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Miryana Grigorova [*]   Marie-Claire Quenez [†]   Agnès Sulem [‡]

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Abstract

We study the superhedging prices and the associated superhedging strategies for American options in a non-linear incomplete market model with default. The points of view of the seller and of the buyer are presented. The underlying market model consists of a risk-free asset and a risky asset driven by a Brownian motion and a compensated default martingale. The portfolio processes follow non-linear dynamics with a non-linear driver f. We give a dual representation of the seller's (superhedging) price for the American option associated with a completely irregular payoff (ξ_t) (not necessarily càdlàg) in terms of the value of a non-linear mixed control/stopping problem. The dual representation involves a suitable set of equivalent probability measures, which we call f-martingale probability measures. We also provide two infinitesimal characterizations of the seller's price process: in terms of the minimal supersolution of a constrained reflected BSDE and in terms of the minimal supersolution of an optional reflected BSDE. Under some regularity assumptions on ξ, we also show a duality result for the buyer's price in terms of the value of a non-linear control/stopping game problem.

Key-words: American options, incomplete markets, non-linear pricing, constrained reflected BSDE, f-expectation, control problems with non-linear expectation, optimal stopping with non-linear expectation, non-linear optional decomposition, pricing-hedging duality

1 Introduction

We consider a financial market with a default time ϑ. The market contains one risky asset whose price dynamics are driven by a one-dimensional Brownian motion and a compensated

[*]School of Mathematics, University of Leeds, Leeds, UK, email: miryana_grigorova@yahoo.fr

[†]LPSM, Université Paris 7 Denis Diderot, Boite courrier 7012, 75251 Paris cedex 05, France, email: quenez@lpsm.paris

[‡]Mathrisk, INRIA Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12, France, and Université Paris-Est, email: agnes.sulem@inria.fr

default martingale. We study the case of a market with imperfections which are encoded in the non-linearity of the portfolio dynamics. Imperfections entering into this framework include different borrowing and lending interest rates (cf. [28], [7]), different repo rates (cf. [4]), the presence of a large investor (cf. [8], [2], [11]), among others. We note that our market is *incomplete* in the sense that not every European contingent claim can be replicated by a portfolio. In this framework, our paper [23] studies the problem of pricing and hedging of European options (from the point of view of the seller and of the buyer). In the present paper, we focus on the problem of pricing and hedging of American options from the point of view of the seller and of the buyer.

We recall that in the case of a *non-linear complete* market, the seller's (superhedging) price of the American option with a càdlàg payoff (ξ_t) is equal to the value of the optimal stopping problem with non-linear f-evaluation/expectation, associated with the given payoff (ξ_t) (cf. [14]). Moreover, the price process admits an infinitesimal characterization as the solution of the reflected BSDE associated with driver f and obstacle (ξ_t): cf. [19] in the Brownian case and for a continuous payoff (ξ_t), and [36] (resp. [14]) in the case of Poisson jumps (resp. a default jump) and a càdlàg process (ξ_t). More recently, these results have been generalized to the case of an irregular payoff (ξ_t) (not necessarily càdlàg) in [20] and [21].

In the *non-linear incomplete* market of the present paper, we provide a dual formulation of the seller's (superhedging) price u_0 of the American option associated with an irregular payoff (ξ_t) (not necessarily càdlàg) in terms of the value of a non-linear mixed control/stopping problem. The dual representation involves a suitable set $\mathcal{Q}$ of equivalent probability measures, which we call f-martingale probability measures. More precisely, we show that u_0 is equal to the supremum, over all f-martingale measures $Q \in \mathcal{Q}$ and over all stopping times τ, of the (f, Q)-evaluation of ξ_τ at time 0, that is,

$$u_0 = \sup_{(Q,\tau)\in\mathcal{Q}\times\mathcal{T}} \mathcal{E}^f_{Q,0,\tau}(\xi_\tau). \tag{1.1}$$

In the case when f is linear and (ξ_t) is càdlàg, our result reduces to the well-known dual representation from the literature on linear incomplete markets (cf. [29]). We also provide two types of infinitesimal characterizations of the (superhedging) price u_0 for the seller: in terms of the minimal supersolution of a *constrained reflected BSDE* with default (associated with the driver f and the obstacle (ξ_t)), and in terms of the minimal supersolution of an *optional reflected BSDE* with default. We note that, even in the linear case (f linear), these results are new, since in the literature, only the càdlàg case has been studied. The treatment of the non càdlàg case requires the introduction of an additional non decreasing process corresponding to the right-hand jumps of the price process. Using some specific techniques of the control theory and the general theory of processes, we show that this process increases only when the price is equal to the payoff. The proofs of the dual representation and of the infinitesimal characterizations rely also on the non-linear optional decomposition of optional (not necessarily càdlàg) processes which are (f, Q)-strong supermartingales for all $Q \in \mathcal{Q}$ (cf. our result established in [23]).

Under some regularity assumptions on the payoff ξ, we also show a dual representation for the buyer's superhedging price at time 0, denoted by $\tilde{u}_0$, in terms of the value of a non-linear control/stopping game problem, which can be written as

$$\tilde{u}_0 = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} \{-\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau)\} = \inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \{-\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau)\}. \tag{1.2}$$

We note that, contrary to the case of a European option (studied in [23]), the buyer's price is not equal to the opposite of the seller's price of the American option associated with the payoff $-\xi$.

The rest of the paper is organized as follows: In Section 2, we introduce some notation and definitions. In Section 3, we first present our market model (Subsection 3.1), we then introduce the set $\mathcal{Q}$ of f-martingale probability measures (Subsection 3.2), we define the the buyer's and seller's superhedging prices of the American option and we discuss no-arbitrage issues (Subsection 3.3). In Section 4, we establish the duality result for the seller's superhedging price. For this, we first study the value Y of the associated non-linear mixed control/stopping problem, which we write as the (essential) supremum of a family of reflected BSDEs. We show in particular that Y is the smallest optional process which is an (f, Q)-strong supermartingale for all $Q \in \mathcal{Q}$, and which dominates the payoff process. We also study the strict value Y^+ of our non-linear mixed control/stopping problem. We show, in particular, that Y^+ can be aggregated by a càdlàg adapted process. We then give the proof of the dual representation of the seller's superhedging price (1.1). In Section 5, we provide the two infinitesimal characterizations of the seller's superhedging price process. Section 6 is devoted to the point of view of the buyer. We first study the value $\tilde{Y}$ of the associated dual problem, which we write as the (essential) infimum of a family of reflected BSDEs. Then, under some regularity assumptions on the payoff, we prove the dual representation of the buyer's superhedging price (1.2). We also introduce and discuss the notion of *buyer's nearly superhedging price*. The Appendix contains some useful technical results and a discussion on reflected BSDEs with a non positive jump at the default time.

2 Notation and definitions

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0, T]$, where ϑ is a random variable which models a default time. We assume that this default can appear after any fixed time, that is $P(\vartheta \geq t) > 0$ for all $t \geq 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the *augmented filtration* generated by W and N. We denote by $\mathcal{P}$ the predictable σ-algebra. We suppose that W is a $\mathbb{G}$-Brownian motion. Let (Λ_t) be the predictable compensator of the nondecreasing process (N_t). Note that $(\Lambda_{t \wedge \vartheta})$ is then the predictable compensator of $(N_{t \wedge \vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t \wedge \vartheta} = \Lambda_t$, $t \geq 0$ a.s. We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$. To simplify the presentation, we suppose that λ is bounded. Since $\Lambda_{t \wedge \vartheta} = \Lambda_t$,

λ vanishes after ϑ. Let M be the compensated martingale given by

$$M_t := N_t - \int_0^t \lambda_s ds\,.$$

Let $T > 0$ be the terminal time. We define the following sets:

- S^2 is the set of $\mathbb{G}$-adapted RCLL processes φ such that $\mathbb{E}[\sup_{0\leq t\leq T}|\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL $\mathbb{G}$-predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes Z such that $\|Z\|^2 := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty\,.$
- $\mathbb{H}^2_\lambda := L^2(\Omega\times[0,T],\mathcal{P},\lambda_t\, dP\otimes dt)$, equipped with the norm $\|U\|^2_\lambda := \mathbb{E}\Big[\int_0^T |U_t|^2\lambda_t dt\Big] < \infty\,.$

Note that, without loss of generality, we may assume that if $U \in \mathbb{H}^2_\lambda$, it vanishes after ϑ.

- We denote by $\mathcal{T}$ the set of stopping times τ such that $\tau \in [0,T]$ a.s.
- For S in $\mathcal{T}$, we denote by $\mathcal{T}_S$ the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

As in [21], the notation $\mathbb{S}^2$ stands for the vector space of $\mathbb{R}$-valued optional (not necessarily cadlag) processes ϕ such that $|||\phi|||^2_{\mathbb{S}^2} := E[ess\sup_{\tau\in\mathcal{T}_0}|\phi_\tau|^2] < \infty$. By Proposition 2.1 in [21], the space $\mathbb{S}^2$ endowed with the norm $|||\cdot|||_{\mathbb{S}^2}$ is a Banach space. We note that the space S^2 is the sub-space of RCLL processes of $\mathbb{S}^2$.

Recall that in this setup, we have a martingale representation theorem with respect to W and M (see [24], [30]).

We give the definition of a λ-admissible driver:

Definition 2.1 (Driver, λ-*admissible* driver)**.** *A function g is said to be a* driver *if* $g : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$; $(\omega,t,y,z,k) \mapsto g(\omega,t,y,z,k)$ *is* $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^3)-$ *measurable, and such that* $g(.,0,0,0) \in \mathbb{H}^2$*. A driver g is called a* λ-admissible driver *if moreover there exists a constant $C \geq 0$ such that $dP\otimes dt$-a.s. , for each* (y_1,z_1,k_1), (y_2,z_2,k_2),

$$|g(t,y_1,z_1,k_1) - g(t,y_2,z_2,k_2)| \leq C(|y_1-y_2| + |z_1-z_2| + \sqrt{\lambda_t}|k_1-k_2|). \tag{2.1}$$

A nonnegative constant C which satisfies this inequality is called a λ-constant *associated with driver g.*

By condition (2.1) and since $\lambda_t = 0$ on $]\vartheta,T]$, g does not depend on k on $]\vartheta,T]$.

Let g be a λ-*admissible driver.* For all $\eta \in L^2(\mathcal{G}_T)$, there exists a unique solution $(X(T,\eta),Z(T,\eta),K(T,\eta))$ (denoted simply by (X,Z,K)) in $S^2\times\mathbb{H}^2\times\mathbb{H}^2_\lambda$ of the following BSDE (see [11]):

$$-dX_t = g(t,X_t,Z_t,K_t)dt - Z_t dW_t - K_t dM_t;\quad X_T = \eta. \tag{2.2}$$

We call g-conditional expectation, denoted by $\mathcal{E}^g$, the operator defined for each $T' \in [0,T]$ and for each $\eta \in L^2(\mathcal{G}_{T'})$ by $\mathcal{E}^g_{t,T'}(\eta) := X_t(T',\eta)$ a.s. for all $t \in [0,T']$.

We introduce the following assumption :

Assumption 2.2. *Assume that there exists a bounded map*

$$\gamma : \Omega \times [0,T] \times \mathbb{R}^4 \to \mathbb{R}\,;\, (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^4)$-measurable and satisfying $dP \otimes dt$-a.s. , for all $(y,z,k_1,k_2) \in \mathbb{R}^4$,

$$g(t,y,z,k_1) - g(t,y,z,k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \tag{2.3}$$

and

$$\gamma_t^{y,z,k_1,k_2} > -1. \tag{2.4}$$

Assumption 2.2 ensures the strict monotonicity of the operator $\mathcal{E}^g$ with respect to terminal condition (see [11, Section 3.3]).

Definition 2.3. *Let $Y \in \mathbb{S}^2$. The process (Y_t) is said to be a strong $\mathcal{E}^g$-supermartingale* [1] *(resp. martingale) if $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \leq Y_\sigma$ (resp. $= Y_\sigma$) a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Note that, by the flow property of BSDEs, for each $\tau \in \mathcal{T}_0$ and for each $\eta \in L^2(\mathcal{G}_\tau)$, the process $\mathcal{E}^g_{\cdot,\tau}(\eta)$ is an $\mathcal{E}^g$-martingale.

3 Market model, f-martingale measures and super-hedging prices

3.1 Market model $\mathcal{M}^f$

We now consider a financial market which consists of one risk-free asset, whose price process $S^0 = (S^0_t)_{0 \leq t \leq T}$ satisfies $dS^0_t = S^0_t r_t dt$, and one risky asset with price process S which admits a discontinuity at time ϑ. Throughout the sequel, we consider that the price process $S = (S_t)_{0 \leq t \leq T}$ evolves according to the equation

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t). \tag{3.1}$$

All the processes σ r, μ, β are supposed to be predictable (that is $\mathcal{P}$-measurable), satisfying $\sigma_t > 0$ $dP \otimes dt$ a.s. and $\beta_\vartheta > -1$ a.s., and such that σ, λ, σ^{-1}, β are bounded.

We consider an investor with an initial wealth equal to x, who can invest his/her wealth in the two assets of the market. At each time t, the investor chooses the amount φ_t of wealth invested in the risky asset. A process $\varphi_\cdot = (\varphi_t)_{0 \leq t \leq T}$ is called a *portfolio strategy* if it belongs to $\mathbb{H}^2$. The value of the associated portfolio (also called *wealth*) at time t is denoted by $V_t^{x,\varphi}$ (or simply V_t).

[1] In the case where Y is moreover RCLL (that is, $Y \in S^2$), we often omit the term "strong".

We assume now that the dynamics of the wealth is *non-linear*. More precisely, let $x \in \mathbb{R}$ be an initial wealth and let φ in $\mathbb{H}^2$ be a portfolio strategy. We suppose that the associated *wealth* process $V_t^{x,\varphi}$ (or simply V_t) satisfies the following (forward) dynamics:

$$-dV_t = f(t, V_t, \varphi_t \sigma_t)dt - \varphi_t \sigma_t dW_t - \varphi_t \beta_t dM_t, \tag{3.2}$$

with $V_0 = x$, where f is a *non-linear* λ-admissible driver which does not depend on k, such that $f(t, 0, 0) = 0$. [2]

We recall the following lemma (cf. Lemma 3.1 in [23]).

Lemma 3.1. *For each $x \in \mathbb{R}$ and each φ in $\mathbb{H}^2$, the associated wealth process $(V_t^{x,\varphi})$ is an $\mathcal{E}^f$-martingale.*

We emphasize that for an arbitrary random variable $\eta \in L^2$, there does not necessarily exist a pair of processes (X, φ) such that $(X_t, \varphi_t\sigma_t, \varphi_t\beta_t)$ is solution of the BSDE with default jump associated with driver f and terminal condition η, that is, such that (X, φ) satisfies the dynamics (3.2) with $X_T = \eta$. In other terms, the market is incomplete.

In the sequel, we will often use the following change of variables which maps a process $\varphi \in \mathbb{H}^2$ to $Z \in \mathbb{H}^2$ defined by $Z_t = \varphi_t \sigma_t$. With this change of variables, the wealth process $V = V_t^{x,\varphi}$ (for a given $x \in \mathbb{R}$) is the unique process satisfying

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \sigma_t^{-1} \beta_t dM_t, \; V_0 = x. \tag{3.3}$$

We recall that in the *classical linear* case, we have $f(t, y, z) = -r_t y - z\theta_t$, where $\theta_t := \dfrac{\mu_t - r_t}{\sigma_t}$.

3.2 f-martingale measures

We recall the notion of f-evaluation under Q (denoted by $\mathcal{E}_Q^f$), the notion of $\mathcal{E}_Q^f$-martingale, and the notion of f-martingale measure (cf. [23]).

Let Q be a probability measure, equivalent to P. From the $\mathbb{G}$-martingale representation theorem (cf. [30], [25]), its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1, \tag{3.4}$$

where (α_t) and (ν_t) are predictable processes with $\nu_{\vartheta\wedge T} > -1$ a.s. By Girsanov's theorem, the process $W_t^Q := W_t - \int_0^t \alpha_s ds$ is a Brownian motion under Q, and the process $M_t^Q := M_t - \int_0^t \nu_s \lambda_s ds$ is a martingale under Q.

