] - \frac{\partial}{\partial x_j} \left(\frac{a_{jk}}{2} \Omega_{ki} \right), \quad (18)$$

247 and

$$\frac{\partial w_i}{\partial x_i} = \frac{1}{2} \frac{\partial^2 a_{jk}}{\partial x_j \partial x_k}. \quad (19)$$

248 The terms that depend on variance tensor account for the influence of the SUS
249 on the resolved scales. A physical interpretation of such terms is proposed:

250 **Effective advection:** the advection velocity is corrected by an inhomogeneous
251 turbulence contribution. As pointed out by Resseguier et al. (2017a), it
252 corresponds to a velocity induced by the unresolved eddies, that is linked
253 to the *turbophoresis* phenomenon detectable in geophysical flows; i.e. the
254 tendency of fluid-particle to migrate in the direction of less energetic tur-
255 bulence.

256 **Diffusion due to SUS:** the last two terms on the right-hand side of equation
257 (18) account for the turbulent diffusion; the variance tensor plays the
258 role of a diffusion tensor similar to a generalised eddy-viscosity coefficient.
259 Both the deformation rate and rotation-rate contribute to diffusion, unlike
260 in the classical eddy-viscosity model in which fluid rotation-rate is assumed
261 to be irrelevant in turbulent modelling.

262 **Turbulent compressibility:** the continuity equation (19) suggests that the
263 flow is turbulent-compressible; i.e. the unresolved turbulence induces a
264 local fluid compression or expansion.

265 The variance tensor (3) is the key parameter of the pseudo-stochastic model.
266 It has the physical dimension of a dynamic viscosity [m^2/s], and carries infor-
267 mation on the intensity of the SUS. The role played in governing equations (4)
268 and in kinetic energy budgets (8)-(12), suggests that a_{ij} can be interpreted as
269 a generalised eddy-viscosity parameter. Implicitly, this leads to the hypothesis
270 that the SUS influences the resolved flow as an alteration (increasing or possi-
271 bly decreasing) of fluid viscosity, which is an empirical consideration largely
272 accepted.

273 The divergence of the variance tensor is hereafter named turbulent advection
274 velocity:

$$u_{\text{TA},i} = - \frac{1}{2} \frac{\partial a_{ij}}{\partial x_j}; \quad (20)$$

275 the divergence of the turbulent advection velocity measures the turbulent com-
 276 pressibility:

$$\Phi_{\text{TC}} = \frac{1}{2} \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}, \quad (21)$$

277 and it is directly proportional to the isotropic turbulent term appearing in equa-
 278 tion (6).

279 3.2. Comparison with LES eddy-viscosity models

280 In the classical framework, the fluid velocity $u(x, t)$ is a deterministic func-
 281 tion of time and space. Adopting the LES approach, the computational grid act
 282 on the governing equations as an implicit spatial filter (denoted by an over-bar)
 283 depending on the local cell width $\Delta = (\Delta x \Delta y \Delta z)^{1/3}$, see [Sagaut \(2000\)](#) and
 284 [Piomelli \(2001\)](#) for extended reviews. Filtering the Navier-Stokes equations, the
 285 sub-grid scale (SGS) stress tensor $\tau_{ij} = (\overline{u_i u_j} - \overline{u_i} \overline{u_j})$ appears:

$$\begin{cases} \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} \\ \frac{\partial \overline{u_i}}{\partial x_i} = 0 \end{cases} \quad (22)$$

286 and it has to be modelled to close the system: a popular choice is to use the
 287 eddy-viscosity models. They are a class of turbulent models relying on the
 288 Boussinesq assumption, where the anisotropic part of τ_{ij} is proportional to the
 289 resolved strain-rate tensor through ν_{SGS} the SGS viscosity parameter:

$$\tau_{ij}^R = \tau_{ij} - \frac{\tau_{kk}}{3} \delta_{ij} = -2\nu_{\text{SGS}} \overline{S}_{ij}, \quad (23)$$

290 while the isotropic part is included in the pressure term and does not contribute
 291 to the motion. This parameter has to be specified by additional models; the
 292 classical constant Smagorisky model is here analysed:

$$\nu_{\text{SGS}} = c_s^2 \Delta^2 |\overline{S}|, \quad (24)$$

293 where $|\overline{S}|$ is the norm of the filtered strain-rate tensor, and the parameter c_s^2
 294 is set constant and can be evaluated from experiments, direct numerical simu-
 295 lations or analytical considerations, e.g. see [Lilly \(1967\)](#). The main drawback
 296 of this approach is to rely on the homogeneous turbulence assumption. This
 297 hypothesis is violated in many, even simple, cases. For example, close to solid
 298 surfaces where the turbulent length-scales decrease. To cope with this short-
 299 coming, a damping function is usually introduced in order to account for the
 300 reduction of turbulence intensity. After the first work of [van Driest \(1956\)](#) sev-
 301 eral modifications of the original damping function have been proposed, e.g. see
 302 [Piomelli et al. \(1989\)](#) [Cabot and Moin \(2000\)](#). They can be summarised in the
 303 following expression:

$$\tilde{\Delta} = \min \left\{ \frac{\kappa y}{C_\delta} \left[1 - e^{\left(-\frac{y^+}{A^+} \right)^n} \right]^m, \Delta \right\}, \quad (25)$$

304 where $\kappa = 0.41$ is the von Kármán constant. The original formulation by van
305 Driest (1956) prescribe $n = m = 1$, $A^+ = 0.26$ and $C_\delta = 1.00$.

