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# Inflation of digitally convex polyominoes

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## 1. Introduction

In the literature, many papers introduced and studied different convexity notions. For example, Kim and Rosenfeld in [1] investigated different notions of discrete convex sets, where a set in Euclidean geometry is convex if and only if for any pair of points  $p_1, p_2$  in a region  $R$ , the line segment joining them is completely included in  $R$ . In discrete geometry on square grids, this notion refers to the digitally convex convexity. We recall that a *polyomino* is a finite 4-connected set of unit squares in the lattice  $\mathbb{Z}^2$ . If  $P$  is a polyomino and if for all  $p_1, p_2$  inside  $P$  and such that the discret segment joining them is completely included in  $P$  then  $P$  is a digitally convex polyomino. Digitally convex polyominoes are also the discretization of convexes of  $\mathbb{R}^2$ , except when this discretization is not 4-connected. It implies that the intersection of two such polyominoes is also digitally convex, as soon as it is 4-connected.

In this talk, we would like to study discrete geometrical constructions to deflate or inflate digitally convex polyominoes.

## 2. How to deflate a digitally convex polyomino?

We first study how to deflate a digitally convex polyomino  $P$ . This polyomino  $P$  can be decomposed into four paths. Indeed, a polyomino  $P$  is a finite set thus we define the minimal bounded box which is a rectangle such that  $P$  touches the four sides of the rectangle. Those paths start at the first unit square of intersection with each side of the rectangle. We denote the unit squares of intersection by  $W$  (the lowest unit square on the leftmost side),  $N$  (the leftmost unit square in the top side),  $E$  (the highest unit square on the rightmost side) and  $S$  (the rightmost unit square on the bottom side). The contour of a convex polyomino is then the union of the four (clockwise) paths  $WN$ ,  $NE$ ,  $ES$  and  $SW$ .

We use the result given by Brlek et al. [2] where they introduced a link between convexity and combinatorics on words by encoding the contour of the convex polyomino. Their result was based on *Christoffel* and *Lyndon words*. They considered each path of the convex polyomino and used the alphabet  $A = \{0, 1, \bar{0}, \bar{1}\}$  to code the boundary of each polyomino where  $0, 1, \bar{0}, \bar{1}$  encode  $\rightarrow, \uparrow, \leftarrow, \downarrow$  respectively.

The main result in [2] states that a convex polyomino is characterized by the fact that the  $WN$  path admits a unique *Lyndon factorization*  $\ell_1^{n_1} \dots \ell_k^{n_k}$  where the  $\ell_i$ 's have decreasing slopes and they are primitive Christoffel words, and similar results for the three other paths.

It is not difficult to deflate the digitally convex polyomino. Indeed, we look at the set  $\{s_1, s_2, \dots, s_{k+1}\}$  of unit squares (called corners) in the boundary of the polyomino such that each  $s_i$  are exactly the unit square corresponding to the extremities of each factor  $\ell_j^{n_j}$  in the factorization given before (see Figure 1), or equivalently have a common unit square with the convex hull of the polyomino. Remove any square  $s_i$ , then the new polyomino is still digitally convex, and this process can be iterated.

This property has two consequences.

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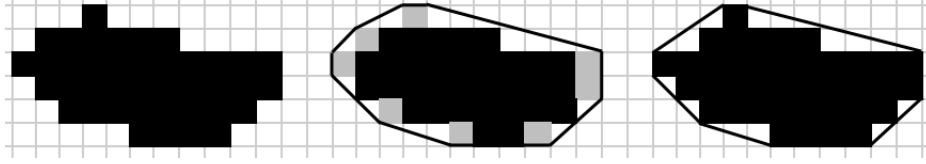


Figure 1: A digitally convex polyomino, its convex hull and its corners (grey unit squares) and a smaller digitally convex polyomino.