As in [23], we define the spaces S_Q^2, $\mathbb{H}_Q^2$ and $\mathbb{H}_{Q,\lambda}^2$ similarly to $S^2, \mathbb{H}^2$ and $\mathbb{H}_\lambda^2$, but under probability Q instead of P.

[2] so that $\mathcal{E}^f_{\cdot,T'}(0) = 0$ for all $T' \in [0, T]$.

Definition 3.2. *We call f-evaluation under Q, or (f, Q)-evaluation in short, denoted by $\mathcal{E}_Q^f$, the operator defined for each $T' \in [0, T]$ and for each $\eta \in L_Q^2(\mathcal{G}_{T'})$ by $\mathcal{E}_{Q,t,T'}^f(\eta) := X_t$ for all $t \in [0, T']$, where (X, Z, K) is the solution in $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$ of the BSDE under Q associated with driver f, terminal time T' and terminal condition η, and driven by W^Q and M^Q, that is* [3]

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_{T'} = \eta.$$

We note that $\mathcal{E}_P^f = \mathcal{E}^f$.

Definition 3.3. *Let $Y \in S_Q^2$. The process (Y_t) is said to be a (strong) $\mathcal{E}_Q^f$-martingale, or an (f, Q)-martingale, if $\mathcal{E}_{Q,\sigma,\tau}^f(Y_\tau) = Y_\sigma$ a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

We now introduce the concept of f-martingale probability measure.

Definition 3.4. *A probability measure Q equivalent to P is called an* f-martingale probability measure *if for all $x \in \mathbb{R}$ and for all $\varphi \in \mathbb{H}^2 \cap \mathbb{H}_Q^2$, the wealth process $V^{x,\varphi}$ is a strong $\mathcal{E}_Q^f$-martingale, or in other terms an (f, Q)-martingale.*

We note that P is an *f-martingale probability measure* (cf. Lemma 3.1).

As in [23], we denote by $\mathcal{Q}$ the set of f-martingale probability measures Q such that the coefficients (α_t) and (ν_t) associated with its density with respect to P (see equation (3.4)) are bounded. We note that $P \in \mathcal{Q}$.

We denote by $\mathcal{V}$ be the set of bounded predictable processes ν such that $\nu_{\vartheta \wedge T} > -1$ a.s., which is equivalent to $\nu_t > -1$ for all $t \in [0, T]$ $\lambda_t dP \otimes dt$-a.s. (cf. Remark 9 in [11]).

Proposition 3.5. *(Characterization of $\mathcal{Q}$) Let Q be a probability measure equivalent to P, such that the coefficients α and ν of its density* (3.4) *with respect to P are bounded. The two following assertions are equivalent:*
(i) $Q \in \mathcal{Q}$, that is, Q is an f-martingale probability measure.
(ii) there exists $\nu \in \mathcal{V}$ such that $Q = Q^\nu$, where Q^ν is the probability measure which admits ζ_T^ν as density with respect to P on $\mathcal{G}_T$, where ζ^ν satisfies

$$d\zeta_t^\nu = \zeta_{t^-}^\nu(-\nu_t \lambda_t \beta_t \sigma_t^{-1} dW_t + \nu_t dM_t); \zeta_0^\nu = 1. \tag{3.5}$$

3.3 Superhedging prices and no-arbitrage

Let us consider an American option associated with maturity T and a payoff given by a process $(\xi_t) \in \mathbb{S}^2$.

The *superhedging price for the seller* of the American option at time 0, denoted by u_0, is classically defined as the minimal initial capital which enables the seller to be superhedged no matter what the exercise time chosen by the buyer is. More precisely, we have the following definition.

[3] We note that since we have a representation theorem for $(Q, \mathbb{G})$-martingales with respect to W^Q and M^Q (see e.g. Proposition 6 in the appendix of [11]), this BSDE admits a unique solution (X, Z, K) in $S_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_{Q,\lambda}^2$.

Definition 3.6. *A* superhedge for the seller *against the American option with initial price $x \in \mathbb{R}$ is a portfolio strategy $\varphi \in \mathbb{H}^2$ such that $V_t^{x,\varphi} \geq \xi_t$, $0 \leq t \leq T$ a.s.*

For each $x \in \mathbb{R}$, we denote by $\mathcal{A}(x)$ the set of all *superhedges for the seller* associated with initial price x.

The *superhedging price for the seller* of the American option at time 0 is thus defined by

$$u_0 := \inf\{x \in \mathbb{R},\ \exists \varphi \in \mathcal{A}(x)\}. \tag{3.6}$$

We now consider the American option from the point of view of the buyer.

Definition 3.7. *A* superhedge for the buyer *against the American option with initial price $z \in \mathbb{R}$ is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2$ such that $V_\tau^{-z,\varphi} + \xi_\tau \geq 0$ a.s.*

For each $z \in \mathbb{R}$, we denote by $\mathcal{B}(z)$ the set of all *superhedges for the buyer* associated with initial price z.

We now define the *buyer's price* $\tilde{u}_0$ of the American option as the supremum of the initial prices which allow the buyer to be superhedged, that is [4]

$$\tilde{u}_0 = \sup\{z \in \mathbb{R},\ \exists (\tau, \varphi) \in \mathcal{B}(z)\}. \tag{3.7}$$

We now introduce the definitions of an *arbitrage opportunity for the seller and for the buyer* of the American option.

Definition 3.8. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, and let φ in $\mathbb{H}^2$. We say that (y, φ) is an* arbitrage opportunity for the seller *of the American option with initial price x if*

$$y < x \quad \textit{and} \quad V_\tau^{y,\varphi} - \xi_\tau \geq 0 \quad \textit{a.s.} \quad \textit{for all } \tau \in \mathcal{T}.$$

Definition 3.9. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, let $\tau \in \mathcal{T}$ and let $\varphi \in \mathbb{H}^2$. We say that (y, τ, φ) is an* arbitrage opportunity for the buyer *of the American option with initial price x, if*

$$y > x \quad \textit{and} \quad V_\tau^{-y,\varphi} + \xi_\tau \geq 0 \quad \textit{a.s.}$$

Proposition 3.10. *Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller (resp. for the buyer) of the American option with price x if and only if $x > u_0$ (resp. $x < \tilde{u}_0$).*

The proof, which relies on the same arguments as those of the proof of Proposition 5.11 in [14] (see also [26]) is omitted.

Definition 3.11. *A real number x is called an* arbitrage-free price for the American option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer.*

By Propositions 3.10, we get

Proposition 3.12. *If $u_0 < \tilde{u}_0$, there does not exist any arbitrage-free price for the American option. If $u_0 \geq \tilde{u}_0$, the interval $[\tilde{u}_0, u_0]$ is the set of all arbitrage-free prices. We call it the* arbitrage-free *interval for the American option.*

[4] Note that u_0, $\tilde{u}_0 \in \overline{\mathbb{R}}$. We shall see below that, under the assumption that $|\xi_t|$ is smaller than or equal to the value of a portfolio sufficiently integrable (cf. (4.1)), u_0 is finite (cf. Theorem 4.1), and that, under this assumption and some regularity conditions on ξ, $\tilde{u}_0$ is also finite (cf. Theorem 6.1).

4 Duality for the seller's superhedging price

From now on, we assume that the payoff (ξ_t) is such that there exist $x \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ satisfying

$$|\xi_t| \;\leq\; V_t^{x,\psi} = x - \int_0^t f(s, V_s^{x,\psi}, \sigma_s\psi_s)ds + \int_0^t \psi_s\sigma_s dW_s + \int_0^t \beta_s\psi_s dM_s, \quad 0 \leq t \leq T, \quad \text{a.s.} \tag{4.1}$$

We will establish the following dual characterization of the seller's superhedging price (in terms of the f-martingale measures from Definition 3.4).

Theorem 4.1 (*Duality for the seller's superhedging price*)**.** *Let (ξ_t) satisfy Assumption (4.1) with $\psi \in \cap_{\nu\in\mathcal{V}}\mathbb{H}^2_{Q^\nu}$. The superhedging price for the seller u_0 of the American option with payoff (ξ_t) satisfies the equality*

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^f_{Q^\nu,0,\tau}(\xi_\tau).$$

Remark 4.2. *In the linear case, this result reduces to the well-known dual representation of the superhedging price for the seller of the American option in an incomplete (linear) market (cf. [29]).*

In order to prove Theorem 4.1, we will work under the primitive probability P, which will allow us to solve the problem under weaker integrability conditions.
To this aim, we introduce a family of drivers $(f^\nu, \nu \in \mathcal{V})$, which will be used in the sequel.

Definition 4.3 (Driver f^ν and $\mathcal{E}^\nu$-expectation)**.** *For $\nu \in \mathcal{V}$, we define*

$$f^\nu(\omega, t, y, z, k) := f(\omega, t, y, z) + \nu_t(\omega)\lambda_t(\omega)\big(k - \beta_t(\omega)\sigma_t^{-1}(\omega)z\big).$$

The mapping f^ν is a λ-admissible driver [5]*.*
The associated non-linear family of operators, denoted by $\mathcal{E}^{f^\nu}$ or, simply, $\mathcal{E}^\nu$, is defined as follows: for each $T' \leq T$ and each $\eta \in L^2(\mathcal{G}_{T'})$, we have $\mathcal{E}^\nu_{\cdot,T'}(\eta) := X^\nu_\cdot$, where (X^ν, Z^ν, K^ν) is the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE

$$-dX^\nu_t = \big(f(t, X^\nu_t, Z^\nu_t) + \nu_t\lambda_t(K^\nu_t - \beta_t\sigma_t^{-1}Z^\nu_t)\big)\,dt - Z^\nu_t dW_t - K^\nu_t dM_t;\; X^\nu_{T'} = \eta. \tag{4.2}$$

Remark 4.4. *By Proposition 3.5, for each $\nu \in \mathcal{V}$, for all $T' \leq T$ and $\eta \in L^2(\mathcal{G}_{T'}) \cap L^2_{Q^\nu}(\mathcal{G}_{T'})$, we derive that the (f^ν, P)-evaluation of η is equal to its (f, Q^ν)-evaluation, that is,*

$$\mathcal{E}^\nu_{\cdot,T'}(\eta) = \mathcal{E}^f_{Q^\nu,\cdot,T'}(\eta).$$

[5] Since ν is a predictable process, f^ν is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^3)-$ measurable. As, moreover, ν is bounded, f^ν is a λ-admissible driver.

4.1 Non-linear problem of control and stopping. The value family (Y(S)).

Establishing the dual representation for the seller's superheding price is based on the study of the following non-linear problem of control and stopping.
For each $S \in \mathcal{T}$, let $Y(S)$ be the $\mathcal{G}_S$-measurable random variable defined by

$$Y(S) := ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau). \tag{4.3}$$

Remark 4.5. *We note that for each $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and each $\nu \in \mathcal{V}$, the random variable $\mathcal{E}^{\nu}_{S,\tau}(\xi_\tau)$ depends on the control ν only through the values of ν on the interval $[S,\tau]$. For each $S \in \mathcal{T}$, let $\mathcal{V}_S$ be the set of bounded predictable processes ν defined on $[S,T]$, such that $\nu_t > -1$, $S \le t \le T$, $dP \otimes dt$-a.s. We thus have*

$$Y(S) = ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}_S} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau) \quad \text{a.s.}$$

Definition 4.6. *We say that a family $X = (X(S),\, S \in \mathcal{T})$ is* admissible *if it satisfies the following conditions*

In order to facilitate the study of the *non-linear problem of control and stopping* (4.3), we introduce the following *auxiliary non-linear optimal stopping problem*: for $\nu \in \mathcal{V}$, for $S \in \mathcal{T}$,

$$Y^{\nu}(S) = ess \sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau) \tag{4.4}$$

We know from [21] that the value family $(Y^{\nu}(S))_{S\in\mathcal{T}}$ of the auxiliary optimal stopping problem can be aggregated by an optional process $(Y^{\nu}_t)_{t\in[0,T]} \in \mathbb{S}^2$ which is a strong $\mathcal{E}^{\nu}$-supermartingale.
From the definitions and Remark 4.5, we have, for all $S \in \mathcal{T}$,

$$Y(S) = ess \sup_{\nu\in\mathcal{V}} Y^{\nu}_S = ess \sup_{\nu\in\mathcal{V}_S} Y^{\nu}_S \text{ a.s.} \tag{4.5}$$

Let us note also that $Y(S) \ge Y^0_S$ a.s., as $0 \in \mathcal{V}$. Moreover, since $|\xi_t| \le V^{x,\psi}_t$, $0 \le t \le T$ a.s. it follows that for all $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, $\mathcal{E}^{\nu}_{S,\tau}(\xi_\tau) \le \mathcal{E}^{\nu}_{S,\tau}(|\xi_\tau|) \le \mathcal{E}^{\nu}_{S,\tau}(V^{x,\psi}_\tau) = V^{x,\psi}_S$ a.s. Hence, taking the essential supremum over $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$ in this inequality, we derive that $Y(S) \le V^{x,\psi}_S$ a.s. Since $Y^0 \in \mathbb{S}^2$ and $V^{x,\psi} \in \mathbb{S}^2$, it follows that $E[ess \sup_{S\in\mathcal{T}} Y(S)^2] < +\infty$.

Lemma 4.7. (admissibility) *The value family $(Y(S))_{S\in\mathcal{T}}$ of the non-linear problem of control and stopping is an admissible family, that is,*
1. For all $S \in \mathcal{T}$, $Y(S)$ is a real-valued $\mathcal{G}_S$-measurable random variable.
2. For all $S, S' \in \mathcal{T}$, $Y(S) = Y(S')$ a.s. on $\{S = S'\}$.

Proof. By definition (4.5), for each $S \in \mathcal{T}$, $Y(S)$ is $\mathcal{G}_S$-measurable as the essential supremum of $\mathcal{G}_S$-measurable random variables. Let $S, S' \in \mathcal{T}$ be such that $S = S'$ a.s. We have $Y^\nu_S = Y^\nu_{S'}$ a.s. for all $\nu \in \mathcal{V}$. Hence, $ess\sup_{\nu\in\mathcal{V}} Y^\nu_S = ess\sup_{\nu\in\mathcal{V}} Y^\nu_{S'}$ a.s. From this, together with (4.5), we get $Y(S) = Y(S')$ a.s. The admissibility of the value family is thus proven. $\square$

Proposition 4.8. (Maximizing sequence) *Let $S \in \mathcal{T}$. There exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$ with $\nu^n \in \mathcal{V}_S$, for all n, such that the sequence $(Y^{\nu^n}_S)_{n\in\mathbb{N}}$ is non- decreasing and satisfies:*

$$Y(S) \quad = \lim_{n\to\infty} \uparrow Y^{\nu^n}_S \quad \text{a.s.} \tag{4.6}$$

Proof. We show that the set $\{Y^\nu_S, \nu \in \mathcal{V}_S\}$ is stable under pairwise maximization. Indeed, let $\nu, \nu' \in \mathcal{V}_S$. Set $A := \{\, Y^{\nu'}_S \le Y^\nu_S \,\}$. We have $A \in \mathcal{F}_S$. Set $\tilde\nu := \nu\mathbf{1}_A + \nu'\mathbf{1}_{A^c}$. Then $\tilde\nu \in \mathcal{V}_S$. We have $Y^{\tilde\nu}_S\mathbf{1}_A = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{\tilde\nu}_{S,\tau}(\xi_\tau)\mathbf{1}_A = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{f^{\tilde\nu}\mathbf{1}_A}_{S,\tilde\tau}(\xi_{\tilde\tau}\mathbf{1}_A) = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^{f^\nu\mathbf{1}_A}_{S,\tau}(\xi_\tau\mathbf{1}_A) = ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^\nu_{S,\tau}(\xi_\tau)\mathbf{1}_A = Y^\nu_S\mathbf{1}_A$ a.s. and similarly on A^c. It follows that $Y^{\tilde\nu}_S = Y^\nu_S\mathbf{1}_A + Y^{\nu'}_S\mathbf{1}_{A^c} = Y^\nu_S \vee Y^{\nu'}_S$ a.s. The result of the proposition follows by a classical result on essential suprema (cf. Neveu (1975)). $\square$