306 *Remarks on eddy-viscosity model*

307 Notice that the eddy-viscosity equation (23) implies that the Boussinesq’s
308 hypotheses are satisfied: (a) the anisotropic Reynolds stress tensor is aligned
309 with the mean strain-rate tensor; (b) the two are directly proportional through
310 a single parameter, equal for all the six independent components of τ_{ij}^R .

311 The pseudo-stochastic model is equivalent to an eddy-viscosity model if the
312 variance tensor is expressed by $a_{ij} = 2\nu_{\text{SUS}}\delta_{ij}$, i.e. assuming that the SUS induce
313 an (isotropic) increasing of fluid viscosity. In this sense, the pseudo-stochastic
314 model can be considered as a generalisation of the eddy-viscosity model. The
315 comparison between the two models points out some theoretical advantages of
316 the former:

- 317 (i) The effects of unresolved scales of motion are given by a_{ij} , without im-
318 posing any constraints on the directions along with the SUS acts on the
319 resolved flow. Hence, hypothesis (a) is not required.
- 320 (ii) The tensor form of a_{ij} allows reproducing the anisotropy of unresolved
321 turbulence, i.e. different turbulent contributions along different directions.
322 Thus, hypothesis (b) is not required.
- 323 (iii) The extra terms in the governing equations account for turbulent effects
324 usually not considered in the classical models, namely turbulent advection
325 and turbulent compressibility.

326 The eddy-viscosity models are reasonable for simple shear flows and it is largely
327 applied in computational fluid dynamics. However, most of their shortcomings
328 derive from the fact that hypotheses (a) and (b) are not generally satisfied; see
329 Pope (2000) for an overview on this issue.

330 It is worth mentioning that the eddy-viscosity parameter a_{ij} comes directly
331 from the basic assumption of the velocity decomposition (2); whereas it is in-
332 troduced in LES equations through an *ad hoc* physical assumption. Overall,
333 the pseudo-stochastic model represents a general approach that overcomes the
334 limitations of the Boussinesq assumption and includes turbulent effects not con-
335 sidered in the classical LES sub-grid scales models.

336 *Remarks on Smagorinsky model*

337 Expanding the pseudo-stochastic isotropic model (13), it can be shown that
338 it reduces to the LES Smagorinsky model under two approximations:

- 339 (i) the rotation-rate does not contribute to turbulence effects on the mean
340 flow;
- 341 (ii) the norm of the strain-rate tensor is almost harmonic (Laplacian is close
342 to zero).

343 Notice that with the latter hypothesis the continuity equation (4-second) boils
344 down to the classical solenoidal constraint. Therefore, the LES Smagorinsky

345 model can be interpreted as a particular case of the pseudo-stochastic isotropic
346 model.

347 Approximation (i) is valid if the turbulent energy is mainly concentrated in
348 the region where the irrotational strain dominates vorticity. Exceptions on this
349 behaviour have been found and have motivated the development of alternative
350 models, like the *wall adaptive local-eddy viscosity* (WALE) model of Nicoud and
351 Ducros (1999) or the structure function model of Métais and Lesieur (1992).
352 Approximation (ii) implies that the flow deformation rate can be represented by
353 a linear function in each spatial point; thus it is a particularly regular function.
354 This is equivalent to neglect the turbulent correction on advective velocity and
355 continuity equation, hence the associated physical phenomena of turbophoresis
356 and turbulent compressibility are not reproduced.

357 4. Simulation methodologies

358 The LU near-wall model (17) is validated on turbulent plain channel flow
359 at $Re_\tau = 395$. Subsequently, the pseudo-stochastic model is studied in detail
360 on channel flow at $Re_\tau = 590$. Several simulations are performed changing the
361 computational grid resolution, and the results of *pseudo-stochastic simulation*
362 (PSS) are compared with a LES and the *direct numerical simulation* (DNS) of
363 Moser et al. (1999).

364 4.1. Methodology and implementation

365 Simulations are performed taking advantage of the open-source software
366 OpenFOAM v6. This is a C++ library for computational fluid dynamics and
367 uses the finite volume method.

368 The LESs are carried out using the solver `pisoFoam` included in the stan-
369 dard software distribution. The implementation details can be found in the
370 OpenFOAM documentation and in Jasak et al. (1999). The filtered classi-
371 cal Navier-Stokes equations are closed by the Smagorinsky model (24), with
372 $c_s = 0.65$. The van Driest function (25) for near-wall damping is used unless
373 otherwise specified. The optimal parameters are set as $n = m = 1$, $A^+ = 0.26$,
374 $C_\delta = 0.158$, which lead to a formulations similar to the original one by van
375 Driest (1956).

