



**HAL**  
open science

## Persistent Random Walks. I. Recurrence Versus Transience

Peggy Cénac, Arnaud Le Ny, Basile de Loynes, Yoann Offret

► **To cite this version:**

Peggy Cénac, Arnaud Le Ny, Basile de Loynes, Yoann Offret. Persistent Random Walks. I. Recurrence Versus Transience. *Journal of Theoretical Probability*, 2018, 31 (1), pp.232-243. 10.1007/s10959-016-0714-4 . hal-02039422v1

**HAL Id: hal-02039422**

**<https://hal.science/hal-02039422v1>**

Submitted on 4 Nov 2019 (v1), last revised 7 Nov 2019 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Persistent random walks I : recurrence *versus* transience

Peggy Cénac<sup>1</sup> . Basile de Loynes<sup>2</sup> . Arnaud Le Ny<sup>3</sup> . Yoann Offret<sup>1</sup>

<sup>1</sup>Institut de Mathématiques de Bourgogne (IMB) - UMR CNRS 5584,  
Université de Bourgogne Franche-Comté, 21000 Dijon, France

<sup>2</sup>Institut de Recherche Mathématique Avancée (IRMA) - UMR CNRS 7501,  
Université de Strasbourg, 67084 Strasbourg, France

<sup>3</sup>Laboratoire d'Analyse et de Mathématiques Appliquées (LAMA) - UMR CNRS 8050,  
Université Paris Est, 94010 Créteil Cedex, France

---

**Abstract** We consider a walker on the line that at each step keeps the same direction with a probability which depends on the discrete time already spent in the direction the walker is currently moving. More precisely, the associated left-infinite sequence of jumps is supposed to be a Variable Length Markov Chain (VLMC) built from a probabilized context tree given by a double-infinite comb. These walks with memories of variable length can be seen as generalizations of Directionally Reinforced Random Walks (DRRW) introduced in [1, Mauldin & al., Adv. Math., 1996] in the sense that the persistence times are anisotropic. We give a complete characterization of the recurrence and the transience in terms of the probabilities to persist in the same direction or to switch. We point out that the underlying VLMC is not supposed to admit any stationary probability. Actually, the most fruitful situations emerge precisely when there is no such invariant distribution. In that case, the recurrent and the transient property are related to the behaviour of some embedded random walk with an undefined drift so that the asymptotic behaviour depends merely on the asymptotics of the probabilities of change of directions unlike the other case in which the criterion reduces to a drift condition. Finally, taking advantage of this flexibility, we treat the case of more general probabilized context trees built by grafting subtrees onto the double-infinite comb.

**Key words** Persistent random walk . Directionally reinforced random walk . Variable length Markov chain . Variable length memory . Probabilized context tree . Recurrence . Transience . Random walk with undefined mean or drift

**Mathematics Subject Classification (2000)** 60G50 . 60J15 . 60G17 . 60J05 . 37B20 . 60K35 . 60K37

---

## 1 Introduction

Classical random walks are usually defined from a sequence of independent and identically distributed (*i.i.d.*) increments  $\{X_k\}_{k \geq 1}$  by

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{k=1}^n X_k \quad \text{for all integers } n \geq 1. \quad (1.1)$$

When the jumps are defined as a (finite-order) Markov chain, a short memory in the dynamics of the stochastic paths is introduced and the random walk  $\{S_n\}_{n \geq 0}$  itself is no longer Markovian. Such a process

is called in the literature a *persistent* random walk, a *Goldstein-Kac* random walk or also a *correlated* random walk. Concerning the genesis of the theory, we allude to [2–7] as regards the discrete-time situation but also its connections with the continuous-time telegraph process.

In this paper, we aim at investigating the asymptotic behavior of one-dimensional random walks for which the increments are driven by a Variable Length Markov Chain (VLMC). The use of VLMCs is motivated by the fact that it furnishes an extended model for the dependence of the increments of the persistent random walk (compared to the finite-order Markov dependency).

Let us recall briefly the probabilistic presentation of the Variable Length Markov Chains (VLMC) given in this paper comes from [8] (*c.f.* [9] for the seminal paper and [10, pp. 117-134] for an overview on VLMC). Introduce the set  $\mathcal{L} = \mathcal{A}^{-\mathbb{N}}$  of left-infinite words on the alphabet  $\mathcal{A} := \{d, u\}$  and consider a complete tree on this alphabet, *i.e.* a tree such that each node has 0 or 2 children, whose leaves  $\mathcal{C}$  are words (possibly infinite) on  $\mathcal{A}$ . To each leaf  $c \in \mathcal{C}$ , called a context, is attached a probability distribution  $q_c$  on  $\mathcal{A}$ . Endowed with this probabilistic structure, such a tree is named a probabilized context tree. The related VLMC is defined as the Markov Chain on  $\mathcal{L}$  whose transitions are given by

$$\mathbb{P}(U_{n+1} = U_n \ell | U_n) = q_{\overleftarrow{\text{pref}}(U_n)}(\ell), \quad (1.2)$$

where  $\overleftarrow{\text{pref}}(w) \in \mathcal{C}$  is defined as the shortest prefix of  $w \in \mathcal{L}$ , when  $w$  is read from right to left, appearing as a leaf of the context tree. Then the increments  $\{X_k\}_{k \geq 1}$  of the persistent random walks are defined as an observable of the VLMC, namely,  $X_k := +1$  if  $U_k = U_{k-1}u$  whereas  $X_k := -1$  if  $U_k = U_{k-1}d$ .