1. Given two digitally convex polyomino  $C_1 \subset C_2$ ,  $C_2$  can be deflated to  $C_1$  in such a way that at each step one unit square only is canceled and such that at each step we have a digitally convex polyomino: as the polyomino  $C_2$  is bigger than  $C_1$  it contains a corner which is not in  $C_1$ , then cancel this corner.
2. It is possible to inflate step by step from  $C_1$  to  $C_2$ , by the reverse process. However, it does not give a practical way to choose the unit square that we must add at each step.

### 3. How to inflate a digitally convex polyomino?

It is more difficult to get effective methods to inflate a convex polyomino  $C_1$  to a convex polyomino  $C_2$  where  $C_1 \subset C_2$  with the constraint that we must add a single unit square at each step and maintain the digitally convex property at each step until reaching  $C_2$ .

#### 3.1. The spiral and strate constructions

First of all we take  $C_1$  as a unit square anywhere inside  $C_2$ .

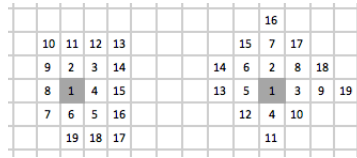


Figure 2: The spiral construction

We first investigate the spiral construction by adding a corner around the polyomino from a unit square polyomino by adding at each step a corner in a clockwise order (see Figure 2). This construction leads to an octogonal shape digitally convex polyomino. However, by keeping only those unit squares which are contained in  $C_2$  (see Figure 3 left and center), we get a global construction of  $C_2$  such that we maintain the convexity property at each step.

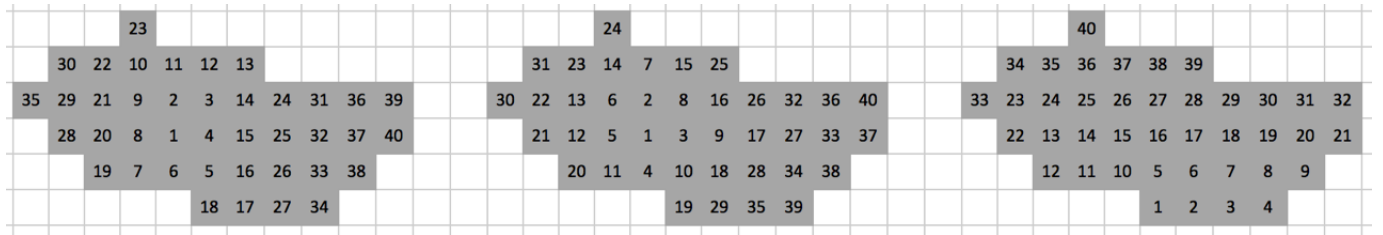


Figure 3: Two spiral and a strate construction of a digitally convex polyomino

In fact, we have many variants of this construction, by considering 4-connected spiral or 8-connected spiral or strate construction (see Figure 3). The strate construction consists in taking at first the lowest unit squares from the left to the right, and to continue to the second row in a correct order, and so on (see Figure 3, right). It has many variants.

If we try to inflate some digitally convex polyomino by the spiral method, it works only when we take special octogones (see Figure 4, left). In the general case, the convexity property disappears at some steps (see Figure 4, right, unit square number 8).

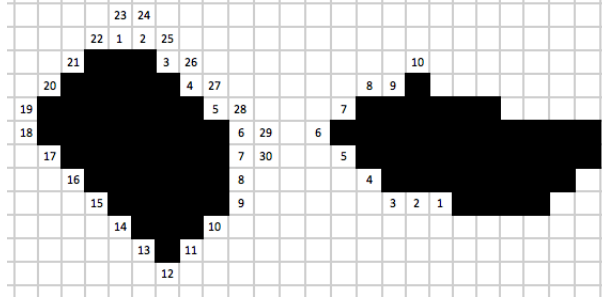


Figure 4: The spiral inflation

Thus now we must add unit squares using one construction of the paper of Dulio et al.[4], the authors introduces the *split* operator (based on the Borel Laubie standard factorization of Christoffel words [3]) in order to add unit square to the border of a digitally convex polyomino. By this operation the Christoffel property of words along the four paths is maintained and thus the difficult part of the inflation remains to maintain the monotony of the slopes of each Christoffel word along the four paths.