376 The PSSs are carried out using the home-made solver `pseudoStochasticPisoFoam`,
377 developed by the authors at the Fluminance research group at INRIA Rennes
378 (France). The pseudo-stochastic equations (4) are solved employing the Pressure-
379 Implicit with Splitting of Operators (PISO) algorithm proposed by Issa et al.
380 (1986) and Oliveira and Issa (2001). The variance tensor is expressed by the
381 isotropic constant model (13), corrected by the near-wall damping function
382 (17) unless otherwise specified. The model constant is set to be $c_m = 2c_s^2$
383 in analogy with the Smagorinsky model. The damping parameter is set to be
384 $y_B^+ = 2/\tilde{\kappa} = 12.7$ after a theoretical estimation, confirmed by several test sim-
385 ulations. In order to regularise the damped profile of a_{yy} , a smoothing filter is
386 applied to the variance tensor.

Re_τ	MESH	grid points	$y_{wall}^+ \div \Delta y_{max}^+$	Δx^+	Δz^+	λ
395	FINE	$50 \times 80 \times 80$	$0.71 \div 25$	50	23	5.00
590	FINE	$96 \times 96 \times 96$	$0.71 \div 36$	40	20	5.25
	COARSE	$64 \times 64 \times 64$	$1.14 \div 48$	58	29	5.20
	VERY COARSE	$32 \times 64 \times 32$	$1.14 \div 48$	116	58	5.20

Table 1: Computational grid settings for numerical simulations of turbulent channel flow. The y_{wall}^+ is the coordinate of the first point near the wall.

387 Variables are discretised in space with a second-order central difference
388 scheme, while time integration is performed using an implicit Euler backward
389 scheme. Such a scheme employs the variables at the previous two time steps,
390 leading to a second order accuracy. Globally, numerical solvers are second-
391 order accurate in time and space. The time advancement fulfils the Courant-
392 Friedrichs-Lewy condition $Co < 0.5$. For LES, the Courant number is computed
393 as $Co = \Delta t |u| / \delta x$, where Δt is the time step, $|u|$ is the velocity magnitude
394 through the cell, δx is the cell length. For PSS, the definition is modified in
395 order to account for the effective advection velocity: $Co = \Delta t |w^*| / \delta x$.

396 4.2. Case geometry and settings

397 The channel is composed of two horizontal and parallel walls between which
398 a shear flow develops. The dimensions in stream-wise (x), vertical (y) and span-
399 wise (z) directions are $2\pi\delta \times \delta \times \pi\delta$, respectively. Several discretisation meshes
400 are employed, whose parameters are summarised in Table 1. The computational
401 points are uniformly distributed in streamwise and spanwise directions, while
402 the grid is stretched along the vertical direction. The stretching is symmetric
403 with respect to the channel center plane $y = \delta$, and it is obtained with a double-
404 side stretching function based on hyperbolic tangent:

$$y(\xi) = \frac{1}{2} \left(1 + \frac{\tanh(\lambda(\xi - 1/2))}{\tanh(\lambda/2)} \right), \quad (26)$$

405 where ξ is the vertical coordinate of uniform point distribution. The fine meshes
406 are such that the first cell is within $y^+ = 1$ and with 9 cells in $y^+ \leq 11$, and
407 the cell width in the wall-parallel plane are sufficient to ensure an accurate
408 resolution of the boundary layer. The coarse and very coarse meshes still have
409 a good vertical resolution but the streamwise and spanwise discretisation is
410 reduced.

411 Cyclic boundary conditions are set at the vertical boundaries, while veloc-
412 ity no-slip condition and pressure zero-gradient are imposed at the horizontal
413 walls. All the cases are initialised with the instantaneous fields provided by a
414 preliminary LES with the constant Smagorinsky SGS model, that has reached
415 the statistical steady state.

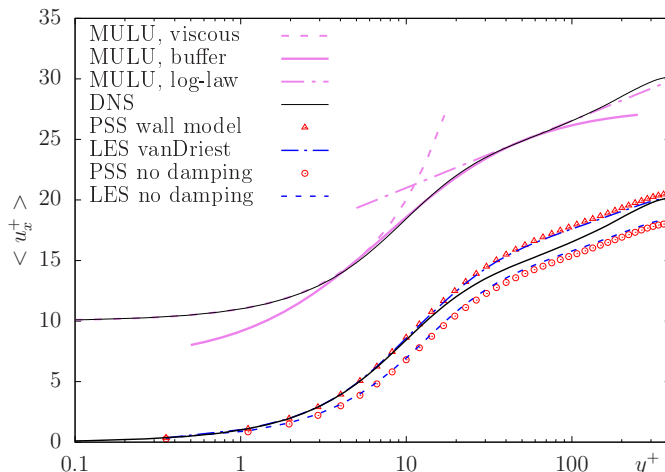


Figure 1: Non-dimensional mean velocity profiles along wall-normal direction for turbulent channel at $Re_\tau = 395$. Solid black, DNS by Moser et al. (1999). Top profiles: dash violet, analytical profile (17) derived from LU by Pinier et al. (2019). Bottom profiles: red symbols, PSS with near-wall model; red lines, LES with van Driest damping; blue symbols, PSS without near-wall model; blue lines, LES without van Driest damping.