Different context trees lead to different probabilistic impacts on the asymptotic behavior of the resulting persistent random walk. Besides, the characterization of the recurrent *versus* transient behavior, the so-called type problem, is difficult for general context trees. In this paper, a criterion characterizing the type of the persistent random walk defined from the VLMC build from the *double-comb* probabilized context tree introduced in [11]. This persistent random walk can be seen as the anisotropic extension of the model of Directionally Reinforced Random Walks introduced in [1]. Surely, the considered DDRW are not restricted to the dimension one, still, persistent random walks can be easily extended to higher dimensions.

In Section 2, we introduce the persistent random walk built from the double-comb VLMC. Under mild conditions on the double-comb probabilized context tree, a renewal pattern for the resulting VLMC is stated in [11]. This renewal property materializes for the persistent random walk to make infinitely many U-turns almost-surely. As a result, one may define the walk more directly and forget the underlying VLMC structure. However, in Section 4, we make use of comparison Lemma 3.2 and ?? to derive a recurrence/transience criteria and SLLN for a large class of probabilized context trees motivating the use of VLMC. The preceding Section 3 is devoted to the statement related to the recurrence or transience of the double-comb persistent random walk. In addition, a extension of the SLLN shown in [11] is also given.

## 2 Settings and assumptions

Foremost, we refer to Figure 2.1 that illustrates our notations and assumptions by a realization of a linear interpolation of our persistent random walk  $\{S_n\}_{n \geq 0}$ , built from a double infinite comb.

### 2.1 Renewal hypothesis

In order to avoid trivial cases, we assume the persistent random walk  $S$  can not be frozen in one of the two directions with a positive probability whatever the initial state  $U_0$  is. In other words, the persistent random walk makes infinitely many U-turns almost-surely. In the sequel, we denote by  $\{B_n\}_{n \geq 0}$  the ordered sequence of positive random times around which  $S$  makes such a U-turn (note that  $B_0 \geq 1$  necessarily).

B: raccourcir l'hypothèse principale ?

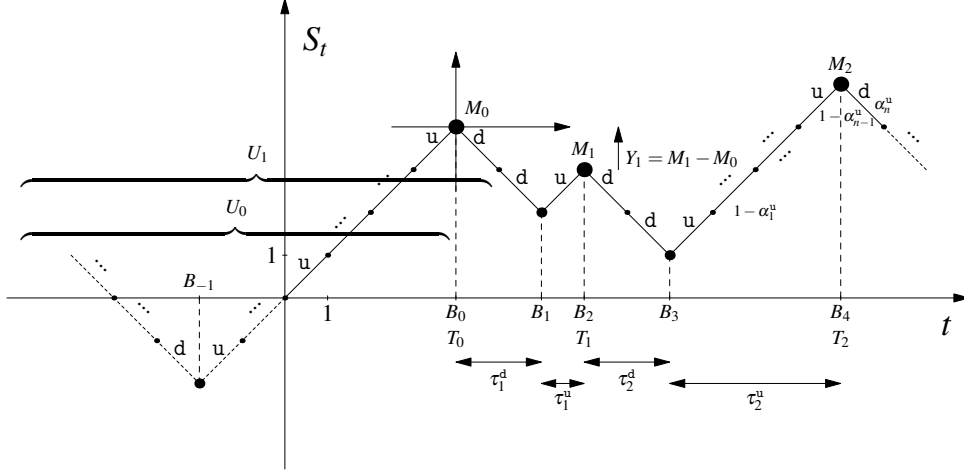


Figure 2.1: Persistent random walk

For the sake of simplicity, throughout this paper, we deal implicitly with the conditional probability  $\mathbb{P}(\cdot | (X_1, X_2) = (1, -1))$ . In particular,  $B_0 = 1$  with probability one. Intuitively, it corresponds to suitable time change so that at time one,  $S$  admits a local maximum. Obviously, there is no loss of generality supposing this and the long time behavior of  $S$  is not affected as well.

With these notations, the renewal property of the double-comb VLMC implies that, for any  $k \geq 1$ , the conditional probabilities

$$\mathbb{P}(X_{n+k+1} = 1 | X_{n+j} = -1, 1 \leq j \leq k, X_n = 1) \quad \text{and} \quad \mathbb{P}(X_{n+k+1} = -1 | X_{n+j} = 1, 1 \leq j \leq k, X_n = -1)$$

are independent of  $n \geq 1$  and shall be denoted by  $\alpha_k^d$  and  $\alpha_k^u$  respectively. Intuitively,  $\alpha_k^u$  (resp.  $\alpha_k^d$ ) stands for the probability of changing direction after  $k$  rises (resp.  $k$  descents). In order to be consistent with the renewal property of the VLMC, one have to assume the following

