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The constraint equations in the presence of a scalar field - the case of the conformal method with volumetric drift

Caterina Vâlcu*

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Abstract

In this paper we establish the existence in low dimensions of solutions to the constraint equations in the case of the conformal system recently proposed by David Maxwell [14], with the added presence of a scalar field and under suitable smallness assumptions on its parameters.

1 Introduction

The field of general relativity deals with the study of spacetime, an object defined as the equivalence class, up to an isometry, of Lorentzian manifolds (\tilde{M}, \tilde{g}) of dimension $n + 1$ satisfying the Einstein field equations

$$Ric_{\alpha\beta}(\tilde{g}) - \frac{1}{2}R(\tilde{g})\tilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$\alpha, \beta = \overline{1, n+1}$. Here, $R(\tilde{g})$ is the scalar curvature of \tilde{g} , Ric the Ricci curvature and $T_{\alpha\beta}$ the stress-energy tensor describing the presence of matter and energy. For example, $T_{\alpha\beta} = 0$ describes vacuum. Our interest focuses on the more general case

$$T_{\alpha\beta} = \nabla_{\alpha}\tilde{\psi}\nabla_{\beta}\tilde{\psi} - \left(\frac{1}{2}|\nabla\tilde{\psi}|_{\tilde{g}}^2 + V(\tilde{\psi})\right)\tilde{g}_{\alpha\beta},$$

which models the existence within the spacetime of a scalar field $\tilde{\psi} \in \mathcal{C}^{\infty}(M)$ having potential $V \in \mathcal{C}^{\infty}(\mathbb{R})$. Thus, $\tilde{\psi} = 0$ and $V = \Lambda$ yield the vacuum with cosmological constant Λ , while $V = \frac{1}{2}m\tilde{\psi}^2$ corresponds to the Einstein-Klein-Gordon setting.

For a globally hyperbolic spacetime, we define its initial data $(M, \hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$. They consist of an n -dimensional Riemannian manifold (M, \hat{g}) , which models

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the spacetime at a particular moment in time, a symmetric 2-tensor \hat{K} , corresponding to its second fundamental form, the scalar field $\hat{\psi}$ in M , and its temporal derivative $\hat{\pi}$. The associated spacetime development takes the form $(M \times \mathbb{R}, \tilde{g}, \tilde{\psi})$, where \tilde{g} is a Lorentzian metric that verifies $\tilde{g}|_M = \hat{g}$ and $\tilde{\psi}$ is a scalar field such that $\tilde{\psi}|_M = \hat{\psi}$ and $\partial_t \tilde{\psi}|_M = \hat{\pi}$.

Initial data in general relativity may not be freely specified, unlike their Newtonian counterparts. Instead, they must verify the Gauss and Codazzi equations,

$$\begin{cases} R(\hat{g}) + (tr_{\hat{g}} \hat{K})^2 - \|\hat{K}\|_{\hat{g}}^2 &= \hat{\pi}^2 + |\hat{\nabla} \hat{\psi}|_{\hat{g}}^2 + 2V(\hat{\psi}) \\ \partial_i (tr_{\hat{g}} \hat{K}) - \hat{K}_{i,j}^j &= \hat{\pi} \partial_i \hat{\psi}, \end{cases}$$

which are referred to as the *constraint equations*. The work of Choquet-Bruhat [8] establishes, once and for all, that the constraint equations are not only necessary but sufficient conditions for the (local) existence of a solution. Later, Choquet-Bruhat and Geroch [3] prove that the maximal development of initial data is unique, up to an isometry. Globally hyperbolic spacetimes may rigorously be studied in the context of mathematical analysis as the result of an evolution problem. The above system is clearly under-determined, which allows for considerable freedom in choosing a solution $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$.

Using the conformal method introduced by Lichnerowicz [13], the constraint equations may be transformed into a determined system of equations by fixing well-chosen quantities (see Choquet-Bruhat, Isenberg and Pollack [4]). The appeal of such a method lies in that it provides a characterisation of the resulting initial data by fixed quantities. Essentially, it maps a space of parameters to the space of solutions.

Given an initial data set $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$, the classical choice of parameters is $(\mathbf{g}, \mathbf{U}, \tau, \psi, \pi; \alpha)$: in this case, the conformal class \mathbf{g} is represented by a Riemannian metric g , the smooth function $\tau = \hat{g}^{ab} \hat{K}_{ab}$ is a mean curvature and the conformal momentum \mathbf{U} measured by a volume form α (volume gauge) is a 2-tensor that is both trace-free and divergence-free with respect to g (a transverse-traceless tensor). We sometimes prefer to indicate the volume gauge by the densitized lapse

$$\tilde{N}_{g,\alpha} := \frac{\alpha}{dV_g}.$$

Note that this quantity depends on the choice of representative g , unlike the volume gauge α which does not. The standard conformal method implicitly fixes $\tilde{N}_{g,\alpha} = 2$; in the present paper, we prefer to make use of the freedom of choosing $\tilde{N}_{g,\alpha}$ as needed. We often refer to a parameter set by indicating the representative metric g and the corresponding densitized lapse $\tilde{N}_{g,\alpha}$ instead of giving the conformal class and volume gauge. However, these quantities can immediately be reconstructed from our data. We refer to Maxwell [14] for an introduction to the conformal method in our context.

Starting from the parameter set $(g, U, \tau, \psi, \pi; \tilde{N})$, the corresponding (physical) initial data is pinpointed by solving a resulting system, comprising the

Lichnerowicz-type equation and the momentum constraints, for a smooth positive function (or conformal factor) u and a smooth vector field W in M ,

$$\begin{cases} \Delta_g u + \mathcal{R}_\psi = -\mathcal{B}_{\tau,\psi,V} u^{q-1} + \frac{\mathcal{A}_{\pi,U}(W)}{u^{q+1}}, \\ \Delta_{g,\text{conf}} W = \frac{n-1}{n} u^q \nabla \tau + \pi \nabla \psi, \end{cases} \quad (1)$$

where

$$\begin{aligned} \mathcal{R}_\psi &= \frac{n-2}{4(n-1)} (R(g) - |\nabla \psi|_g^2), \\ \mathcal{B}_{\tau,\psi,V} &= \frac{n-2}{4(n-1)} \left(\frac{n-1}{n} \tau^2 - 2V(\psi) \right), \\ \mathcal{A}_{\pi,U}(W) &= \frac{n-2}{4(n-1)} (|U + \mathcal{L}_g W|_g^2 + \pi^2). \end{aligned}$$

If (u, W) solves the above system, then the initial data we've been searching for are

$$\hat{g} = u^{q-2} g, \quad \hat{K} = u^{-2} \left(U + \frac{\tilde{N}}{2} \mathcal{L}_g W \right) + \frac{\tau}{n} \hat{g}, \quad \hat{\psi} = \psi, \quad \hat{\pi} = u^{-q} \pi.$$

Note also that the solutions generated by $(g, U, \tau; \tilde{N})$ and

$$(\varphi^{q-2} g, \varphi^{-2} U, \tau, \psi, \varphi^{-q} \pi; \varphi^q \tilde{N})$$

are the same, where φ is a smooth positive function. The notations above are similar to those of Choquet-Bruhat, Isenberg and Pollack ([4], [5]). The following quantities often appear throughout the present paper: $q = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of H^1 in Lebesgue spaces, $\Delta_g = -\text{div}_g \nabla$ denotes the Laplace-Beltrami operator taken with non-negative eigenvalues, $\Delta_{g,\text{conf}} W = \text{div}_g(\mathcal{L}_g W)$ is the Lamé operator and \mathcal{L}_g is the conformal Killing operator with respect to g ,

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} \text{div}_g W g_{ij}.$$

Conformal Killing fields are defined as vector fields in the kernel of \mathcal{L}_g .

The conformal method is particularly successful in finding solutions when the mean curvature τ is constant as the system (1) becomes uncoupled, but it is unclear how well the method functions when the mean curvature is far from being constant: see Maxwell [15] and [16], where a given set of parameters point to no or to an infinite number of solutions. We emphasize that any failing of the system does not necessarily translate to a singularity in the space of solutions to the constraints system, but may instead derive from a poor choice of mapping. This motivates the study of variations to standard conformal methods.

The drift method introduced by Maxwell replaces the mean curvature τ as a parameter by a pair (τ^*, \tilde{V}) , where τ^* is a unique constant called volumetric

momentum and \tilde{V} a vector field related to the drift. They verify an analogue of York splitting, namely

$$\tau = \tau^* + \tilde{N}_{\hat{g},\alpha} \operatorname{div}_{\hat{g}} \tilde{V} = \tau^* + \frac{\tilde{N}_{g,\alpha}}{u^{2q}} \operatorname{div}_g(u^q \tilde{V}), \quad (2)$$

the notation \tilde{V} being specific to this paper in order to avoid confusion with the potential V . Interestingly, $\tau^* = 0$ holds true for all counterexamples found by Maxwell ([15], [16]). This suggests that the volumetric momentum may play an important role in characterizing the space of initial data. Ideally, we would like to know as soon as we fix a set of parameters $(g, U, \tau, \psi, \pi; \tilde{N})$ if we find ourselves in the case $\tau^* = 0$. However, τ^* cannot be directly calculated by (2) without first solving (1), which somewhat defeats the purpose. This motivates a new choice of parameters, even at the risk of working with an analytically more complicated system. The idea of Maxwell ([15], [16]) is thus to choose τ^* as an additional parameter that is to be fixed in the place of τ : the hope is thus to avoid the aforementioned problem. As well as τ^* , Maxwell added \tilde{N} and \tilde{V} as parameters, for geometric and physical reasons. Therefore, instead of fixing τ , we fix τ^* , \tilde{N} and \tilde{V} .

Intuitively, the drift is a geometric quantity describing infinitesimal motion in the space of metrics modulo the group of diffeomorphisms connected to the identity such that the conformal class and volume are preserved. For any given drift, the choice of a representative vector field \tilde{V} is unique up to conformal Killing fields and vector fields which are divergence-free with respect to the initial metric \hat{g} . Given g an arbitrary representative of the conformal class, it is not clear whether two vector fields are indicative of the same drift class defined for \hat{g} ; this problem is discussed at length in the paper of Mike Holst, David Maxwell and Rafe Mazzeo [11] for conformal systems where the critical non-linearity is non-focusing, or negative. Our analysis treats systems with focusing (that is to say positive) non-linearities stemming from the presence of a scalar field with positive potential.

The following system corresponds to Problem 12.1 of [14] in the presence of a scalar field, where g admits no non-trivial conformal Killing fields:

$$\left\{ \begin{array}{l} \Delta_g u + \frac{n-2}{4(n-1)} (R(g) - |\nabla \psi|_g^2) u - \frac{(n-2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n-1)u^{q+1}} \\ - \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\tilde{N} \operatorname{div}_g(u^q \tilde{V})}{u^{2q}} \right)^2 \right] u^{q-1} = 0 \\ \operatorname{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) - \frac{n-1}{n} u^q \mathbf{d} \left(\frac{\tilde{N} \operatorname{div}_g(u^q \tilde{V})}{2u^{2q}} \right) - \pi \nabla \psi = 0. \end{array} \right. \quad (3)$$

We denote the exterior derivative by \mathbf{d} . The unknowns are a smooth positive scalar function u defined on M and a smooth vector field W on M . The parameters are $(g, U, \tau^*, \tilde{V}, \psi, \pi; \tilde{N})$. Maxwell's new set of parameters include τ^* , which could not be calculated *a priori* in the classical method. The initial data

of the constraint equations verify

$$\begin{aligned} \hat{g} &= u^{q-2}g, & \hat{K} &= u^{-2} \left(U + \frac{\tilde{N}}{2} \mathcal{L}_g W \right) + \frac{1}{n} \left(\tau^* + \frac{\tilde{N}}{u^{2q}} \operatorname{div}(u^q \tilde{V}) \right) \hat{g}, \\ \hat{\psi} &= \psi, & \hat{\pi} &= u^{-q}\pi. \end{aligned} \quad (4)$$

The following is a more general system than (3). The central result of the paper consists in showing that it admits solutions. Let (M, g) be a closed Riemannian manifold of dimension $n \in \{3, 4, 5\}$, and g has no non-trivial conformal Killing fields. Let $b, c, d, f, h, \rho_1, \rho_2, \rho_3$ be smooth functions on M and let Y and Ψ be smooth vector fields defined on M . Let $0 < \gamma < 1$. Assume that $\Delta_g + h$ is coercive, in the sense that its first eigenvalue is positive. Assume that $f > 0, \rho_1 > 0$ and $|\nabla \rho_3| < (2C_1)^{-1}$, where C_1 is a dimensional constant - see (101). Consider the system

$$\begin{cases} \Delta_g u + hu &= fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ & - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \\ \operatorname{div}_g(\rho_3 \mathcal{L}_g W) &= \mathcal{R}(u). \end{cases} \quad (5)$$

Here \mathcal{R} is an operator verifying

$$\mathcal{R}(u) \leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{\mathcal{C}^2}^2}{(\inf_M u)^2} \right)$$

for a constant $C_{\mathcal{R}} > 0$.

A supersolution of the Lichnerowicz-type equation is a smooth function u verifying that

$$\begin{aligned} \Delta_g u + hu &\geq fu^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} - \frac{b}{u} \\ &\quad - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}}. \end{aligned}$$

Similarly, a subsolution satisfies an inequality of opposite sign. Whenever the inequality is strict, we say u is a strict subsolution or a strict supersolution respectively.

We fix

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (6)$$

and

$$T = \max(\|f\|_{\mathcal{C}^{1,\gamma}}, \|\rho_1\|_{\mathcal{C}^{0,\gamma}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (7)$$

Here is the main result of our paper:

Theorem 1.1. *There exists a constant $C = C(n, h)$, $C > 0$ such that if ρ_1 verifies*

$$\|\rho_1\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (8)$$

and there exists a constant

$$\delta = \delta(\theta, T) > 0$$

such that, if

$$\|b\|_{C^{0,\gamma}} + \|Y\|_{C^{0,\gamma}} + \|\Psi\|_{C^{0,\gamma}} + \|\rho_2\|_{C^{0,\gamma}} + C_{\mathcal{R}} \leq \delta, \quad (9)$$

then the system (5) admits a solution.

Remark 1.1. For a slightly more detailed expression of the smallness assumptions, see Section 4. The constant $C(n, h) = \frac{C(n)}{S_h^{n-1}}$ appears explicitly in a paper by Hebey, Pacard and Pollack ([9], Corollary 3.1). By S_h we understand the Sobolev constant which is defined as the smallest constant $S_h > 0$ such that

$$\int_M |v|^q dv_g \leq S_h \left(\int_M (|\nabla v|^2 + hv^2) dv_g \right)^{\frac{q}{2}}$$

for all $v \in H^1(M)$.

The following corollary deals with the existence of solutions to the conformal system. It suffices to take

$$\begin{cases} h = \frac{n-2}{4(n-1)} (\mathcal{R}_g - |\nabla\psi|_g^2), & f = \frac{n-2}{4(n-1)} [2V(\psi) - \frac{n-1}{n}(\tau^*)^2], \\ \rho_1 = \frac{n-2}{4(n-1)} \left(\pi - \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \right), & \rho_2 = \sqrt{\frac{n-2}{4(n-1)} \frac{\tilde{N}}{2}}, \quad \Psi = \sqrt{\frac{n-2}{4(n-1)}} U, \\ b = \frac{n-2}{2n} \tau^* \tilde{N} \operatorname{div}_g(\tilde{V}), & c = 2\sqrt{\frac{n-2}{4n}}, \quad d = \tau^* \\ Y = \sqrt{\frac{n}{n-2}} \tilde{N} \tilde{V}, & \rho_3 = \ln \tilde{N}, \\ \mathcal{R} = \frac{n-1}{n} \operatorname{div}_g(\tilde{V}) \nabla \ln \tilde{N} + \frac{n-1}{n} \nabla(\operatorname{div}_g(\tilde{V})) + \frac{\pi \delta_i \psi}{\tilde{N}} \\ \quad + 2\langle \tilde{V}, \frac{\nabla u}{u} \rangle \nabla \ln \tilde{N} - 2\frac{n-1}{n+1} \langle \tilde{V}, \frac{\nabla u}{u^2} \rangle - \frac{n-1}{n} \langle \tilde{V}, \frac{\Delta_g u}{u} \rangle \end{cases}$$

in (5). It is a direct application of Theorem 1.1.

Corollary 1.1. Let $\Delta_g + \frac{n-2}{4(n-1)} (\mathcal{R}_g - |\nabla\psi|_g^2)$ be a coercive operator. Assume that

$$2V(\psi) > \frac{n-1}{n}(\tau^*)^2, \quad \pi > \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \quad \text{and} \quad |\nabla \ln \tilde{N}| < C_1^{-1}, \quad (10)$$

where C_1 depends on n and g (see (101) for more details). Moreover, assume that

$$\|\pi - \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V})\|_{L^1} \leq C(n, g, h) \|2V(\psi) - \frac{n-1}{n}(\tau^*)^2\|_{L^\infty}^{1-n}. \quad (11)$$

Then there exists a constant

$$\begin{aligned} \delta = \delta \left(\inf_M \frac{n-2}{4(n-1)} [2V(\psi) - \frac{n-1}{n}(\tau^*)^2], \inf_M \frac{n-2}{4(n-1)} \left(\pi - \frac{n-1}{n}(\tilde{N})^2 \operatorname{div}_g(\tilde{V}) \right) \right. \\ \left. \tau^*, \|\pi\|_{C^{0,\gamma}}, \|\mathcal{R}_g - |\nabla\psi|_g^2\|_{C^{0,\gamma}}, \|2V(\psi)\|_{C^{1,\gamma}} \right) > 0 \end{aligned} \quad (12)$$

such that, if

$$\|U\|_{\mathcal{C}^{0,\gamma}} + \|\pi\|_{\mathcal{C}^{0,\gamma}} + \|\nabla\psi\|_{\mathcal{C}^{0,\gamma}} + \|\ln \tilde{N}\|_{\mathcal{C}^{1,\gamma}} + \|\tilde{V}\|_{\mathcal{C}^{1,\gamma}} \leq \delta, \quad (13)$$

then (3) admits a solution (u, W) , where u is a smooth positive function on M and W a smooth vector field on M .