4.3. Non-dimensional parameters

Quantities are made non-dimensional by the friction velocity u_τ and molecular viscosity ν as follow: space $x^+ = xu_\tau/\nu$, time $t^+ = tu_\tau^2/\nu$, velocity $u^+ = u/u_\tau$, variance tensor $a_{ij}^+ = a_{ij}/\nu$.

The flow is driven by a constant pressure gradient $\frac{\partial p}{\partial x} = -\rho u_\tau/\delta$; Reynolds number is set to $Re_\tau = u_\tau\delta/\nu$. The characteristic flow time is estimated to be $t_0 = U_0/2\pi\delta$, where U_0 is the bulk velocity in stream-wise direction, while the large-eddy turn over time is estimated to be $t^* = tu_\tau/\delta$.

5. Results and discussion

The following notation is adopted: if ϕ is a generic variable, then $\langle\phi\rangle$ is the time and space averaged over x, z -directions, $\phi' = \phi - \langle\phi\rangle$ is the instantaneous fluctuation and $[\phi]_{rms} = \sqrt{\langle\phi'^2\rangle}$ is the root-mean square. After the statistical steady state is reached, statistics are collected in an interval of $30t^* \sim 3t_0$ every $0.1t^*$.

5.1. Near-wall model assessment

The LU near-wall model for variance tensor is validated in the plane channel flow $Re_\tau = 395$. The computational grid is described in Table 1, and ensures a high resolution of the flow. Four simulations are performed: PSS that enforce the near-wall model, LES with van Driest damping, PSS and LES switching off the near-wall models.

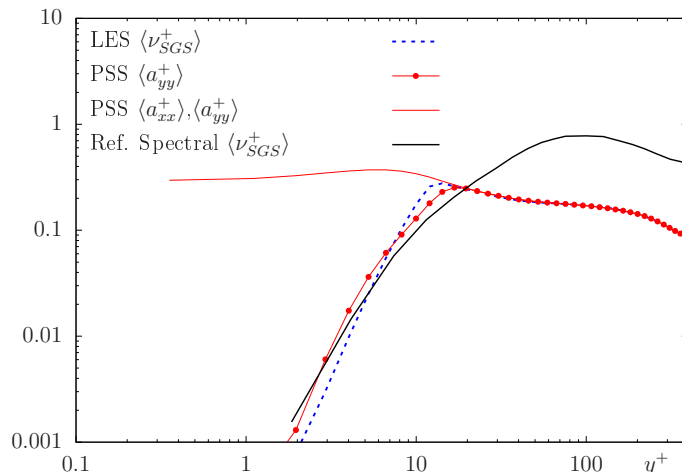


Figure 2: Non-dimensional mean eddy-viscosity parameters along wall-normal direction for turbulent channel at $Re_\tau = 395$. Dash blue, LES with van Driest damping; solid red, PSS wall-parallel components of a_{ij} ; circle red, PSS wall-normal component of a_{ij} ; solid black, well-resolved LES in [Armenio and Piomelli \(2000\)](#) with spectral code.

436 Figure 1 shows the non-dimensional mean streamwise velocity along ver-
437 tical direction. In the top-plot, analytical expression (14) for mean velocity
438 is compared with the DNS data: in all the three boundary layer regions, the
439 velocity profile is correctly described. Particularly, there is a good agreement
440 between the hyperbolic function and the reference data in the buffer layer. In the
441 bottom-plot, the results of the simulations with and without near-wall models
442 are reported. The data of the PSS and LES collapse one onto the other; hence
443 they are discussed together. As expected, when the near-wall models are dis-
444 abled, the velocity profile is underestimated. This is caused by a non-physical
445 high level of eddy-viscosity near the wall (see also discussion of Figure 2), that
446 induces a large energy dissipation. When the near-wall models are activated,
447 velocity is well captured in the viscous and buffer layer.

448 Figure 2 presents the non-dimensional mean eddy-viscosity parameters for
449 LES and PSS, respectively $\nu_{SGS}^+ = \nu_{SGS}/\nu$ and $a_{ij}^+ = a_{ij}/\nu$. Simulations are
450 compared with the SGS eddy-viscosity profile reported in [Armenio and Piomelli](#)
451 [\(2000\)](#). Such a profile is obtained from LES of the channel at $Re_\tau = 395$ with a
452 spectral code described in [Sarghini et al. \(1999\)](#). The size and the discretisation
453 of the computational domain are comparable to the one used here. The spectral
454 code implements the Lagrangian dynamic model of [Meneveau et al. \(1996\)](#),
455 where the eddy-viscosity is computed cell-by-cell, by comparing two scales of
456 motion and minimising the model error along a fluid particle trajectory. The
457 near-wall model has a crucial role in the correct damping of the eddy-viscosity
458 close to the wall, both for LES and PSS. The LU near-wall model appears to
459 accurately reproduce the slope in the region $5 < y^+ < y_B^+$, while the van Driest
460 model exhibits a larger deflection.