**Assumption 2.1.** 1. For any  $\ell \in \{u, d\}$  and  $r \geq 1$ ,  $\alpha_\infty^\ell \neq 0$  and

$$\prod_{k=r}^{\infty} (1 - \alpha_k^\ell) = 0. \quad (2.1)$$

2. For any  $\ell \in \{u, d\}$  and  $r \geq 1$ ,  $\alpha_\infty^\ell \neq 0$  and, either there exists  $n \geq r$  such that  $\alpha_n^\ell = 1$ , or

$$\sum_{k=r}^{\infty} \alpha_k^\ell = \infty. \quad (2.2)$$

This assumption disallows a too strong reinforcement, that is a too fast decreasing rate for the probabilities of change of directions. Sequences of transition satisfying this assumption are said to be admissible. Below are given typical examples for which the assumption holds or fails.

## 2.2 Persistence times and embedded random walk

Defining the length of rises ( $\tau_n^u$ ) and of descents ( $\tau_n^d$ ) for all  $n \geq 1$  by  $\tau_n^d := B_{2n-1} - B_{2n-2}$  and  $\tau_n^u := B_{2n} - B_{2n-1}$  respectively, the renewal property implies that  $(\tau_n^d)$  and  $(\tau_n^u)$  are independent sequences of *i.i.d.* random variables whose distribution tails are given by  $\mathcal{F}_\ell(n) := \mathcal{F}_\ell(n) := \mathbb{P}(\tau_1^\ell \geq n) = \prod_{k=1}^{n-1} (1 - \alpha_k^\ell)$ ,  $\ell \in \{u, d\}$ .

At this stage, we exclude for simplicity the situation of almost surely constant length of runs which trivializes the analysis of the underlying persistent random walk.

In order to deal with a more tractable random walk built with possibly unbounded but *i.i.d.* increments, we define the underlying skeleton random walk  $\{M_n\}_{n \geq 0}$  associated with the even breaking times (up-down breaking times) and its almost-sure drift  $\mathbf{d}_M$  by

$$M_n := S_{B_{2n}} = \sum_{k=1}^n (\tau_k^u - \tau_k^d), \quad n \geq 0, \quad \mathbf{d}_M := \mathbb{E}[\tau_1^u] - \mathbb{E}[\tau_1^d],$$

the latter being meaningful whenever  $\mathbb{E}[\tau_1^u]$  or  $\mathbb{E}[\tau_1^d]$  is finite. Also, we can set (extended by continuity whenever necessary)

$$\mathbf{d}_S := \frac{\mathbb{E}[\tau_1^u] - \mathbb{E}[\tau_1^d]}{\mathbb{E}[\tau_1^u] + \mathbb{E}[\tau_1^d]} \in [-1, 1]. \quad (2.3)$$

In regards to the convergence (3.3), the latter quantity is naturally termed the (almost sure) drift of  $S$ .

### 3 Recurrence and transience

#### 3.1 Equivalent criteria and comparison lemma

B: réc/trans  
connu

As stated by [12, Theorem 1., Chap. XII and Theorem 4., Chap. VI], any non constant  $\mathbb{Z}$ -valued random walk  $M$  is either oscillating, its limit point set is  $\mathbb{Z} \cup \{\pm\infty\}$ , or drifting toward  $+\infty$  (*resp.*  $-\infty$ ), and its limit point set is  $\{+\infty\}$  (*resp.*  $-\infty$ ). Moreover, whenever  $M$  admits a SLLN, the different cases are characterized with the help of a standard condition on the drift  $\mathbf{d}_M$ .

The recurrence *versus* transience behavior of the persistent random walk  $S$  reduces to the oscillating *versus* drifting behavior of  $M$  as shown in the following lemma.

B: résumé en  
une phrase :  
trois cas pour  
l'ensemble  
des points  
limites  
 $\mathbb{Z} \cup \{\pm\infty\}$ ,  
 $\{+\infty\}$  ou  
 $\{-\infty\}$  ;  
SLLN  
connue

**Lemma 3.1.** *The persistent random walk  $S$  is either recurrent or transient accordingly as skeleton random walk  $M$  is oscillating or drifting. More precisely, one has:*

- 1)  $S$  is recurrent if and only if  $M$  is oscillating.
- 2)  $S$  is transient to  $\infty$  (*resp.*  $-\infty$ ) if and only if  $M$  is drifting to  $\infty$  (*resp.*  $-\infty$ ).

*Proof of Lemma 3.1.* First, when  $M$  is oscillating,  $S$  is recurrent. Next if  $M$  is drifting to  $-\infty$ , then  $S$  is transient to  $-\infty$  since the trajectory of  $S$  is always under the broken line formed by the  $M_n$ 's. Finally, with Theorem [12, Theorem 1., Chap. XII and Theorem 4., Chap. VI], the oscillating and drifting to  $\pm\infty$  behaviour form, up to a null set, a partition of the universe. Therefore, it only remains to prove that if  $M$  is drifting to  $\infty$ , then  $S$  is transient to  $\infty$ . It is worth to note that we assume the initial time to be an up-down breaking time as in Figure 2.1 so that the geometric argument considered above does not apply straightforwardly. Nonetheless, the expected assertion follows by remarking that, up to an independent random variable, the skeleton random walk at odd breaking times (down-up breaking times) is equal in distribution to  $M$  which ends the proof of the lemma.  $\square$

Let us end this part with a comparison lemma which is necessary to prove the extended SLLN in Proposition 3.1 but also very useful in practice.