Definition 4.9. ($\mathcal{E}^g$-(super)martingale family) *Let g be a λ-admissible driver satisfying Assumption 2.2. Let $(X(S),\ S \in \mathcal{T})$ be an admissible family such that $\mathbb{E}[ess\sup_{S\in\mathcal{T}}(X(S))^2] < \infty$. We say that $(X(S),\ S \in \mathcal{T})$ is an $\mathcal{E}^g$*-supermartingale *(resp. $\mathcal{E}^g$-* submartingale*, $\mathcal{E}^g$-*martingale*)* family *if for all $S, S' \in \mathcal{T}$ such that $S \ge S'$ a.s., $\mathcal{E}^g_{S',S}(X(S)) \le$ (resp. $\ge$, $=$) $X(S')$ a.s.*

Proposition 4.10. *The family $(Y(S))$ satisfies the following properties: $(Y(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$ and $Y(S) \ge \xi_S$ a.s. for all $S \in \mathcal{T}$. Moreover, $(Y(S))$ is the smallest family satisfying these properties.*

Proof. For all $S \in \mathcal{T}$, for all $\nu \in \mathcal{V}_S$, $Y^\nu_S \ge \xi_S$ a.s. Hence, for all $S \in \mathcal{T}$, $Y(S) \ge \xi_S$ a.s. Let $S, S' \in \mathcal{T}$ be such that $S \ge S'$ a.s. By Proposition 4.8, there exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$, with ν_n in $\mathcal{V}_S$ for all n, such that $Y(S) \quad = \lim_{n\to\infty} \uparrow Y^{\nu^n}_S$ a.s. Let $\nu \in \mathcal{V}$. By the continuity property of $\mathcal{E}^\nu$ with respect to terminal condition, we have $\mathcal{E}^\nu_{S',S}(Y(S)) = \lim_{n\to\infty} \mathcal{E}^\nu_{S',S}(Y^{\nu_n}_S)$ a.s. For each n, we set $\tilde\nu^n_t := \nu_t\mathbf{1}_{]S',S]}(t) + \nu^n_t\mathbf{1}_{]S,T]}(t)$. We note that $\tilde\nu^n \in \mathcal{V}_{S'}$; hence, $f^{\tilde\nu^n}$ is λ-admissible. We have $f^{\tilde\nu^n} = f^\nu\mathbf{1}_{]S',S]} + f^{\nu^n}\mathbf{1}_{]S,T]}$. Moreover, $Y^{\nu_n}_S = Y^{\tilde\nu_n}_S$ (as $f^{\tilde\nu^n} = f^{\nu^n}$ on $[S,T]$, $dt \otimes dP$-a.e.). From these observations, we deduce

$$\mathcal{E}^\nu_{S',S}(Y^{\nu_n}_S) = \mathcal{E}^{\tilde\nu^n}_{S',S}(Y^{\tilde\nu_n}_S) \le Y^{\tilde\nu_n}_{S'},$$

where the (last) inequality is due to the fact that $Y^{\tilde\nu_n}$ is a strong $\mathcal{E}^{\tilde\nu^n}$-supermartingale. We thus get $\mathcal{E}^\nu_{S',S}(Y(S)) = \lim\limits_{n\to\infty} \mathcal{E}^\nu_{S',S}(Y^{\nu_n}_S) \le \liminf\limits_{n\to\infty} Y^{\tilde\nu_n}_{S'} \le Y(S')$ a.s. , where the last inequality follows from (4.5). As $\nu \in \mathcal{V}$ is arbitrary, we conclude that the family $(Y(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$.

Let us prove the second statement. Let $(Y'(S), S \in \mathcal{T})$ be an admissible family satisfying the properties: $(Y'(S))$ is an $\mathcal{E}^\nu$-supermartingale family for all $\nu \in \mathcal{V}$ and $Y'(S) \ge \xi_S$ a.s. for all $S \in \mathcal{T}$. Let $\nu \in \mathcal{V}$. By the properties of Y', for all $S \in \mathcal{T}$, for all $\tau \in \mathcal{T}_S$, $Y'(S) \ge \mathcal{E}^\nu_{S,\tau}(Y'(\tau)) \ge \mathcal{E}^\nu_{S,\tau}(\xi_\tau)$ a.s. By taking the essential supremum over $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, we get $Y'(S) \ge Y(S)$ a.s. $\square$

Corollary 4.11. *There exists an r.u.s.c. process* $(Y_t) \in \mathbb{S}^2$ *which aggregates the value family* $(Y(S))$ *of the problem of control and stopping* (4.3)*. The process* (Y_t) *is a strong* $\mathcal{E}^\nu$*-supermartingale for all* $\nu \in \mathcal{V}$ *and* $Y_t \geq \xi_t$*, for all* $t \in [0,T]$*, a.s. Moreover, the process* (Y_t) *is the smallest process in* $\mathbb{S}^2$ *satisfying these properties.*

Proof. The above Proposition 4.10 implies in particular that the value family $(Y(S))$ is a strong $\mathcal{E}^0$-supermartingale family. By Lemma A.1 in [23], there exists an r.u.s.c. process (Y_t) in $\mathbb{S}^2$ aggregating the family $(Y(S))$. The other properties of the aggregating process (Y_t) follow directly from Proposition 4.10. □

Corollary 4.12 (The right-continuous case)**.** *Assume moreover that the process* (ξ_t) *in problem* (4.3) *is* RCLL*. Then, the process* (Y_t) *is RCLL. Moreover,* (Y_t) *is the smallest RCLL process in* $\mathbb{S}^2$ *satisfying the properties: for each* $\nu \in \mathcal{V}$*,* (Y_t) *is a (strong) RCLL* $\mathcal{E}^\nu$*-supermartingale greater than or equal to* (ξ_t)*.*

Proof. This result follows directly from Corollary 4.11, together with Remark A.7 in [23]. □

4.2 The strict value family $(Y^+(S))$

Let S be a stopping time in $\mathcal{T}_0$. We denote by $\mathcal{T}_{S^+}$ the set of stopping times $\theta \in \mathcal{T}_0$ with $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$. The strict value $Y^+(S)$ (at time S) is defined by

$$Y^+(S) := ess \sup_{(\tau,\nu) \in \mathcal{T}_{S^+} \times \mathcal{V}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau). \tag{4.7}$$

We note that (as for $Y(S)$) the set $\mathcal{V}$ in the above problem can be replaced with the set $\mathcal{V}_S$ without changing the value of the problem.
We note also that $Y^+(S) = \xi_T$ a.s. on $\{S = T\}$.
Let S be a stopping time in $\mathcal{T}_0$ and let $\nu \in \mathcal{V}$. We introduce the following *auxiliary (strict) optimal stopping problem* (to be compared with (4.4)):

$$Y^{\nu,+}(S) := ess \sup_{\tau \in \mathcal{T}_{S^+}} \mathcal{E}^\nu_{S,\tau}(\xi_\tau). \tag{4.8}$$

We know from [21] (cf. Proposition 9.1) that there exists a strong $\mathcal{E}^\nu$-supermartingale, denoted by $(Y^{\nu,+}_t)$, which aggregates the value family $(Y^{\nu,+}(S))$ of the above (strict) optimal stopping problem. Note that we have

$$Y^+(S) = ess \sup_{\nu \in \mathcal{V}} Y^{\nu,+}_S = ess \sup_{\nu \in \mathcal{V}_S} Y^{\nu,+}_S \quad \text{a.s.} \tag{4.9}$$

Using the above representation and the same type of arguments as those used above for the value family $(Y(S))_{S \in \mathcal{T}_0}$, we show that the strict value family $(Y^+(S))_{S \in \mathcal{T}_0}$ is an admissible family, satisfying the integrability condition $\mathbb{E}[ess \sup_{S \in \mathcal{T}_0} (Y^+(S))^2] < \infty$ and the following properties:

Proposition 4.13. *For each $S \in \mathcal{T}_0$, there exists a maximizing sequence $(\nu^n) = (\nu^n(S)) \in \mathcal{V}_S^n$ for the optimal control problem from equation (4.9), that is, $Y_S^+ = \lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}$ a.s.*

Proposition 4.14. *The family $(Y^+(S))_{S\in\mathcal{T}_0}$ is an $\mathcal{E}^\nu$-supermartingale family for each $\nu \in \mathcal{V}$.*

As above, we deduce the following

Corollary 4.15. *There exists a process $(Y_t^+) \in \mathbb{S}^2$ which aggregates the strict value family $(Y^+(S))_{S\in\mathcal{T}_0}$. The process (Y_t^+) is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$.*

Moreover, the following result holds true. The result is based on the representation (4.9) and on properties of the strict value process $(Y_t^{\nu,+})$ of the auxiliary optimal stopping problem (4.8).
We recall that (Y_{t+}) denotes the process of right limits of the process (Y_t). We recall also that (Y_{t+}) is well-defined as (Y_t) is a strong $\mathcal{E}^\nu$-supermartingale, and hence, has right (and left) limits.
We recall that (Y_{t+}^ν) denotes the process of right limits of the process (Y_t^ν).

Theorem 4.16. (i) *The strict value process (Y_t^+) is right-continuous.*

(ii) *For all $S \in \mathcal{T}_0$, $Y_S^+ = Y_{S+}$ a.s. (in other words, the strict value process (Y_t^+) coincides with the process of right limits (Y_{t+})).*

(iii) *For all $S \in \mathcal{T}_0$, $Y_S = Y_{S+} \vee \xi_S$ a.s.*

We have the following intermediary result:

Proposition 4.17. *For all $S \in \mathcal{T}_0$,*

$$\mathbb{E}[Y_S^+] = \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}].$$

Proof. From the representation (4.9), we deduce $\mathbb{E}[Y_S^+] = \mathbb{E}[ess\sup_{\nu\in\mathcal{V}} Y_S^{\nu,+}] \geq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$. We now show the converse inequality. By Proposition 4.13, there exists a sequence $(\nu_n) = (\nu_n(S))$ in $\mathcal{V}_S^n$ such that $Y_S^+ = \lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}$. We thus have $\mathbb{E}[Y_S^+] = \mathbb{E}[\lim_{n\to\infty} \uparrow Y_S^{\nu_n,+}] = \lim_{n\to\infty} \uparrow \mathbb{E}[Y_S^{\nu_n,+}]$, where we have used dominated convergence to exchange limit and expectation. For all n, we have $\mathbb{E}[Y_S^{\nu_n,+}] \leq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$.[6] We conclude that $\mathbb{E}[Y_S^+] \leq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_S^{\nu,+}]$. The proposition is thus proved. □

We are now ready to prove Theorem 4.16.
Proof of Theorem 4.16. To prove statement (i), we first show that the process (Y_t^+) is right-lowersemicontinuous along stopping times in expectation. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim \downarrow S_n = S$ a.s. By Proposition 4.17, we have $\mathbb{E}[Y_{S_n}^+] = \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y_{S_n}^{\nu,+}]$, for all $n \in \mathbb{N}$. Hence, $\liminf_{n\to\infty} \mathbb{E}[Y_{S_n}^+] =$

[6] Indeed, each process $\nu \in \mathcal{V}_S$ can be seen as a process $\tilde{\nu}$ in $\mathcal{V}$ by setting $\tilde{\nu} = \nu$ on $[S,T]$ and $\tilde{\nu} = 0$ on $[0,S)$.

$\liminf_{n\to\infty} \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y^{\nu,+}_{S_n}] \geq \sup_{\nu\in\mathcal{V}} \liminf_{n\to\infty} \mathbb{E}[Y^{\nu,+}_{S_n}]$. Now, for all $\nu \in \mathcal{V}$, the process $(Y^{\nu,+})$ is right-continuous (cf. Theorem 9.2 in [21]), hence right-continuous along stopping times (cf. [9]); by dominated convergence, we thus have $\liminf_{n\to\infty} \mathbb{E}[Y^{\nu,+}_{S_n}] = \lim_{n\to\infty} \mathbb{E}[Y^{\nu,+}_{S_n}] = \mathbb{E}[Y^{\nu,+}_S]$. This, together with the above computation, gives $\liminf_{n\to\infty} \mathbb{E}[Y^+_{S_n}] \geq \sup_{\nu\in\mathcal{V}} \mathbb{E}[Y^{\nu,+}_S] = \mathbb{E}[Y^+_S]$, where the last equality holds due to Proposition 4.17. We conclude that the process (Y^+_t) is right-lowersemicontinuous along stopping times in expectation. On the other hand, we know already that the process (Y^+_t) is right-uppersemicontinuous along stopping times, and hence right-uppersemicontinuous along stopping times in expectation (due to its integrability). Hence, (Y^+_t) is right-continuous along stopping times in expectation. We deduce that (Y^+_t) is right-continuous (cf. e.g. [10]). We now show (ii). Let $S \in \mathcal{T}_0$. One inequality, namely the inequality $Y_{S+} \geq Y^+_S$ a.s., follows from the right-continuity of (Y^+_t), established in (i). Indeed, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S^+}$ with $\lim \downarrow S_n = S$ a.s. We know that $Y_\tau \geq Y^+_\tau$ a.s., for all $\tau \in \mathcal{T}_0$. Hence, $Y_{S_n} \geq Y^+_{S_n}$ a.s., for all n. By taking the limit when $n \to \infty$ and by using the right-continuity of (Y^+_t), we get $Y_{S+} \geq Y^+_S$ a.s. For the converse inequality, we first show

$$\mathcal{E}^0_{S,S^n}(Y_{S_n}) \leq Y^+_S \text{ a.s. for all } n. \tag{4.10}$$

To prove this, we fix n and we take $(\tau^p, \nu^p) \in \mathcal{T}_{S_n} \times \mathcal{V}$ an optimizing sequence for the problem with value Y_{S_n}, i.e. $Y_{S_n} = \lim_{p\to\infty} \mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})$. We have

$$\mathcal{E}^0_{S,S_n}(Y_{S_n}) = \mathcal{E}^0_{S,S_n}(\lim_{p\to\infty} \mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) = \lim_{p\to\infty} \mathcal{E}^0_{S,S_n}(\mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) \text{ a.s.,} \tag{4.11}$$

where we have used the continuity property of $\mathcal{E}^0_{S,S^n}(\cdot)$ with respect to the terminal condition (recall that here n is fixed). We set $\bar{\nu}^p_t := \nu^p_t \mathbb{I}_{\{t>S_n\}}$ (hence, $\bar{\nu}^p_t = 0$ on $\{t \leq S_n\}$). We note that $\bar{\nu} \in \mathcal{V}$. Using the definition of $\bar{\nu}$ and the consistency property of $\mathcal{E}$-expectations, we get $\mathcal{E}^0_{S,S_n}(\mathcal{E}^{\nu^p}_{S_n,\tau_p}(\xi_{\tau_p})) = \mathcal{E}^{\bar{\nu}}_{S,\tau_p}(\xi_{\tau_p}) \leq Y^+_S$ a.s. (where for the inequality we have used that $\tau_p \in \mathcal{T}_{S^+}$). From this, together with equation (4.11), we derive the desired inequality (4.10). From (4.10) and using the continuity of $\mathcal{E}$-expectations with respect to the terminal time and the terminal condition, we derive $Y^+_S \geq \lim_{n\to\infty} \mathcal{E}^0_{S,S^n}(Y_{S_n}) = \mathcal{E}^0_{S,S}(Y_{S+}) = Y_{S+}$ a.s. Hence, $Y^+_S \geq Y_{S+}$ a.s., which, together with the previously shown converse inequality, proves the equality.

We now show (iii). Using successively statement (ii), relation (4.9), Theorem 9.2 (iii) in [21], and relation (4.5), we get

$$Y_{S+} \vee \xi_S = Y^+_S \vee \xi_S = ess\sup_{\nu\in\mathcal{V}} \left(Y^{\nu,+}_S \vee \xi_S\right) = ess\sup_{\nu\in\mathcal{V}} Y^\nu_S = Y_S \quad \text{a.s.} \ \square$$

4.3 Proof of the dual representation

We will now provide a dual representation for the seller's superhedging price u_0 in terms of the value (at time 0) of the non-linear problem of control and stopping studied above. We also give a superhedging strategy for the seller. From this result, we will deduce the dual representation (in terms of the f-martingale probability measures) stated in Theorem 4.1.

To this aim, we first give the following lemma.