A few remarks on the results of the present paper. The classical system of constraint equations obtained by the conformal method (without the modifications proposed by Maxwell [14]) was studied by Bruno Premoselli ([19], [20]) in the presence of a scalar field. Second, the above system is the subject of a paper by Mike Holst, David Maxwell and Rafe Mazzeo [11] - in their case, certain conditions are imposed on the presence of the matter field. We treat the separate and delicate case wherein the dominant non linearity is focusing and leads to possible loss of compactness. It is interesting to note that the size of n plays a role; as Premoselli proves in his paper, while his system may be well-behaved in low dimensions ($3 \leq n \leq 5$), it most certainly fails to do so in higher dimensions ($n \geq 6$). Even if our results are similar to those of Premoselli, they are considerably more difficult to obtain. This is mainly due to the presence of a $|\nabla u|^2$ term in the scalar equation, a term which is not compact a priori.

Outline of the paper. Section 2 is devoted to the study of the first equation in (5), the so-called Lichnerowicz equation. We prove the existence of stable solutions under suitable assumptions.

Section 3 deals with a priori estimates for solutions of the Lichnerowicz equation. A careful blow-up analysis is carried out. As already mentioned, the term $|\nabla u|^2$ poses additional difficulty: blow-up can occur at the \mathcal{C}^1 level, even if the solution is bounded in L^∞ .

Section 4 is devoted to the proof of Theorem 1.1 and Corollary 1.1, which relies heavily on the a priori estimates obtained in Section 3. At the end of Section 4, we also explain how to extend Corollary 1.1 in the presence of conformal Killing vector fields.

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2 Existence of minimal solutions of the scalar equation

We study the Lichnerowicz-type scalar equation in (5). The following theorem states that, given the existence of supersolutions, one may use an iterative procedure to obtain a sequence which converges in \mathcal{C}^1 norm to a solution. We draw the reader's attention to the fact that this solution is uniquely determined by its construction. The proof contains some similarities with that of Premoselli [19], but some new difficulties appear. The main difference here comes from the presence of non-linearities containing gradient terms, which force us to further

refine the analysis. These gradient terms lead to difficulties in obtaining a priori estimates on solutions of the equation, which in turn lead to problems of stability. The existence result we prove in this section reads as follows:

Theorem 2.1. *Let (M, g) be a closed Riemannian manifold. Let a, b, c, d, f, h be smooth functions on M and Y be a smooth vector field on M . Assume that $a > 0$ and $f > 0$. The equation*

$$\Delta_g u + hu - fu^{q-1} - \frac{a}{u^{q+1}} + \frac{b}{u} + \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} + c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) = 0 \quad (14)$$

admits a smooth positive solution u as soon as it admits a supersolution.

Remark 2.1. *The solution obtained by the construction below is unique. Moreover, it is stable (see Lemma 2.1 at the end of this section.)*

Proof of Theorem 2.1: We begin by fixing a supersolution and a subsolution to serve as upper and lower bounds respectively for the iterative process. Let ψ be a positive supersolution of (14). Let $\varepsilon_0 > 0$ be a small constant such that

$$\varepsilon_0 < \inf_M \psi, \quad \left(\sup_M h \right) \varepsilon_0^{q+2} < \frac{\inf_M a}{2} \quad \text{and} \quad \left(\sup_M b \right) \varepsilon_0^q < \frac{\inf_M a}{2}.$$

The last two bounds ensure that $u_0 = \varepsilon_0$ is a strict subsolution of (14) since $f > 0$. We let

$$F(t, x) = -f(x)t^{q-1} - \frac{a(x)}{t^{q+1}} + \frac{b(x)}{t} - Kt$$

for $x \in M$ and $t > 0$. We choose $K > 0$ large enough such that

$$\frac{\partial}{\partial t} \left[F(t, x) + \frac{A(x)^2}{t^{q+3}} + c(x)A(x) \left(\frac{d(x)}{t^2} + \frac{1}{t^{q+2}} \right) \right] \leq 0 \quad (15)$$

for all $x \in M$ and all $\varepsilon_0 \leq t \leq \sup_M \psi$, whatever $A(x)$ is. It is sufficient to take

$$K \geq \sup_{x \in M, \varepsilon_0 \leq t \leq \sup_M \psi} \left[-(q-1)f(x)t^{q-2} + (q+1)\frac{a(x)}{t^{q+2}} - \frac{b(x)}{t^2} + \frac{c(x)^2 \left(\frac{2d(x)}{t^3} + \frac{d+2}{t^{q+3}} \right)^2 t^{q+4}}{4(q+3)} \right]. \quad (16)$$

Up to choosing K larger, we may also assume that $h + K > 0$ and that $F(t, x)$ is negative for $\varepsilon_0 \leq t \leq \sup_M \psi$.

We shall now consider a sequence $(u_i)_{i \in \mathbb{N}}$ defined by induction by $u_0 \equiv \varepsilon_0$ and

$$(E_i) : \quad \Delta_g u_i + (h + K)u_i + F(u_{i-1}(x), x) + \frac{\langle \nabla u_i, Y \rangle^2}{u_{i-1}^{q+3}} + c \langle \nabla u_i, Y \rangle \left(\frac{d}{u_{i-1}^2} + \frac{1}{u_{i-1}^{q+2}} \right) = 0. \quad (17)$$

We prove in Step 1 below that the sequence is well defined. In Step 2, we prove that the sequence is pointwise increasing and uniformly bounded. At last,

Step 3 is devoted to the proof that the sequence (u_i) converges to a solution of (17).

Step 1: We prove that (u_i) is well defined. We consider the more general equation

$$\Delta_g u + Hu + \theta_1 \langle \nabla u, Z \rangle^2 + \theta_2 \langle \nabla u, Z \rangle + \theta_3 = 0, \quad (18)$$

with $H, \theta_1, \theta_2, \theta_3$ smooth functions on M and Z a smooth vector field on M such that $\theta_1 > 0, H > 0, \theta_3 < 0$. We claim that (18) admits a unique smooth positive solution.

Proof of Step 1: We shall use the fixed point theorem as stated in Evans [7], Section 9.2.2, Theorem 4. Let us define the operator $T : \mathcal{C}^{1,\gamma}(M) \rightarrow \mathcal{C}^{1,\gamma}(M)$ such that

$$\Delta_g T(u) + HT(u) + \theta_1 \langle \nabla u, Z \rangle^2 + \theta_2 \langle \nabla u, Z \rangle + \theta_3 = 0.$$

If we can prove that there exists $C > 0$ such that

$$\forall 0 \leq \tau \leq 1, \quad w = \tau T(w) \quad \Rightarrow \quad \|w\|_{\mathcal{C}^{1,\gamma}(M)} \leq C, \quad (19)$$

then the operator T will have a fixed point, leading to a solution of (18). Note that this solution will be unique. Indeed, assume that w_1 and w_2 are two solutions of (18), then at a point of maximum x_0 of $w_1 - w_2$, we have that $\nabla w_1(x_0) = \nabla w_2(x_0)$ and $\Delta_g w_1(x_0) \geq \Delta_g w_2(x_0)$ so that (18) gives

$$H(x_0)(w_1(x_0) - w_2(x_0)) \leq 0.$$

Since $H > 0$, this leads to $w_1 \leq w_2$. By symmetry, uniqueness is proved. Note that the fixed point of T is smooth and positive by the standard regularity theory and the maximum principle.

Thus we are left with the proof of (19). Let $0 \leq \sigma_m \leq 1$ and let $w_m \in \mathcal{C}^{1,\gamma}(M)$ be such that

$$w_m = \sigma_m T(w_m).$$

Multiplying (18) by σ_m , we obtain that

$$\Delta_g w_m + Hw_m + \sigma_m \theta_1 \langle \nabla w_m, Z \rangle^2 + \sigma_m \theta_2 \langle \nabla w_m, Z \rangle + \sigma_m \theta_3 = 0.$$

First, the L^∞ bounds on w_m exist *a priori*. Indeed, consider $x_0 \in M$ a minimum of w_m . Since $\Delta_g w_m(x_0) \leq 0$ and $\nabla w_m(x_0) = 0$, which holds true for all minima, then $Hw_m(x_0) \geq -\sigma_m \theta_3(x_0)$. By applying the same procedure to the study of maxima, we obtain that

$$\inf_M \frac{-\sigma_m \theta_3}{H} \leq w_m \leq \sup_M \frac{-\sigma_m \theta_3}{H}. \quad (20)$$

Assume now that $\|\nabla w_m\|_{L^\infty(M)} \rightarrow \infty$. Let

$$\mu_m := \frac{1}{\|\nabla w_m\|_{L^\infty(M)}} \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

and $(x_m)_m \subset M$ be such that

$$\|\nabla w_m\|_{L^\infty(M)} = |\nabla w_m(x_m)|.$$

Consider the domains $\Omega_m := B_0\left(\frac{i_g(M)}{2\mu_m}\right)$, where $i_g(M)$ is the injectivity radius of M , and the rescaled quantities

$$v_m(x) := w_m(\exp_{x_m}(\mu_m x)) \quad \text{and} \quad g_m(x) := (\exp_{x_m}^* g)(\mu_m x).$$

Clearly, $\|\nabla v_m\|_{L^\infty} \leq 1$ and $|\nabla v_m(0)| = 1$. The L^∞ bounds remain unchanged. In $(\Omega_m)_{m \geq 1}$, we have that

$$\begin{aligned} & \Delta_{g_m} v_m + \mu_m^2 H(\exp_{x_m}(\mu_m \cdot)) v_m + \sigma_m \mu_m^2 \theta_3(\exp_{x_m}(\mu_m \cdot)) \\ & \quad + \sigma_m \theta_1(\exp_{x_m}(\mu_m \cdot)) \langle \nabla v_m, Z(\exp_{x_m}(\mu_m \cdot)) \rangle^2 \\ & \quad + \mu_m \sigma_m \theta_2(\exp_{x_m}(\mu_m \cdot)) \langle \nabla v_m, Z(\exp_{x_m}(\mu_m \cdot)) \rangle = 0 \end{aligned}$$

Note that $g_m \rightarrow \xi$ in $\mathcal{C}_{loc}^2(\mathbb{R}^n)$. By standard elliptic theory, $(v_m)_m$ is bounded in $\mathcal{C}_{loc}^{1,\eta}(\mathbb{R}^n)$, with $\eta \in (0, 1)$. We may extract, up to a subsequence, $v_\infty = \lim_{m \rightarrow \infty} v_m$, $x_\infty = \lim_{m \rightarrow \infty} x_m$ and $\sigma_\infty := \lim_{m \rightarrow \infty} \sigma_m$. From this it follows that $\|\nabla v_\infty\|_{L^\infty} = 1$ and that the *a priori* bounds (20) become

$$\inf_M \frac{-\sigma_\infty \theta_3}{H} \leq v_\infty \leq \sup_M \frac{-\sigma_\infty \theta_3}{H}. \quad (21)$$

Moreover, v_∞ solves the limit equation

$$\Delta v_\infty + (\partial_1 v_\infty)^2 = 0$$

in \mathbb{R}^n , where we have let

$$\partial_1 v_\infty := \sqrt{\sigma_\infty \theta_1(0)} \nabla v_\infty \cdot Z(x_0).$$

If $\sigma_\infty = 0$, then v_∞ is a bounded harmonic function, and thus a constant. Let us assume that $\sigma_\infty \neq 0$. Note that, for $\alpha \in \mathbb{R}$,

$$\begin{aligned} \Delta v_\infty^{-\alpha} &= -\frac{\alpha}{v_\infty^{\alpha+1}} \Delta v_\infty - \frac{\alpha(\alpha+1)|\nabla v_\infty|^2}{v_\infty^{\alpha+2}} \\ &\leq \frac{\alpha|\nabla v_\infty|^2}{v_\infty^{\alpha+1}} \left(\sigma_\infty \theta_1(0) |Z(0)|^2 - \frac{\alpha+1}{v_\infty} \right). \end{aligned}$$

This and (21) imply that, for α sufficiently large, $v_\infty^{-\alpha}$ is subharmonic. We then apply Lemma 5.1 (see annex) to get that v_∞ must be constant. Whichever the case, $\nabla v_\infty \equiv 0$ leads to a contradiction. The $\mathcal{C}^{1,\gamma}$ bound follows from an elliptic regularity argument. This ends the proof of Step 1. \square

Step 2: We claim that

$$\varepsilon_0 \leq u_i(x) \leq u_{i+1}(x) \leq \psi(x)$$

for all $x \in M$ and all $i \leq 0$.

Proof of Step 2: We proceed by induction. We prove first that

$$\forall i \geq 0, \quad u_i \text{ is a subsolution of } (E_{i+1}) \text{ and } u_i \leq \psi. \quad (22)$$

Note that

$$(22) \Rightarrow u_i \leq u_{i+1}. \quad (23)$$

Indeed, let $x_0 \in M$ be a maximum point of $u_i - u_{i+1}$. Then $\nabla u_i(x_0) = \nabla u_{i+1}(x_0)$ and we can use the fact that u_i is a subsolution of (E_{i+1}) and u_{i+1} a solution of (E_{i+1}) to write that

$$\Delta_g(u_i - u_{i+1})(x_0) + (h(x_0) + K)(u_i - u_{i+1})(x_0) \leq 0$$

which implies that $u_i(x_0) \leq u_{i+1}(x_0)$ since $\Delta_g(u_i - u_{i+1})(x_0) \geq 0$ and $h + K > 0$. This proves (23).

We now prove (22) by induction. For $i = 0$, it follows from the choice of ε_0 we made. Assume that (22) holds for some $i \geq 0$. We need to prove that u_{i+1} is a subsolution of (E_{i+2}) . It suffices to show that

$$\begin{aligned} & \Delta_g u_{i+2} + (h + K)u_{i+2} + F(u_{i+1}(x), x) + \frac{\langle \nabla u_{i+2}, Y \rangle^2}{u_{i+1}^{q+3}} \\ & + c \langle \nabla u_{i+2}, Y \rangle \left(\frac{d}{u_{i+1}^2} + \frac{1}{u_{i+1}^{q+2}} \right) \leq \Delta_g u_{i+1} + (h + K)u_{i+1} + F(u_{i+1}(x), x) \\ & + \frac{\langle \nabla u_{i+1}, Y \rangle^2}{u_{i+1}^{q+3}} + c \langle \nabla u_{i+1}, Y \rangle \left(\frac{d}{u_{i+1}^2} + \frac{1}{u_{i+1}^{q+2}} \right), \end{aligned}$$

since u_{i+1} is defined as a solution of (E_{i+1}) . This is equivalent to showing that

$$\begin{aligned} & F(u_{i+1}) + c \langle \nabla u_{i+1}, Y \rangle \left(\frac{d}{u_{i+1}^2} + \frac{1}{u_{i+1}^{q+2}} \right) + \frac{\langle \nabla u_{i+1}, Y \rangle^2}{u_{i+1}^{q+3}} \\ & \leq F(u_i) + c \langle \nabla u_{i+1}, Y \rangle \left(\frac{d}{u_i^2} + \frac{1}{u_i^{q+2}} \right) + \frac{\langle \nabla u_{i+1}, Y \rangle^2}{u_i^{q+3}}. \end{aligned}$$

And this is a consequence of (15) with $A(x) = \langle \nabla u_{i+1}, Y \rangle$, since (23) implies that $u_{i+1} \geq u_i$ by induction hypothesis. Thus, u_{i+1} is a subsolution of (E_{i+2}) .

Finally, so as to check the last point, assume there exists $x_0 \in M$ such that $u_{i+1}(x_0) > \psi(x_0)$ and that it corresponds to $\max_M (u_{i+1}(x) - \psi(x))$. Since $\nabla u_{i+1}(x_0) = \nabla \psi(x_0)$ and $\Delta_g u_{i+1}(x_0) \geq \Delta_g \psi(x_0)$, we obtain that

$$\Delta_g(u_{i+1} - \psi)(x_0) + (h + K)(u_{i+1} - \psi)(x_0) > 0.$$

But ψ is a supersolution for (14), so we get that

$$\begin{aligned} & 0 < \Delta_g(u_{i+1} - \psi)(x_0) + (h(x_0) + K)(u_{i+1} - \psi)(x_0) \\ & \leq F(\psi(x_0), x_0) - F(u_i(x_0), x_0) - \langle \nabla \psi(x_0), Y(x_0) \rangle^2 \left(\frac{1}{u_i^{q+3}} - \frac{1}{\psi^{q+3}} \right)(x_0) \\ & \quad - \langle \nabla \psi(x_0), Y(x_0) \rangle \left(\frac{d}{u_i^2} - \frac{d}{\psi^2} + \frac{1}{u_i^{q+2}} - \frac{1}{\psi^{q+2}} \right)(x_0). \end{aligned}$$

Thanks to (15) with $A(x) = \langle \nabla \psi(x), Y(x) \rangle$ and to the induction hypothesis which says that $u_i \leq \psi$, we obtain a contradiction. This wraps up the induction argument and the proof of Step 2. \square

Step 3: The sequence $(u_i)_{i \in \mathbb{N}}$ is uniformly bounded in $\mathcal{C}^1(M)$.