**Lemma 3.2** (comparison lemma). *Let  $S$  and  $\tilde{S}$  be two persistent random walks such that the associated distribution tails of their length of runs satisfy for all  $n \geq 1$ ,*

$$\mathcal{T}_u(n) \leq \tilde{\mathcal{T}}_u(n) \quad \text{and} \quad \mathcal{T}_d(n) \geq \tilde{\mathcal{T}}_d(n). \quad (3.1)$$

*Then there exists a coupling, still denoted by  $(S, \tilde{S})$  up to a slight abuse, such that for all  $n \geq 1$ ,*

$$S_n \leq \tilde{S}_n \quad \text{a.s.} \quad (3.2)$$

*Proof of Lemma 3.2.* Let  $(\tau_n^\ell)$  and  $(\tilde{\tau}_n^\ell)$  be the associated lengths of runs and  $G_\ell$  and  $\tilde{G}_\ell$  be the left continuous inverse of their cumulative distribution functions. Then inequalities in (3.1) yield that for all  $x \in [0, 1]$ ,

$$G_u(x) \leq \tilde{G}_u(x) \quad \text{and} \quad G_d(x) \geq \tilde{G}_d(x).$$

Then we can construct a coupling (see for instance the book [13, Chap. 1.3.]) of the lengths of runs such that, with probability one, for all  $n \geq 1$ ,

$$\tau_n^u \leq \tilde{\tau}_n^u \quad \text{and} \quad \tau_n^d \geq \tilde{\tau}_n^d.$$

To be more specific, considering two independent sequences  $(V_n^\ell)$  of uniform random variables on  $[0, 1]$ , we can set

$$\tau_n^\ell := G_\ell(V_n^\ell) \quad \text{and} \quad \tilde{\tau}_n^\ell := \tilde{G}_\ell(V_n^\ell).$$

Consequently, there exists a coupling of the persistent random walks  $S$  and  $\tilde{S}$  satisfying inequality (3.2) since they are entirely determined by these lengths of runs.  $\square$

With respect to the considerations above, it seems natural to distinguish two cases providing whether one of the mean length of runs between  $\mathbb{E}[\tau_1^u]$  or  $\mathbb{E}[\tau_1^d]$  is finite or both are infinite. The former case correspond to the situation in which the drift of  $M$  is well-defined and is considered in the next section. The latter case, when the definition of the drift in (2.3) is meaningless, is considered apart in Section 3.3.

### 3.2 Well-defined Drift case

In this part, assume that the drift is well defined, that is  $\Theta_u(\infty)$  or  $\Theta_d(\infty)$  is finite so that  $\mathbf{d}_S$  given in (2.3) is well-defined. We will highlight a Strong Law of Large Number (SLLN) for the persistent random walk and we shall prove a null drift recurrence criterion similarly to the classical context of random walks with integrable jumps.

**Proposition 3.1** (recurrence criterium and SLLN). *The persistent random walk  $S$  is recurrent if and only if  $\mathbf{d}_S = 0$  and transient otherwise. Furthermore, one has*

$$\lim_{t \rightarrow \infty} \frac{S_n}{n} = \mathbf{d}_S \in [-1, 1] \quad a.s.. \quad (3.3)$$

*Proof.* First remark that, in this setting, the recurrence criterion is a straightforward consequence of [12, Theorem 1., Chap. XII and Theorem 4., Chap. VI] and Lemma 3.1. Besides, the law of large numbers (3.3) when  $\mathbb{E}[\tau_1^u]$  and  $\mathbb{E}[\tau_1^d]$  are both finite is already proved in [11, Proposition 4.5, p. 33] under the assumption that  $\mathbf{d}_S \in (-1, 1)$ . Then it only remains to prove the SLLN when  $\mathbb{E}[\tau_1^u] = \infty$  and  $\mathbb{E}[\tau_1^d] < \infty$  (and thus  $\mathbf{d}_S = 1$ ). Note that it is sufficient to prove the minoration in (3.3) since  $S_n \leq n$  for all  $n \geq 0$ . To this end, let  $\varepsilon \in (0, 1)$  and set

$$\alpha_n^{u,\varepsilon} := \begin{cases} \alpha_n^u, & \text{when } 1 \leq n \leq N-1, \\ 1, & \text{when } n \geq N, \end{cases} \quad \text{and} \quad \alpha_n^{d,\varepsilon} := \alpha_n^d. \quad (3.4)$$

This defines a persistent random  $S^\varepsilon$  satisfying  $\mathbf{d}_{S^\varepsilon} \geq 1 - \varepsilon$  as soon as  $N$  is chosen sufficiently large in (3.4). The comparison Lemma 3.2, giving a coupling such that  $S^\varepsilon \leq S$  almost-surely, together with the latter SLLN applied to  $S^\varepsilon$  end the proof.  $\square$

B: si on ne supprime pas la preuve, peut-être résumer en 1 ou 2 phrases l'idée

### 3.3 Undefined drift case

In this section we consider the remaining case in which both  $\Theta_u(\infty)$  and  $\Theta_d(\infty)$  are infinite. In this case, the information given by the expectation of one increment of  $M$  is no longer sufficient to discriminate between transience and recurrence.