Lemma 4.18. *($\mathcal{E}^f$-optional decomposition of the value process Y) There exists a unique $Z \in \mathbb{H}^2$, a unique $C \in \mathbb{C}^2$ and a unique nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[h_T^2] < \infty$ such that*

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s) - h_t - C_{t^-},\ 0 \le t \le T \quad \text{a.s.} \tag{4.12}$$

Proof. By Corollary 4.11, the value process Y is the smallest process in $\mathbb{S}^2$, which is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$ such that $Y_t \ge \xi_t$, for all $t \in [0,T]$, a.s. The desired result then follows from the non-linear optional decomposition of strong $\mathcal{E}^\nu$-supermartingales for all $\nu \in \mathcal{V}$ (cf. Theorem B.4 in [23]). □

Theorem 4.19. *The* superhedging price u_0 *of the American option is equal to the value Y_0 (at time 0) of the non-linear problem of control and stopping* (4.3)*, that is*

$$u_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^\nu_{0,\tau}(\xi_\tau). \tag{4.13}$$

Moreover, the portfolio strategy $\varphi^* := \sigma^{-1}Z$, where the process Z is the one from the $\mathcal{E}^f$-optional decomposition of the value process Y from Lemma 4.18, is a superhedging strategy for the seller, that is, $\varphi^* \in \mathcal{A}(u_0)$.

Proof. Let $\mathcal{H}$ be the set of initial capitals which allow the seller to be "superhedged", that is $\mathcal{H} = \{x \in \mathbb{R} : \exists \varphi \in \mathcal{A}(x)\}$. From the definition of u_0 (see (3.6)), we have $u_0 = \inf \mathcal{H}$. We first show that

$$u_0 \ge \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^\nu_{0,\tau}(\xi_\tau). \tag{4.14}$$

Let $x \in \mathcal{H}$. There thus exists $\varphi \in \mathcal{A}(x)$, which implies that for each $\tau \in \mathcal{T}$, we have $V_\tau^{x,\varphi} \ge \xi_\tau$ a.s. Let $\nu \in \mathcal{V}$. By taking the $\mathcal{E}^\nu$-evaluation in the above inequality, using the monotonicity of $\mathcal{E}^\nu$ and the $\mathcal{E}^\nu$-martingale property of the wealth process $V^{x,\varphi}$, we obtain $x = \mathcal{E}^\nu_{0,\tau}(V_\tau^{x,\varphi}) \ge \mathcal{E}^\nu_{0,\tau}(\xi_\tau)$. By arbitrariness of $\tau \in \mathcal{T}$ and $\nu \in \mathcal{V}$, we get $x \ge \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^\nu_{0,\tau}(\xi_\tau)$, which holds for all $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we obtain the desired inequality (4.14). Since, by definition of Y_0, we have $Y_0 = \sup_{(\tau,\nu)\in\mathcal{T}\times\mathcal{V}} \mathcal{E}^\nu_{0,\tau}(\xi_\tau)$, the inequality (4.14) can be written $u_0 \ge Y_0$.

We now show the converse inequality, that is, $Y_0 \ge u_0$. Since $u_0 = \inf \mathcal{H}$, it is sufficient to show that the portfolio strategy $\varphi^* := \sigma^{-1}Z$ is a superhedging strategy for the seller associated with the initial capital Y_0, that is, satisfies

$$\varphi^* \in \mathcal{A}(Y_0). \tag{4.15}$$

We consider the portfolio associated with the initial capital Y_0 and the strategy φ^*. By (3.2)-(3.3), the value of this portfolio $(V_t^{Y_0,\varphi^*})$ satisfies the following forward equation:

$$V_t^{Y_0,\varphi^*} = Y_0 - \int_0^t f(s, V_s^{Y_0,\varphi^*}, Z_s)ds + \int_0^t \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s),\ 0 \le t \le T \quad \text{a.s.} \tag{4.16}$$

Moreover, by the optional $\mathcal{E}^f$-decomposition of the value process (Y_t) (cf. Lemma 4.18), the process (Y_t) satisfies the forward SDE (4.12). Since (h_t) and (C_{t^-}) are nondecreasing, by the comparison result for forward differential equations, we thus get $V_t^{Y_0,\varphi^*} \geq Y_t$, $0 \leq t \leq T$ a.s. Hence, since $Y_t \geq \xi_t$, we get $V_t^{Y_0,\varphi^*} \geq \xi_t$, $0 \leq t \leq T$ a.s., which implies the desired property (4.15). We thus derive that $Y_0 \in \mathcal{H}$, and hence that $Y_0 \geq u_0$. Since $Y_0 \leq u_0$, we deduce the equality $Y_0 = u_0$. Moreover, by (4.15), we derive that $\varphi^* \in \mathcal{A}(u_0)$, which completes the proof. □

Remark 4.20. *From a financial point of view, the process (h_t) from equation* (4.12) *can be interpreted as the cumulative amount the seller withdraws from the hedging portfolio up to time t. More precisely, for each time t, the seller can withdraw the amount dh_t from his/her portfolio between t and $t + dt$. In particular, at time ϑ, the seller can withdraw the amount Δh_ϑ from his/her portfolio, which, by equation* (4.12)*, is equal to*

$$\Delta h_\vartheta = \beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta - \Delta Y_\vartheta \quad \text{a.s.}$$

The term $\beta_\vartheta \sigma_\vartheta^{-1} Z_\vartheta = \beta_\vartheta \varphi_\vartheta^*$ represents the jump at the default time ϑ of the amount invested in the risky asset S (which is equal to the jump of the value of the portfolio). Note that in this case, the value of the hedging portfolio, denoted by $(V_t^{Y_0,\varphi^*,h})$, taking into account these withdrawals, satisfies

$$dV_t^{Y_0,\varphi^*,h} = -f(t, V_t^{Y_0,\varphi^*,h}, \sigma_t \varphi_t^*)dt + \varphi_t^*(\sigma_t dW_t + \beta_t dM_t) - dh_t;\ V_0^{Y_0,\varphi^*,h} = Y_0.$$

We thus have $V_t^{Y_0,\varphi^*} = V_t^{Y_0,\varphi^*,0}$.

Proof of Theorem 4.1: The proof follows from the previous theorem 4.19 and from Remark 4.4. Indeed, under the additional integrability condition $\psi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$ on the process ψ from Assumption (4.1), by Remark 4.4, the above dual representation of the *superhedging price* can be written in terms of the f-martingale probability measures (characterized in Proposition 3.5), that is

$$u_0 = \sup_{(\tau,\nu) \in \mathcal{T} \times \mathcal{V}} \mathcal{E}^f_{Q^\nu,0,\tau}(\xi_\tau),$$

which ends the proof of Theorem 4.1. □

5 Infinitesimal characterizations of the seller's superhedging price process

We now introduce the *seller's (superhedging) price* of the American option at each time/stopping time $S \in \mathcal{T}$. We first define, for each initial wealth $X \in L^2(\mathcal{G}_S)$, a *superhedging strategy* as a portfolio strategy $\varphi \in \mathbb{H}^2$ such that $V_t^{S,X,\varphi} \geq \xi_t$ for all $t \in [0,T]$ a.s., where $V^{S,X,\varphi}$ denotes the wealth process associated with initial time S and initial condition X. Let $\mathcal{A}_S(X)$ be the set of all superhedging strategies associated with initial time S and initial wealth X. The *seller's (superhedging) price at time S* is defined by the random variable

$$u(S) := \operatorname{ess\,inf}\{X \in L^2(\mathcal{G}_S),\ \exists \varphi \in \mathcal{A}_S(X)\}.$$

Using Lemma 4.18 and similar arguments to those used in the proof of Theorem 4.19, one can show that for each $S \in \mathcal{T}$, we have $u(S) = Y_S$ a.s. We call (Y_t) the *seller's (superhedging) price process* of the American option.

Definition 5.1. *Let $\xi \in \mathbb{S}^2$. A process $Y' \in \mathbb{S}^2$ is said to be a* supersolution *of the* constrained reflected BSDE *with driver f and obstacle ξ if there exists a process $(Z', K', A', C') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-\,dY'_t = f(t, Y'_t, Z'_t)dt + dA'_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t; \tag{5.1}$$

$$Y'_T = \xi_T \ \textit{a.s.} \quad \text{and} \quad Y'_t \geq \xi_t \textit{ for all } t \in [0,T] \quad \textit{a.s.}; \tag{5.2}$$

$$(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \ \textit{a.s. for all } \tau \in \mathcal{T}_0; \tag{5.3}$$

$$A'_\cdot + \int_0^\cdot (K'_s - \beta_s \sigma_s^{-1} Z'_s)\lambda_s ds \in \mathcal{A}^2 \quad \text{and} \quad (K'_t - \beta_t \sigma_t^{-1} Z'_t)\lambda_t \leq 0,\ t \in [0,T],\ dP \otimes dt - \text{a.e.}; \tag{5.4}$$

Remark 5.2. *This definition can be extended to the case of a general driver g (which may depend also on k).*

Equation (5.3) is referred to as *Skorokhod condition* for the process C'.

Remark 5.3. *The process A' can be uniquely decomposed as the sum of two nondecreasing processes B' and $\hat{B}$ belonging to $\mathcal{A}^2$ with $dB'_t \perp d\hat{B}_t$,*[7] *such that B' satisfies the* Skorokhod condition*, that is*

$$\int_0^t (Y'_{s^-} - \xi_{s^-}) dB'_s = 0 \quad \text{a.s.} \tag{5.5}$$

Note that the processes B' and $\hat{B}$ are given by $B'_t = \int_0^t \mathbf{1}_{\{Y'_{s^-} = \xi_{s^-}\}} dA'_s$ and $\hat{B}_t = \int_0^t \mathbf{1}_{\{Y'_{s^-} > \xi_{s^-}\}} dA'_s$ for all $t \in [0,T]$. It follows that $Y' \in \mathbb{S}^2$ is a supersolution *of the* constrained reflected BSDE *with driver f and obstacle ξ if and only if there exists a process $(Z', K', B', \hat{B}, C') \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$-\,dY'_t = f(t, Y'_t, Z'_t)dt + dB'_t + d\hat{B}_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t; \tag{5.6}$$

$$Y'_T = \xi_T \ \textit{a.s.} \quad \textit{and} \quad Y'_t \geq \xi_t \textit{ for all } t \in [0,T] \quad \textit{a.s.}; \tag{5.7}$$

$$(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \ \textit{a.s. for all } \tau \in \mathcal{T}_0; \tag{5.8}$$

$$\int_0^t (Y'_{s^-} - \xi_{s^-}) dB'_s = 0 \quad \textit{a.s.} \quad \textit{and} \quad d\hat{B}_t \perp dB'_t, \tag{5.9}$$

[7] in the sense of Definition 2.3 from [12]

and such that the constraints (5.4) *hold, with A' replaced by $B' + \hat{B}$.*
In the particular case when $\hat{B} = 0$, since B' satisfies the Skorokhod condition, *the process* (Y', Z', K', B', C') *is thus a* solution [8] *of the reflected BSDE (with irregular obstacle (ξ_t)), here with the additional constraints* (5.4). *Thus, when passing from the notion of a* solution *of the reflected BSDE to the notion of a* supersolution *of the reflected BSDE, there appears an additional nondecreasing predictable process $\hat{B}$, which increases only when $Y'_{t^-} > \xi_{t^-}$.*

Theorem 5.4. *(Infinitesimal characterization I) The seller's price process (Y_t) is a supersolution of the* constrained reflected BSDE *associated with driver f and obstacle ξ from Definition 5.1, that is, there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that (Y, Z, K, A, C) satisfies Definition 5.1. Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0, T]$ a.s.*

Moreover, the portfolio strategy $\varphi^* := \sigma^{-1} Z$ is a superhedging strategy for the seller, that is, $\varphi^* \in \mathcal{A}(u_0)$.

Remark 5.5. *Suppose here that there is no default in the market. In this case, the filtration $\mathbb{G}$ is the one associated with the Brownian motion W, and in the dynamics of the price process (S_t) and of the wealth process (V_t), $M = 0$ and $\beta = 0$. Hence, the market is complete, and we have $\mathcal{V} = \{0\}$. From this observation, we derive that for each $S \in \mathcal{T}$, $Y_S = Y^0_S = ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}^0_{S,\tau}(\xi_\tau)$ a.s. By Theorem 6.7 in [22], (Y_t) is thus the* solution *of the reflected BSDE associated with driver f and irregular obstacle (ξ_t). In other words, there exists $(Z, K, B, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that equations* (5.6) *to* (5.9) *hold with $\hat{B} = 0$.*

Proof. Since Y is the value process, we have $Y_T = \xi_T$ a.s. and $Y_t \geq \xi_t$ for all $t \in [0, T]$ a.s. Moreover, by Corollary 4.11, the value process Y is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$. Hence, by the non-linear predictable decomposition (cf. Proposition B.1 in [23]), there exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that equation (5.1) and the conditions (5.4) hold. We now show that the process C satisfies the *Skorokhod condition* (5.3). Let $\tau \in \mathcal{T}_0$. By Theorem 4.16 (iii), we have $Y_\tau = Y_{\tau+} \vee \xi_\tau$ a.s.. Hence, $\Delta_+ Y_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta_+ Y_\tau$ a.s. On the other hand, since (Y, Z, K, A, C) satisfies equation (5.1), we have $\Delta C_\tau = -\Delta_+ Y_\tau$ a.s. We conclude that $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ a.s. Hence, the Skorokhod condition (5.3) is satisfied.

It remains to show that (Y_t) is the *minimal* supersolution of the constrained reflected BSDE from Definition 5.1. Let Y' be another supersolution of this constrained reflected BSDE and let (Z', K', A', C') be the associated process (from the definition of a supersolution). We have $Y'_t \geq \xi_t$ for all $t \in [0, T]$ a.s. Let now $\nu \in \mathcal{V}$. Let A'^ν be the process defined by

$$A'^\nu_t := A'_t - \int_0^t (K'_s - \beta_s \sigma_s^{-1} Z'_s) \nu_s \lambda_s ds, \quad 0 \leq t \leq T.$$

Since $\nu \in \mathcal{V}$, we have $\nu_t + 1 > 0$ $dP \otimes dt$-a.s. This together with the second condition from (5.4) imply that $(K'_t - \beta_t \sigma_t^{-1} Z'_t) \lambda_t (1 + \nu_t) \leq 0$ $dP \otimes dt$-a.s. Then, using the first condition

[8] in the sense from Definition 2.3 in [22], which, in the case of a right-continuous obstacle, corresponds to the well-known notion of a solution of a reflected BSDE)

from (5.4) (and the definition of A'^{ν}), we obtain that the process A'^{ν} is nondecreasing. On the other hand, since (Y', Z', K', A', C') satisfies the dynamics from Definition 5.1, we have

$$-dY'_t = \left(f(t, Y'_t, Z'_t) + (K'_t - \beta_t \sigma_t^{-1} Z'_t)\nu_t \lambda_t\right) dt + dA'^{\nu}_t + dC'_{t-} - Z'_t dW_t - K'_t dM_t.$$

Hence, by the $\mathcal{E}^g$-Mertens decomposition of strong $\mathcal{E}^g$-supermartingales, applied with the driver $g := f^{\nu}$ (cf. Theorem 5.1 in [20], or Theorem 7.1 in [21]), we derive that the process Y' is a strong $\mathcal{E}^{\nu}$- supermartingale. Since this holds for all $\nu \in \mathcal{V}$, we derive from Corollary 4.11 that $Y'_t \geq Y_t$, for all $t \in [0, T]$ a.s. □

Remark 5.6. *This result can be extended to any λ-admissible driver (depending also on k).*

Definition 5.7. *Let $\xi \in \mathbb{S}^2$. A process $Y' \in \mathbb{S}^2$ is called a* supersolution *of the* optional reflected BSDE *associated with driver f and obstacle ξ if there exist $Z' \in \mathbb{H}^2$, $C' \in \mathbb{C}^2$ and a nondecreasing optional RCLL process h', with $h'_0 = 0$ and $E[(h'_T)^2] < \infty$ such that*

$$\begin{aligned}
&-dY'_t = f(t, Y'_t, Z'_t)dt - Z'_t \sigma_t^{-1}(\sigma_t dW_t + \beta_t dM_t) + dC'_{t-} + dh'_t; \\
&Y'_T = \xi_T \quad \text{and} \quad Y'_t \geq \xi_t \text{ for all } t \in [0, T] \quad a.s.; \\
&(Y'_\tau - \xi_\tau)(C'_\tau - C'_{\tau-}) = 0 \ a.s. \text{ for all } \tau \in \mathcal{T}_0.
\end{aligned}$$

Remark 5.8. *We call the above equation an* optional *reflected BSDE because the associated non decreasing right-continuous process is optional but not necessarily predictable contrary the reflected BSDEs considered in the literature.*

Note also that when the obstacle ξ is right-continuous, the purely discontinuous non decreasing process C' (corresponding to the right-jumps of Y') is equal to 0.