Proof of Step 3: Thanks to Step 2, we know that $(u_i)_{i \in \mathbb{N}}$ is an increasing sequence bounded by ψ . Thus there exists $u \in \mathcal{C}^0(M)$ such that $u_i \rightarrow u_0$ in $\mathcal{C}^0(M)$.

Assume by contradiction that exists a subsequence $(u_{\phi(m)})_{m \in \mathbb{N}}$ such that $\|\nabla u_{\phi(m)}\|_{L^\infty} \rightarrow \infty$. Let

$$\mu_m := \frac{1}{\|\nabla u_{\phi(m)}\|_{L^\infty}}$$

and let $(x_m)_m \subset M$ be such that

$$|\nabla u_{\phi(m)}(x_m)| = \|\nabla u_{\phi(m)}\|_{L^\infty}.$$

Consider the domains $\Omega_m = B_0\left(\frac{i_g M}{2\mu_m}\right)$ and the rescaled quantities

$$v_m(x) := u_{\phi(m)}(\exp_{x_m}(\mu_m x)) \quad \text{and} \quad g_m(x) := (\exp_{x_m}^* g)(\mu_m x)$$

in Ω_m . We get

$$\begin{aligned} & \Delta_{g_m} v_m + \mu_m^2 (h(\exp_{x_m}(\mu_m \cdot)) + K) v_m(\exp_{x_m}(\mu_m \cdot)) \\ & + \mu_m^2 F(u_{\phi(m)-1}(\exp_{x_m}(\mu_m \cdot))) + \frac{\langle \nabla v_m, Y(\exp_{x_m}(\mu_m \cdot)) \rangle^2}{u_{\phi(m)-1}^{q+3}(\exp_{x_m}(\mu_m \cdot))} \\ & + \mu_m \langle \nabla v_m, Y(\exp_{x_m}(\mu_m \cdot)) \rangle c(\exp_{x_m}(\mu_m \cdot)) \left[\frac{d(\exp_{x_m}(\mu_m \cdot))}{u_{\phi(m)-1}^2(\exp_{x_m}(\mu_m \cdot))} \right. \\ & \left. + \frac{1}{u_{\phi(m)-1}^{q+2}(\exp_{x_m}(\mu_m \cdot))} \right] = 0 \end{aligned} \quad (24)$$

with $(v_m)_{m \in \mathbb{N}}$ bounded in L^∞ , $\|\nabla v_m\|_{L^\infty} = 1$ and $\varepsilon_0 \leq v_m$. By the Sobolev embedding theorem and standard elliptic regularity, there exists a smooth positive limit v_∞ of $(v_m)_{m \in \mathbb{N}}$, up to a subsequence. Recall that $(u_i)_{i \in \mathbb{N}}$ converges everywhere, so $u_{\phi(m)-1}(\mu_m x) \rightarrow v_\infty(x)$ in M . By taking $m \rightarrow \infty$ in (24),

$$\Delta v_\infty + \frac{(\nabla v_\infty \cdot Y(0))^2}{v_\infty^{q+3}} = 0.$$

Note also that

$$\Delta v_\infty^{-\alpha} \leq \frac{\alpha |\nabla v_\infty|^2}{v_\infty^{\alpha+2}} \left(\frac{|Y(0)|^2}{v_\infty^{q+2}} - (\alpha + 1) \right).$$

For α large enough, $v_\infty^{-\alpha}$ is subharmonic. Using Lemma 5.1 (see Annex), we find that v_∞ is constant, which contradicts the fact that $\|\nabla v_\infty\|_{L^\infty} = 1$. This ends the proof of Step 3. \square

Since $(u_i)_{i \in N}$ is uniformly bounded in \mathcal{C}^1 , we conclude by standard elliptic theory that its limit u is a positive smooth function solving equation (14). This ends the proof of the theorem. \square

The solution constructed in the previous proof is uniquely determined as the pointwise limit of $(u_i)_{i \in N}$, where each u_i is the unique solution of (17). Furthermore, the solution is minimal among all supersolutions (including solutions) of (14) with values between ε_0 and $\sup_M \psi$. These bounds were explicitly used in the inductive argument. By construction, $u \leq \psi$, where ψ is the supersolution fixed at the very beginning. Note that the constant K appearing in (17) depend on $\sup_M \psi$ and ε_0 . We would obtain the same iteration were we to use another supersolution $\tilde{\psi}$ and the same K , given that $\varepsilon_0 < \tilde{\psi} < \sup_M \psi$. Therefore, u is smaller than any supersolution between ε_0 and $\sup_M \psi$.

As an immediate consequence of the minimality discussed above, the solutions we found corresponding to different functions a are ordered. Let $0 < a < \tilde{a}$ be two functions, and assume that the equation associated to \tilde{a} admits a solution \tilde{u} . Then \tilde{u} is a supersolution for (14) corresponding to a , and by the previous proof we find a solution $u \leq \tilde{u}$. Moreover, given that \tilde{u} may be viewed as a supersolution to all (14) with $a \leq \tilde{a}$, we obtain a monotonicity of u in a : for $a_1 \leq a_2 \leq \tilde{a}$, then $u_1 \leq u_2 \leq \tilde{u}$.

Finally, the solution u is stable, as defined in the following lemma.

Lemma 2.1. *The operator L resulting from the linearization of (14) at the minimal solution u admits a real, simple eigenvalue $\lambda_0 \geq 0$ such that*

$$\begin{aligned} L\varphi_0 &= \Delta_g \varphi_0 + \left[h - (q-1)fu^{q-2} + (q+1)au^{-q-2} - bu^{-2} \right. \\ &\quad \left. - (q+3)\frac{\langle \nabla u, Y \rangle^2}{u^{q+4}} - c\langle \nabla u, Y \rangle \left(\frac{2d}{u^3} + \frac{q+2}{u^{q+3}} \right) \right] \varphi_0 \\ &\quad + \langle \nabla \varphi_0, Y \rangle \left[c \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) + \frac{2\langle \nabla u, Y \rangle}{u^{q+3}} \right] \\ &= \lambda_0 \varphi_0, \end{aligned}$$

where φ_0 is the corresponding positive eigenfunction. Furthermore, if $\lambda \in \mathbb{C}$ is any other eigenvalue, then $\operatorname{Re}(\lambda) \geq \lambda_0$.

Proof of Lemma 2.1: Notice that L is nonsymmetric; moreover, one may find a large enough constant K such that

$$\begin{aligned} h - (q-1)fu^{q-2} + (q+1)au^{-q-2} - bu^{-2} - (q+3)\frac{\langle \nabla u, Y \rangle^2}{u^{q+4}} \\ - c\langle \nabla u, Y \rangle \left(\frac{2d}{u^3} + \frac{q+2}{u^{q+3}} \right) + K \geq 0. \end{aligned}$$

According to ([7], Section 6.5, Theorem 1) there exists a real, positive eigenvalue $\lambda_K > 0$ of $L + K$, such that any other complex eigenvalue of $L + K$ has a greater real part. Consequently, the operator L admits a minimal real eigenvalue $\lambda_0 > -K$. We now assume that $\lambda_0 < 0$. Let $u_\delta := u_0 - \delta\varphi_0$, $\delta > 0$. By taking

δ small enough, we may ensure that $\varepsilon_0 < u_\delta$. Then

$$\begin{aligned} \Delta_g u_\delta + h u_\delta - f u_\delta^{q-1} - \frac{a}{u_\delta^{q+1}} + \frac{b}{u_\delta} + \frac{\langle \nabla u_\delta, Y \rangle^2}{u_\delta^{q+3}} + c \langle \nabla u_\delta, Y \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) \\ = -\delta \lambda_0 \varphi_0 + o(\delta). \end{aligned}$$

This implies that $\varepsilon_0 < u_\delta < u$ is a supersolution of (14), which cannot be the case, as discussed above. Thus, $\lambda_0 \geq 0$. \square

3 A priori estimates on solutions of the scalar equation in low dimensions

The \mathcal{C}^1 estimates obtained in this section will play a crucial role in the proof of Theorem 1.1, which is based on a fixed-point argument. This section is devoted to the proof of the following theorem:

Theorem 3.1. *Let (M, g) be a closed Riemannian manifold of dimension $n = 3, 4, 5$. Let $\frac{1}{2} < \eta < 1$ and $0 < \alpha < 1$. Let a, b, c, d, f, h be smooth functions on M , let Y be a smooth vector field on the M .*

For any $0 < \theta < T$, there exists $S_{\theta, T}$ such that any smooth positive solution u of (14) with parameters within

$$\begin{aligned} \mathcal{E}_{\theta, T} := \left\{ (f, a, b, c, d, h, Y), \quad f \geq \theta, \quad a \geq \theta, \right. \\ \left. \|f\|_{\mathcal{C}^{1, \eta}} \leq T, \quad \|a\|_{\mathcal{C}^{0, \alpha}}, \|b\|_{\mathcal{C}^{0, \alpha}}, \|c\|_{\mathcal{C}^{0, \alpha}}, \|d\|_{\mathcal{C}^{0, \alpha}}, \|h\|_{\mathcal{C}^{0, \alpha}}, \|Y\|_{\mathcal{C}^{0, \alpha}} \leq T \right\}, \end{aligned}$$

satisfies $\|u\|_{\mathcal{C}^2} \leq S_{\theta, T}$.

Remark 3.1. *For the sake of clarity, we've taken the bounds on the parameters to be of the form θ and T . They can of course be individually specified.*

We proceed by contradiction. We assume the existence of a sequence $(u_\alpha)_{\alpha \in \mathbb{N}}$ of smooth positive solutions of equations (EL_α)

$$\begin{aligned} \Delta_g u_\alpha + h_\alpha u_\alpha - f_\alpha u_\alpha^{q-1} - \frac{a_\alpha}{u_\alpha^{q+1}} + \frac{b_\alpha}{u_\alpha} + \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} \\ + c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left[\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right] = 0 \end{aligned} \quad (25)$$

with parameters $(a_\alpha, b_\alpha, c_\alpha, d_\alpha, f_\alpha, h_\alpha, Y_\alpha)$ in $\mathcal{E}_{\theta, T}$ such that

$$\|u_\alpha\|_{\mathcal{C}^1(M)} \rightarrow \infty, \quad \text{as } \alpha \rightarrow \infty. \quad (26)$$

A *concentration point* is the limit in M of any sequence $(x_\alpha)_\alpha$ where (26) holds. Note that a \mathcal{C}^1 -bound on $(u_\alpha)_\alpha$ automatically gives a \mathcal{C}^2 -bound by elliptic theory. Note also that, up to a subsequence, all parameters converge in $\mathcal{C}^0(M)$.

Let $m_\alpha = \min_{x \in M} u_\alpha(x) = u_\alpha(x_\alpha) > 0$. Since $\nabla u_\alpha(x_\alpha) = 0$ and since $\Delta_g u_\alpha(x_\alpha) \leq 0$, we have thanks to (25) that

$$h_\alpha(x_\alpha)m_\alpha - f_\alpha(x_\alpha)m_\alpha^{q-1} - \frac{a_\alpha(x_\alpha)}{m_\alpha^{q+1}} + \frac{b_\alpha(x_\alpha)}{m_\alpha} \geq 0.$$

Thanks to the definition of $\mathcal{E}_{\theta,T}$, it follows that

$$\frac{\theta}{m_\alpha^{q+1}} \leq T(m_\alpha + \frac{1}{m_\alpha}).$$

Then there exists $\varepsilon = \varepsilon(\theta, T, n) > 0$ such that $m_\alpha \geq \varepsilon$, meaning that

$$u_\alpha > \varepsilon > 0 \quad \text{for all } x \in M \text{ and all } \alpha. \quad (27)$$

The scheme of the proof follows the work of Druet and Hebey [6], with the added difficulty consisting in the gradient terms in (25).

3.1 Concentration points

The first step in finding *a priori* estimates for $(u_\alpha)_\alpha$ is to find all potential concentration points.

Lemma 3.1. *There exists $N_\alpha \in \mathbb{N}^*$ and $\mathcal{S}_\alpha := (x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$ a set of critical points of $(u_\alpha)_\alpha$ such that*

$$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \geq 1 \quad (28)$$

for all $i, j \in \{1, \dots, N_\alpha\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq 1 \quad (29)$$

for all critical points of u_α and such that there exists $C_1 > 0$ such that

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right) \leq C_1 \quad (30)$$

for all $x \in M$ and all $\alpha \in \mathbb{N}$.

Remark 3.2. *Estimate (30) implies that any concentration point of $(u_\alpha)_{\alpha \in \mathbb{N}}$ calls for the existence of a sequence $(x_\alpha)_\alpha \subset (\mathcal{S}_\alpha)_\alpha$ converging to it. We shall focus our analysis in the neighbourhood of \mathcal{S}_α as $\alpha \rightarrow \infty$ to find concentration points.*

Proof of Lemma 3.1: In order to choose $(\mathcal{S}_\alpha)_\alpha$, we make use of a simple result describing any sufficiently regular function on a compact manifold.

Lemma 3.2. *Let u be a positive real-valued C^2 function defined in a compact manifold M . Then there exists $N \in \mathbb{N}^*$ and (x_1, x_2, \dots, x_N) a set of critical points of u such that*

$$d_g(x_i, x_j)^{\frac{n-2}{2}} u(x_i) \geq 1$$

for all $i, j \in \{1, \dots, N\}$, $i \neq j$, and

$$\left(\min_{i=1, \dots, N} d_g(x_i, x) \right)^{\frac{n-2}{2}} u(x) \leq 1$$

for all critical points x of u .

The lemma and its proof may be found in Druet and Hebey's paper [6]. Applying this lemma to (u_α) gives N_α and \mathcal{S}_α as in Lemma 3.1 such that (28) and (29) hold. We need to prove (30). Proceeding by contradiction, assume that there exists a sequence $(x_\alpha)_\alpha$ such that

$$\left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x_\alpha) \right)^{\frac{n-2}{2}} \left(u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \right) \rightarrow \infty \quad (31)$$

as $\alpha \rightarrow \infty$, where

$$\begin{aligned} & \left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x_\alpha) \right)^{\frac{n-2}{2}} \left(u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}} \right) \\ &= \sup_{x \in M} \left(\min_{i=1, \dots, N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right). \end{aligned} \quad (32)$$

Denote

$$\nu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right|^{\frac{n-2}{2}}$$

and see that (31) translates to

$$\frac{d_g(x_\alpha, \mathcal{S}_\alpha)}{\nu_\alpha} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty. \quad (33)$$

Also, since M is compact,

$$\nu_\alpha \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty. \quad (34)$$

Consider the rescaled quantities

$$v_\alpha(x) := \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad \text{and} \quad g_\alpha(x) := (\exp_{x_\alpha}^* g)(\nu_\alpha x)$$

defined in $\Omega_\alpha := B_0\left(\frac{\delta}{\nu_\alpha}\right)$, with $0 < \delta < \frac{1}{2}i_g(M)$. We emphasize that, for any $R > 0$,

$$\limsup_{\alpha \rightarrow \infty} \sup_{B_0(R)} \left(v_\alpha + \left| \frac{\nabla v_\alpha}{v_\alpha} \right|^{\frac{n-2}{2}} \right) = 1 \quad (35)$$

thanks to (32) and (33). However, unlike $(u_\alpha)_\alpha$, the sequence $(v_\alpha)_\alpha$ is not necessarily bounded from below by a small positive constant ε . Instead, we deduce from (35) that

$$|\nabla \ln v_\alpha| \leq 1 + o(1) \quad \text{in } B_0(R)$$

for all $R > 0$ so that

$$v_\alpha(0)e^{-2|x|} \leq v_\alpha(x) \leq v_\alpha(0)e^{2|x|} \quad \text{in } B_0(R) \quad (36)$$

for all $R > 0$ as soon as α is large enough. We rewrite (25) in Ω_α as

$$\begin{aligned} \Delta_{g_\alpha} v_\alpha &= f_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) v_\alpha^{q-1} + \nu_\alpha^{\frac{n+2}{2}} \frac{a_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^{q+1}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \nu_\alpha^2 h_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) v_\alpha - \nu_\alpha^{\frac{n+2}{2}} \frac{b_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \frac{\langle \nabla v_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle^2}{v_\alpha} \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \\ &\quad - \nu_\alpha c_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \langle \nabla v_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle \left[\frac{d_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^2(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right. \\ &\quad \left. + \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right] \end{aligned} \quad (37)$$

Note that the metrics $g_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 as $\alpha \rightarrow \infty$. Because of (27) and (35), the right hand side is bounded, so by standard elliptic theory there exists up to a subsequence a \mathcal{C}^1 limit $U := \lim_{\alpha \rightarrow \infty} v_\alpha$ and $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$.