In fact, following Erickson [14, Theorem 2., p. 372], the oscillating or drifting behaviour of the skeleton random walk  $M$  is characterized through the cumulative distribution function of its increments  $(Y_n) := (\tau_n^u - \tau_n^d)$ , especially if the mean is undefined. Roughly speaking, the criterion of Erickson together with the lemma 3.1 imply that the persistent random walk  $S$  is recurrent if the distribution tails of the positive and negative parts of an increment are comparable, transient otherwise.

However, Erickson's criterion does not suit to our context since the distribution of an increment is not explicitly given by the parameters of the model, but merely by the convolution of two *a priori* known distributions. To circumvent these difficulties, we consider a sequence  $(\xi_n)$  of non-degenerate *i.i.d.* Bernoulli random variables with parameter  $p \in (0, 1)$ , independent of the sequences of length of runs  $(\tau_n^u)$  and  $(\tau_n^d)$ . Then we introduce the following classical random walk defined for all  $n \geq 0$  by

$$M_n^\xi := \sum_{k=1}^n Y_k^\xi, \quad \text{with} \quad Y_k^\xi := \xi_k \tau_k^u - (1 - \xi_k) \tau_k^d. \quad (3.5)$$

The proof of the following lemma is postponed to the end of this part.

**Lemma 3.3** (randomized random walk). *The random walks  $M$  and  $M^\xi$  are simultaneously oscillating or drifting (towards  $\pm\infty$ ).*

Therefore, in order to obtain the oscillating or drifting property of  $M$  we can apply the criterion of Erickson to  $M^\xi$ . It is then not difficult to see that the criterion consists of determining the convergence or divergence of the more tractable series (compare to (??)) given, for any  $\ell_1, \ell_2$  in  $\{u, d\}$ , by

$$J_{\ell_1|\ell_2} := \sum_{n=1}^{\infty} \frac{n \mathbb{P}(\tau^{\ell_1} = n)}{\sum_{k=1}^n \mathbb{P}(\tau^{\ell_2} \geq k)} = \sum_{n=1}^{\infty} \frac{n(-\Delta \mathcal{T}_{\ell_1}(n))}{\sum_{k=1}^n \mathcal{T}_{\ell_2}(k)}, \quad (3.6)$$

where  $\Delta V(n)$  denotes the forward discrete derivative at point  $n$  of the real sequence  $(V_n)$ , *i.e.*

$$\Delta V(n) = V(n+1) - V(n). \quad (3.7)$$

**Theorem 3.1** (recurrence criterium and *a.s.* asymptotics). *The persistent random walk  $S$  is recurrent if and only if*

$$J_{u|d} = \infty \quad \text{and} \quad J_{d|u} = \infty, \quad (3.8)$$

*and transient to  $\infty$  (resp. transient to  $-\infty$ ) if and only if*

$$J_{u|d} = \infty \quad \text{and} \quad J_{d|u} < \infty \quad (\text{resp.} \quad J_{u|d} < \infty \quad \text{and} \quad J_{d|u} = \infty). \quad (3.9)$$

*Moreover, when  $J_{u|d} = \infty$  (resp.  $J_{d|u} = \infty$ ),*

$$\limsup_{t \rightarrow \infty} \frac{S_n}{n} = 1 \quad \left( \text{resp.} \quad \liminf_{t \rightarrow \infty} \frac{S_n}{n} = -1 \right) \quad \text{a.s.} \quad (3.10)$$

*Alternatively, the quantities  $J_{u|d}$  and  $J_{d|u}$  can be substituted with  $K_{u|d}$  and  $K_{d|u}$  respectively.*

This theorem ends the characterization of the type of persistent random walks. In Table 3.1 the conditions for the recurrence and the transience are summarized and we give some applications of these criteria below.

B: la forme alternative est sans intérêt sans les perturbations

	$\Theta_u(\infty) < \infty$		$\Theta_u(\infty) = \infty$
$\Theta_d(\infty) < \infty$	Recurrent $\mathbf{d}_s = 0$	Transient to $+\infty$ $\mathbf{d}_s \in (0, 1)$	Transient to $+\infty$  $\mathbf{d}_s = 1$
		Transient to $-\infty$ $\mathbf{d}_s \in (-1, 0)$	
$\Theta_d(\infty) = \infty$	Transient to $-\infty$  $\mathbf{d}_s = -1$		Recurrent $J_{u d} = J_{d u} = \infty$
			Transient to $+\infty$ $\infty = J_{u d} > J_{d u}$
			Transient to $-\infty$ $\infty = J_{d u} > J_{u d}$

Table 3.1: Recurrence and transience criteria.