From the non-linear optional decomposition (cf. Theorem B.4 in [23]), together with the equivalence of the non-linear predictable and the non-optional decompositions (cf. Proposition B.5. in [23]), we derive the following result:

Theorem 5.9. *(Infinitesimal characterization II) The seller's superhedging price (Y_t) of the American option is a supersolution of the* optional reflected BSDE *from Definition 5.7. Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0, T]$ a.s.*

6 Duality for the buyer's superhedging price

Theorem 6.1 (*Duality for the buyer's superhedging price*)**.** *Let $(\xi_t) \in \mathbb{S}^2$ be such that Assumption (4.1) is satisfied with $\psi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$. Suppose moreover that (ξ_t) is right-continuous and left-uppersemicontinuous along stopping times. The* superhedging price for the buyer $\tilde{u}_0$ *of the American option satisfies*

$$\tilde{u}_0 = \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \{-\mathcal{E}^f_{Q^\nu, 0, \tau}(-\xi_\tau)\} = \sup_{\tau \in \mathcal{T}} \inf_{\nu \in \mathcal{V}} \{-\mathcal{E}^f_{Q^\nu, 0, \tau}(-\xi_\tau)\}.$$

Again, to prove Theorem 6.1, we will work under the primitive probability measure P.

We define $\tilde{f}(t,\omega,y,z) := -f(t,\omega,-y,-z)$.

Let $\nu \in \mathcal{V}$. We denote by $\mathcal{E}^{\tilde{f}^\nu}$ or $\tilde{\mathcal{E}}^\nu$ the non-linear conditional expectation associated with the λ-admissible driver

$$\tilde{f}^\nu(t,y,z,k) := \tilde{f}(t,y,z) + \nu_t \lambda_t (k - \beta_t \sigma_t^{-1} z).$$

Hence, for each $T' \leq T$ and each $\eta \in L^2(\mathcal{G}_{T'})$, we have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = \tilde{X}^\nu$ a.s., where $(\tilde{X}^\nu, \tilde{Z}^\nu, \tilde{K}^\nu)$ be the unique solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the BSDE associated with driver $\tilde{f}^\nu$, terminal time T' and terminal condition η.

Remark 6.2. *Let $\nu \in \mathcal{V}$ and $T' \leq T$. Note that for all $\eta \in L^2(\mathcal{G}_{T'})$, we have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = -\mathcal{E}^\nu_{\cdot,T'}(-\eta)$, since $\tilde{f}^\nu(t,y,z,k) = -f^\nu(t,-y,-z,-k)$.*

Let $\eta \in L^2(\mathcal{G}_{T'}) \cap L^2_{Q^\nu}(\mathcal{G}_{T'})$. By Remark 4.4, $\mathcal{E}^\nu_{\cdot,T'}(\eta) = \mathcal{E}^f_{Q^\nu,\cdot,T'}(\eta)$. We thus have $\tilde{\mathcal{E}}^\nu_{\cdot,T'}(\eta) = -\mathcal{E}^f_{Q^\nu,\cdot,T'}(-\eta)$.

For each $S \in \mathcal{T}$, we define the $\mathcal{F}_S$-measurable random variable $\tilde{Y}(S)$ as follows:

$$\tilde{Y}(S) := ess \inf_{\nu \in \mathcal{V}_S} ess \sup_{\tau \in \mathcal{T}_S} \tilde{\mathcal{E}}^\nu_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{6.1}$$

6.1 First properties of the value family $\tilde{Y}$

Let us first show that $E[ess\sup_{\tau\in\mathcal{T}} \tilde{Y}^2(\tau)] < \infty$.

As $0 \in \mathcal{V}$, we have $\tilde{Y}(S) \geq ess\sup_{\tau\in\mathcal{T}_S} \mathcal{E}^0_{S,\tau}(\xi_\tau) = \tilde{Y}^0_S$ a.s., where $(\tilde{Y}^0_t)$ is the first coordinate of the solution of the reflected BSDE associated with driver $\tilde{f}$ and lower obstacle (ξ_t). Now, since $|\xi_t| \leq V^{x,\psi}_t$, $0 \leq t \leq T$ a.s., we get that for all $S \in \mathcal{T}$, $\tau \in \mathcal{T}_S$ and $\nu \in \mathcal{V}$, $\tilde{\mathcal{E}}^\nu_{S,\tau}(\xi_\tau) = -\mathcal{E}^\nu_{S,\tau}(-\xi_\tau) \geq -\mathcal{E}^\nu_{S,\tau}(|\xi_\tau|) \geq -\mathcal{E}^\nu_{S,\tau}(V^{x,\psi}_\tau) = -V^{x,\psi}_S$ a.s. Hence, taking the essential supremum over $\tau \in \mathcal{T}_S$ and then the essential infimum over $\nu \in \mathcal{V}$ in this inequality, we obtain $\tilde{Y}(S) \geq -V^{x,\psi}_S$ a.s.
Since $\tilde{Y}^0 \in S^2$ and $V^{x,\psi} \in S^2$, it follows that $E[ess\sup_{S\in\mathcal{T}} \tilde{Y}(S)^2] < +\infty$.

Using the characterization of the solution of a reflected BSDE with lower obstacle in terms of an optimal stopping problem with g-expectations (see Theorem 4.2 in [20] when (ξ_t) is right-u.s.c. payoff), we can rewrite the value function of our problem as follows

$$\tilde{Y}(S) = ess \inf_{\nu\in\mathcal{V}_S} \tilde{Y}^\nu_S = ess \inf_{\nu\in\mathcal{V}} \tilde{Y}^\nu_S, \tag{6.2}$$

where $\tilde{Y}^\nu$ is the solution of the reflected BSDE associated with driver $\tilde{f}^\nu$, obstacle $(\xi_t)_{0\leq t<T}$ and terminal condition ξ_T.

Proposition 6.3. (Minimizing sequence) *Let $S \in \mathcal{T}$. There exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$ with $\nu^n \in \mathcal{V}_S$, for all n, such that the sequence $(\tilde{Y}^{\nu^n}_S)_{n\in\mathbb{N}}$ is non-increasing and satisfies:*

$$\tilde{Y}(S) \quad = \lim_{n\to\infty} \downarrow \tilde{Y}^{\nu^n}_S \quad \text{a.s.} \tag{6.3}$$

Proof. Same proof as for Y. We show that the set $\{\tilde{Y}_S^\nu, \nu \in \mathcal{V}_S\}$ is stable under pairwise maximization. The result of the proposition follows by a classical result on essential suprema/infima (cf. Neveu (1975)). $\square$

Proposition 6.4. (Aggregation) *Let $(\xi_t) \in \mathbb{S}^2$ (without any regularity assumption). There exists an r.u.s.c. process $(\tilde{Y}_t) \in \mathbb{S}^2$ which aggregates the value family $(\tilde{Y}(S))$ of the problem of control and stopping* (6.1).

The proof of the proposition uses the following lemma.

Lemma 6.5. *For all $S \in \mathcal{T}_0$,*

$$\mathbb{E}[\tilde{Y}(S)] = \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu].$$

Proof. From the representation (6.2), we deduce $\mathbb{E}[\tilde{Y}(S)] = \mathbb{E}[ess \inf_{\nu \in \mathcal{V}} \tilde{Y}_S^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[Y_S^\nu]$. We now show the converse inequality. By Proposition 6.3, there exists a sequence of controls $(\nu_n) = (\nu_n(S))$ in $\mathcal{V}_S^n$ such that $\tilde{Y}(S) = \lim_{n\to\infty} \downarrow \tilde{Y}_S^{\nu_n}$. We thus have $\mathbb{E}[\tilde{Y}(S)] = \mathbb{E}[\lim_{n\to\infty} \downarrow \tilde{Y}_S^{\nu_n}] = \lim_{n\to\infty} \downarrow \mathbb{E}[\tilde{Y}_S^{\nu_n}]$, where we have used dominated convergence to exchange limit and expectation. For all n, we have $\mathbb{E}[\tilde{Y}_S^{\nu_n}] \geq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu]$.We conclude that $\mathbb{E}[\tilde{Y}(S)] \geq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu]$. The proposition is thus proved. $\square$

We now prove Proposition 6.4.
Proof. To prove the result, we first show that the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times in expectation. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim \downarrow S_n = S$ a.s. By the previous Lemma 6.5, we have $\mathbb{E}[\tilde{Y}(S_n)] = \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_{S_n}^\nu]$, for all $n \in \mathbb{N}$. Hence, $\limsup_{n\to\infty} \mathbb{E}[\tilde{Y}(S_n)] = \limsup_{n\to\infty} \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \limsup_{n\to\infty} \mathbb{E}[\tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu]$, where we have used Fatou's lemma to obtain the last inequality. Now, for all $\nu \in \mathcal{V}$, the process $(\tilde{Y}_t^\nu)$ right-uppersemicontinuous along stopping times, so $\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu \leq \tilde{Y}_S^\nu$. Using this and the above computations, we get $\limsup_{n\to\infty} \mathbb{E}[\tilde{Y}(S_n)] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\limsup_{n\to\infty} \tilde{Y}_{S_n}^\nu] \leq \inf_{\nu \in \mathcal{V}} \mathbb{E}[\tilde{Y}_S^\nu] = \mathbb{E}[\tilde{Y}(S)]$, where the (last) equality is due to Lemma 6.5. We conclude that the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times in expectation. Hence, the family $(\tilde{Y}(S))$ is right-uppersemicontinuous along stopping times (cf. Theorem 12 in [9]). By Corollary 11 in [9], there exists a unique r.u.s.c. optional process $(\tilde{Y}_t)$ which aggregates the family. The process $(\tilde{Y}_t)$ is in $\mathbb{S}^2$, due to the fact that $E[ess \sup_{S \in \mathcal{T}} \tilde{Y}(S)^2] < +\infty$. $\square$

Remark 6.6. *Due to the above aggregation result (Proposition 6.4), we can replace $\tilde{Y}(S)$ by $\tilde{Y}_S$ in the representation* (6.2) *and in Proposition 6.3.*

6.2 Proof of the dual representation for the buyer's superhedging price

We now define the *backward semigroup* of operators $\mathbf{Y}^{g,\xi} = \left(\mathbf{Y}^{g,\xi}_{t,T'}\right)_{0\leq t\leq T'\leq T}$ associated with a reflected BSDE with driver g and obstacle ξ (see e.g. [5] and [13]). Recall that this notion of stochastic backward semigroup was first introduced by Peng [32] and applied to study the dynamic programming principle for stochastic control problems.

Let g be a λ-*admissible* driver. Let $(\xi_t) \in \mathbb{S}^2$.
For each $T' \in [0,T]$ and each $\eta \in L^2(\mathcal{F}_{T'})$, we define

$$\mathbf{Y}^{g,\xi}_{t,T'}(\eta) := Y_t, \quad 0 \le t \le T', \tag{6.4}$$

where $(Y_t)_{0\le t\le T'}$ corresponds to the first component of the solution of the reflected BSDE associated with terminal time T', driver g and (lower) obstacle $(\xi_t \mathbf{1}_{t<T'} + \eta \mathbf{1}_{t=T'})$. Note that (Y_t) can be extended to the whole interval $[0,T]$ by setting $Y_t = \eta$ for all $t \in [T', T]$. [9]

More generally, for each stopping time $\theta \in \mathcal{T}$ and each $\eta \in L^2(\mathcal{F}_\theta)$, we define $\mathbf{Y}^{g,\xi}_{\cdot,\theta}(\eta) := Y_\cdot$, where $Y_\cdot$ is the first component of the solution of the reflected BSDE associated with terminal time T, terminal condition η, driver $g\mathbf{1}_{t\le\theta}$, and obstacle $(\xi_t \mathbf{1}_{t<\theta} + \eta \mathbf{1}_{t\ge\theta})$.

For each $\nu \in \mathcal{V}$, we consider the *backward semigroup* of operators $\mathbf{Y}^{\tilde{f}^\nu,\xi} = \left(\mathbf{Y}^{\tilde{f}^\nu,\xi}_{t,T'}\right)$. To abbreviate the notation, we denote it by $\mathbf{Y}^{\nu,\xi} = \left(\mathbf{Y}^{\nu,\xi}_{t,T'}\right)$.
Note that $\tilde{Y}^\nu_t = \mathbf{Y}^{\nu,\xi}_{\mathbf{t},\mathbf{T}}(\xi_\mathbf{T})$, for all $t \in [0,T]$, a.s.

Proposition 6.7. (Dynamic Programming Principle) *The value process $(\tilde{Y}_t)$ satisfies the following Dynamic Programming Principle: for all S, S' in $\mathcal{T}_0$ such that $S \le S'$ a.s., we have*

$$\tilde{Y}_S = ess \inf_{\nu \in \mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) \ \text{a.s.} \tag{6.5}$$

Proof. Let $S, S' \in \mathcal{T}$ be such that $S \le S'$ a.s. By Proposition 6.3, there exists a sequence of controls $(\nu^n)_{n\in\mathbb{N}}$, with ν_n in $\mathcal{V}_{S'}$ for all n, such that $\tilde{Y}_{S'} \ = \lim_{n\to\infty} \downarrow \tilde{Y}^{\nu^n}_{S'}$ a.s. Let $\nu \in \mathcal{V}_S$. By the continuity property of Reflected BSDEs with respect to the terminal condition (cf. last assertion of Lemma 7.1), we have $\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) = \mathbf{Y}^{\nu,\xi}_{S,S'}(\lim_{n\to\infty} \tilde{Y}^{\nu^n}_{S'}) = \lim_{n\to\infty} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'})$ a.s. For each n, we set $\bar{\nu}^n_t := \nu_t \mathbf{1}_{]S,S']}(t) + \nu^n_t \mathbf{1}_{]S',T]}(t)$. We have $\tilde{f}^{\bar{\nu}^n} = \tilde{f}^\nu \mathbf{1}_{]S,S']} + \tilde{f}^{\nu^n} \mathbf{1}_{]S',T]}$ and $\tilde{Y}^{\nu_n}_{S'} = \tilde{Y}^{\bar{\nu}_n}_{S'}$. We deduce

$$\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'}) = \mathbf{Y}^{\bar{\nu}^n,\xi}_{S,S'}(\tilde{Y}^{\bar{\nu}_n}_{S'}) = \tilde{Y}^{\bar{\nu}_n}_S \text{ a.s.},$$

where the last equality follows the flow (or semi-group) property of reflected BSDEs. We thus get
$\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) = \lim_{n\to\infty} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^{\nu_n}_{S'}) \ = \ \lim_{n\to\infty} \tilde{Y}^{\bar{\nu}_n}_S \ge \tilde{Y}_S$ a.s., where the (last) inequality follows from (6.2). As $\nu \in \mathcal{V}_S$ is arbitrary, we derive $ess \inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'}) \ge \tilde{Y}_S$ a.s.

We now prove the converse inequality. Let $\nu \in \mathcal{V}_S$. By the flow property of reflected BSDEs, we have $\tilde{Y}^\nu_S = \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'})$ a.s. On the other hand, $\tilde{Y}^\nu_{S'} \ge \tilde{Y}_{S'}$ a.s. (cf. property (6.2)). From this, by the comparison theorem for reflected BSDEs (cf.), we deduce $\mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'}) \ge \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. Hence, $\tilde{Y}^\nu_S = \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}^\nu_{S'}) \ge \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. By taking the essential infimum

[9] Recall that, by the flow property for reflected BSDEs, the family of operators $\mathbf{Y}^{g,\xi} = \left(\mathbf{Y}^{g,\xi}_{t,T'}\right)_{0\le t\le T'\le T}$ satisfies a *semi-group* property.

over $\nu \in \mathcal{V}_S$, we get $ess\inf_{\nu\in\mathcal{V}_S} \tilde{Y}_S^\nu \geq ess\inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s. But, $\tilde{Y}_S = ess\inf_{\nu\in\mathcal{V}_S} \tilde{Y}_S^\nu$ a.s. (cf. (6.2)). Hence, $\tilde{Y}_S \geq ess\inf_{\nu\in\mathcal{V}_S} \mathbf{Y}^{\nu,\xi}_{S,S'}(\tilde{Y}_{S'})$ a.s., which is the desired inequality. As both inequalities hold, we have the equality (6.5). The proof is complete. $\square$

We recall that, by Proposition 6.4, the value process $(\tilde{Y}_t)$ is right upper-semicontinuous (without any regularity assumption on the process (ξ_t)).