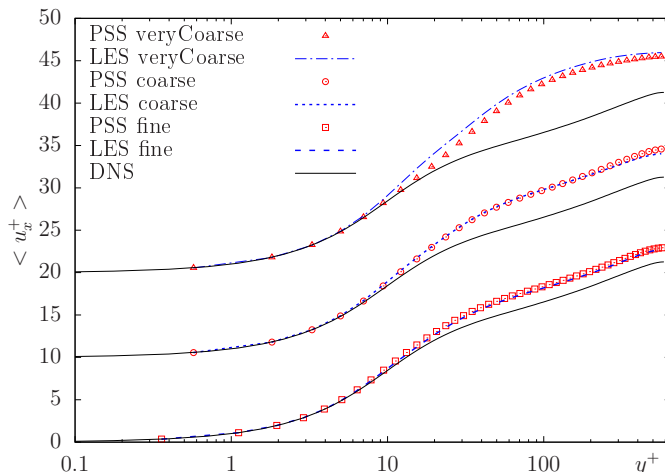


Figure 3: Non-dimensional mean velocity profiles along wall-normal direction of turbulent channel at $Re_\tau = 590$, for the three meshes described in Table 1. Solid lines, DNS by Moser et al. (1999); symbols, PSS; dash lines, LES.

461 Overall, the PSS with the near-wall model is able to reproduce the velocity
 462 profile as well as the LES. The eddy-viscosity profile is correctly reproduced
 463 near the wall, where the LU damping is in a good agreement with reference
 464 data. Notice that the damping is imposed only on the wall-normal component
 465 of the generalised eddy-viscosity tensor (a_{yy}), while no modification are required
 466 for the wall-parallel components a_{xx} , a_{zz} . Moreover, the only parameter to be
 467 set is the damping point y_B^+ . On the contrary, the class of van Driest functions
 468 (25) are applied to all the velocity components and required to choose several
 469 empirical parameters, which leads to larger empirical content.

470 5.2. Channel flow analysis

471 The PSS with near-wall damping is compared with LES van Driest damping
 472 on three different meshes with a decreasing resolution in wall-parallel directions
 473 (see Table 1) at $Re_\tau = 590$.

474 Figure 3 displays the non-dimensional streamwise velocity component. PSS
 475 and LES practically collapse on the same values. They exhibit accurate results in
 476 the inner-region ($y^+ < 50$) for all the meshes; whereas they tend to overestimate
 477 velocity in the outer-region ($50 < y^+$). Such overestimation increases as the
 478 computational grid degrades. For a very coarse grid, the PSS shows a slightly
 479 better profile with respect to LES in the buffer layer ($10 < y^+ < 30$), as a
 480 consequent of a different damping (see Figure 5)

481 Figure 4 reports the root-mean square (RMS) of velocity components. In
 482 general, no significant differences are detectable between PSS and LES. As expected,
 483 the profiles are more accurate as the mesh resolution increases. The
 484 streamwise RMS is overestimated and the peak moves from the buffer layer towards
 485 the log-law region as the mesh becomes coarser and coarser. Notice that

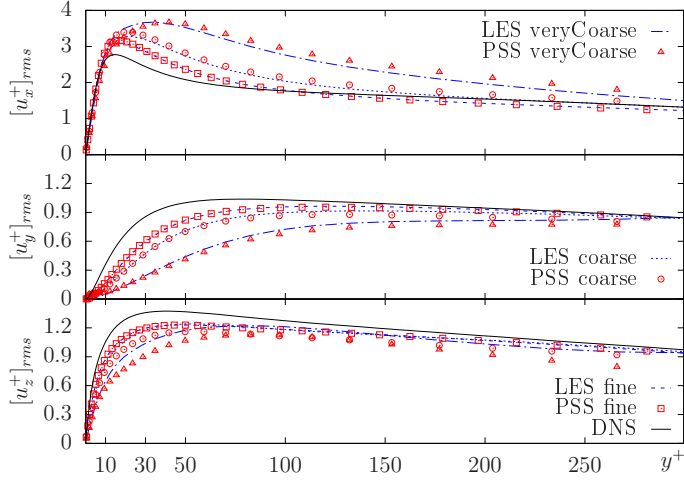


Figure 4: Non-dimensional root-mean square of velocity components along wall-normal direction of turbulent channel at $Re_\tau = 590$. Simulations with the three meshes described in Table 1. Same labels as in Figure 3.

486 in very coarse case, they assume lower values in the range ($10 < y^+ < 30$) for
 487 PSS than LES. The wall-parallel RMS are globally underestimated.