### 3.2. The split operator

From now on, we are working on the  $WN$  path. The split operator, as mentioned in [4], considers the furthest point of the Christoffel word  $\ell_i$  with respect to its line segment and switches the factor 01 to 10. Using this operator and replacing the factors is equivalent to adding a unit square to the polyomino. We obtain in this case two new Christoffel words  $\ell_i^+$  and  $\ell_i^-$ , where  $\ell_i = \ell_i^- \ell_i^+$  is the standard factorization of the Christoffel word, and  $\ell_i^- < \ell_i < \ell_i^+$  both for slopes and lexical order. The split operator consists in replacing  $\ell_i$  by  $\ell_i^+ \ell_i^-$  in the coding word of the path. Then the convexity property remains true in the simple case :

$$\ell_{i+1} \leq \ell_i^- < \ell_i^+ \leq \ell_{i-1}.$$

but these inequalities can fail.

So we might have different cases and situations, that we will present in the next section depending on the property of each unit square to add.

### 3.3. Adding one unit square:

We start by considering the case, where we will add one unit square to the convex  $C_1(j)$  for a given  $j$  ( $C_1(j)$  represents the  $j$ th step of the construction that is  $C_1$  with  $j$  added unit squares). Which means that with this case we reach the step  $C_1(j+1)$ . In fact, three different cases can occur:

1. The first case, where we add a unit square on the Christoffel word  $\ell_i$  with  $n_i = 1$  of a given path using the split operator and no problems are faced. Which means the monotone order of slopes is maintained, i.e., as the split operator consists in replacing  $\ell_i$  by  $\ell_i^+ \ell_i^-$  in the coding word of the path:

$$\ell_{i+1} \leq \ell_i^- < \ell_i^+ \leq \ell_{i-1}.$$

and the convexity is conserved.

2. The second case also needs  $n_i = 1$ . It is if we add a unit square to  $\ell_i$  of a given path we keep the Lyndon factorization property, but this factorization is not as in the first case

$$\ell_1^{n_1} \dots (\ell_{i-1}^{n_{i-1}} \ell_i^+) \ell_i^- \ell_{i+1}^{n_{i+1}} \dots \ell_k^{n_k}.$$

The convexity is conserved in this second case.

3. The third case that we can face, is if we add a unit square to  $\ell_i$  of a given path we loose the Lyndon factorization property; which means the monotone order of slopes is no longer correct, and the new polyomino is not digitally convex.

However, we can get a characterization of the first case. We give this characterization in the simplest case, which corresponds to  $n_i = 1$ . Going back to the factorization  $\ell_1^{n_1} \dots \ell_k^{n_k}$  of the  $WN$  path, cases 1 or 2 occur when the following conditions are satisfied.

1.  $l_{i+1} \leq l_i^-$  or  $l_{i+1} = l_i^{-k} \ell$  for some positive integer  $k$  and some Christoffel word  $\ell$ ;
2.  $l_{i-1} \geq l_i^+$  or  $l_{i-1} = \ell l_i^{+k'}$  for some positive integer  $k'$  and some Christoffel word  $\ell$ .

### 3.4. An inflation method

We consider as before two distinct digitally convex polyominoes  $C_1 \subset C_2$ . Then there always exists in each of the four paths describing  $C_1$  at least one Christoffel word corresponding to the cases 1 or 2 before. More precisely, we can choose the longest Christoffel word of the path, among those which does not correspond to some side of  $C_2$ . It gives an effective method to inflate  $C_1$  to  $C_2$  and conserving the convex property at each step.

Some other methods based on completely different considerations could be investigated. For example we can use a continuous and increasing deformation of the border of the convex hull of  $C_1$  to the border of the convex hull of  $C_2$ , such that this deformation contains at any time at most one integer point. An effective construction of such a deformation can be made, using simple arguments but encountering some technical difficulties.

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