Let $(\bar{\xi}_t)$ be the right-u.s.c. process defined by

$$\bar{\xi}_t := \limsup_{s\downarrow t, s\geq t} \xi_s, \quad \text{for all} \quad t \in [0,T[^{10}. \tag{6.6}$$

We now show that if the process $(\bar{\xi}_t)$ is right-lowersemicontinuous (right-l.s.c.), then $(\tilde{Y}_t)$ is also right-l.s.c., hence right-continuous.

Proposition 6.8 (The case when $(\bar{\xi}_t)$ is right-lowersemicontinuous)**.** *Let $(\xi_t) \in \mathbb{S}^2$. Suppose that $(\bar{\xi}_t)$ is right-l.s.c. (which is satisfied if, for example, the process ξ is right-limited and right-l.s.c.[11]). Then, the value process $(\tilde{Y}_t)$ of the problem of control and stopping* (6.1) *is* right-l.s.c.

Proof. Let $S \in \mathcal{T}_0$, let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_S$ with $\lim_{n\to+\infty} S_n = S$ a.s. and for all $n \in \mathbb{N}$, $S_n > S$ a.s. on $\{S < T\}$, and such that $\lim_{n\to+\infty} \tilde{Y}_{S_n}$ exists a.s. Since $0 \in \mathcal{V}_S$, by the dynamic programming principle, we have $\tilde{Y}_S \leq \mathbf{Y}^{0,\xi}_{S,S_n}(\tilde{Y}_{S_n})$ a.s. Hence, by the continuity property of Reflected BSDEs with respect to the pair terminal time-terminal condition [12] (cf. Lemma 7.1), we thus get

$$\tilde{Y}_S \leq \lim_{n\to\infty} \mathbf{Y}^{0,\xi}_{S,S_n}(\tilde{Y}_{S_n}) = \mathbf{Y}^{0,\xi}_{S,S}(\lim_{n\to\infty} \tilde{Y}_{S_n}) = \lim_{n\to+\infty} \tilde{Y}_{S_n} \quad \text{a.s.}$$

By Proposition 2 of Dellacherie and Lenglart [9], the process $\tilde{Y}$ is thus right-lowersemicontinuous. The proof is thus complete. $\square$

Remark 6.9. *The above proof also shows the following property: Let $S \in \mathcal{T}$. If $(\bar{\xi}_t)$ is right-l.s.c. at S (which is satisfied if, for example, the process ξ is right-limited and right-l.s.c. at S), then, the value process $(\tilde{Y}_t)$ is* right-l.s.c *at S.*

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[10]Note that $\bar{\xi}_t = \max(\hat{\xi}_t, \xi_t)$, where $(\hat{\xi}_t)$ denotes the *right upper-semicontinuous envelope* of the process (ξ_t), defined by $\hat{\xi}_t := \limsup_{s\downarrow t, s>t} \xi_s$, for all $t \in [0,T[$ in [15, page 133]. Note also that $(\bar{\xi}_t)$ is a right-u.s.c. progressive process.

[11]Indeed, in this case, we have $\bar{\xi}_t = \max(\xi_{t^+}, \xi_t)$. Moreover, the right-l.s.c. property of ξ is equivalent to the condition $\xi_{t^+} \geq \xi_t$, which is equivalent to $\bar{\xi}_t = \xi_{t^+}$

[12]We note that the conditions from Lemma 7.1 are satisfied here, that is, the condition (7.1), which is written here as the condition $\lim_{n\to\infty} \tilde{Y}_{S_n} \geq \bar{\xi}_S$, holds: indeed, since $\tilde{Y}$ is right-u.s.c., we have $\tilde{Y}_{S_n} \geq \bar{\xi}_{S_n}$ a.s. for all n; hence, $\lim_{n\to\infty} \tilde{Y}_{S_n} \geq \liminf_{n\to\infty} \bar{\xi}_{S_n} \geq \bar{\xi}_S$ a.s., where we have used the assumption of right-lowersemicontinuity of $\bar{\xi}$ for the last inequality.

Lemma 6.10. *Let (ξ_t) be a process in $\mathbb{S}^2$. We define the following stopping times:*

$$\tilde{\tau} := \inf\{t \in [0,T] : \tilde{Y}_t = \xi_t\} \tag{6.7}$$

$$\textit{For } \varepsilon > 0, \qquad \tilde{\tau}_\varepsilon := \inf\{t \in [0,T] : \tilde{Y}_t \leq \xi_t + \varepsilon\} \tag{6.8}$$

We note that $\tilde{\tau}^\varepsilon \leq \tilde{\tau}$ a.s.

(i) *If (ξ_t) is left-uppersemicontinuous along stopping times at $\tilde{\tau}$, then, for all $\nu \in \mathcal{V}$, the value process $(\tilde{Y}_t)$ is a strong $\tilde{\mathcal{E}}^\nu$-submartingale on $[0, \tilde{\tau}]$.*

(ii) *For all $\epsilon > 0$, for all $\nu \in \mathcal{V}$, the value process $(\tilde{Y}_t)$ is a strong $\tilde{\mathcal{E}}^\nu$-submartingale on $[0, \tilde{\tau}_\varepsilon]$.*

Proof. We show (i). Let $\nu \in \mathcal{V}$. Let S, τ in $\mathcal{T}$ be such that $0 \leq S \leq \tau \leq \tilde{\tau}$ a.s. We show that $\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) \geq \tilde{Y}_S$. By the representation (6.2) and Proposition 6.3, there exists a minimizing sequence for $\tilde{Y}_\tau$, that is, there exists $\nu^p := \nu^p(\tau) \in \mathcal{V}_\tau$ such that $\tilde{Y}_\tau = \lim_{p\to\infty} \downarrow \tilde{Y}^{\nu^p}_\tau$. Hence, $\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) = \tilde{\mathcal{E}}^\nu_{S,\tau}(\lim_{p\to\infty} \tilde{Y}^{\nu^p}_\tau) = \lim_{p\to\infty} \tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}^{\nu^p}_\tau)$, where we have used the continuity property of the non-linear expectation $\tilde{\mathcal{E}}^\nu(\cdot)$ with respect to terminal condition. For all $p \in \mathbb{N}$, we set $\bar{\nu}^p_t := \nu^p_t \mathbb{I}_{\{t>\tau\}} + \nu_t \mathbb{I}_{\{t\leq\tau\}}$. We have $\bar{\nu}^p \in \mathcal{V}$. We thus get $\lim_{p\to\infty} \tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}^{\nu^p}_\tau) = \lim_{p\to\infty} \tilde{\mathcal{E}}^{\bar{\nu}^p}_{S,\tau}(\tilde{Y}^{\bar{\nu}^p}_\tau) \geq ess\inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^\mu_{S,\tau}(\tilde{Y}^\mu_\tau)$. Putting together the above computations gives

$$\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^\mu_{S,\tau}(\tilde{Y}^\mu_\tau). \tag{6.9}$$

For all $\mu \in \mathcal{V}$, we set $\tau^\mu := \inf\{t \in [0,T] : \tilde{Y}^\mu_t = \xi_t\}$. We notice that, for all $\mu \in \mathcal{V}$, $\tilde{\tau} \leq \tau^\mu$ a.s.; this follows from the definitions of $\tilde{\tau}$ and τ^μ and from the fact that $\xi_t \leq \tilde{Y}_t \leq \tilde{Y}^\mu_t$ for all t a.s. By Lemma 4.1 in [21], for all $\mu \in \mathcal{V}$, the process $(\tilde{Y}^\mu_t)$ is a strong $\tilde{\mathcal{E}}^\mu$-martingale on $[0, \tau^\mu]$; hence, also a strong $\tilde{\mathcal{E}}^\mu$-martingale on $[0, \tilde{\tau}]$ (as $\tilde{\tau} \leq \tau^\mu$ a.s.). Hence, for all $\mu \in \mathcal{V}$, $\tilde{\mathcal{E}}^\mu_{S,\tau}(\tilde{Y}^\mu_\tau) = \tilde{Y}^\mu_S$ (recall that $0 \leq S \leq \tau \leq \tilde{\tau}$ a.s.) Using this and (6.9), we get $\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) \geq ess\inf_{\mu\in\mathcal{V}} \tilde{Y}^\mu_S = \tilde{Y}_S$, where the (last) equality is due to the representation (6.2). Property (i) is thus proved.

Let us show (ii). Let $\varepsilon > 0$. Let S, τ in $\mathcal{T}$ be such that $0 \leq S \leq \tau \leq \tilde{\tau}_\varepsilon$ a.s. By exactly the same arguments as in part (i), we get

$$\tilde{\mathcal{E}}^\nu_{S,\tau}(\tilde{Y}_\tau) \geq ess \inf_{\mu\in\mathcal{V}} \tilde{\mathcal{E}}^\mu_{S,\tau}(\tilde{Y}^\mu_\tau). \tag{6.10}$$

For all $\mu \in \mathcal{V}$, we set $\tau^\mu_\varepsilon := \inf\{t \in [0,T] : \tilde{Y}^\mu_t \leq \xi_t + \varepsilon\}$. We note that, for all $\mu \in \mathcal{V}$, $\tilde{\tau}_\varepsilon \leq \tau^\mu_\varepsilon$ a.s. By Lemma 4.1 in [21], for all $\mu \in \mathcal{V}$, the process $(\tilde{Y}^\mu_t)$ is a strong $\tilde{\mathcal{E}}^\mu$-martingale on $[0, \tau^\mu_\varepsilon]$; hence, also a strong $\tilde{\mathcal{E}}^\mu$-martingale on $[0, \tilde{\tau}_\varepsilon]$ (as $\tilde{\tau}_\varepsilon \leq \tau^\mu_\varepsilon$ a.s.). From this and (6.10), we conclude as in part (i). □

We will now give a dual representation for the buyer's superhedging price $\tilde{u}_0$ in terms of the value (at time 0) of the non-linear problem of control and stopping studied above. We also give a superhedge for the buyer. From this result, we will deduce the dual representation (in terms of the f-martingale probability measures) stated in Theorem 6.1.

Theorem 6.11 (*Buyer's superhedging price*)**.** *Let* $(\xi_t) \in \mathbb{S}^2$.
Suppose that (ξ_t) *is right-continuous and left-u.s.c. along stopping times.*
The buyer's price $\tilde{u}_0$ *of the American option satisfies*

$$\tilde{u}_0 = \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau). \tag{6.11}$$

Let $\tilde{\tau} := \inf\{t \in [0,T] : \tilde{Y}_t = \xi_t\}$*. There exists a portfolio strategy* $\tilde{\varphi} \in \mathbb{H}^2$ *such that* $(\tilde{\tau}, \tilde{\varphi})$ *is a superhedge for the buyer, that is, such that* $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$.

Remark 6.12. *This result still holds under the following (weaker) local assumptions:*
(i) the processes $\tilde{Y}$ *and* ξ *are respectively right-l.s.c. and right-u.s.c. at* $\tilde{\tau}$ *(which, by Proposition 6.9, is satisfied if, for example,* ξ *is right-continuous at* $\tilde{\tau}$*).* [13]
(ii) ξ *is left-u.s.c. along stopping times at* $\tilde{\tau}$*.*[14]

Proof. It is sufficient to show that $\tilde{u}_0 = \tilde{Y}_0$ and that there exists $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0)$.
Let $\mathcal{S}$ be the set of initial prices which allow the buyer to be "superhedged", that is $\mathcal{S} = \{x \in \mathbb{R} : \exists (\tau, \varphi) \in \mathcal{B}(x)\}$. Note that $\tilde{u}_0 = \sup \mathcal{S}$.

Let us first show that $\tilde{u}_0 \leq \tilde{Y}_0$. Let $x \in \mathcal{S}$. By definition of $\mathcal{S}$, there exists $(\theta, \varphi) \in \mathcal{B}(x)$, that is, such that $V^{-x,\varphi}_\theta \geq -\xi_\theta$ a.s. Let $\nu \in \mathcal{V}$. By taking the $\mathcal{E}^\nu$-evaluation in the above inequality, using the monotonicity of $\mathcal{E}^\nu$ and the $\mathcal{E}^\nu$-martingale property of the process $V^{-x,\varphi}$, we derive that $-x = \mathcal{E}^\nu_{0,\theta}(V^{-x,\varphi}_\theta) \geq \mathcal{E}^\nu_{0,\theta}(-\xi_\theta) = -\tilde{\mathcal{E}}^\nu_{0,\theta}(\xi_\theta)$, where the last equality follows from the first assertion of Remark 6.2. We deduce $x \leq \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^\nu_{0,\tau}(\xi_\tau)$. Since $\nu \in \mathcal{V}$ is arbitrary, we get

$$x \leq \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau) = \tilde{Y}_0,$$

which holds for any $x \in \mathcal{S}$. By taking the supremum over $x \in \mathcal{S}$, we get $\tilde{u}_0 \leq \tilde{Y}_0$.

Let us now show that $\tilde{Y}_0 \leq \tilde{u}_0$. To this aim, we prove that $\tilde{Y}_0 \in \mathcal{S}$, that is, there exists a portfolio strategy $\tilde{\varphi} \in \mathbb{H}^2$ such that

$$(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0). \tag{6.12}$$

Since ξ is left-u.s.c. along stopping times at $\tilde{\tau}$, by the first assertion from Lemma 6.10, the process $(\tilde{Y}_{t \wedge \tilde{\tau}})$ is a strong $\tilde{\mathcal{E}}^\nu$-submartingale for all $\nu \in \mathcal{V}$. This together with the first assertion from Remark 6.2 implies that $(-\tilde{Y}_{t \wedge \tilde{\tau}})$ is a strong $\mathcal{E}^\nu$-supermartingale for all $\nu \in \mathcal{V}$.

By the optional $\mathcal{E}^f$-decomposition of strong $\mathcal{E}^\nu$-supermartingale for each $\nu \in \mathcal{V}$ (cf. Theorem B.4 in [23]), there exists a unique pair $(\tilde{Z}, \tilde{C}) \in \mathbb{H}^2 \times \mathbb{C}^2$ and a unique nondecreasing optional RCLL process $\tilde{h}$, with $\tilde{h}_0 = 0$ and $E[\tilde{h}^2_T] < \infty$ such that

$$-\tilde{Y}_t = -\tilde{Y}_0 - \int_0^t f(s, -\tilde{Y}_s, \tilde{Z}_s) ds + \int_0^t \tilde{Z}_s \sigma_s^{-1} (\sigma_s dW_s + \beta_s dM_s) - \tilde{h}_t - \tilde{C}_{t^-}, \; 0 \leq t \leq \tilde{\tau} \text{ a.s.} \tag{6.13}$$

[13] Indeed, these assumptions are sufficient to ensure the inequality (6.16).
[14] Indeed, this assumption is sufficient to apply the first assertion from Lemma 6.10, which is used in the proof.

We now consider the portfolio associated with the initial capital $-\tilde{Y}_0$ and the strategy

$$\tilde{\varphi} := \sigma^{-1}\tilde{Z}. \tag{6.14}$$

By (3.2)-(3.3), the value of the portfolio process $(V_t^{-\tilde{Y}_0,\tilde{\varphi}})$ satisfies:

$$V_t^{-\tilde{Y}_0,\tilde{\varphi}} = -\tilde{Y}_0 - \int_0^t f(s, V_s^{-\tilde{Y}_0,\tilde{\varphi}}, \tilde{Z}_s)ds + \int_0^t \tilde{Z}_s\sigma_s^{-1}(\sigma_s dW_s + \beta_s dM_s),\ 0 \leq t \leq T. \tag{6.15}$$

By (6.13) and (6.15) and the comparison result for forward differential equations, we get $-\tilde{Y}_t \leq V_t^{-\tilde{Y}_0,\tilde{\varphi}}$, $0 \leq t \leq \tilde{\tau}$ a.s. We thus have $V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} + \tilde{Y}_{\tilde{\tau}} \geq 0$ a.s.
Now, since ξ is right-continuous, by Proposition 6.8, $\tilde{Y}$ is right-l.s.c. (and even right-continuous). Hence, by the definition of $\tilde{\tau}$ (and since ξ is right-u.s.c.), we get

$$\tilde{Y}_{\tilde{\tau}} \leq \xi_{\tilde{\tau}} \quad \text{a.s.} \tag{6.16}$$

We thus conclude that

$$V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} + \xi_{\tilde{\tau}} \geq 0 \quad \text{a.s.},$$

which implies the desired property (6.12). We thus have $\tilde{Y}_0 \leq \tilde{u}_0$. It follows that $\tilde{u}_0 = \tilde{Y}_0$. By (6.12), we get $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$, which completes the proof. □

Remark 6.13. *We emphasize that the superhedging portfolio strategy $\tilde{\varphi}$ is given by* (6.14) *via the optional decomposition* (6.13) *of $\tilde{Y}$ on* $[0, \tilde{\tau}]$.