488 Figure 5 shows the non-dimensional mean eddy-viscosity for LES and vari-
 489 ance tensor components for PSS. The SGS eddy-viscosity and the wall-normal
 490 component a_{yy} have similar profiles, and they are discussed together below.
 491 They display common features for all the meshes used: in the viscous sublayer
 492 ($y^+ < 5$), they are practically null; in the buffer layer ($5 < y^+ < 30$), they
 493 rapidly increase and reach a peak in the range $y^+ \in [10, 15]$, after which a
 494 smooth decay starts. In the log-law region ($30 < y^+ < 150$), the profiles for
 495 fine and coarse meshes decrease moderately and they eventually reduce to low
 496 values at the channel centre; the profile for very coarse mesh reports a more
 497 regular slope. For the coarse meshes, ν_{SGS} shows slightly higher values than a_{yy}
 498 near the channel center. This is possibly caused by the particular numerical
 499 implementation of the LES Samgorinsky model. However, this does not affect
 500 velocity statistics. Their values are moderate for the fine and the coarse grids
 501 (maximum 50% of the molecular viscosity), whereas they are of the same order
 502 of molecular viscosity for very coarse mesh. Hence, the SUS/SGS model plays
 503 a crucial role in this last case. The coordinates of the peaks correspond to the
 504 points where the damping starts. This is set to a fixed value $y_B^+ = 12.7$ for
 505 the LU near-wall model, while it is variable for the van Driest model. With
 506 respect to the former, the latter is activated slightly closer to the wall in the
 507 fine mesh case, about at the same point in the coarse case, and slightly further
 508 from the wall in the very coarse case. In this last case, PSS provides a higher
 509 level of eddy-viscosity which reflects on the mean velocity and the streamwise
 510 RMS profiles (see Figures 3 and 4), which are closer to the reference data in the
 511 buffer region. These results validate the pertinence of the LU wall-law model.

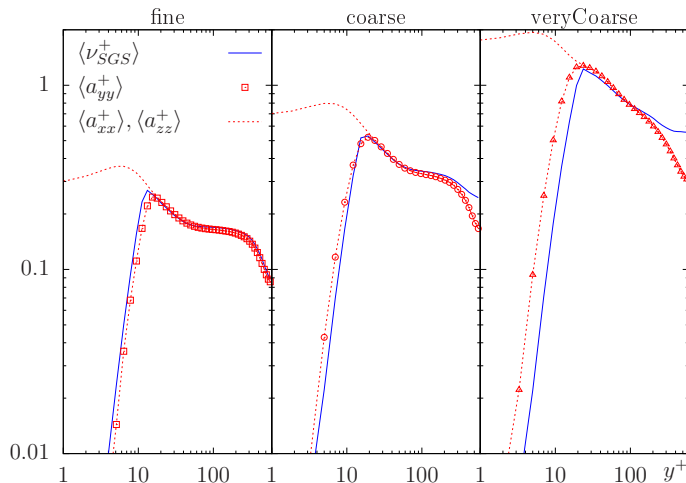


Figure 5: Non-dimensional mean eddy-viscosity parameters along wall-normal direction of turbulent channel at $Re_\tau = 590$. Simulations with the three meshes described in Table 1. Solid line, sub-grid scale eddy viscosity from LES with van Driest damping; symbols, wall-normal variance tensor component from PSS with near-wall damping; dash lines, wall-parallel variance tensor components from PSS with near-wall damping.

512 Figure 6 reports selected terms of the resolved TKE budget (12), averaged
 513 in time and wall-parallel planes, for the three meshes used. The time variation
 514 of TKE is made non-dimensional by the molecular viscosity. The equation for
 515 LES is obtained from (12) setting $a_{ij} = 0$, except in the dissipation term where
 516 $a_{ij} = \nu_{SGS}\delta_{ij}$ in order to account for the dissipative effect of the sub-grid model.
 517 The dissipation profiles of the PSS and LES are identical except in the region
 518 $y^+ < 20$ close to the wall, where the PSSs have lower values. This is mainly due
 519 to the fact that the wall-parallel component of variance tensor are not damped,
 520 but contribute to the energy dissipation term in equation (12). Dissipation is
 521 higher when the mesh degrades. The production terms are similar for PSS and
 522 LES: in the fine mesh case, they peak at $y^+ \simeq 15$ and $y^+ \simeq 13$ (respectively),
 523 in the coarse case they both peak at $y^+ \simeq 19$, while in the very coarse one at
 524 $y^+ \simeq 35$. It is worth to note that the PSS for the very coarse case yields a
 525 lower production close to the wall ($5 < y^+ < 20$), probably as an effect of the
 526 lower streamwise RMS (see also discussion Figure 5). The loss of energy due to
 527 SUS is only present in the pseudo-stochastic model; it assumes non-negligible
 528 negative values close to the wall ($10 < y^+$), and increases in magnitude as
 529 the mesh become coarser. It contributes to the total TKE dissipation. The
 530 turbulent compression/expansion term due to SUS is practically zero and does
 531 not contribute to the TKE budget.

532 5.3. Turbulent advection and compressibility

533 The additional terms that characterise the pseudo-stochastic model are here
 534 scrutinised. Turbulent advection (20) and the turbulent compressibility (21) are