**Corollary 3.1.** *In the setting of Theorem 3.1, the criterion can be alternatively stated in terms of quantities*

$$K_{\ell_1|\ell_2} := \sum_{n=1}^{\infty} \left( 1 - \frac{n\mathcal{T}_{\ell_2}(n)}{\sum_{k=1}^n \mathcal{T}_{\ell_2}(k)} \right) \frac{\mathcal{T}_{\ell_1}(n)}{\sum_{k=1}^n \mathcal{T}_{\ell_2}(k)} \quad (3.11)$$

in place of the quantities  $J_{u|d}$  and  $J_{d|u}$ .

These alternative quantities only involves distribution tails making the criterion more transparent.

*Proof.* For the alternative form of the theorem, it remains to prove that  $J_{u|d} = \infty$  if and only if  $K_{u|d} = \infty$ . Summing by parts (the so called *Abel transformation*) we can write for any  $r \geq 1$ ,

$$\sum_{n=1}^r \frac{n(-\Delta \mathcal{T}_u(n))}{\sum_{k=1}^n \mathcal{T}_d(k)} = \left[ 1 - \frac{(r+1)\mathcal{T}_u(r+1)}{\sum_{k=1}^{r+1} \mathcal{T}_d(k)} \right] + \sum_{n=1}^r \Delta \left( \frac{n}{\sum_{k=1}^n \mathcal{T}_d(k)} \right) \mathcal{T}_u(n+1). \quad (3.12)$$

Besides, a simple computation gives

$$\Delta \left( \frac{n}{\sum_{k=1}^n \mathcal{T}_d(k)} \right) = \frac{\sum_{k=1}^n \mathcal{T}_d(k) - n\mathcal{T}_d(n+1)}{\sum_{k=1}^{n+1} \mathcal{T}_d(k) \sum_{k=1}^n \mathcal{T}_d(k)} = \frac{\mathbb{E}[\tau^d \mathbb{1}_{\tau^d \leq n}]}{\sum_{k=1}^{n+1} \mathcal{T}_d(k) \sum_{k=1}^n \mathcal{T}_d(k)} \geq 0. \quad (3.13)$$

It follows that

$$\sum_{n=1}^r \frac{n(-\Delta \mathcal{T}_u(n))}{\sum_{k=1}^n \mathcal{T}_d(k)} = \left[ 1 - \frac{(r+1)\mathcal{T}_u(r+1)}{\sum_{k=1}^{r+1} \mathcal{T}_d(k)} \right] + \sum_{n=1}^r \left( 1 - \frac{n\mathcal{T}_d(n+1)}{\sum_{k=1}^n \mathcal{T}_d(k)} \right) \frac{\mathcal{T}_u(n+1)}{\sum_{k=1}^{n+1} \mathcal{T}_d(k)}. \quad (3.14)$$

Due to the non-negativeness of the forward discrete time derivative (3.13) and due to our assumption for  $\Theta_u(\infty)$  and  $\Theta_d(\infty)$  to be infinite, the general term of the series in the right-hand side of the latter equation is non-negative and (up to a shift) equivalent to that of  $K_{u|d}$ . Moreover, we get again from (3.13) that

$$\frac{r\mathcal{T}_u(r)}{\sum_{k=1}^r \mathcal{T}_d(k)} = \sum_{m=r+1}^{\infty} \frac{r(-\Delta \mathcal{T}_u(m))}{\sum_{k=1}^r \mathcal{T}_d(k)} \leq \sum_{m=r+1}^{\infty} \frac{m(-\Delta \mathcal{T}_u(m))}{\sum_{k=1}^m \mathcal{T}_d(k)}. \quad (3.15)$$

Thus, if  $J_{u|d}$  is infinite then so is  $K_{u|d}$ . Conversely, the finiteness of  $J_{u|d}$  together with the estimate (3.15) implies the first term on the right-hand side in (3.14) remains bounded achieving the proof.  $\square$



*Proof of Theorem 3.1.* First, the statements (3.8) and (3.9) related to the recurrence and transience properties are direct consequences of Erickson's criteria [14] and of lemma 3.3 since the two-sided distribution tails of the increments of the random walk  $M^\xi$  given in (3.5) satisfies for all  $n \geq 1$ ,

$$\mathbb{P}(Y_1^\xi \geq n) = \mathbb{P}(\xi_1 = 1)\mathbb{P}(\tau^u \geq n) \quad \text{and} \quad \mathbb{P}(Y_1^\xi \leq -n) = \mathbb{P}(\xi_1 = 0)\mathbb{P}(\tau^d \geq n).$$

Besides, from the equalities

$$T_n = \sum_{k=1}^n \tau_k^u + \sum_{k=1}^n \tau_k^d \quad \text{and} \quad S_{T_n} = \sum_{k=1}^n \tau_k^u - \sum_{k=1}^n \tau_k^d$$

we can see that (3.10) is satisfied if for all  $c > 0$ ,

$$\mathbb{P}\left(\tau_n^u \geq c \sum_{k=1}^n \tau_k^d \quad i.o.\right) = 1. \quad (3.16)$$