First case: If $u_\alpha(x_\alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$, then we can pass to the limit in equation (37) to get

$$\Delta U = f(x_0)U^{q-1}$$

in \mathbb{R}^n . The exact form of these solutions is found in a paper by Caffarelli, Gidas and Spruck [2]:

$$U(x) = \left(1 + \frac{f(x_0)|x - y_0|^2}{n(n-2)} \right)^{1 - \frac{n}{2}}$$

with $y_0 \in \mathbb{R}^n$ the unique maximum point of the function. There exist therefore $(y_\alpha)_\alpha$ local maxima of $(u_\alpha)_\alpha$ such that

$$d_g(x_\alpha, y_\alpha) = O(\nu_\alpha) \quad (38)$$

and

$$\nu_\alpha^{\frac{n-2}{2}} u_\alpha(y_\alpha) \rightarrow 1 \quad \text{as } \alpha \rightarrow \infty. \quad (39)$$

Since $(y_\alpha)_\alpha$ are critical points, (29) implies that

$$d_g(\mathcal{S}_\alpha, y_\alpha)^{\frac{n-2}{2}} u_\alpha(y_\alpha) \leq 1$$

for all $\alpha \in \mathbb{N}$, so by (39), $d_g(\mathcal{S}_\alpha, y_\alpha) = O(\nu_\alpha)$; together with (38), this leads to $d_g(\mathcal{S}_\alpha, x_\alpha) = O(\nu_\alpha)$, which contradicts (33).

Second case: Assume that, up to a subsequence, $u_\alpha(x_\alpha) \rightarrow l < \infty$. From (34) we deduce that $|\nabla u_\alpha(x_\alpha)| \rightarrow \infty$ and that $v_\alpha(0) \rightarrow 0$ as $\alpha \rightarrow \infty$. Let us set

$$w_\alpha(x) := \frac{v_\alpha(x)}{v_\alpha(0)}.$$

These functions are bounded from below, since by (27),

$$w_\alpha(x) = \frac{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(x_\alpha)} \geq \frac{\varepsilon}{l} + o(1) \quad (40)$$

in $B_0(R)$ for all $R > 0$. Moreover, (36) implies that

$$w_\alpha(x) \leq e^{2|x|}$$

in $B_0(R)$ for α large. Multiply (37) by $v_\alpha(0)^{-1}$ to get

$$\begin{aligned} \Delta_g w_\alpha = & f_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) w_\alpha^{q-1} v_\alpha^{q-2}(0) + \nu_\alpha^2 \frac{a(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^{q+1}(\exp_{x_\alpha}(\nu_\alpha \cdot)) u_\alpha(x_\alpha)} \\ & - \nu_\alpha^2 h_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) w_\alpha - \nu_\alpha^2 \frac{b_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) u_\alpha(x_\alpha)} \\ & - \frac{\langle \nabla w_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle^2}{w_\alpha^{q+2}} \frac{1}{u_\alpha(x_\alpha)^{q+2}} \\ & - \nu_\alpha c_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \langle \nabla w_\alpha, Y_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot)) \rangle \left[\frac{d_\alpha(\exp_{x_\alpha}(\nu_\alpha \cdot))}{u_\alpha^2(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right. \\ & \left. + \frac{1}{u_\alpha^{q+2}(\exp_{x_\alpha}(\nu_\alpha \cdot))} \right]. \end{aligned}$$

By standard elliptic theory, we find that there exists $w := \lim_{\alpha \rightarrow \infty} w_\alpha$ in \mathcal{C}^1 solving:

$$\Delta w = - \frac{1}{l^{q+2}} \frac{\langle \nabla w, Y(x_0) \rangle^2}{w^{q+3}}$$

in \mathbb{R}^n . Note that, since $w \geq \frac{\varepsilon}{l}$ by (40),

$$\Delta w^{-\alpha} \leq \alpha \frac{|\nabla w|^2}{w^{\alpha+2}} \left[\frac{|Y(x_0)|^2}{\varepsilon^{q+2}} - (\alpha + 1) \right],$$

so $w^{-\alpha}$ is subharmonic for α large. By applying Lemma 5.1 (see the Annex), we deduce that w is constant, which in turn implies that $U = 0$, and so $\nabla U = 0$. This implies that

$$\lim_{\alpha \rightarrow \infty} \nu_\alpha \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} = 0,$$

as it contradicts the choice of ν_α above, which may be rewritten as

$$\nu_\alpha^{\frac{n-2}{2}} \left(u_\alpha(x_\alpha) + \left| \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right| \right) = 1,$$

since we are in the case where $\nu_\alpha^{\frac{n-2}{2}} u_\alpha(x_\alpha) = v_\alpha(0) \rightarrow 0$ as $\alpha \rightarrow 0$. \square

The following is a Harnack-type inequality. It holds whenever an estimate like (30) is verified, that is when there exists a constant C_2 and a sequence $(x_\alpha, \rho_\alpha)_\alpha$ such that

$$d_g(x_\alpha, x)^{\frac{n-2}{2}} \left[u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right] \leq C_2, \quad \forall x \in B_{x_\alpha}(7\rho_\alpha). \quad (41)$$

Lemma 3.3. *Let $(x_\alpha, \rho_\alpha)_\alpha$ be a sequence such that (41) holds. Then there exists a constant $C_3 > 1$ such that for any sequence $0 < s_\alpha \leq \rho_\alpha$, we get*

$$s_\alpha \|\nabla u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C_3 \sup_{\Omega_\alpha} u_\alpha \leq C_3^2 \inf_{\Omega_\alpha} u_\alpha,$$

where $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$.

Proof of Lemma 3.3: Estimate (41) implies that

$$\left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right| \leq C_2 d_g(x_\alpha, x)^{-1} \quad (42)$$

in Ω_α , and therefore

$$s_\alpha |\nabla \ln u_\alpha(x)| \leq 6C_2 \quad (43)$$

in Ω_α . Taking $C_3 \geq 6C_2$, we get the first inequality from (42). Then, from (43) and from the fact that the domain is an annulus $\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)$, we estimate that

$$\sup_{\Omega_\alpha} \ln u_\alpha - \inf_{\Omega_\alpha} \ln u_\alpha \leq l_\alpha(\Omega_\alpha) \|\nabla \ln u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq 42C_2,$$

where $l_\alpha(\Omega_\alpha)$ is the infimum of the length of a curve in Ω_α drawn between a maximum and a minimum of u_α . Equivalently

$$\sup_{\Omega_\alpha} u_\alpha \leq e^{42C_2} \inf_{\Omega_\alpha} u_\alpha,$$

so it suffices to take $C_3 = e^{42C_2}$. \square

3.2 Local blow-up analysis

In order to show that u_α is bounded in \mathcal{C}^1 , we define a *blow-up sequence* $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ as follows: let $(x_\alpha)_\alpha$ be critical points of $(u_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ positive numbers such that they verify the following three conditions:

$$0 < \rho_\alpha < \frac{1}{7} i_g(M), \quad (44)$$

$$\rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha \rightarrow \infty, \quad (45)$$

and

$$d_g(x_\alpha, x)^{\frac{n-2}{2}} \left[u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right] \leq C_2 \quad \forall x \in B_{x_\alpha}(7\rho_\alpha), \quad (46)$$

where C_2 is a constant. In the rest of the section, we denote

$$\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha).$$

Remark 3.3. *The limit as $\alpha \rightarrow \infty$ of a blow-up sequence is a concentration point, as seen from (45).*

Remark 3.4. *Any sequence $(x_\alpha)_\alpha \subset (S_\alpha)_\alpha$ qualifies as a blow-up sequence as soon as (45) is verified. In this case, $(\rho_\alpha)_\alpha$ can be chosen as*

$$\rho_\alpha := \min \left(\frac{1}{7} i_g(M), \frac{1}{2} \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}) \right)$$

and $C_2 = C_1$.

Given any blow-up sequence, the following proposition gathers the central results of our local analysis for the reader's convenience: namely, it states the exact asymptotic profile of $(u_\alpha)_\alpha$ at distance $(\rho_\alpha)_\alpha$ of $(x_\alpha)_\alpha$ and it gives sharp pointwise asymptotic estimates on balls of radius ρ_α . This is the result we shall point to whenever we want to describe the local asymptotic behaviour of $(u_\alpha)_\alpha$ around a concentration point corresponding to local maximum points x_α . Note that in this case $\nabla u_\alpha(x_\alpha) = 0$.

Proposition 3.1. *Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then there exists $C_4 > 0$ such that*

$$u_\alpha(x) + d_g(x_\alpha, x) |\nabla u_\alpha(x)| \leq C_4 \mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, x)^{2-n} \quad (47)$$

for all $x \in B_{x_\alpha}(6\rho_\alpha) \setminus \{x_\alpha\}$. Moreover, we see that up to a subsequence, the asymptotic profile of $(u_\alpha)_\alpha$ is

$$u_\alpha(x_\alpha) \rho_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(\rho_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x) \quad (48)$$

in $C_{loc}^2(B_0(5) \setminus \{0\})$, where H is some harmonic function in $B_0(5)$ satisfying $H(0) = 0$. Here $R_0^2 = \frac{n(n-2)}{f(x_0)}$ where $x_0 = \lim_{\alpha \rightarrow \infty} x_\alpha$.

The proof of this proposition is the subject of this section. It will follow from Lemma 3.6 and Lemma 3.7 below. We first describe the asymptotic profile at distance $(\mu_\alpha)_\alpha$ of $(x_\alpha)_\alpha$ as $\alpha \rightarrow \infty$ of any blow-up sequence.

Lemma 3.4. *Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. We have*

$$\mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x)) \rightarrow \left(1 + \frac{f(x_0)|x|^2}{n(n-2)} \right)^{1-\frac{n}{2}} \quad (49)$$

in $\mathcal{C}_{loc}^1(\mathbb{R}^n)$ as $\alpha \rightarrow \infty$, with $\mu_\alpha^{1-\frac{n}{2}} := u_\alpha(x_\alpha)$ and $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$, up to a subsequence.

Proof of Lemma 3.4: The proof involves similar arguments to the ones used for Lemma 3.1. Let $y_\alpha \in B_{x_\alpha}(6\rho_\alpha)$ be such that

$$u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}} = \sup_{B_{x_\alpha}(6\rho_\alpha)} \left(u_\alpha(x) + \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right|^{\frac{n-2}{2}} \right)$$

and let

$$\nu_\alpha^{1-\frac{n}{2}} := u_\alpha(y_\alpha) + \left| \frac{\nabla u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \right|^{\frac{n-2}{2}}.$$

Conditions (45) and (46) imply that

$$\frac{\rho_\alpha}{\nu_\alpha} \rightarrow \infty$$

and

$$d_g(x_\alpha, y_\alpha) \leq C_2^{\frac{2}{n-2}} \nu_\alpha.$$

It follows that the coordinates of y_α in the exponential chart around x_α defined as $\tilde{y}_\alpha := \nu^{-1} \exp_{x_\alpha}^{-1}(y_\alpha)$ are bounded by $C_2^{\frac{2}{n-2}}$. Up to a subsequence, we may choose a finite limit $\tilde{y}_0 := \lim_{\alpha \rightarrow \infty} \tilde{y}_\alpha$. We denote

$$v_\alpha(x) = \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\nu_\alpha x)) \quad \text{and} \quad g_\alpha(x) = (\exp_{x_\alpha}^* g)(\exp_{x_\alpha}(\nu_\alpha x))$$

for $x \in \Omega_\alpha := B_0\left(\frac{\rho_\alpha}{\nu_\alpha}\right)$. As before, $g_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 , $v_\alpha = O(1)$, and $\left| \frac{\nabla v_\alpha}{v_\alpha} \right| = O(1)$. By applying the same analysis as in the proof of Lemma 3.1, we get that, up to passing to a subsequence, there exists $U := \lim_{\alpha \rightarrow \infty} v_\alpha$ in $\mathcal{C}_{loc}^1(\mathbb{R}^n)$, with $x_0 := \lim_{\alpha \rightarrow \infty} x_\alpha$,

$$\Delta U = f(x_0)U^{q-1}.$$

where

$$U(x) = \left(1 + \frac{f(x_0)|x - \tilde{y}_0|^2}{n(n-2)} \right)^{1-\frac{n}{2}}.$$

We know that x_α are local maxima for u_α , so both 0 and \tilde{y}_0 are maxima of U . However, since U admits a unique maximum, we conclude that $\tilde{y}_0 = 0$. \square

We recall the aim of this section is to show that concentration points do not exist for the system (5). So far, we have obtained a pointwise estimate (30) that holds everywhere on M and an asymptotic profile in the neighbourhood of x_α , a blow-up sequence; we aim to also find estimates around x_α . We defined $(\rho_\alpha)_\alpha$ as the quantity describing the sphere of dominance of the blow-up sequence $(x_\alpha)_\alpha$. However, the influence of other blow-up sequences may be felt earlier. Let $\varphi_\alpha : (0, \rho_\alpha) \rightarrow \mathbb{R}^+$ be the average of u_α defined as

$$\varphi_\alpha(r) := \frac{1}{|\partial B_{x_\alpha}(r)|_g} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g.$$

It follows from Lemma 3.4 that

$$(\mu_\alpha r)^{\frac{n-2}{2}} \varphi_\alpha(\mu_\alpha r) \rightarrow r^{\frac{n-2}{2}} \left(1 + \frac{f(x_0)r^2}{n(n-2)}\right)^{1-\frac{n}{2}} \quad (50)$$

in $C_{loc}^1([0, +\infty))$. We define

$$r_\alpha := \sup_{r \in (2R_0\mu_\alpha, \rho_\alpha)} \left\{ s^{\frac{n-2}{2}} \varphi_\alpha(s) \text{ is non-increasing in } (2R_0\mu_\alpha, r) \right\} \quad (51)$$

with

$$R_0^2 := \frac{n(n-2)}{f(x_0)}.$$

Note that

$$\text{if } r_\alpha < \rho_\alpha, \quad \text{then } \left(r^{\frac{n-2}{2}} \varphi_\alpha(r) \right)'(r_\alpha) = 0.$$

Since $r^{\frac{n-2}{2}} U$ is non-increasing in $[2R_0, \infty)$, then (50) implies

$$\frac{r_\alpha}{\mu_\alpha} \rightarrow \infty. \quad (52)$$

so $\mu_\alpha = o(r_\alpha)$.

Remark 3.5. *Considering that the asymptotic profile of blow-up sequences $(x_\alpha)_\alpha$, which is a bump function with a unique maximum, the quantity r_α is an indicator of the beginning of the influence of neighbouring blow-up sequences within the sphere of dominance, as the average of u_α is no longer decreasing.*

Let

$$\eta_\alpha := \sup_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}r_\alpha)} u_\alpha. \quad (53)$$

Note that, by Lemma 3.3,

$$\frac{1}{C_3} \sup_{B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)} u_\alpha \leq \varphi_\alpha(s_\alpha) \leq C_3 \inf_{B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)} u_\alpha$$

for $0 < s_\alpha \leq r_\alpha$ and all α . By (50), we obtain the estimate

$$\lim_{R \rightarrow \infty} \limsup_{\alpha \rightarrow \infty} \sup_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(R\mu_\alpha)} d_g(x_\alpha, x)^{\frac{n-2}{2}} u_\alpha = 0. \quad (54)$$

Thus,

$$r_\alpha^2 \eta_\alpha^{q-2} \rightarrow 0 \quad (55)$$

as $\alpha \rightarrow \infty$. It is important to note that this implies that

$$r_\alpha \rightarrow 0 \quad (56)$$

as $\alpha \rightarrow \infty$ since $u_\alpha \geq \varepsilon$ by (27). We now prove a pointwise asymptotic estimate for u_α in $B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$.

Lemma 3.5. *Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then, for any $0 < \varepsilon < \frac{1}{2}$, there exists $C_\varepsilon > 0$ such that*

$$u_\alpha(x) \leq C_\varepsilon \left(\mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)} d_g(x_\alpha, x)^{(n-2)(1-\varepsilon)} + \eta_\alpha \left(\frac{r_\alpha}{d_g(x_\alpha, x)} \right)^{(n-2)\varepsilon} \right)$$

for all $x \in B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$.

Proof of Lemma 3.5: Let G be a Green function for the Laplace operator Δ_g on M with $G > 0$. Recall the following estimates, that can be found in Aubin [1]:

$$\begin{aligned} \left| d_g(x_\alpha, y)^{n-2} G(x, y) - \frac{1}{(n-2)\omega_{n-1}} \right| &\leq \tau(d_g(x, y)) \\ \left| d_g(x_\alpha, y)^{n-1} |\nabla G(x, y)| - \frac{1}{\omega_{n-1}} \right| &\leq \tau(d_g(x, y)) \end{aligned} \quad (57)$$

where $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying $\tau(0) = 0$. For a fixed ε , let

$$\Phi_\alpha^\varepsilon(x) := \mu_\alpha^{\frac{n-2}{2}(1-2\varepsilon)} G(x_\alpha, x)^{1-\varepsilon} + \eta_\alpha r_\alpha^{(n-2)\varepsilon} G(x_\alpha, x)^\varepsilon \quad (58)$$

and let $y_\alpha \in \overline{B_{x_\alpha}(6r_\alpha)} \setminus \{x_\alpha\}$ be such that

$$\sup_{B_{x_\alpha}(6r_\alpha)} \frac{u_\alpha}{\Phi_\alpha^\varepsilon} = \frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)}, \quad (59)$$

We continue by studying the following two cases, separately.

First case: Assume that the relative size of $d_g(x_\alpha, y_\alpha)$ with respect to μ_α is

$$R := \lim_{\alpha \rightarrow \infty} \frac{d_g(x_\alpha, y_\alpha)}{\mu_\alpha} \quad \text{with } R \in [0, \infty). \quad (60)$$

Thanks to Lemma 3.4,

$$\mu_\alpha^{\frac{n-2}{2}} u_\alpha(y_\alpha) = \left(1 + \frac{R^2}{R_0^2} \right)^{1-\frac{n}{2}} + o(1),$$

so that, whenever $R \in [0, \infty)$, using (52), (57) and (60), it is easily shown that

$$\frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} \rightarrow ((n-2)\omega_{n-1})^{1-\varepsilon} R^{(n-2)(1-\varepsilon)} \left(1 + \frac{R^2}{R_0^2} \right)^{1-\frac{n}{2}}$$

as $\alpha \rightarrow \infty$.