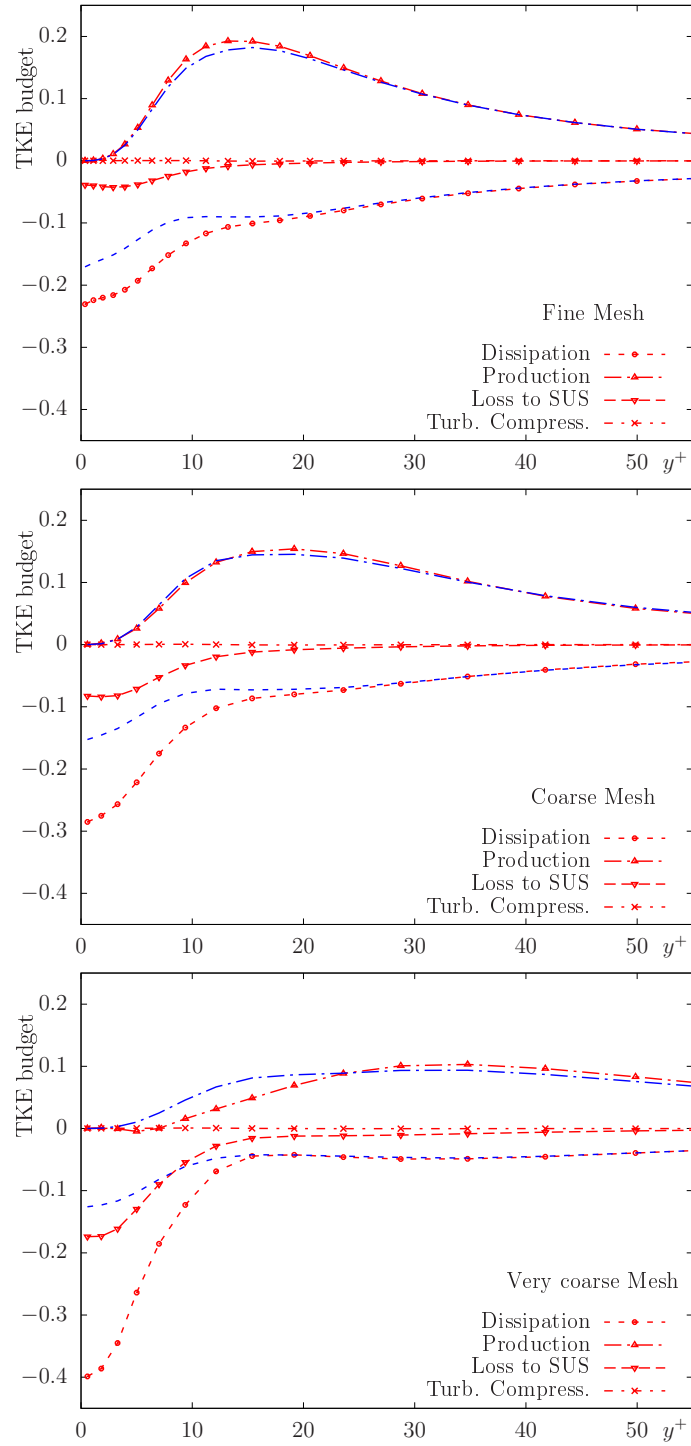


Figure 6: Non-dimensional Turbulent Kinetic Energy (TKE) budget (12) along the wall-normal direction. Simulations with the three meshes described in Table 1: from top to bottom: fine, coarse and very coarse mesh. Red lines with symbols, PSS. Blue lines without symbols, LES.

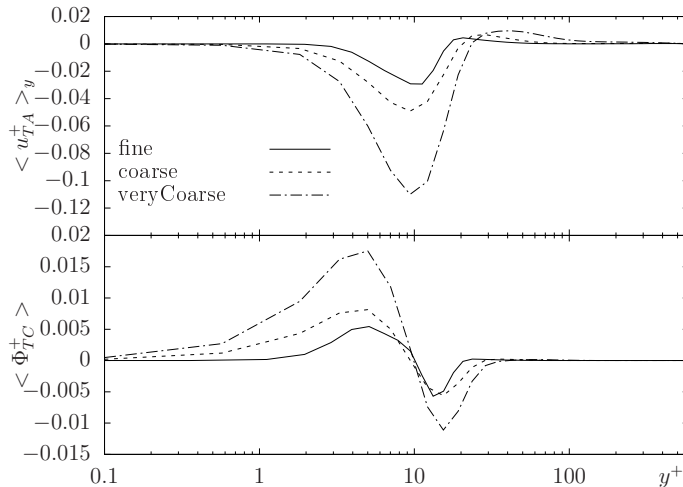


Figure 7: Non-dimensional turbulent advection (wall-normal component) u_{TA}^+ and non-dimensional turbulent compressibility Φ_{TC}^+ along wall distance. Average in time and wall-parallel directions. Simulations with the three meshes described in Table 1.

535 strongly connected with the variance tensor behaviour in Figure 5.

536 Figure 7 displays the wall-normal component of non-dimensional turbulent
 537 advection $u_{TA,y}^+ = u_{TA,y}/u_\tau$ and the non-dimensional turbulent compressibility
 538 $\Phi_{TC}^+ = \Phi_{TC}\nu/u_\tau^2$ along wall distance. The other components of u_{TA} are almost
 539 zero; thus, they are not reported. Globally, the magnitude of both quantities
 540 increases when the discretisation points decrease, since a larger part of the flow
 541 turbulence has to be modelled. In all the cases, wall-normal turbulent advection
 542 peaks at $y^+ = 10$ and is almost zero in the viscous sublayer and log-law region.
 543 The magnitude is quite small compared with the mean streamwise velocity: in
 544 the very coarse case, the peak of the vertical turbulent advection is 1.4% of
 545 the mean streamwise velocity at the same point. However, it generates a non-
 546 negligible vertical velocity that drives the flow towards the wall. This qualifies
 547 u_{TA} as a turbophoresis velocity, that advects the flow from a region of high
 548 to low turbulence level (quantified by the RMS velocity intensity). Turbulent
 549 compressibility presents a maximum at the end of the viscous sublayer $y^+ = 5$,
 550 and a minimum at y_B^+ . It assumes moderate values. When positive (negative),
 551 it can be associated with a sort of fluid expansion (contraction) of the fluid due
 552 to turbulence.