In fact, using the Kolmogorov's zero-one law, we only need to prove that this probability is not zero. To this end, we can see that [15, Theorem 5., p. 1190] applies and it follows that

$$\limsup_{n \rightarrow \infty} \frac{(Y_n^\xi)^+}{\sum_{k=1}^n (Y_k^\xi)^-} = \limsup_{n \rightarrow \infty} \frac{\xi_n \tau_n^u}{\sum_{k=1}^n (1 - \xi_k) \tau_k^d} = \infty \quad a.s.. \quad (3.17)$$

Roughly speaking, this theorem states that the position of a one-dimensional random walk with an undefined mean is essentially given by the last big jump. Introducing the counting process given for all  $n \geq 1$  by

$$N_n := \#\{1 \leq k \leq n : \xi_k = 0\}, \quad (3.18)$$

we shall prove that

$$\left\{ \sum_{k=1}^n (1 - \xi_k) \tau_k^d \right\}_{n \geq 1} \stackrel{\mathcal{L}}{=} \left\{ \sum_{k=1}^{N_n} \tau_k^d \right\}_{n \geq 1}. \quad (3.19)$$

For this purpose, we will see that the sequences of increments consists of independent random variables and are equal in distribution in the following sense

$$\{(1 - \xi_n) \tau_n^d\}_{n \geq 1} \stackrel{\mathcal{L}}{=} \{(1 - \xi_n) \tau_{N_n}^d\}_{n \geq 1}. \quad (3.20)$$

First note that for any  $n \geq 1$ ,

$$\mathbb{P}((1 - \xi_n) \tau_{N_n}^d = 0) = \mathbb{P}(\xi_1 = 1) = \mathbb{P}((1 - \xi_n) \tau_n^d = 0). \quad (3.21)$$

Moreover, up to a null set, we have  $\{\xi_n = 0\} = \{N_n = N_{n-1} + 1\}$  and  $N_{n-1}$  is independent of  $\xi_n$  and of the lengths of runs. We deduce that for any  $k \geq 1$ ,

$$\mathbb{P}((1 - \xi_n) \tau_{N_n}^d = k) = \mathbb{P}(\xi_1 = 0, \tau_1^d = k) = \mathbb{P}((1 - \xi_n) \tau_n^d = k). \quad (3.22)$$

Hence the increments of the random walks in (3.19) are identically distributed. Since the increments on left-hand side in (3.20) are independent, it only remains to prove the independence of those on the right-hand side in (3.20) to obtain the equality in distribution in (3.19). Let us fix  $n \geq 1$  and set for any non-negative integers  $k_1, \dots, k_n \geq 0$ ,

$$I_n := \{1 \leq j \leq n : k_j \neq 0\} \quad \text{and} \quad m_n := \text{card}(I_n).$$

Remark that  $\ell \mapsto m_\ell$  is increasing on  $I_n$  and up to a null set,

$$\bigcap_{\ell \notin I_n} \{\xi_\ell = 1\} \cap \bigcap_{\ell \in I_n} \{\xi_\ell = 0\} \subset \{N_n = m_n\}.$$

Then using (3.21) and (3.22) together with the independence properties we can see that

$$\mathbb{P} \left( \bigcap_{j=1}^n \{(1 - \xi_j) \tau_{N_j}^d = k_j\} \right) = \mathbb{P} \left( \bigcap_{\ell \notin I_n} \{\xi_\ell = 1\} \cap \bigcap_{\ell \in I_n} \{\xi_\ell = 0, \tau_{m_\ell}^d = k_\ell\} \right) = \prod_{j=1}^n \mathbb{P}((1 - \xi_j) \tau_{N_j}^d = k_j),$$

which ends the proof of (3.19).

Next, by the standard LLN for *i.i.d.* sequences, we obtain that for any integer  $q$  greater than  $1/p$ , with probability one, the events  $\{N_n \geq \lfloor n/q \rfloor\}$  hold for all sufficiently large  $n$ . We deduce by (3.19) and (3.17) that

$$\mathbb{P} \left( \tau_n^u \geq c \sum_{k=1}^{\lfloor n/q \rfloor} \tau_k^d \quad i.o. \right) = 1.$$

As a consequence,

$$\mathbb{P} \left( \bigcup_{\ell=0}^{q-1} \left\{ \tau_{qn+\ell}^u \geq c \sum_{k=1}^n \tau_k^d \right\} \quad i.o. \right) = 1. \quad (3.23)$$

Again, applying the Kolmogorov's zero-one law, we get that the  $q$  sequences of events (having the same distribution) in the latter equation occur infinitely often with probability one. We deduce that (3.16) is satisfied and this achieves the proof of (3.10).  $\square$

*Proof Lemma 3.3.* Deeply exploiting the Theorem ?? stating that any non-constant random walk is, with probability one, either (trichotomy) oscillating or drifting to  $\pm\infty$ , the proof is organized as follows:

- 1) At first, we shall prove the result in the symmetric case  $p = 1/2$ .
- 2) Secondly, we shall deduce the statement for any arbitrary  $p \in (0, 1)$  from the latter particular case.