Second case: It remains to study the case

$$\lim_{\alpha \rightarrow \infty} \frac{d_g(x_\alpha, y_\alpha)}{\mu_\alpha} \rightarrow \infty \quad \text{as } \alpha \rightarrow \infty.$$

If $(y_\alpha)_\alpha$ sits on the outer boundary $\partial B_{x_\alpha}(6r_\alpha)$, then by (56), (57), and (58),

$$\frac{u_\alpha(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} \leq (6^{n-2}(n-2)\omega_{n-1})^\varepsilon + o(1).$$

Otherwise, if up to a subsequence $y_\alpha \in B_{x_\alpha}(6r_\alpha)$, then

$$\frac{\Delta_g u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \geq \frac{\Delta_g \Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)}$$

as a consequence of the fact that y_α is the maximum of $\frac{u_\alpha}{\Phi_\alpha^\varepsilon}$. On the other hand, taking note of the sign of the dominant gradient term in equations (25), we see that

$$\begin{aligned} \Delta_g u_\alpha &= -h_\alpha u_\alpha + f_\alpha u_\alpha^{q-1} + \frac{a_\alpha}{u_\alpha^{q+1}} - \frac{b_\alpha}{u_\alpha} - \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} \\ &\quad - c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left[\frac{d_g}{u_\alpha} + \frac{1}{u_\alpha^{q+2}} \right] \\ &\leq C u_\alpha^{q-1}, \end{aligned} \tag{61}$$

where C is a constant depending on θ and T . Here we used (27). Finally, thanks to (54),

$$d_g(x_\alpha, y_\alpha)^2 \frac{\Delta_g u_\alpha(y_\alpha)}{u_\alpha(y_\alpha)} \leq C d_g(x_\alpha, y_\alpha)^2 u_\alpha^{q-2}(y_\alpha) \rightarrow 0.$$

To conclude, (56) and (57) imply that

$$d_g(x_\alpha, y_\alpha)^2 \frac{\Delta_g \Phi_\alpha^\varepsilon(y_\alpha)}{\Phi_\alpha^\varepsilon(y_\alpha)} = \varepsilon(1 - \varepsilon)(n - 2)^2 + o(1).$$

We deduce that $u_\alpha(y_\alpha) = O(\Phi_\alpha^\varepsilon(y_\alpha))$. The study of the previous two cases ends the proof of the lemma. \square

The following lemma improves the estimate we've just obtained and gives a very important bound on the size of r_α .

Lemma 3.6. *Let $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then there exists $C_4 > 0$ such that*

$$u_\alpha(x) + d_g(x_\alpha, x) |\nabla u_\alpha(x)| \leq C_4 \mu_\alpha^{\frac{n-2}{2}} (d_g(x_\alpha, x) + \mu_\alpha)^{2-n} \tag{62}$$

for all $x \in B_{x_\alpha}(6r_\alpha) \setminus \{x_\alpha\}$. Moreover, $r_\alpha^2 = O(\mu_\alpha)$.

Proof of Lemma 3.6: It suffices to prove the estimate for u_α ; the rest follows as an immediate consequence of Lemma 3.3. We start by showing that for any sequence $z_\alpha \in \overline{B_{x_\alpha}(6r_\alpha)} \setminus \{x_\alpha\}$, there holds

$$u_\alpha(z_\alpha) = O\left(\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, z_\alpha)^{2-n} + \eta_\alpha\right). \tag{63}$$

First, if $d_g(x_\alpha, z_\alpha) = O(\mu_\alpha)$, it falls within the range described in Lemma 3.4. On the other hand, when $r_\alpha = O(d_g(x_\alpha, z_\alpha))$, we use Lemma 3.3 together with (53). It remains to consider the intermediary case:

$$\frac{d_g(x_\alpha, z_\alpha)}{\mu_\alpha} \rightarrow \infty \quad \text{and} \quad \frac{d_g(x_\alpha, z_\alpha)}{r_\alpha} \rightarrow 0 \quad \text{at } \alpha \rightarrow \infty.$$

According to the Green representation formula,

$$u_\alpha(z_\alpha) = O\left(\int_{B_{x_\alpha}(6r_\alpha)} d_g(z_\alpha, x)^{2-n} \Delta_g u_\alpha(x) dv_g\right) + O(\eta_\alpha),$$

where the second term corresponds to the boundary element. Recall that

$$\Delta_g u_\alpha \leq C u_\alpha^{q-1}$$

because of the sign of the dominant gradient term, see (61). Using (27), (55) and Lemma 3.5, we can write that

$$\begin{aligned} & \int_{B_{x_\alpha}(6r_\alpha)} d_g(z_\alpha, x)^{2-n} u_\alpha^{q-1}(x) dv_g \\ &= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) \\ &+ O\left(\mu_\alpha^{\frac{n+2}{2}(1-2\varepsilon)} \int_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\mu_\alpha)} d_g(z_\alpha, x)^{2-n} d_g(x_\alpha, x)^{-(n+2)(1-\varepsilon)} dv_g\right) \\ &+ O\left(\eta_\alpha^{q-1} r_\alpha^{(n+2)\varepsilon} \int_{B_{x_\alpha}(6r_\alpha) \setminus B_{x_\alpha}(\mu_\alpha)} d_g(z_\alpha, x)^{2-n} d_g(x_\alpha, x)^{-(n+2)\varepsilon} dv_g\right) \\ &= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) + O(\eta_\alpha^{q-1} r_\alpha^2) \\ &= O\left(\mu_\alpha^{\frac{n}{2}-1} d_g(x_\alpha, z_\alpha)^{2-n}\right) + O(\eta_\alpha). \end{aligned}$$

In order to get estimate (62), it suffices to show that

$$\eta_\alpha = O\left(\mu_\alpha^{\frac{n-2}{2}} r_\alpha^{2-n}\right). \quad (64)$$

For any fixed $0 < \delta < 1$, taking α large enough, then the monotonicity of $r^{\frac{n-2}{2}} \varphi_\alpha(r)$ expressed in the definition of r_α , see (51), and the fact that $\mu_\alpha = o(r_\alpha)$, see (52), imply that

$$r_\alpha^{\frac{n-2}{2}} \varphi_\alpha(r_\alpha) \leq (\delta r_\alpha)^{\frac{n-2}{2}} \varphi_\alpha(\delta r_\alpha) \quad \text{for all } 0 < \delta < 1,$$

so by Lemma 3.3,

$$\frac{1}{C_3} \eta_\alpha \leq \delta^{\frac{n-2}{2}} \sup_{\partial B_{x_\alpha}(\delta r_\alpha)} u_\alpha.$$

According to estimate (63), this leads to

$$\eta_\alpha \leq C \left(\mu_\alpha^{\frac{n-2}{2}} \delta^{2-n} r_\alpha^{2-n} + \eta_\alpha \right) \delta^{\frac{n-2}{2}},$$

where C is independent of δ and α . Choosing δ small enough leads to (64).

Estimates (27) and (64) imply that

$$r_\alpha^2 = O(\mu_\alpha). \quad (65)$$

This ends the proof of the lemma. \square

Given a blow-up sequence $(x_\alpha)_\alpha$ with $(\rho_\alpha)_\alpha$, the following lemma gives the exact asymptotic profile of $(u_\alpha)_\alpha$ at distance $(\rho_\alpha)_\alpha$ of $(x_\alpha)_\alpha$.

Lemma 3.7. *Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be a blow-up sequence. Then we have that $r_\alpha = \rho_\alpha$, where r_α is as in (51). Up to a subsequence, we have*

$$u_\alpha(x_\alpha)\rho_\alpha^{n-2}u_\alpha(\exp_{x_\alpha}(\rho_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x) \quad (66)$$

in $C_{loc}^2(B_0(5)\setminus\{0\})$, where H is some harmonic function in $B_0(5)$ satisfying $H(0) = 0$.

Proof of Lemma 3.7: First, we prove that, up to a subsequence,

$$u_\alpha(x_\alpha)r_\alpha^{n-2}u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rightarrow \frac{R_0^{n-2}}{|x|^{n-2}} + H(x). \quad (67)$$

Let us define the following rescaled quantities:

$$\hat{u}_\alpha(x) = \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \quad \text{and} \quad \hat{g}_\alpha(x) = \exp_{x_\alpha}^* g(\exp_{x_\alpha}(r_\alpha x)).$$

Then

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha = \hat{F}_\alpha,$$

in $B_0(\delta r_\alpha^{-1})$ for some $\delta > 0$ small enough, with

$$\begin{aligned} \hat{F}_\alpha = & -\mu_\alpha^{1-\frac{n}{2}} r_\alpha^n h_\alpha(\exp_{x_\alpha}(r_\alpha x)) u_\alpha(\exp_{x_\alpha}(r_\alpha x)) \\ & + \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n f_\alpha(\exp_{x_\alpha}(r_\alpha x)) u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) \\ & + \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{a_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha^{q+1}(\exp_{x_\alpha}(r_\alpha x))} - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{b_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))} \\ & - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n \frac{\langle \nabla u_\alpha(\exp_{x_\alpha}(r_\alpha x)), Y_\alpha(\exp_{x_\alpha}(r_\alpha x)) \rangle^2}{u_\alpha^{q+3}(\exp_{x_\alpha}(r_\alpha x))} \\ & - \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n c_\alpha(\exp_{x_\alpha}(r_\alpha x)) \langle \nabla u_\alpha, Y_\alpha \rangle(\exp_{x_\alpha}(r_\alpha x)) \left[\frac{d_\alpha(\exp_{x_\alpha}(r_\alpha x))}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))^2} \right. \\ & \left. + \frac{1}{u_\alpha(\exp_{x_\alpha}(r_\alpha x))^{q+2}} \right] \end{aligned} \quad (68)$$

Thanks to Lemma 3.6, we know that $|\hat{u}_\alpha| \leq C_K$ and $|\hat{F}_\alpha| = O(1)$ on any compact $K \subset B_0(5)\setminus\{0\}$. By standard elliptic theory, up to a subsequence,

$$\hat{u}_\alpha \rightarrow \hat{U} \quad \text{in } C_{loc}^1(B_0(5)\setminus\{0\})$$

with

$$\Delta_\xi \hat{U} = 0 \quad \text{in } B_0(5)\setminus\{0\}.$$

Separate \hat{U} into the sum of a regular harmonic function and a singular part

$$\hat{U} = \frac{\lambda}{|x|^{n-2}} + H(x),$$

where $\lambda \geq 0$.

To get (67), it remains to show that $\lambda = R_0^{n-2}$. For any $\delta > 0$:

$$\int_{B_0(\delta)} \hat{F}_\alpha dv_{\hat{g}_\alpha} = - \int_{\partial B_0(\delta)} \partial_\nu \hat{u}_\alpha d\sigma_{\hat{g}_\alpha}. \quad (69)$$

Using the equation (68), we estimate the left hand side of (69). In particular,

$$\begin{aligned} & \int_{B_0(\delta)} f_\alpha(\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) dx \\ &= \int_{B_0(\delta \frac{r_\alpha}{\mu_\alpha})} f_\alpha(\exp_{x_\alpha}(\mu_\alpha z)) \mu_\alpha^{\frac{n+2}{2}} u_\alpha^{q-1}(\exp_{x_\alpha}(\mu_\alpha z)) dz \end{aligned}$$

where $z = \frac{r_\alpha}{\mu_\alpha} x$, and by Lemma 3.4 and Lemma 3.6,

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} \int_{B_0(\delta)} f_\alpha(\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n u_\alpha^{q-1}(\exp_{x_\alpha}(r_\alpha x)) dx \\ &= f(x_0) \int_{R^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx. \end{aligned}$$

The gradient terms are controlled with the estimate (30), and together with (27), we obtain that the dominant gradient term of (68) verifies

$$\begin{aligned} \int_{B_0(\delta)} \frac{|\nabla u_\alpha|^2}{u_\alpha^{q+3}}(\exp_{x_\alpha}(r_\alpha x)) \mu_\alpha^{1-\frac{n}{2}} r_\alpha^n dx &\leq C \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} \int_{B_0(\delta)} |x|^{-2} dx \\ &\leq C \omega_{n-1} \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} \delta^{n-2}. \end{aligned} \tag{70}$$

As $\mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} = O(1)$, the integral does not vanish as $\alpha \rightarrow \infty$; its size depends on δ . The remaining terms in (68) are negligible. Thus

$$\int_{B_0(\delta)} \hat{F}_\alpha dv_{\hat{g}_\alpha} = f(x_0) \int_{R^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx + o(1) + O(\delta^{n-2})$$

for any $\delta > 0$. It follows that

$$f(x_0) \int_{R^n} \left(1 + \frac{|x|^2}{R_0^2}\right)^{-1-\frac{n}{2}} dx = (n-2)\omega_{n-1} R_0^{n-2}.$$

Note also that the right hand side of (69) verifies

$$\begin{aligned} - \int_{\partial B_0(\delta)} \partial_\nu \hat{u}_\alpha d\sigma_{\hat{g}_\alpha} &= - \int_{\partial B_0(\delta)} \partial_\nu \hat{U} + o(1) \\ &= \lambda(n-2)\omega_{n-1} + o(1). \end{aligned}$$

since H is smooth and harmonic. Since

$$\lambda(n-2)\omega_{n-1} = R_0^{n-2}(n-2)\omega_{n-1} + O(\delta^{n-2}) + o(1),$$

for any $\delta > 0$, we get that $\lambda = R_0^{n-2}$.

Finally, let us prove that $H(0) = 0$. The equation's dominant terms are invariant by rescaling, which leads us to use a Pohozaev identity to obtain

new estimates for the remaining terms. Let Ω_α correspond to $B_0(\delta r_\alpha)$ in the exponential chart at $x_\alpha \in M$ and let $X_\alpha = \frac{1}{2}\nabla d_g(x_\alpha, x)^2$ be the vector field of coordinates. Using integration by parts,

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(X_\alpha) \Delta_g u_\alpha dv_g &= \int_{\Omega_\alpha} \langle \nabla(\nabla u_\alpha(X_\alpha)), \nabla u_\alpha \rangle dv_g \\ &\quad - \int_{\partial\Omega_\alpha} \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha d\sigma_g \\ &= \int_{\Omega_\alpha} \nabla^\# \nabla u_\alpha(X_\alpha, \nabla u_\alpha) + \nabla^\# X_\alpha(\nabla u_\alpha, \nabla u_\alpha) dv_g \\ &\quad - \int_{\partial\Omega_\alpha} \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha d\sigma_g, \end{aligned}$$

where $(\nabla^\# X_\alpha) = (\nabla^i X_\alpha)^j$. Since

$$\begin{aligned} \int_{\Omega_\alpha} |\nabla u_\alpha|^2 \operatorname{div}_g X_\alpha dv_g + \int_{\Omega_\alpha} \langle \nabla(|\nabla u_\alpha|^2), X_\alpha \rangle dv_g \\ = \int_{\partial\Omega_\alpha} |\nabla u_\alpha|^2 \langle X_\alpha, \nu \rangle d\sigma_g, \end{aligned}$$

we can write that

$$\begin{aligned} \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \Delta_g u_\alpha dv_g \\ = \int_{\Omega_\alpha} (\nabla^\# X_\alpha(\nabla u_\alpha, \nabla u_\alpha) - \frac{1}{2} (\operatorname{div}_g X_\alpha) |\nabla u_\alpha|^2) dv_g \\ + \int_{\partial\Omega_\alpha} \left(\frac{1}{2} \langle X_\alpha, \nu \rangle |\nabla u_\alpha|^2 - \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha - \frac{n-2}{2} u_\alpha \partial_\nu u_\alpha \right) d\sigma_g. \end{aligned} \quad (71)$$

We begin by analyzing the right-hand side of (71). By our choice of X_α , $(\nabla^\# X_\alpha)^{ij} = g^{ij} + O(d_g(x_\alpha, x)^2)$, and consequently

$$\begin{aligned} \int_{\Omega_\alpha} \left(\nabla^\# X_\alpha(\nabla u_\alpha, \nabla u_\alpha) - \frac{1}{2} (\operatorname{div}_g X_\alpha) |\nabla u_\alpha|^2 \right) dv_g \\ = O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 dv_g \right). \end{aligned}$$

According to (62),

$$\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 dv_g \leq C \int_{\Omega_\alpha} \mu_\alpha^{n-2} (d_g(x_\alpha, x) + \mu_\alpha)^{4-2n} dv_g,$$

so

$$\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 |\nabla u_\alpha|^2 dv_g \leq \begin{cases} O(\mu_\alpha r_\alpha) & \text{if } n = 3 \\ O\left(\mu_\alpha^2 \ln \frac{1}{\mu_\alpha}\right) & \text{if } n = 4 \\ O(\mu_\alpha^2) & \text{if } n = 5 \end{cases}$$

In all these three cases, thanks to (65), the integral is of the order $o(\mu_\alpha^{n-2} r_\alpha^{2-n})$. From (66),

$$\begin{aligned} \int_{\partial\Omega_\alpha} \left(\frac{1}{2} \langle X_\alpha, \nu \rangle |\nabla u_\alpha|^2 - \nabla u_\alpha(X_\alpha) \partial_\nu u_\alpha - \frac{n-2}{2} u_\alpha \partial_\nu u_\alpha \right) d\sigma_g \\ = \left(\frac{(n-2)^2}{2} \omega_{n-1} R_0^{n-2} H(0) + o(1) \right) \mu_\alpha^{n-2} r_\alpha^{2-n}. \end{aligned}$$