Proof of Theorem 6.1: The proof follows from the previous Theorem 6.11 and from Remark 6.2. We note first that, since (ξ_t) is supposed to be right-continuous and left-uppersemicontinuous along stopping times, it follows that $\bar{\xi} = \xi$, and the assumptions of Theorem 6.11 hold.

Under the additional integrability condition $\psi \in \cap_{\nu \in \mathcal{V}} \mathbb{H}^2_{Q^\nu}$ on the process ψ from Assumption (4.1), by Remark 6.2, the dual representation (6.11) can be written in terms of the f-martingale probability measures, that is

$$\tilde{u}_0 = \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \{-\mathcal{E}^f_{Q^\nu,0,\tau}(-\xi_\tau)\}.$$

The fact that the infimum and the supremum can be interchanged follows from Proposition 6.19 (shown under weaker regularity assumptions on ξ). The proof of Theorem 6.1 is thus complete. □

6.3 Buyer's nearly superhedging price

We now consider the case when ξ does not satisfy any regularity assumption on the left.
We introduce the definition of an ε-*superhedge for the buyer*:

Definition 6.14. *For each initial price z and for each $\varepsilon > 0$, an ε–*superhedge for the buyer *against the American option is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a portfolio strategy $\varphi \in \mathbb{H}^2$ such that*

$$V_\tau^{-z,\varphi} \geq -\xi_\tau - \varepsilon \quad \text{a.s.}$$

For each $z \in \mathbb{R}$ and each $\varepsilon > 0$, we denote by $\mathcal{B}_\varepsilon(z)$ the set of all ε– superhedges for the buyer associated with initial price z.

We introduce the *nearly superhedging price $\bar{u}_0$ of the American option for the buyer* as the supremum of the initial prices which allow the buyer to be ε–superhedged for all $\varepsilon > 0$, that is,

$$\bar{u}_0 = \sup\{z \in \mathbb{R},\ \forall \varepsilon > 0, \exists (\tau, \varphi) \in \mathcal{B}_\varepsilon(z)\}. \tag{6.17}$$

Theorem 6.15 (*Buyer's nearly superhedging price*)**.** *Let $(\xi_t) \in \mathbb{S}^2$ supposed to be right-continuous. The* buyer's nearly superhedging price $\bar{u}_0$ *of the American option satisfies*

$$\bar{u}_0 = \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^\nu_{0,\tau}(\xi_\tau). \tag{6.18}$$

*For each $\varepsilon > 0$, let $\tilde{\tau}_\varepsilon := \inf\{t \in [0,T] :\ \tilde{Y}_t \leq \xi_t + \varepsilon\}$. There exists a portfolio strategy $\tilde{\varphi} \in \mathbb{H}^2$ such that, for each $\varepsilon > 0$, the pair $(\tilde{\tau}_\varepsilon, \tilde{\varphi})$ is an ε-*superhedge for the buyer *(associated with the initial price $\bar{u}_0$).*

Remark 6.16. *We note that when ξ is left-u.s.c. along stopping times at $\tilde{\tau}$, the* buyer's nearly superhedging price *is equal to the* buyer's superhedging price.

Remark 6.17. *The result from Theorem 6.15 still holds under the following (weaker) local assumptions: the processes $\tilde{Y}$ and ξ are respectively right-l.s.c.and right-u.s.c. at $\tilde{\tau}_{\varepsilon_n}$, for a sequence (ε_n) tending to 0 (which, by Proposition 6.9, is satisfied if, for example, ξ is right-continuous at $\tilde{\tau}_{\varepsilon_n}$ for all n).*[15]

Proof. Let us first show that $\bar{u}_0 \leq \tilde{Y}_0$.
Let $z \in \mathbb{R}$ be such that, for each $\varepsilon > 0$, there exists $(\tau_\varepsilon, \varphi^\varepsilon) \in \mathcal{B}_\varepsilon(z)$.
Fix now $\varepsilon > 0$. By definition of $\mathcal{B}_\varepsilon(z)$, there exists $(\tau_\varepsilon, \varphi^\varepsilon) \in \mathcal{T} \times \mathbb{H}^2$ such that $V_{\tau_\varepsilon}^{-z,\varphi^\varepsilon} \geq -\xi_{\tau_\varepsilon} - \varepsilon$ a.s. Let $\nu \in \mathcal{V}$. By taking the $\mathcal{E}^\nu$-evaluation in the above inequality, using the monotonicity of $\mathcal{E}^\nu$ and the $\mathcal{E}^\nu$-martingale property of the process $V^{-z,\varphi^\varepsilon}$, we derive that

$$-z = \mathcal{E}^\nu_{0,\tau_\varepsilon}(V_{\tau_\varepsilon}^{-z,\varphi^\varepsilon}) \geq \mathcal{E}^\nu_{0,\tau_\varepsilon}(-\xi_{\tau_\varepsilon} - \varepsilon). \tag{6.19}$$

Now, by Lemma 7.3, we get

$$\mathcal{E}^\nu_{0,\tau_\varepsilon}(-\xi_{\tau_\varepsilon}) - \mathcal{E}^\nu_{0,\tau_\varepsilon}(-\xi_{\tau_\varepsilon} - \varepsilon) = E[e^{\int_0^{\tau_\varepsilon} f_y(s)ds} H_{0,\tau_\varepsilon}\, \varepsilon] \leq e^{CT} \varepsilon\, E[H_{0,\tau_\varepsilon}] = e^{CT}\varepsilon \quad \text{a.s.},$$

where the inequality and the last equality follow from the fact that the process $f_y(\cdot)$ is uniformly bounded by the Lispchitz constant C of f, and the fact that, since $\nu \in \mathcal{V}$, the process $H_{0,\cdot}$ is a nonnegative martingale. We thus have

$$-\mathcal{E}^\nu_{0,\tau_\varepsilon}(-\xi_{\tau_\varepsilon} - \varepsilon) \leq e^{CT}\varepsilon - \mathcal{E}^\nu_{0,\tau_\varepsilon}(-\xi_{\tau_\varepsilon}) = e^{CT}\varepsilon + \tilde{\mathcal{E}}^\nu_{0,\tau_\varepsilon}(\xi_{\tau_\varepsilon}), \tag{6.20}$$

[15] Indeed, these assumptions are sufficient to ensure the inequality (6.21) at $\tilde{\tau}_{\varepsilon_n}$ for all n.

where the last equality follows from Remark 6.2.
Using (6.19), we deduce $z \leq e^{CT}\varepsilon + \tilde{\mathcal{E}}^{\nu}_{0,\tau_\varepsilon}(\xi_{\tau_\varepsilon}) \leq e^{CT}\varepsilon + \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau)$. Since this inequality holds for all $\varepsilon > 0$, we get $z \leq \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau)$. As $\nu \in \mathcal{V}$ is arbitrary, we deduce

$$z \leq \inf_{\nu\in\mathcal{V}} \sup_{\tau\in\mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau) = \tilde{Y}_0.$$

Using the definition of $\bar{u}_0$ as a supremum (cf. (6.17)), we get $\bar{u}_0 \leq \tilde{Y}_0$.

We now show that $\tilde{Y}_0 \leq \bar{u}_0$. Let $\varepsilon > 0$. By the second assertion of Lemma 6.10, the process $(\tilde{Y}_{t\wedge\tilde{\tau}_\varepsilon})$ is a strong $\tilde{\mathcal{E}}^{\nu}$-submartingale for all $\nu \in \mathcal{V}$. This together with the first assertion from Remark 6.2 implies that $(-\tilde{Y}_{t\wedge\tilde{\tau}_\varepsilon})$ is a strong $\mathcal{E}^{\nu}$-supermartingale for all $\nu \in \mathcal{V}$. By the optional $\mathcal{E}^{f}$-decomposition of strong $\mathcal{E}^{\nu}$-supermartingale for each $\nu \in \mathcal{V}$ (see Theorem B.4 in [23] applied to the right-continuous process $-\tilde{Y}$), there exists a unique process $\tilde{Z} \in \mathbb{H}^2$ and a unique nondecreasing optional RCLL process $\tilde{h}$, with $\tilde{h}_0 = 0$ and $E[\tilde{h}_T^2] < \infty$ such that (6.13) holds on $[0, \tilde{\tau}_\varepsilon]$. Moreover, the wealth $V_{\cdot}^{-\tilde{Y}_0,\tilde{\varphi}}$ associated with the initial capital $-\tilde{Y}_0$ and the strategy $\tilde{\varphi} := \sigma^{-1}\tilde{Z}$ is the solution of the *forward* differential equation (6.15). By the comparison result for forward differential equations, it follows that $-\tilde{Y}_t \leq V_t^{-\tilde{Y}_0,\tilde{\varphi}}$, $0 \leq t \leq \tilde{\tau}_\varepsilon$ a.s.
Now, since ξ is right-continuous, by Proposition 6.8, $\tilde{Y}$ is right-l.s.c. (and even right-continuous). Hence, by the definition of $\tilde{\tau}_\varepsilon$ (and the right-uppersemicontinuity of ξ), we get

$$\tilde{Y}_{\tilde{\tau}_\varepsilon} \leq \xi_{\tilde{\tau}_\varepsilon} + \varepsilon \quad \text{a.s.} \tag{6.21}$$

We thus obtain the inequality $V_{\tilde{\tau}_\varepsilon}^{-\tilde{Y}_0,\tilde{\varphi}} \geq -\tilde{Y}_{\tilde{\tau}_\varepsilon} \geq -\xi_{\tilde{\tau}_\varepsilon} - \varepsilon$ a.s.
Hence, for each $\varepsilon > 0$, the pair $(\tilde{\tau}_\varepsilon, \tilde{\varphi})$ is an *ε-superhedge for the buyer* associated with the initial price $\tilde{Y}_0$, that is

$$(\tilde{\tau}_\varepsilon, \tilde{\varphi}) \in \mathcal{B}_\varepsilon(\tilde{Y}_0). \tag{6.22}$$

Using the definition of $\bar{u}_0$, we get $\tilde{Y}_0 \leq \bar{u}_0$. It follows that $\bar{u}_0 = \tilde{Y}_0$. By (6.22), we derive that for each $\epsilon > 0$, $(\tilde{\tau}_\varepsilon, \tilde{\varphi}) \in \mathcal{B}_\varepsilon(\bar{u}_0)$, which completes the proof. □

Remark 6.18. *In the complete case, Theorem 6.15 still holds without the assumption that $(\bar{\xi}_t)$ is right-l.s.c. Indeed, in this case, $\mathcal{Q} = \{P\}$ and $\tilde{Y} = \tilde{Y}^0$, where $\tilde{Y}^0$ is the solution of the reflected BSDE associated with driver $\tilde{f}$, lower obstacle ξ and terminal time T, which, by the Skorokhod condition, implies that $\tilde{Y}^0_{\tilde{\tau}_\varepsilon+} - \tilde{Y}^0_{\tilde{\tau}_\varepsilon} = 0$ a.s. on the set $\{\tilde{Y}^0_{\tilde{\tau}_\varepsilon} > \xi_{\tilde{\tau}_\varepsilon}\}$. This allows us to obtain the inequality* (6.21) *even if $\tilde{Y}^0$ is not right-l.s.c. at $\tilde{\tau}_\varepsilon$ (see the proof of Lemma 4.1 in [20] for details).*

We now show that the operations of infimum and supremum in the dual representation (6.11) (resp. (6.18)) of the buyer's superhedging (resp. nearly superhedging) price can be interchanged.

Proposition 6.19. *Let $(\xi_t) \in \mathbb{S}^2$, supposed to be right-continuous.*[16] *We have*

$$\inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau) = \sup_{\tau \in \mathcal{T}} \inf_{\nu \in \mathcal{V}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau).$$

Proof. We clearly have the inequality $\sup_{\tau \in \mathcal{T}} \inf_{\nu \in \mathcal{V}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau) \leq \inf_{\nu \in \mathcal{V}} \sup_{\tau \in \mathcal{T}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau)$. It remains to show the converse inequality, that is

$$\tilde{Y}_0 \leq \sup_{\tau \in \mathcal{T}} \inf_{\nu \in \mathcal{V}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau). \tag{6.23}$$

By Lemma 6.10 (ii), for all $\epsilon > 0$, for all $\nu \in \mathcal{V}$, the value process $(\tilde{Y}_t)$ is a strong $\tilde{\mathcal{E}}^{\nu}$-submartingale on $[0, \tilde{\tau}_\varepsilon]$, where $\tilde{\tau}_\varepsilon$ is defined by (6.8). Hence, $\tilde{Y}_0 \leq \tilde{\mathcal{E}}^{\nu}_{0,\tilde{\tau}_\varepsilon}(\tilde{Y}_{\tilde{\tau}_\varepsilon})$. Recall that $\tilde{Y}_{\tilde{\tau}_\varepsilon} \leq \xi_{\tilde{\tau}_\varepsilon} + \varepsilon$ a.s. (cf. (6.21)). By the same arguments as those used in the proof of Theorem 6.15 to show the estimate (6.20), we derive that for all $\nu \in \mathcal{V}$,

$$\tilde{Y}_0 \leq \tilde{\mathcal{E}}^{\nu}_{0,\tilde{\tau}_\varepsilon}(\tilde{Y}_{\tilde{\tau}_\varepsilon}) \leq \tilde{\mathcal{E}}^{\nu}_{0,\tilde{\tau}_\varepsilon}(\xi_{\tilde{\tau}_\varepsilon} + \varepsilon) \leq e^{CT}\varepsilon + \tilde{\mathcal{E}}^{\nu}_{0,\tilde{\tau}_\varepsilon}(\xi_{\tilde{\tau}_\varepsilon}). \tag{6.24}$$

By taking the infimum over $\nu \in \mathcal{V}$, we obtain

$$\tilde{Y}_0 \leq e^{CT}\varepsilon + \inf_{\nu \in \mathcal{V}} \tilde{\mathcal{E}}^{\nu}_{0,\tilde{\tau}_\varepsilon}(\xi_{\tilde{\tau}_\varepsilon}) \leq e^{CT}\varepsilon + \sup_{\tau \in \mathcal{T}} \inf_{\nu \in \mathcal{V}} \tilde{\mathcal{E}}^{\nu}_{0,\tau}(\xi_\tau).$$

Since this inequality holds for all $\varepsilon > 0$, we get the inequality (6.23). The proof is thus complete. $\square$

7 Appendix

In this Appendix, we provide some useful results.

We first show that the non-linear operator $\mathbf{Y}^{g,\xi}$ induced by the reflected BSDE with driver g and obstacle $(\xi_t)_{t<T}$, defined by (6.4), simply denoted by $\mathbf{Y}^g$, is continuous with respect to the terminal condition. Moreover, for each $\theta \in \mathcal{T}_0$ and each $\eta \in L^2(\mathcal{G}_\theta)$, $\mathbf{Y}^g$ is continuous with respect to the pair terminal time-terminal condition at the point (θ, η) under an additional assumption on (ξ_t) and η on a right neighborhood of θ.

Lemma 7.1. *Let g be a λ-admissible driver satisfying Assumption 2.3. Let $(\xi_t) \in \mathbb{S}^2$. Let $(\bar{\xi}_t)$ be the right-u.s.c. process defined by $\bar{\xi}_t := \limsup_{s \downarrow t, s \geq t} \xi_s$, for all $t \in [0, T[$.*
Let $(\theta^n)_{n \in \mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}_0$, converging a.s. to θ. Let $(\eta^n)_{n \in \mathbb{N}}$ be a sequence of random variables such that $E[\sup_n (\eta^n)^2] < +\infty$, and for each n, η^n is $\mathcal{G}_{\theta^n}$-measurable. Suppose that the sequence (η^n) converges a.s. to an $\mathcal{G}_\theta$-measurable random variable η. We also assume the following condition:

$$\bar{\xi}_\theta \leq \eta \quad \text{a.s.} \tag{7.1}$$

Then, for each $S \in \mathcal{T}_0$, $\lim_{n \to +\infty} \mathbf{Y}^g_{S,\theta^n}(\eta^n) = \mathbf{Y}^g_{S,\theta}(\eta)$ a.s.