To this end, assume that the supremum limit of  $M^\xi$  is *a.s.* infinite. Following exactly the same lines as in the proof of (3.16), we obtain that  $M$  is non-negative infinitely often with probability one. Applying Theorem ?? we deduce that the supremum limit of  $M$  is also *a.s.* infinite. Thereafter, again from the latter theorem and by symmetry, we only need to prove that if  $M^\xi$  is drifting, then so is  $M$ . When the *i.i.d.* Bernoulli random variables  $(\xi_n)$  are symmetric, that is  $p = 1/2$ , it is a simple consequence of the equalities

$$M^\xi \stackrel{\mathcal{L}}{=} M^{1-\xi} \quad \text{and} \quad M = M^\xi + M^{1-\xi}. \quad (3.24)$$

At this stage it is worth noting that the lemma is proved in the symmetric situation.  $\square$

**Remark 3.1.** *Contrary to the well-defined drift case for which a small perturbation on the parameters of a recurrent persistent random walk leads in general to a transient behaviour, in the case of an undefined drift the persistent random walk may stay recurrent as long as the perturbation remains asymptotically controlled. To put it in a nutshell, the criterion is global in the former case and asymptotic in the latter case.*

B: en fait, pour la réc/trans,  $p = 1/2$  est suffisant dans ts les cas

## 4 Perturbations results

## References

- [1] R. Daniel Mauldin, Michael Monticino, and Heinrich von Weizsäcker. Directionally reinforced random walks. *Adv. Math.*, 117(2):239–252, 1996. URL: <http://dx.doi.org/10.1006/aima.1996.0011>, doi:10.1006/aima.1996.0011. 1, 2
- [2] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math.*, 4:129–156, 1951. 2
- [3] Mark Kac. A stochastic model related to the telegrapher’s equation. *Rocky Mountain J. Math.*, 4:497–509, 1974. Reprinting of an article published in 1956, Papers arising from a Conference on Stochastic Differential Equations (Univ. Alberta, Edmonton, Alta., 1972). 2
- [4] Eric Renshaw and Robin Henderson. The correlated random walk. *J. Appl. Probab.*, 18(2):403–414, 1981. 2
- [5] George H. Weiss. *Aspects and applications of the random walk*. Random Materials and Processes. North-Holland Publishing Co., Amsterdam, 1994. 2
- [6] Eugene C. Eckstein, Jerome A. Goldstein, and Mark Leggas. The mathematics of suspensions: Kac walks and asymptotic analyticity. In *Proceedings of the Fourth Mississippi State Conference on Difference Equations and Computational Simulations (1999)*, volume 3 of *Electron. J. Differ. Equ. Conf.*, pages 39–50. Southwest Texas State Univ., San Marcos, TX, 2000. 2
- [7] George H. Weiss. Some applications of persistent random walks and the telegrapher’s equation. *Phys. A*, 311(3-4):381–410, 2002. URL: [http://dx.doi.org/10.1016/S0378-4371\(02\)00805-1](http://dx.doi.org/10.1016/S0378-4371(02)00805-1), doi:10.1016/S0378-4371(02)00805-1. 2
- [8] Peggy Cénac, Brigitte Chauvin, Frédéric Paccaut, and Nicolas Pouyanne. Context trees, variable length Markov chains and dynamical sources. In *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Math.*, pages 1–39. Springer, Heidelberg, 2012. URL: [http://dx.doi.org/10.1007/978-3-642-27461-9\\_1](http://dx.doi.org/10.1007/978-3-642-27461-9_1), doi:10.1007/978-3-642-27461-9\_1. 2
- [9] Jorma Rissanen. A universal data compression system. *IEEE Trans. Inform. Theory*, 29(5):656–664, 1983. URL: <http://dx.doi.org/10.1109/TIT.1983.1056741>, doi:10.1109/TIT.1983.1056741. 2
- [10] Imre Csiszár. Festschrift in honor of jorma rissanen on the occasion of his 75th birthday edited by peter grünwald, petri myllymäki, ioan tabus, marcelo weinberger, bin yu. *International Statistical Review*, 76(3):436–437, 2008. URL: [http://dx.doi.org/10.1111/j.1751-5823.2008.00062\\_1.x](http://dx.doi.org/10.1111/j.1751-5823.2008.00062_1.x), doi:10.1111/j.1751-5823.2008.00062\_1.x. 2
- [11] P. Cénac, B. Chauvin, S. Herrmann, and P. Vallois. Persistent random walks, variable length Markov chains and piecewise deterministic Markov processes. *Markov Process. Related Fields*, 19(1):1–50, 2013. 2, 5
- [12] William Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons Inc., New York, 1971. 4, 5
- [13] Hermann Thorisson. *Coupling, stationarity, and regeneration*. Probability and its Applications (New York). Springer-Verlag, New York, 2000. URL: <http://dx.doi.org/10.1007/978-1-4612-1236-2>, doi:10.1007/978-1-4612-1236-2. 5

- [14] K. Bruce Erickson. The strong law of large numbers when the mean is undefined. *Trans. Amer. Math. Soc.*, 185:371–381 (1974), 1973. [6](#), [8](#)
- [15] Harry Kesten. The limit points of a normalized random walk. *Ann. Math. Statist.*, 41:1173–1205, 1970. [8](#)