Note that the boundary term does not depend on δ , and as a result

$$\begin{aligned} \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \Delta_g u_\alpha dv_g \\ = \left(\frac{(n-2)^2}{2} \omega_{n-1} R_0^{n-2} H(0) + o(1) \right) \mu_\alpha^{n-2} r_\alpha^{2-n} \end{aligned} \quad (72)$$

We now analyse the right hand side of (71) by using (25):

$$\begin{aligned} \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \Delta_g u_\alpha dv_g \\ = \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) f_\alpha u_\alpha^{q-1} dv_g \\ - \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \langle \nabla u_\alpha, Y_\alpha \rangle^2 u_\alpha^{-q-3} dv_g \\ - \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle (d_\alpha u_\alpha^{-2} + u_\alpha^{-q-2}) dv_g \\ + \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) (a_\alpha u_\alpha^{-q-1} - b_\alpha u_\alpha^{-1} - h_\alpha u_\alpha) dv_g \end{aligned} \quad (73)$$

and we look at each term in turn. By the estimates (27) and (30), we get

$$\begin{aligned} \left| \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) \langle \nabla u_\alpha, Y_\alpha \rangle^2 u_\alpha^{-q-3} dv_g \right| &\leq C \int_{\Omega_\alpha} d_g(x_\alpha, x)^{-2} dv_g \\ &\leq C(\delta r_\alpha)^{n-2} \end{aligned} \quad (74)$$

and, similarly,

$$\left| \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle (d_\alpha u_\alpha^{-2} + u_\alpha^{-q-2}) dv_g \right| \leq C(\delta r_\alpha)^{n-1}. \quad (75)$$

We also have that

$$\left| \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) (a_\alpha u_\alpha^{-q-1} - b_\alpha u_\alpha^{-1}) \right| \leq C r_\alpha^n. \quad (76)$$

From (62), we obtain that

$$\begin{aligned} \left| \int_{\Omega_\alpha} h_\alpha \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) u_\alpha dv_g \right| \\ = O \left(\int_{\Omega_\alpha} \mu_\alpha^{n-2} (\mu_\alpha + d_g(x_\alpha, x))^{4-2n} dx \right) \\ = o(\mu_\alpha^{n-2} r_\alpha^{2-n}) \end{aligned} \quad (77)$$

for $3 \leq n \leq 5$. Using integration by parts,

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(X_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} \int_{\partial\Omega_\alpha} f_\alpha r_\alpha u_\alpha^q d\sigma_g \\ &\quad - \frac{1}{q} \int_{\Omega_\alpha} \operatorname{div}_g X_\alpha f_\alpha u_\alpha^q dv_g \\ &\quad - \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g. \end{aligned} \quad (78)$$

Thus we can write that

$$\begin{aligned} \int_{\Omega_\alpha} \left(\nabla u_\alpha(X_\alpha) + \frac{n-2}{2} u_\alpha \right) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} r_\alpha \int_{\partial\Omega_\alpha} f_\alpha u_\alpha^q d\sigma_g \\ &\quad + \int_{\Omega_\alpha} \left(-\frac{1}{q} \operatorname{div}_g(X_\alpha) + \frac{n-2}{2} \right) f_\alpha u_\alpha^q dv_g \\ &\quad - \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g. \end{aligned}$$

Since $\operatorname{div}_g(X_\alpha) = n + O(d_g(x_\alpha, x)^2)$, this leads to

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= \frac{1}{q} r_\alpha \int_{\partial\Omega_\alpha} f_\alpha u_\alpha^q d\sigma_g \\ &+ O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^2 u_\alpha^q dv_g\right) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g. \end{aligned}$$

Using lemmas 3.4 and 3.6, this leads to

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= O(\mu_\alpha^n r_\alpha^{-n}) + O(\mu_\alpha^2) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g \end{aligned}$$

so that, thanks to (65),

$$\begin{aligned} \int_{\Omega_\alpha} (\nabla u_\alpha(X_\alpha) + \frac{n-2}{2}u_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= o(\mu_\alpha^{n-2} r_\alpha^{2-n}) \\ &- \frac{1}{q} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g \end{aligned} \quad (79)$$

if $3 \leq n \leq 5$.

We claim that

$$\int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g = o(\mu_\alpha^{n-2} r_\alpha^{2-n}). \quad (80)$$

Thanks to (72), (74), (75), (76), (77), (79) and (80), we see that

$$H(0) = o(1) + \delta^4 \quad (81)$$

for any $\delta > 0$, so by taking $\delta \rightarrow 0$ we see that $H(0) = 0$.

In order to prove (80), we can first use Lemma 3.6 to write that

$$\int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g = O(\mu_\alpha)$$

which leads to (80) if $n = 3, 4$ thanks to (65), but is not enough for $n = 5$. In order to improve the estimate in the case $n = 5$, note that

$$\begin{aligned} \int_{\Omega_\alpha} \nabla f_\alpha(X_\alpha) u_\alpha^q dv_g &= \partial_i f(x_\alpha) \int_{\Omega_\alpha} x^i u_\alpha^q dv_g \\ &+ O\left(\int_{\Omega_\alpha} d_g(x_\alpha, x)^{1+\eta} u_\alpha^q dv_g\right) \\ &= o(\mu_\alpha |\nabla f_\alpha(x_\alpha)|) + O(\mu_\alpha^{1+\eta}) \\ &= o(\mu_\alpha |\nabla f_\alpha(x_\alpha)|) + o(\mu_\alpha^3 r_\alpha^{-3}). \end{aligned}$$

with $\eta > \frac{1}{2}$. Thus it remains to prove that

$$|\nabla f_\alpha(x_\alpha)| = O(\mu_\alpha^2 r_\alpha^{-3}). \quad (82)$$

As before, we use a Pohozaev-type identity. We make use of the equation's symmetry by translation, with $Z = Z^i$ a constant vector field in the exponential chart of x_α . We can write that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) \Delta_g u_\alpha dv_g = O \left(\int_{\Omega_\alpha} d_g(x_\alpha, x) |\nabla u_\alpha|^2 dv_g + \int_{\partial\Omega_\alpha} |\nabla u_\alpha|^2 d\sigma_g \right), \quad (83)$$

which is $o(\mu_\alpha^2 r_\alpha^{-3})$. On the left-hand side, we use (25). Lemma 3.3 and (27) imply that

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) \frac{\langle \nabla u_\alpha, Y_\alpha \rangle^2}{u_\alpha^{q+3}} dv_g &\leq \frac{1}{\varepsilon^q} \int_{\Omega_\alpha} \left| \frac{\nabla u_\alpha}{u_\alpha} \right|^3 dv_g \\ &\leq \frac{1}{\varepsilon^q} \int_{\Omega_\alpha} d_g(x_\alpha, x)^{-3} dv_g \\ &= O(r_\alpha^2) = o(\mu_\alpha^2 r_\alpha^{-3}). \end{aligned} \quad (84)$$

We see that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) c_\alpha \langle \nabla u_\alpha, Y_\alpha \rangle \left(\frac{d_\alpha}{u_\alpha^2} + \frac{1}{u_\alpha^{q+2}} \right) dv_g = O(r_\alpha^2) = o(\mu_\alpha^2 r_\alpha^{-3}), \quad (85)$$

and that the same holds for the terms corresponding to h_α , b_α and c_α . So (83), (84) and (85) imply that

$$\int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) f_\alpha u_\alpha^{q-1} dv_g = o(\mu_\alpha^2 r_\alpha^{-3}).$$

Furthermore,

$$\begin{aligned} \int_{\Omega_\alpha} \nabla u_\alpha(Z_\alpha) f_\alpha u_\alpha^{q-1} dv_g &= O \left(\int_{\partial\Omega_\alpha} u_\alpha^q d\sigma_g \right) \\ &\quad - \frac{1}{q} \int_{\Omega_\alpha} \operatorname{div}_g(Z_\alpha) f_\alpha u_\alpha^q dv_g \\ &\quad - \frac{1}{q} \nabla f_\alpha(Z_\alpha) \int_{\Omega_\alpha} u_\alpha^q dv_g, \end{aligned}$$

which leads us to conclude the claim in (80). Note that

$$\operatorname{div}_g(Z_\alpha) = O(d_g(x_\alpha, x)^\eta)$$

and

$$\int_{\Omega_\alpha} u_\alpha^q dv_g \rightarrow \int_{R^n} \left(1 + \frac{|x|^2}{R_0^2} \right)^{-5} dx < \infty.$$

Finally, we are in the position to remark that $\rho_\alpha = r_\alpha$. Remember that

$$\varphi(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int_{\partial B_0(r)} \hat{U} = \left(\frac{R_0}{r}\right)^{n-2} + H(0)$$

and that $\left(r^{\frac{n-2}{2}}\varphi(r)\right)'(1) = 0$, so if $r_\alpha < \rho_\alpha$, then $H(0) = R_0^{n-2}$, which contradicts (81). Thus (67) implies (66), and this wraps up the proof of the lemma. \square

Moreover, $\rho_\alpha = r_\alpha$ means that $\rho_\alpha \rightarrow 0$ because $r_\alpha = O\left(\mu_\alpha^{\frac{1}{2}}\right)$ thanks to (65). As an important consequence, there do not exist any isolated bubbles. Otherwise, if a bubble were isolated, then we could choose a blow-up sequence with $0 < \delta < \rho_\alpha$, contradicting the previous result.

3.3 Proof of the stability theorem

We are now in the position to prove Theorem 3.1. Let

$$\delta_\alpha := \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha}).$$

For any $R > 0$, let $1 \leq M_{R,\alpha}$ be such that

$$\begin{aligned} d_g(x_{1,\alpha}, x_{i_\alpha,\alpha}) &\leq R\delta_\alpha \quad \text{for } i_\alpha \in \{1, \dots, M_{R,\alpha}\}, \text{ and} \\ d_g(x_{1,\alpha}, x_{j_\alpha,\alpha}) &> R\delta_\alpha \quad \text{for } j_\alpha \in \{M_{R,\alpha} + 1, \dots, N_\alpha\}. \end{aligned}$$

We consider the rescaled quantities

$$\check{u}_\alpha(x) := \delta_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_{1,\alpha}}(\delta_\alpha x)) \quad \text{and} \quad \check{g}_\alpha(x) := \left(\exp_{x_{1,\alpha}}^* g\right)(\delta_\alpha x)$$

and the coordinates $\check{x}_{i,\alpha} := \delta_\alpha^{-1} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha})$ in the exponential chart. It's obvious that $|\check{x}_{2,\alpha}| = 1$ and $|\check{x}_{i,\alpha}| \geq 1$.

The following lemma is a direct consequence of Lemma 3.3.

Lemma 3.8. *For all $R > 0$, there exists $C_R > 0$ such that the Harnack-type inequality*

$$\|\nabla \check{u}_\alpha\|_{L^\infty(\Omega_R)} \leq C_R \sup_{\Omega_R} \check{u}_\alpha \leq C_R^2 \inf_{\Omega_R} \check{u}_\alpha$$

holds, where $\Omega_R = B_0(R) \setminus \bigcup_{i=1}^{M_{2R,\alpha}} B_{\check{x}_{i,\alpha}}\left(\frac{1}{R}\right)$.

Note that, for $1 \leq i < j \leq M_{R,\alpha}$, $B_{x_{i,\alpha}}\left(\frac{\delta_\alpha}{4}\right)$ and $B_{x_{j,\alpha}}\left(\frac{\delta_\alpha}{4}\right)$ are disjoint, which is equivalent to saying that $B_{\check{x}_{i,\alpha}}\left(\frac{1}{4}\right)$ and $B_{\check{x}_{j,\alpha}}\left(\frac{1}{4}\right)$ are also disjoint.

At this point, we are finally able to prove Theorem 3.1, which we stated at the very beginning of this section. We define two possible types of concentration points, according to how \check{u}_α explodes. We prove that, within a cluster, we can only find one type or the other, but never both. Finally, we see that the existence of either type leads to contradictions, which implies that \check{u}_α admits no concentration points whatsoever.

Proof of Theorem 3.1: Consider the cluster around $(x_{1,\alpha})_\alpha$, for some $R > 0$. There are two possible cases. The first type of concentration point corresponds to

$$\sup_{B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} \left(\tilde{u}_\alpha(x) + \left| \frac{\nabla \tilde{u}_\alpha(x)}{\tilde{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right) = O(1). \quad (86)$$

In this case, note that $(\tilde{u}_\alpha)_\alpha$ is uniformly bounded in \mathcal{C}_{loc}^1 . Moreover, we find a lower bound, as by (28) from Lemma 3.1,

$$|\tilde{x}_{i,\alpha}|^{\frac{n-2}{2}} \tilde{u}_\alpha(\tilde{x}_{i,\alpha}) \geq 1.$$

There exists $\delta_i > 0$ such that

$$\inf_{B_{\tilde{x}_{i,\alpha}}(\delta_i)} \tilde{u}_\alpha \geq \frac{1}{2} |\tilde{x}_{i,\alpha}|^{1-\frac{n}{2}},$$

which leads to the existence of $\delta_0 > 0$ where

$$\inf_{B_{\tilde{x}_{i,\alpha}}(\delta_0)} \tilde{u}_\alpha \geq \frac{1}{2}. \quad (87)$$

The second type is defined by

$$\sup_{B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} \left(\tilde{u}_\alpha(x) + \left| \frac{\nabla \tilde{u}_\alpha(x)}{\tilde{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right) \rightarrow \infty. \quad (88)$$

In this case, either

$$\sup_{B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} \tilde{u}_\alpha(x) \leq M \quad \text{and} \quad \sup_{B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} |\nabla \tilde{u}_\alpha(x)| \rightarrow \infty, \quad (89)$$

or

$$\sup_{B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} \tilde{u}_\alpha(x) \rightarrow \infty. \quad (90)$$

We show (89) is not actually possible. Assume it holds true. Then there exist $(\tilde{x}_\alpha)_\alpha \subset (B_{\tilde{x}_{i,\alpha}}(\frac{1}{2}))_\alpha$ and $(\tilde{\nu}_\alpha)_\alpha$ such that

$$\tilde{\nu}_\alpha^{1-\frac{n}{2}} := \tilde{u}_\alpha(\tilde{x}_\alpha) + \left| \frac{\nabla \tilde{u}_\alpha(\tilde{x}_\alpha)}{\tilde{u}_\alpha(\tilde{x}_\alpha)} \right|^{\frac{n-2}{2}} = \sup_{x \in B_{\tilde{x}_{i,\alpha}}(\frac{1}{2})} \left(\tilde{u}_\alpha(x) + \left| \frac{\nabla \tilde{u}_\alpha(x)}{\tilde{u}_\alpha(x)} \right|^{\frac{n-2}{2}} \right)$$

with

$$\tilde{\nu}_\alpha \rightarrow 0. \quad (91)$$

We define the rescaled quantities

$$\check{\nu}_\alpha(x) := \tilde{\nu}_\alpha^{\frac{n-2}{2}} \tilde{u}_\alpha(\exp_{\tilde{x}_\alpha}(\tilde{\nu}_\alpha x)) \quad \text{and} \quad \check{h}_\alpha(x) := (\exp_{\tilde{x}_\alpha}^* \check{g})(\tilde{\nu}_\alpha x)$$

respectively, defined in $\Omega_\alpha := B_0\left(\frac{1}{2\check{\nu}_\alpha}\right)$. For any $R > 0$ and α large enough so that $R < \frac{1}{2\check{\nu}_\alpha}$,

$$\limsup_{\alpha \rightarrow \infty} \sup_{B_0(R)} \left(\check{\nu}_\alpha + \left| \frac{\nabla \check{\nu}_\alpha}{\check{\nu}_\alpha} \right| \right) = 1. \quad (92)$$

Thus

$$|\nabla \ln \check{\nu}_\alpha| \leq 1$$

and

$$\check{\nu}_\alpha(0)e^{-x} \leq \check{\nu}_\alpha(x) \leq \check{\nu}_\alpha(0)e^x. \quad (93)$$

Note that the metrics $\check{h}_\alpha \rightarrow \xi$ in \mathcal{C}_{loc}^2 as $\alpha \rightarrow \infty$. Assume that, up to a subsequence, $u_\alpha(\check{x}_\alpha) \rightarrow l < \infty$. We also deduce that $\check{\nu}_\alpha(0) \rightarrow 0$. Let $\check{x}_0 := \lim_{\alpha \rightarrow \infty} \check{x}_\alpha$ and let us denote

$$\check{w}_\alpha(x) := \frac{\check{\nu}_\alpha(x)}{\check{\nu}_\alpha(0)}.$$

These functions are bounded from below,

$$\check{w}_\alpha(x) \geq \frac{\varepsilon}{l} + o(1) > 0. \quad (94)$$

Moreover,

$$\check{w}_\alpha(x) \leq e^{|x|}.$$

By standard elliptic theory, we find that there exists $\check{w} := \lim_{\alpha \rightarrow \infty} \check{w}_\alpha$ in \mathcal{C}^1 solving:

$$\Delta \check{w} = -\frac{1}{l^{q+2}} \frac{\langle \nabla \check{w}, Y(\check{x}_0) \rangle^2}{\check{w}^{q+3}}.$$

Note that

$$\Delta \check{w}^{-\alpha} \leq \alpha \frac{|\nabla \check{w}|^2}{\check{w}^{\alpha+2}} \left[\frac{\|Y(\check{x}_0)\|_{L^\infty}^2}{\varepsilon^{q+2}} - (\alpha + 1) \right],$$

so $\check{w}^{-\alpha}$ is subharmonic for α large, and so Lemma 5.1 (see the Annex) implies that \check{w} is constant, which in turn implies that $\check{U} = 0$ and $\nabla \check{U} = 0$, which is false (see proof of Lemma 3.2). Therefore, the second subcase cannot be true. This essentially means that when a concentration point is of the second type, then

$$\sup_{B_{\check{x}_i, \alpha}\left(\frac{1}{2}\right)} \check{u}_\alpha(x) \rightarrow \infty$$

and so

$$\check{u}_\alpha(\check{x}_{i, \alpha}) \rightarrow \infty.$$

Let us denote $\check{x}_i := \lim_{\alpha \rightarrow \infty} \check{x}_{i, \alpha}$ up to a subsequence. According to Proposition 3.1,

$$\check{u}_\alpha(\check{x}_{i, \alpha}) \check{u}_\alpha(x) \mapsto \frac{\lambda_i}{|x - \check{x}_i|^{n-2}} + H_i(x) \quad (95)$$

in \mathcal{C}^1 in $B_{\tilde{x}_i}(\frac{1}{2}) \setminus \{\tilde{x}_i\}$, with $\lambda_i > 0$, where H_i is a harmonic function in $B_{\tilde{x}_i, \alpha}(\frac{1}{2})$, $H(\tilde{x}_i) = 0$.