When for each n, $\theta_n = \theta$ a.s., the result still holds without any assumption on (ξ_t).

[16] Note that this result still holds under the assumptions from Remark 6.12 or those from Remark 6.17.

Remark 7.2. *We note that the condition* (7.1) *is necessary in general to ensure this continuity property of the reflected operator* $\mathbf{Y}^{g,\xi}$.
When the obstacle (ξ_t) *is right-u.s.c., the condition* (7.1) *reduces to* $\xi_\theta \leq \eta$ *a.s.*
When the obstacle (ξ_t) *is right-continuous, we recover the continuity result shown in [13] (cf. [13, Lemma A.6]).*

Proof. In the particular case when for each n, $\theta_n = \theta$ a.s., the result follows from the a priori estimates for reflected BSDEs with irregular obstacles (cf. Theorem 5.5 in [21]), which do not require any additional assumption on (ξ_t).
Let us now consider the general case. Using the same arguments as those used in the proof of Lemma A.6 in [13], we show that $\liminf_{n\to\infty} \mathbf{Y}^{g,\xi}_{\theta,\theta_n}(\eta^n) \geq \eta$ a.s.

It thus remains to show that $\limsup_{n\to\infty} \mathbf{Y}^{g,\xi}_{\theta,\theta_n}(\eta^n) \leq \eta$ a.s. By the monotonicity property of reflected BSDEs with respect to the obstacle, for each $n \in \mathbb{N}$, we have $\mathbf{Y}^{g,\xi}_{\theta,\theta_n}(\eta^n) \leq \mathbf{Y}^{g,\bar{\xi}}_{\theta,\theta_n}(\eta^n)$ a.s., since $\xi \leq \bar{\xi}$. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$. Recall that $\mathbf{Y}^{g,\bar{\xi}}_{\cdot,\theta_n}(\eta^n)$ is the solution of the reflected BSDE associated with terminal time θ_n and the obstacle $(\bar{\xi}_t \mathbf{1}_{t<\theta_n} + \eta^n \mathbf{1}_{t\geq\theta_n})$, which is right-u.s.c. (since $(\bar{\xi}_t)$ is right-u.s.c.). Hence, by Theorem 4.2 in [20], there exists $\tau_n^\varepsilon \in \mathcal{T}_\theta$ such that

$$\mathbf{Y}^{g,\bar{\xi}}_{\theta,\theta_n}(\eta^n) \leq \mathcal{E}^g_{\theta,\tau_n^\varepsilon\wedge\theta_n}(\bar{\xi}_{\tau_n^\varepsilon}\mathbf{1}_{\tau_n^\varepsilon<\theta_n} + \eta^n \mathbf{1}_{\tau_n^\varepsilon\geq\theta_n}) + \varepsilon \quad \text{a.s.} \tag{7.2}$$

Now, by the right-uppersemicontinuity of $\bar{\xi}$ and the condition (7.1), we have $\limsup_{n\to\infty} \bar{\xi}_{\tau_n^\varepsilon\wedge\theta_n} \leq \bar{\xi}_\theta \leq \eta$ a.s., which implies that $\limsup_{n\to\infty}(\bar{\xi}_{\tau_n^\varepsilon}\mathbf{1}_{\tau_n^\varepsilon<\theta_n} + \eta^n \mathbf{1}_{\tau_n^\varepsilon\geq\theta_n}) \leq \eta$ a.s. Hence, using the Fatou property for BSDEs with respect to the pair terminal time-terminal condition (cf. e.g. Lemma A.5 in [13]), we derive that

$$\limsup_{n\to\infty} \mathcal{E}^g_{\theta,\tau_n^\varepsilon\wedge\theta_n}(\bar{\xi}_{\tau_n^\varepsilon}\mathbf{1}_{\tau_n^\varepsilon<\theta_n} + \eta^n \mathbf{1}_{\tau_n^\varepsilon\geq\theta_n}) \leq \mathcal{E}^g_{\theta,\theta}(\eta) = \eta \quad \text{a.s.}$$

Hence, by (7.2), we get $\limsup_{n\to\infty} \mathbf{Y}^{g,\xi}_{\theta,\theta_n}(\eta^n) \leq \eta + \varepsilon$ a.s. The desired result follows. □

We state the following result which provides, for each $\nu \in \mathcal{V}$, a useful representation of the difference of the solutions of two BSDEs associated with the driver f^ν in terms of the spread between the difference of the terminal conditions.

Lemma 7.3. *Let* η_1 *and* $\eta_2 \in L^2(\mathcal{G}_T)$. *Let* $\nu \in \mathcal{V}$. *For* $i = 1, 2$, *let* (X^i, Z^i, K^i) *be the solution in* $S^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ *of the BSDE associated with driver* f^ν *(defined in Definition 4.3), terminal time* T *and terminal condition* η_i. *Let* $\bar{X}_s := X^1_s - X^2_s$; $\bar{Z}_s := Z^1_s - Z^2_s$; $\bar{K}_s := K^1_s - K^2_s$. *Let* $f_y(s) := \dfrac{f(s, X^1_{s^-}, Z^1_s) - f(s, X^2_{s^-}, Z^1_s)}{\bar{X}_{s^-}}$ *if* $\bar{X}_{s^-} \neq 0$, *and* 0 *otherwise, and let* $f_z(s) := \dfrac{f(s, X^2_{s^-}, Z^1_s) - f(s, X^2_{s^-}, Z^2_s)}{\bar{Z}_s}$ *if* $\bar{Z}_s \neq 0$, *and* 0 *otherwise. We have*

$$\bar{X}_t = E[e^{\int_t^T f_y(s)ds} H_{t,T}(\eta_1 - \eta_2) \mid \mathcal{G}_t], \; 0 \leq t \leq T, \quad a.s.$$

where $H_{t,\cdot}$ *is solution of the following SDE*

$$dH_{t,s} = H_{t,s^-}\left[(f_z(s) - \nu_s\lambda_s\beta_s\sigma_s^{-1})dW_s + \nu_s dM_s\right]; \;\; H_{t,t} = 1. \tag{7.3}$$

Proof. By a classical "linearization procedure" (as the one used at the beginning of the proof of Theorem 3 in [11]), we derive the desired result. □

A result on reflected BSDEs with a non positive jump at the default time ϑ: Let $\mathcal{V}$ be the set of bounded predictable processes ν such that $\nu_t \geq 0$ $dP \otimes dt$-a.e.
Let g be a λ-admissible driver and let (δ_t) be a bounded predictable process.

For each $\nu \in \mathcal{V}$, we define

$$g^{\nu}(\omega, t, y, z, k) := g(\omega, t, y, z, k) + \nu_t(\omega)\lambda_t(\omega)\big(k - \delta_t(\omega)z\big)$$

Note that g^{ν} is a λ-admissible driver. For each $S \in \mathcal{T}$, the value $Y(S)$ at time S is defined by

$$Y(S) := ess \sup_{(\tau,\nu)\in\mathcal{T}_S\times\mathcal{V}} \mathcal{E}^{\nu}_{S,\tau}(\xi_\tau), \tag{7.4}$$

where $\mathcal{E}^{\nu} = \mathcal{E}^{g^{\nu}}$. By the same arguments as before (cf. the proof of Corollary 4.11), there exists an r.u.s.c. process $(Y_t) \in \mathbb{S}^2$ which aggregates the value family $(Y(S))$, which is a strong $\mathcal{E}^{\nu}$-supermartingale for all $\nu \in \mathcal{V}$ and $Y_t \geq \xi_t$, for all $t \in [0, T]$, a.s. Moreover, the process (Y_t) is the smallest process in $\mathbb{S}^2$ satisfying these properties.

By similar arguments as those used in the proof of Theorem 5.4, it can be shown that the value process (Y_t) is a supersolution of the *constrained reflected BSDE* from Definition 5.1 with f replaced by g and the constraints (5.4) replaced by the constraint (7.6) hereafter. We thus have the following result.

Proposition 7.4. *There exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that*

$$\begin{aligned}
&-dY_t = g(t, Y_t, Z_t, K_t)dt + dA_t + dC_{t-} - Z_t dW_t - K_t dM_t; &(7.5)\\
&Y_T = \xi_T \;\; a.s. \quad \text{and} \quad Y_t \geq \xi_t \; for\ all\ t \in [0, T] \;\; a.s.\,; &\\
&(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0 \;\; a.s.\ for\ all\ \tau \in \mathcal{T}_0; &\\
&(K_t - \delta_t Z_t)\lambda_t \leq 0,\;\; t \in [0, T],\;\; dP \otimes dt - \text{a.e.} &(7.6)
\end{aligned}$$

In other words, the value process (Y_t) is a supersolution of the above constrained reflected BSDE. *Moreover, it is the minimal one, that is, if (Y'_t) is another supersolution, then $Y'_t \geq Y_t$ for all $t \in [0, T]$ a.s.*

Note that when $\delta = 0$, the constraint (7.6) means that the jump of the process (X_t) at the default time ϑ is non-positive. In the case when $\delta = 0$ and the obstacle is right-continuous, our result gives the existence of a minimal supersolution of the reflected BSDE with driver g, obstacle ξ and with non positive jumps, which correponds to a result shown in [6] by using a penalization approach. Moreover, our result provides a dual representation (with non linear expectation) of this minimal supersolution.

References

[1] Bayraktar E. and Yao S., On the Robust Optimal Stopping Problem, *SIAM Journal on Control and Optimization*, 52(5), 3135-3175, 2014.

[2] Bank, P. and Kramkov D. (2015): A model for a large investor trading at market indifference prices. II: Continuous-time case, *The Annals of Applied Probability* No. 5, 2708-2742.

[3] Bouchard B., D. Possamaï, and X. Tan (2016): A general Doob-Meyer-Mertens decomposition for g-supermartingale system, *Electronic Journal of Probability* 21, paper no. 36, 21 pages.

[4] Brigo, D., Franciscello, M., and Pallavicini, A. Analysis of non-linear valuation equations under credit and funding effects. In *Innovations in Derivative Markets, Springer Proceedings in Mathematics and Statistics*, Vol. 165, Glau, K., Grbac, Z., Scherer, M., and Zagst, R. (Eds.), Springer, Heidelberg, 2016, 37-52.

[5] Buckdahn, R. and Li, J. (2009), Probabilistic interpretation for systems of Isaacs equations with two reflecting barriers, *non-linear Differ.Equ. Appl.* 16, pp. 381-420.

[6] Choukroun S., Cosso A. and Pham H., Reflected BSDEs with nonpositive jumps, and controller-and-stopper games, *Stochastics Processes and their Applications*, vol. 125(2), 597-633, 2015.

[7] Crepey, S, Bilateral Counterparty Risk under Funding Constraints- Part 1 CVA. Pricing. *Mathematical Finance* 25 (1), 1-22, 2015. 113, 2015.

[8] Cvitanić J. and Ma, J. , Hedging options for a large investor and forward-backward SDEs, *Annals of applied Prob.* 1996. **6**, n.2 370-398.

[9] Dellacherie C. and E. Lenglart (1981): Sur des problèmes de régularisation, de recollement et d'interpolation en théorie des processus, *Sém. de Proba. XVI*, lect. notes in Mathematics, 920, 298-313, Springer-Verlag.

[10] Dellacherie, C. and Meyer, P.-A. (1975). *Probabilités et Potentiel, Chap. I-IV.* Nouvelle édition. Hermann. **MR**0488194

[11] Dumitrescu, R., Grigorova, M., Quenez M.C., Sulem A. (2018), BSDEs with default jump, in *Computation and Combinatorics in Dynamics, Stochastics and Control - Abel Symposium, August 2016*, E. Celledoni et al. (Eds) vol 13. Springer, p. 233–263, https://doi.org/10.1007/978-3-030-01593-0_9,
available at https://hal.inria.fr/hal-01799335.

[12] Dumitrescu R., M.-C. Quenez, and A. Sulem (2016), Generalized Dynkin Games and Doubly reflected BSDEs with jumps, *Electronic Journal of Probability*, 21(64), 32p.

[13] Dumitrescu, R., Quenez M.C., Sulem A., Mixed Generalized Dynkin Games and Stochastic control in a Markovian framework, *Stochastics* (2017), 89, 400–429.

[14] Dumitrescu R., Quenez M-C., and A. Sulem , American options in an imperfect complete market with default, *ESAIM Proceedings & Surveys*, Vol 64, pp 93–110, 2018.

[15] El Karoui, N., *Les aspects probabilistes du contrôle stochastique*, in: Ecole d'été de Probabilités de Saint-Flour IX-1979, in: Lect. Notes in Math., vol. 876, 1981, pp. 73–238. MR0637469.

[16] Nutz M. and Zhang J. (2015), Optimal stopping under adverse nonlinear expectation and related games, *Annals of Applied Probability* 25 (5), 2503–2534.

[17] El Karoui N., Peng S. and M.C. Quenez, "Backward Stochastic Differential Equations in Finance", *Mathematical Finance*, 7, 1 (January 1997), 1-71.

[18] El Karoui N. and M.C. Quenez (1995), Dynamic programming and pricing of contingent claims in an incomplete market, *SIAM J. Control and Optimization,* 33, 29-66.

[19] El Karoui N. and M.C. Quenez (1996), "Non-linear Pricing Theory and Backward Stochastic Differential Equations", Lectures Notes in Mathematics 1656, Bressanone, 1996, Editor:W.J.Runggaldier, collection Springer,1997.

[20] Grigorova, M., Imkeller, P., Offen, E., Ouknine, Y., Quenez, M.-C. (2017): Reflected BSDEs when the obstacle is not right-continuous and optimal stopping, *the Annals of Applied Probability*, 27(5), 3153-3188.

[21] Grigorova, M., Imkeller, P., Ouknine, Y., Quenez, M.-C. (2016): Optimal stopping with f-expectations: the irregular case, arXiv:1611.09179, submitted.

[22] Grigorova M., P. Imkeller, Y. Ouknine, and M.-C. Quenez (2016): Doubly Reflected BSDEs and $\mathcal{E}^f$-Dynkin games: beyond the right-continuous case, to appear in *Electron. J. Probab.*

[23] Grigorova M., Quenez, M.-C., and Sulem, A (2018): European options in a non-linear incomplete market with default, preprint.

[24] Jeanblanc M. and Song S., Martingale representation theorem in progressively enlarged filtrations *Stochastic Processes and their Applic.*, 125, 4242-4271, 2015.

[25] Jeanblanc, M., Yor M., and Chesney M. (2009): Mathematical methods for financial markets, Springer Finance.

[26] Karatzas I. and Kou S., Hedging American contingent claims with constrained portfolios, *Finance and Stochastics* 2(3) 215-258, 1998.

[27] Karatzas I. and Shreve S.E., Methods of Mathematical Finance. Applications of Mathematics (New York) 39. Springer, New York, 1998.

[28] Korn R. (1995): Contingent claim valuation in a market with different interest rates, *Mathematical Methods of Operation Research* 42, 255-274.

[29] Kramkov D., Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets, *Probab. Theory Relat. Fields*, 105, 459-479, 1996.

[30] Kusuoka, S., A remark on default risk models, *Adv. Math. Econ.*, 1, 69–82, 1999.

[31] Li H., Peng S. and A. Soumana Hima, Reflected solutions of backward stochastic differential equations driven by G–Brownian motion, *A. Sci. China Math.* (2018) **61** (1), 1–26.

[32] Peng, S., A generalized dynamic programming principle and Hamilton-Jacobi-Bellman-Equation, *Stochastics and Stochastics Reports*, 38, 1992.

[33] Peng, S. and M. Xu, Reflected BSDEs with a constraint and its applications in an incomplete market, *Bernoulli*, 2010, **16** (3), 614–640.

[34] Peng, S. (2004), non-linear expectations, non-linear evaluations and risk measures, 165-253, *Lecture Notes in Math.*, Springer, Berlin.

[35] Protter, P. (2004) Stochastic integration and differential equations, second edition, Springer-Verlag Berlin Heidelberg New York.

[36] Quenez M.-C. and Sulem A., Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps, *Stochastic Processes and Applications* 124, (2014) 3031–3054.

[37] Riedel, F. (2009), Optimal stopping with multiple priors, *Econometrica* 77, 857–908.