553 Figure 8 shows the Φ_{TC}^+ isosurfaces of negative (blue) and positive (orange)
 554 near the bottom wall, at the last time configuration. They are organised in
 555 spots elongated in the streamwise direction, confined in the buffer and viscous
 556 layer. In accordance with the mean profile, positive spots are closer to the wall,
 557 while the negative ones are immediately above. The shape and the position
 558 of these structures suggest a possible correlation with the streaks turbulent
 559 structures (e.g. see Chernyshenko and Baig (2005)); however, an additional

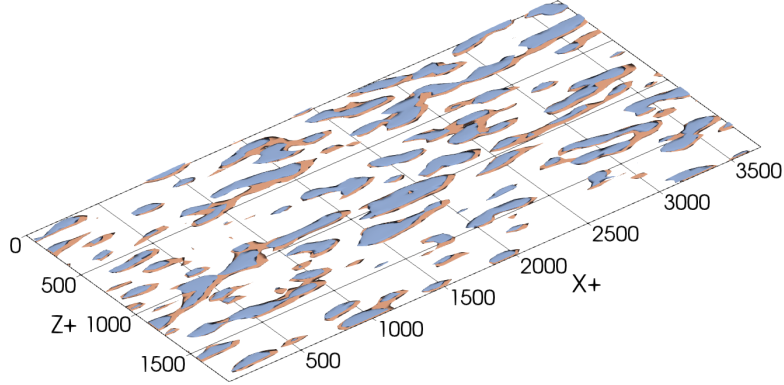


Figure 8: Positive and negative isosurfaces of Φ_{TC}^+ near the bottom wall. Orange: isosurfaces $\Phi_{TC}^+ = 0.5 \max(\Phi_{TC}^+) = 0.013$. Blue: isosurfaces $\Phi_{TC}^+ = 0.5 \min(\Phi_{TC}^+) = 0.0095$.

560 study is required to better investigate such a correlation.

561 6. Conclusion

562 The pseudo-stochastic model introduced by [Mémín \(2014\)](#) is investigated
 563 both mathematically and numerically. Such a model is shown to be a general-
 564 ization of the classical eddy-viscosity turbulent models, where the variance a_{ij}
 565 plays the role of an eddy-viscosity tensor. Turbulence effects are not limited
 566 to energy dissipation, but induce additional phenomena as turbulent advection
 567 and compressibility that are not usually considered. Moreover, it does not rely
 568 on the restrictive physical assumptions related to Boussinesq’s hypotheses. The
 569 turbulent kinetic energy budget is derived and presented, along with a near-
 570 wall model for a_{ij} that is inferred from the analysis of boundary layer by [Pinier
 571 et al. \(2019\)](#). A simple isotropic constant model for a_{ij} is adopted for numeri-
 572 cal simulation of turbulent channel flows, which are directly compared with an
 573 equivalent large-eddy simulation with the Smagorinsky model. In both cases, a
 574 near-wall damping function is used to correct the turbulent model in the prox-
 575 imity of the solid boundaries: the latter uses the classical van Driest function,
 576 the former employs the LU near-wall model here developed.

577 First, the LU near-wall model is successfully validated in the channel flow at
 578 $Re_\tau = 395$. The eddy-viscosity tensor is correctly damped and exhibits a better
 579 agreement than the van Driest function with a reference solution obtained by
 580 a highly accurate simulations model. It is worth noticing that the LU model
 581 acts only on the wall-normal direction and depends on one single parameter
 582 (theoretically estimated); in contrast to the classical model that damps eddy-
 583 viscosity for all the velocity components and requires to set several parameters.
 584 Hence, the former appears to impose a minimal correction and to have reduced

585 empirical content. Second, the channel flow at $Re_\tau = 590$ is simulated using
586 fine, coarse and very coarse meshes. Overall, the pseudo-stochastic simulations
587 with the LU near-wall model are as accurate as the classical techniques. They
588 show slightly better results when the computational grid is very coarse, since
589 the van Driest model tends to excessively damp the eddy-viscosity. The PSS
590 model is more effective in dissipating turbulent kinetic energy near the wall. A
591 weak turbulent advection velocity is detectable between the viscous and buffer
592 layer; such a velocity slightly advects the flow near the wall, from regions at
593 high to low turbulent level (with respect to velocity RMS intensity). Hence,
594 it is qualified as a turbophoresis phenomenon. In the same region, turbulent
595 compressibility displays moderate positive and negative values; the visualisation
596 of instantaneous isosurfaces suggests a possible link with the streaks turbulent
597 structures.

598 Finally, the pseudo-stochastic model is a promising alternative approach for
599 turbulent modelling, that generalises the classical models and describes a richer
600 physics. Mathematical and numerical investigations demonstrate the potential
601 of the model.

602 References

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