Let U be a connected open set of \mathbb{R}^n , $U_R \subset B_0(R+1)$, containing no other point of the cluster apart from \tilde{x}_i and \tilde{x}_j . For any $0 < r < \frac{1}{8}$, we set

$$V_{r,R} = U_R \setminus \left(\overline{B_{\tilde{x}_i}(r) \cup B_{\tilde{x}_j}(r)} \right).$$

For a fixed $x \in B_{\tilde{x}_i}(\frac{1}{4}) \setminus V_{r,R}$, (95) implies that $\check{u}_\alpha(x) \rightarrow 0$ as $\alpha \rightarrow \infty$. It follows from Lemma 3.8 and (87) that all points of a cluster must be of the same type.

Assuming all points in the cluster are of the first type, then

$$\check{u}_\alpha(0) + \left| \frac{\nabla \check{u}_\alpha(0)}{\check{u}_\alpha(0)} \right|^{\frac{n-2}{2}} = O(1),$$

then by standard elliptic theory there exists $\check{u} := \lim_{\alpha \rightarrow \infty} \check{u}_\alpha$ in $\mathcal{C}^1(B_0(R))$, $R > 0$. Repeating the reasoning of Lemma 3.1 or Lemma 3.4, we know that

$$\Delta_\xi \check{u} = f(x_1) \check{u}^{q-1}.$$

However, \check{u} must have at least two separate maxima, at 0 and \tilde{x}_2 , which leads to a contradiction by the classification result of Caffarelli, Gidas and Spruck [2].

Therefore $\check{u}_\alpha(\tilde{x}_{i,\alpha}) \rightarrow \infty$, for any $i = \overline{1, M_{2R,\alpha}}$. Up to a subsequence

$$\frac{\check{u}_\alpha(\tilde{x}_{i,\alpha})}{\check{u}_\alpha(0)} \rightarrow \mu_i > 0 \quad \text{as } \alpha \rightarrow \infty.$$

We fix $R > 0$ and assume, without loss of generality, that $(M_{2R,\alpha})_\alpha$ is a constant denoted by M_{2R} . Using Lemma 3.8 and standard elliptic theory, we pass to a subsequence and get

$$\check{u}_\alpha(0) \check{u}_\alpha(x) \rightarrow \check{G}(x)$$

in $\mathcal{C}_{loc}^1(B_0(R) \setminus \{\tilde{x}_i\}_{i=\overline{1, N_{2R}}})$ for $\alpha \rightarrow \infty$, with

$$\begin{aligned} \check{G}(x) &= \sum_{i=1}^p \frac{\lambda_i}{\mu_i |x - \tilde{x}_i|^{n-2}} + \check{H}(x) \\ &= \frac{\lambda_1}{|x|^{n-2}} + \left(\sum_{i=2}^p \frac{\lambda_i}{\mu_i |x - \tilde{x}_i|^{n-2}} + \check{H}(x) \right) \end{aligned}$$

Here, \check{H} is harmonic on $B_0(R)$, and $2 \leq p \leq M_{2R}$ such that $|\tilde{x}_p| \leq R$ as $|\tilde{x}_{p+1}| > R$. If we apply Proposition 3.1 to the blow-up sequence $x_\alpha = x_{1,\alpha}$ with $\rho_\alpha = \frac{1}{16} d_\alpha$, we obtain

$$\hat{H}(0) := \sum_{i=2}^p \frac{\lambda_i}{\mu_i |\tilde{x}_i|^{n-2}} + \check{H}(0) = 0$$

Since $\hat{H}(x) - \frac{\lambda_2}{\mu_2|x-\hat{x}_2|} = \check{G}(x) - \frac{\lambda_1}{|x|^{n-2}} - \frac{\lambda_2}{\mu_2|x-\hat{x}_2|^{n-2}}$ is harmonic in the ball $B_0(R) \setminus \{\check{x}_i\}_{i \in \{2, N_{2R}\}}$ and $\check{G} \geq 0$, then as a consequence of the maximum principle, by considering a minimum on $\partial B_0(R)$, we see that

$$\hat{H}(0) \geq \frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{R^{n-2}} - \frac{\lambda_2}{\mu_2(R-1)^{n-2}}.$$

Choosing $R > 0$ large enough, we ensure that $\hat{H} > 0$, which contradicts Theorem 3.7. Consequently, u_α admits no concentration points and is therefore uniformly bounded in C^1 . \square

Lemma 3.9. *Assuming equation (14) associated to \tilde{a} admits a supersolution and that $\Delta_g + h$ is coercive, then for any $0 < T < \inf_M \tilde{a}$ and any equation with parameters in $\mathcal{E}_{\theta, T}$ (as in Theorem 3.1), there exists a constant $C_{\theta, T} = C(n, \theta, T) > 0$ such that, for any $\|Y\|_{L^\infty} \leq C_{\theta, T}$ and $\|b\|_{L^\infty} \leq C_{\theta, T}$, we may find a smallest real eigenvalue $\lambda_0 > 0$, where λ_0 is as in Lemma 2.1.*

Proof. Given any parameters (f, a, b, c, d, h, Y) in $\mathcal{E}_{\theta, T}$ and additionally asking for Y and b to be sufficiently small in L^∞ norm with respect to θ and T , we aim to prove that minimal solutions to the Lichnerowicz-type equation change continuously with their parameters. In order to do this, we study the sign of the smallest real eigenvalue associated to the linearisation around a minimal solution and show that it is positive by comparing it to the smallest real eigenvalue at $b = 0$ and $Y = 0$. Indeed, let $s > 0$ a real number and E_s the equation

$$E_s(u_s) := \Delta_g u_s + h u_s - f u_s^{q-1} - \frac{a}{u_s^{q+1}} + \frac{sb}{u_s} + c \langle \nabla u_s, sY \rangle \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) + \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+3}} = 0, \quad (96)$$

with u_s its minimal solution. Let L_s be the linearisation of E_s around u_s ,

$$\Delta_g \varphi_s + \left[h - (q-1) f u_s^{q-2} + (q+1) \frac{a}{u_s^{q+2}} - \frac{sb}{u_s} - c \langle \nabla u_s, sY \rangle \left(\frac{2d}{u_s^3} + \frac{q+2}{u_s^{q+3}} \right) - (q+3) \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+4}} \right] \varphi_s + \langle \nabla \varphi_s, sY \rangle \left[c \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) + \frac{2 \langle \nabla u_s, sY \rangle}{u_s^{q+3}} \right] = \lambda_s \varphi_s, \quad (97)$$

with $\lambda_s \geq 0$ the smallest real eigenvalue, $\varphi_s > 0$ the associated eigenfunction, normalised such that $\|\varphi_s\|_{L^2} = 1$. Note that the linear equations L_s are stable, in the sense that φ_s is *a priori* uniformly bounded in C^1 . This follows from the fact that the u_s is uniformly bounded. We may also suppose that λ_s is uniformly bounded, because if $\lambda_s \rightarrow \infty$, then it is clear that $\lambda_s > 0$.

As Premoselli proved by way of a variational argument [19], the equation E_0 is strictly stable, in the sense that its corresponding smallest real eigenvalue is positive. It uses the coerciveness of $\Delta_g + h$. We emphasize that his argument makes use of the fact that E_0 is symmetric, which is not the case for our more general equations. The strict stability implies continuity, *i.e.* that $u_s \rightarrow u_0$, with u_0 the minimal value. Indeed, let $u_s \rightarrow \tilde{u}$ another solution of E_0 . Clearly,

$\tilde{u} > u_0$. Let $\tilde{u}_\delta = u_0 + \delta\varphi_0$. Note that

$$\begin{aligned} E_s(\tilde{u}_\delta) &= E_0(\tilde{u}_\delta) + \frac{sb}{u_\delta} + c\langle \nabla u_\delta, sY \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) + \frac{\langle \nabla u_\delta, sY \rangle^2}{u_\delta^{q+3}} \\ &= E_0(u_0) + \lambda_0\delta\varphi_0 + o(\delta) + \frac{sb}{u_\delta} + sc\langle \nabla u_\delta, Y \rangle \left(\frac{d}{u_\delta^2} + \frac{1}{u_\delta^{q+2}} \right) \\ &\quad + s^2 \frac{\langle \nabla u_\delta, Y \rangle^2}{u_\delta^{q+3}} \end{aligned}$$

If we fix $\delta > 0$ sufficiently small, the error terms $|o(\delta)| \leq \frac{\lambda_0\delta\varphi_0}{3}$ and $\tilde{u}_\delta < \tilde{u} \leq u_s$, $\forall s$. Then, by taking s sufficiently close to 0, we get that the rest of the terms are also smaller in absolute size than $\frac{\lambda_0\delta\varphi_0}{3}$. Consequently, $E_s(\tilde{u}_\delta) > 0$, so \tilde{u}_δ is a supersolution of E_s that is smaller than the minimal solution u_s .

Since $u_s \rightarrow u_0$, we also get that $\lambda_s \rightarrow \lambda_0$, so for s small, the first eigenvalue $\lambda_s > 0$. We would like to obtain that there exists $s_{\theta,T} > 0$ such that, for any $0 \leq s < s_{\theta,T}$, the minimal eigenvalue corresponding to L_s is positive, where $(a, b, c, d, f, h, Y) \in \mathcal{E}_{\theta,T}$. In other words, we attempt to set a size for Y and b , depending on θ and T (and n), such that the resulting equations are strictly stable.

First, there exists $\delta_{\theta,T} > 0$ such that if $Y = 0$, $b = 0$ and the equation's parameters are found in $\mathcal{E}_{\theta,T}$, then $\lambda_0 > \delta_{\theta,T}$. We let $u_s = u_0 + \varepsilon_s v_s$ such that $\|v_s\|_{L^2} = 1$, $\varepsilon_s \in \mathbb{R}$. Note that $\varepsilon_s \rightarrow 0$ as $s \rightarrow 0$.

We begin by analyzing the difference in size between ε_s and s , or equivalently between $\|u_s - u_0\|_{L^\infty}$ and s . Let

$$E_s = E_0 + sM_s,$$

where

$$M_s(u_s) = \frac{b}{u_s} + c\langle \nabla u_s, Y \rangle \left(\frac{d}{u_s^2} + \frac{1}{u_s^{q+2}} \right) + s \frac{\langle \nabla u_s, Y \rangle^2}{u_s^{q+3}}.$$

Recall that

$$E_0(u_s) = -sM_s(u_s) \tag{98}$$

where

$$E_0(u_s) = L_0(u_s - u_0) + O(\|u_s - u_0\|^2).$$

Since u_0 is a solution of E_0 and the operator L_0 is coercive, with minimal eigenvalue λ_0 , then by testing (98) against $(u_s - u_0)$, we see that

$$\lambda_0(1 + o(1))\|u_s - u_0\|_{L^2}^2 \leq -s \int_M M_s(u_s)(u_s - u_0) \leq s\|M_s(u_s)\|_{L^2}\|u_s - u_0\|_{L^2}.$$

The size of $M_s(u_s)$ is determined by a constant depending on θ and T . Therefore, we may write

$$(1 + o(1))\varepsilon_s = (1 + o(1))\|u_s - u_0\|_{L^2} \leq s \frac{C}{\lambda_0}$$

Finally, in order to compare λ_s to λ_0 , extract the terms of order s from the quantity $\int_M \varphi_0 L_s(\varphi_s) - \varphi_s L_0(\varphi_0)$,

$$\begin{aligned} & - \int_M \frac{sb}{u_0} \varphi_0 \varphi_s - \int_M c \langle \nabla u_0, sY \rangle \left(\frac{2d}{u_0^3} + \frac{q+3}{u_0^{q+3}} \right) \varphi_s \varphi_0 - \int_M (q+3) \frac{\langle \nabla u_s, sY \rangle^2}{u_s^{q+4}} \varphi_s \varphi_0 \\ & \quad \varepsilon_s \int_M \left[(q-1)(q-2) f u_0^{q-3} - (q+1)(q+2) \frac{a}{u_0^{q+3}} \right] v_s \varphi_s \varphi_0 \\ & \quad + O(s^2) = (\lambda_\theta - \lambda_0) \int_M \varphi_s \varphi_0, \end{aligned} \tag{99}$$

so there exists a constant C depending on θ and T such that

$$|\lambda_s - \lambda_0| \leq sC \left(\int_M \varphi_s \varphi_0 \right)^{-1}. \tag{100}$$

As $\varphi_s = \varphi_0 + o(1)$ and the L^2 norm of φ_0 is 1, we may choose s small enough so that $|\lambda_s - \lambda_0| \geq \frac{\delta_{\theta,T}}{2}$, and thus $\lambda_s > 0$. \square

4 Existence of solutions to the system

4.1 The proof of the main theorem

The following is a useful estimate we can find in [12]; it plays a crucial role in ensuring the necessary compactness of the sequence W_α in the main theorem.

Proposition 4.1. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ such that g has no conformal Killing fields. Let X be a smooth vector field in M . Then there exists a unique solution W of*

$$\Delta_{g, \text{conf}} W = X.$$

Also, for $0 < \gamma < 1$, there exists a constant $C_0 > 0$ that depends only on n and g such that

$$\|W\|_{C^{1,\gamma}} \leq C_0 \|X\|_\infty.$$

Remark 4.1. *As a consequence, there exists a constant $C_1 = C_1(n, g)$ such that*

$$\|\mathcal{L}_g W\|_{C^{0,\gamma}} \leq C_1 \|X\|_\infty \tag{101}$$

Let (M, g) be a closed Riemannian manifold of dimension $n \in \{3, 4, 5\}$ such that g has no conformal Killing fields. Let $b, c, d, f, h, \rho_1, \rho_2, \rho_3$ be smooth functions on M and let Y and Ψ be smooth vector fields defined on M . Let $0 < \gamma < 1$.

Assume that $\Delta_g + h$ is coercive. Assume that $f > 0, \rho_1 > 0$ and $|\nabla \rho_3| < (2C_1)^{-1}$, where C_1 is defined in (101).

Consider the coupled system

$$\begin{cases} \Delta_g u + hu & = f u^{q-1} + \frac{\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_g^2}{u^{q+1}} \\ & - \frac{b}{u} - c \langle \nabla u, Y \rangle \left(\frac{d}{u^2} + \frac{1}{u^{q+2}} \right) - \frac{\langle \nabla u, Y \rangle^2}{u^{q+3}} \\ \text{div}_g(\rho_3 \mathcal{L}_g W) & = \mathcal{R}(u), \end{cases} \tag{102}$$

where \mathcal{R} is an operator verifying

$$\mathcal{R}(u) \leq C_{\mathcal{R}} \left(1 + \frac{\|u\|_{\mathcal{C}^2}^2}{(\inf_M u)^2} \right) \quad (103)$$

for a constant $C_{\mathcal{R}} > 0$.

We fix

$$\theta = \min(\inf_M \rho_1, \inf_M f), \quad (104)$$

and

$$T = \max(\|f\|_{\mathcal{C}^{1,\eta}}, \|\rho_1\|_{\mathcal{C}^{0,\gamma}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (105)$$

Let

$$M = \ln S_{\theta, 2T}, \quad (106)$$

with $S_{\theta, 2T}$ a constant as in Theorem 3.1. The following theorem is the main result of the present paper.

Theorem 4.1. *Assume there exists a smooth positive function \tilde{a} for which*

$$\Delta_g \tilde{u} + h \tilde{u} = f \tilde{u}^{q-1} + \frac{\tilde{a}}{\tilde{u}^{q+1}} \quad (107)$$

admits a positive supersolution \tilde{u} . Assume that $\rho_1 < \tilde{a}$ and let $\omega = \inf_M (\tilde{a} - \rho_1)$. Then there exists

$$\delta = \delta(\omega, \theta, T) > 0 \quad (108)$$

such that if

$$\|b\|_{\mathcal{C}^{0,\gamma}} + \|Y\|_{\mathcal{C}^{0,\gamma}} + \|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta, \quad (109)$$

the system (102) admits a solution (u, W) , with u a smooth positive function and W a smooth vector field.

Remark 4.2. *We can use a result by Hebey, Pacard and Pollack ([9], Corollary 3.1) in order to ensure the existence of a supersolution \tilde{u} . There exists a constant $C = C(n, h)$, $C > 0$ such that if \tilde{a} is a smooth positive function verifying*

$$\|\tilde{a}\|_{L^1(M)} \leq C(n, h) \left(\max_M |f| \right)^{1-n}, \quad (110)$$

then (107) accepts a smooth positive solution.

Proof of Theorem 4.1: The proof of the theorem consists of a fixed-point argument. Formally, we define the operator

$$\Phi : \varphi \rightarrow \ln u(\mathcal{L}_g W(e^\varphi)),$$

where $W(e^\varphi)$ solves the second equation of (102) for a fixed $u = e^\varphi$ and where $u(\mathcal{L}_g W(e^\varphi))$ is the solution of the scalar equation of (102) constructed in Section 2 for a fixed $W(e^\varphi)$. In order to apply Schauder's fixed point theorem, we show

that $\Phi : B_M \rightarrow B_M$, $B_M := \{\varphi \in \mathcal{C}^2(M), \|\varphi\|_{\mathcal{C}^2} \leq M\}$, where M is defined as in (106), and that Φ is continuous and compact.

We first want to prove that

$$\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 < \tilde{a} \quad (111)$$

to ensure that $\Phi(\varphi)$ is well defined, with \tilde{u} from (107) a supersolution. By (101), we have

$$\|\mathcal{L}_g W(e^\varphi)\|_{\mathcal{C}^{0,\gamma}} \leq C_1 (\|\nabla \rho_3\|_{L^\infty} \|\mathcal{L}_g W(e^\varphi)\|_{L^\infty} + \|\mathcal{R}(e^\varphi)\|_{L^\infty}) \quad (112)$$

and thanks to (103) we see that

$$\begin{aligned} \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 &\leq \rho_1 + 2\|\Psi\|_{L^\infty}^2 \\ &+ 2 \left(\frac{C_1 C_{\mathcal{R}} \|\rho_2\|_{L^\infty}}{1 - C_1 \|\nabla \rho_3\|_{L^\infty}} \right)^2 \left(1 + \frac{M^2 e^{2M}}{e^{2\varepsilon}} \right)^2, \end{aligned} \quad (113)$$

where ε is the lower bound of any solution corresponding to $\mathcal{E}_{\theta, 2T}$ from Theorem 3.1. There exists

$$\delta_1 = \delta_1(\omega, \theta, T) > 0 \quad (114)$$

such that if

$$\|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta_1, \quad (115)$$

then (111) holds.

In order to use the *a priori* estimate of Section 3 to see that $\Phi : B_M \rightarrow B_M$, we need to prove that

$$\theta \leq \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2 \quad \text{and} \quad \|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{\mathcal{C}^{0,\gamma}} \leq 2T. \quad (116)$$

From (104) we deduce that

$$\theta \leq \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2. \quad (117)$$

and thanks to (112) we see that

$$\begin{aligned} \|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{\mathcal{C}^{0,\gamma}} &\leq \|\rho_1\|_{\mathcal{C}^{0,\gamma}} + 2\|\Psi\|_{\mathcal{C}^{0,\gamma}}^2 \\ &+ 2 \left(\frac{C_1 C_{\mathcal{R}} \|\rho_2\|_{\mathcal{C}^{0,\gamma}}}{1 - C_1 \|\nabla \rho_3\|_{L^\infty}} \right)^2 \left(1 + \frac{M^2 e^{2M}}{e^{2\varepsilon}} \right)^2. \end{aligned} \quad (118)$$

There exists

$$\delta_2 = \delta_2(\omega, \theta, T) > 0 \quad (119)$$

such that if

$$\|\Psi\|_{\mathcal{C}^{0,\gamma}} + \|\rho_2\|_{\mathcal{C}^{0,\gamma}} + C_{\mathcal{R}} \leq \delta_2, \quad (120)$$

then

$$\|\rho_1 + |\Psi + \rho_2 \mathcal{L}_g W(e^\varphi)|_g^2\|_{\mathcal{C}^{0,\gamma}} \leq 2T. \quad (121)$$

Thanks to (117) and (121), the *a priori* estimates in Section 3 imply that

$$\|u(\mathcal{L}_g W(e^\varphi))\|_{\mathcal{C}^2} \leq S_{\theta, 2T},$$

so

$$\Phi(\varphi) \leq M,$$

where M is as in (106). We have thus proved that Φ is well-defined and that $\Phi : B_M \rightarrow B_M$.

In order to show that Φ is continuous, we want to check that it holds true for $a \mapsto u(a)$, where $u(a)$ is the minimal solution constructed in Section 2. For all $a < \tilde{a}$, we've established monotony, which ensures that the minimal solutions exist. For $t > 0$ small, let us denote by u_t the solutions corresponding to $a(1+t) < \tilde{a}$. Let u_0 be the limit of u_t as $t \rightarrow 0$; it is also a solution of the Lichnerowicz-type equation associated to a . If $u_0 \neq u$, then $u < u_0$. According to Section 3, there exists $C_{\theta, 2T} > 0$ such that

$$\|b\|_{C^{0,\gamma}} + \|Y\|_{C^{0,\gamma}} \leq C_{\theta, 2T} \quad (122)$$

implies that u is strictly stable. We ask that

$$\delta \leq \min(\delta_1, \delta_2, C_{\theta, 2T}) \quad (123)$$

where δ_1 is defined in (114) and δ_2 is defined in (119). We choose $\mu > 0$ small enough such that $u < \hat{u}_\mu < u_0$, where $\hat{u}_\mu := u + \mu\psi$, ψ a positive eigenfunction at u corresponding to the smallest real eigenvalue. But \hat{u}_μ is a supersolution for $a(1+\epsilon)$, $\epsilon > 0$ small, which contradicts the monotonicity. Therefore, Φ is continuous.

Lastly, B_M being a closed convex set in \mathcal{C}^2 , it remains to show that $\Phi(B_M)$ is compact to conclude. From the previous discussion, $\Phi(B_M) \subset B_M$, and is thus bounded in \mathcal{C}^2 . By standard elliptic theory, we conclude the proof of Theorem 1.1. \square

4.2 The case of a metric with conformal Killing fields

Let us consider the case of a metric g with non-trivial conformal Killing fields associated to it. For \tilde{V} a representative of the drift, the equation

$$\operatorname{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) = \frac{n-1}{n} u^q \mathbf{d} \left(u^{-2q} \tilde{N} \operatorname{div}_g(u^q \tilde{V}) \right) + \pi \nabla \psi \quad (124)$$

admits a solution W if and only if

$$\frac{n-1}{n} \int_M u^{-2q} \tilde{N} \operatorname{div}_g(u^q \tilde{V}) \operatorname{div}_g(u^q P) = \int_M \langle \pi \nabla \psi, P \rangle \quad (125)$$

for all P conformal Killing fields. Moreover, the solution W is unique up to the addition of a conformal Killing field. Note that the drift is defined modulo conformal Killing fields, so \tilde{V} and $\tilde{V} + P$ are representatives of the same drift for all P conformal Killing fields. We claim that given a vector field \tilde{V} there exists a conformal Killing field \tilde{Q} which is unique up to a true Killing field and such that

$$\frac{n-1}{n} \int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + \tilde{Q}) \right) \operatorname{div}_g(u^q P) = \int_M \langle \pi \nabla \psi, P \rangle. \quad (126)$$

By analyzing the homogeneous operator associated to the equation above,

$$\int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + \tilde{Q}') \right) \operatorname{div}_g (u^q P) = 0, \quad (127)$$

we check that it is positive definite, thus invertible. Consider the functional

$$F(P) = \int_M u^{-2q} \tilde{N} \operatorname{div}_g \left(u^q (\tilde{V} + P) \right)^2 dv_g \quad (128)$$

on the finite-dimensional space of conformal Killing fields and note that \tilde{Q}' is stationary for F . Since F is quadratic and non-negative definite, stationary points are associated to minimizers. If \bar{g} does not admit any nontrivial true Killing fields, then every conformal Killing field P satisfies $\operatorname{div} P \neq 0$ and the quadratic term of F is positive definite. On the other hand, if g admits proper Killing fields, then F descends to a functional on the quotient space and its quadratic order term is again positive definite. So the minimum of F is unique up to a true Killing field.

The conformal system proposed by Maxwell [14] becomes in this framework

$$\begin{cases} \Delta_g u + \frac{n-2}{4(n-1)} (R(g) - |\nabla \psi|_g^2) u - \frac{(n-2)}{4(n-1)} \frac{|U + \mathcal{L}_g W|^2 + \pi^2}{u^{q+1}} \\ - \frac{n-2}{4(n-1)} \left[2V(\psi) - \frac{n-1}{n} \left(\tau^* + \frac{\tilde{N} \operatorname{div}_g (u^q (\tilde{V} + \tilde{Q}))}{u^{2q}} \right) \right]^2 u^{q-1} = 0 \\ \operatorname{div}_g \left(\frac{\tilde{N}}{2} \mathcal{L}_g W \right) = \frac{n-1}{n} u^q \mathbf{d} \left(\frac{\tilde{N} \operatorname{div}_g (u^q (\tilde{V} + \tilde{Q}))}{2u^{2q}} \right) + \pi \nabla \psi, \end{cases} \quad (129)$$

whose solution (u, W, \tilde{Q}) is a smooth positive function u , a smooth vector field W , defined up to a conformal Killing field, and \tilde{Q} a conformal Killing field defined up to a true Killing field.

The existence of solutions to (129) follows from Theorem 1.1 and is similar to Corollary 1.1, with slight modifications. Here,

$$\begin{aligned} \rho_1 &= \frac{n-2}{4(n-1)} \left[\pi - \frac{n-1}{n} \tilde{N}^2 \operatorname{div}_g (\tilde{Q} + \tilde{V}) \right], \quad b = -\tau^* \tilde{N} \operatorname{div}_g (\tilde{V} + \tilde{Q}), \\ Y &= \sqrt{\frac{n}{n-2}} \tilde{N} (\tilde{V} + \tilde{Q}) \end{aligned} \quad (130)$$

and

$$C_{\mathcal{R}} = C_{\mathcal{R}} (\|\tilde{Q}\|_{\mathcal{C}^2}). \quad (131)$$

Moreover, we define θ and T as in (6) and (7) respectively, but without the dependency on $\rho_1 = \rho_1(\tilde{Q})$, *i.e.*

$$\theta = \min(\inf_M f), \quad (132)$$

and

$$T = \max(\|f\|_{\mathcal{C}^{1,\eta}}, \|c\|_{\mathcal{C}^{0,\gamma}}, \|d\|_{\mathcal{C}^{0,\gamma}}, \|h\|_{\mathcal{C}^{0,\gamma}}). \quad (133)$$

First of all, the stability of the first equation still holds, as in Lemma 2.1 and Lemma 3.9. In order to apply the last theorem, we need to check that: $\rho_1(\tilde{Q}) > \theta$, $\|\rho_1(\tilde{Q})\|_{C^{0,\gamma}} < 2T$, $\|b\|_{C^{0,\gamma}} \leq C_{\theta,2T}$ and $\|Y\|_{C^{0,\gamma}} \leq C_{\theta,2T}$. This translates to

$$\operatorname{div}_g \tilde{Q} < \left(\frac{n-2}{4(n-1)}\pi - \theta \right) \frac{n}{n-1} \tilde{N}^{-2} - \operatorname{div}_g \tilde{V}, \quad (134)$$

$$\frac{n-2}{4(n-1)} \|\pi\|_{C^{0,\gamma}} + \frac{n-1}{n} \|\tilde{N} \operatorname{div}_g(\tilde{V} + \tilde{Q})\|_{C^{0,\gamma}} \leq 2T, \quad (135)$$

$$\|\tau^* \tilde{N} \operatorname{div}_g(\tilde{V} + \tilde{Q})\|_{C^{0,\gamma}} \leq C_{\theta,2T}, \quad (136)$$

and

$$\sqrt{\frac{n}{n-2}} \|\tilde{N}(\tilde{V} + \tilde{Q})\| \leq C_{\theta,2T}. \quad (137)$$

We find bounds on \tilde{Q} depending on π , ψ , \tilde{N} , \tilde{V} from (126), thereby proving the necessary compactness. Finally, the continuity $(a, b, Y) \rightarrow u_{a,b,Y}$ doesn't pose any problem, and the proof mirrors our previous argument for the continuity of $a \rightarrow u(a)$. This shows the existence of solutions (u, W, \tilde{Q}) .

5 Annex

We used the following result repeatedly throughout the paper.

Lemma 5.1. *Let u be a bounded subharmonic function defined on \mathbb{R}^n . If there exists $0 < \varepsilon \leq u$ which bounds u from below and $\alpha > 0$ such that $u^{-\alpha}$ is a subharmonic function, then u is a constant.*

Proof of Lemma 5.1: Let us denote

$$\bar{u}_x(R) := \frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_x(R)} u(y) dy$$

the average of a smooth function u over the sphere $\partial B_x(R)$. We will sometimes use the simplified notation $\bar{u}(R)$. Recall that, given any subharmonic function u , $x \in \mathbb{R}^n$ and for any two radii $R \leq \tilde{R}$, then

$$\bar{u}_x(R) \leq \bar{u}_x(\tilde{R}). \quad (138)$$

This follows from

$$r^{n-1} \bar{u}'(r) = \frac{1}{\omega_{n-1}} \int_{\partial B_x(r)} \partial_\nu u(y) dy = -\frac{1}{\omega_{n-1}} \int_{B_x(r)} \Delta u(y) dy \geq 0$$

where $r > 0$ and ν is the exterior normal.

Note that $u^{-\alpha} \leq \varepsilon^{-\alpha}$ implies that the average of $u^{-\alpha}$ on arbitrary subsets is uniformly bounded. Let us fix $x \in \mathbb{R}^n$. Since $u^{-\alpha}$ is bounded, there exists a constant $M > 0$ and a sequence of radii $R_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$M^{-\alpha} := \lim_{i \rightarrow \infty} \overline{u^{-\alpha}}_x(R_i). \quad (139)$$

In fact, because the averages are decreasing (138), any sequence $R \rightarrow \infty$ around any point in \mathbb{R}^n leads to the same limit M , since one may always find a subsequence of R_i such that $B_x(R_i)$ includes the new sequence.

As $u^{-\alpha}$ is subharmonic,

$$u^{-\alpha}(x) \leq \overline{u^{-\alpha}}_x(R)$$

and therefore $u^{-\alpha}(x) \leq M^{-\alpha}$, or equivalently

$$M \leq u(x). \quad (140)$$

For $z \in \mathbb{R}^n$, let $R := |z - x|$ and $\tilde{R} > R$. By Green's representation theorem, we get

$$\begin{aligned} u(z) &\leq \int_{\partial B_x(\tilde{R})} u(y) \frac{\tilde{R}^2 - R^2}{\omega_{n-1} \tilde{R} |z - y|^n} dy \\ &\leq \frac{(\tilde{R} + R) \tilde{R}^{n-2}}{(\tilde{R} - R)^{n-1}} \bar{u}_x(\tilde{R}). \end{aligned} \quad (141)$$

For $\delta > 0$, we denote

$$\Omega_{\delta,R} := \{z \in \partial B_x(R), u(z) \geq M + \delta\}$$

a subset of $\partial B_x(R)$ and let

$$\theta_{\delta,R} := \frac{|\Omega_{\delta,R}|}{|\partial B_x(R)|} \in [0, 1]$$

be the corresponding relative size of its volume. Note that $\theta_{\delta,R} \rightarrow 0$ as $R \rightarrow \infty$. Otherwise, if there exists $\varepsilon \in (0, 1]$ such that

$$\limsup_{R \rightarrow \infty} \frac{|\{z \in \partial B_x(R), u(z) \geq M + \delta\}|}{|\partial B_x(R)|} = \varepsilon$$

then

$$\limsup_{R \rightarrow \infty} \overline{u^{-\alpha}}_x(R) \leq \varepsilon(M + \delta)^{-\alpha} + (1 - \varepsilon)M^{-\alpha} < M^{-\alpha}$$

which contradicts our definition (139) of M .

By choosing R large, $\theta_{\delta,R} \leq \delta$. Let

$$\lambda_{\delta,i} := \bar{u}_x(2^i R)$$

Note that, by (141), $\lambda_{\delta,i} \leq 3 \times 2^{n-2} \lambda_{\delta,i+1}$. Since

$$u(x) \leq \lambda_{\delta,i} \leq (M + \delta)(1 - \theta_{\delta,2^i R}) + \lambda_{\delta,i+1} \times \theta_{\delta,2^i R}$$

then, by induction,

$$u(x) \leq (M + \delta) \frac{1 - \delta^l}{1 - \delta} + \lambda_l \delta^l$$

for all $l \in \mathbb{N}$. As we take $l \rightarrow \infty$,

$$u(x) \leq (M + \delta) \frac{1}{1 - \delta}$$

for any $\delta > 0$, and therefore $u(x) \leq M$. By (140), $u(x) \equiv M$.

We may apply the same argument to any other $\tilde{x} \in \mathbb{R}^n$ and obtain the same value $u(\tilde{x}) = M$. Indeed, assuming that

$$\tilde{M}^{-\alpha} := \lim_{\tilde{R} \rightarrow \infty} \overline{u^{-\alpha}_{\tilde{x}}(\tilde{R})}$$

so that $\tilde{M}^{-\alpha} \geq M^{-\alpha}$, then for \tilde{R} large, $\overline{u^{-\alpha}_{\tilde{x}}(\tilde{R})} \geq M^{-\alpha}$. But, at the same time, given any fixed \tilde{R} , then for R sufficiently large, by (141), $\overline{u^{-\alpha}_{\tilde{x}}(\tilde{R})} \leq \overline{u^{-\alpha}_x(R)}$. Thus we obtain that $u \equiv M$ in \mathbb{R}^n . \square

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