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Box-Cox Gamma-G family of distributions: Theory and Applications

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Abstract

This paper is devoted to a new family of distributions called the Box-Cox Gamma-G family of distributions. It can be viewed as a natural generalization of the useful RB-G family of distributions, containing a wide variety of power Gamma-G distributions, including the odd one. The key tool of this generalization is the use of the Box-Cox transformation. Some mathematical properties of the new family of distributions are derived. Then a specific member with three parameters, based on the half-Cauchy distribution as baseline, is studied and considered as a statistical model. The estimation of its parameters is discussed by the method of maximum likelihood. A simulation study is provided to support the theoretical convergence of the estimators. Finally, two real data sets are considered to show the power of adjustment of the new model compared to other competitive models.

Keywords: Generalized distribution; Box-Cox transformation; Mathematical properties; Maximum likelihood estimation; Half-Cauchy distribution.

2000 MSC: 60E05, 62E15, 62F10.

1. Introduction

The standard probability distributions do not offer a convincing statistical models for a large panel of practical data sets. This fact has been the motor of studies dedicated to the creation of new probability distributions by various approaches. One of the most popular approach is the use of generators of probability distributions. The most popular ones include the Marshall-Olkin-G by Marshall and Olkin (1997), the exp-G by Gupta *et al.* (1998), the beta-G by Eugene *et al.* (2002), the gamma-G by Zografos and Balakrishnan (2009), the Kumaraswamy-G by Cordeiro and de Castro (2011), the RB-G (also called gamma-G type 2) by Ristić and Balakrishnan (2012), the exponentiated generalized-G by Cordeiro *et al.* (2013), the logistic-G by Torabi and Montazeri (2014), the TX-G by Alzaatreh *et al.* (2013), the Weibull-G by Bourguignon *et al.* (2014), the exponentiated half-logistic-G by Cordeiro *et al.* (2014), the odd generalized exponential family by Tahir *et al.* (2015), the odd Burr III-G by Jamal *et al.* (2017), the cosine-sine-G by Chesneau *et al.* (2018), the generalized odd Gamma-G by Hosseini *et al.* (2018) and the extended odd-G family by Bakouch *et al.* (2019).

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In this paper, we propose a new family of distributions, based on a new generator, with interesting features for the statistician. For a first approach, we can say that generalize, in some sense, the RB-G by Ristić and Balakrishnan (2012) by the use of the Box-Cox transformation. Let us now briefly present the RB-G. Let $G(x; \phi)$ be a cumulative distribution function (cdf), where ϕ denotes one parameter or several parameters. Then the RB-G is characterized by the following cdf:

$$F(x; \delta, \phi) = 1 - \frac{1}{\Gamma(\delta)} \gamma(\delta, -\log[G(x; \phi)]), \quad (1)$$

where $\gamma(\delta, u)$ denotes the lower incomplete gamma function defined by $\gamma(\delta, u) = \int_0^u t^{\delta-1} e^{-t} dt$. The RB-G family of distributions is very rich, providing successful solutions for modeling various kinds of data. We refer to the extensive survey of Cordeiro and Bourguignon (2016), and the references therein. The success of the RB-G family of distributions is a motivation to introduce natural extensions, with a great potential of applicability, as the one studied in this paper. Our idea is to introduce an additional parameter $\lambda > 0$ and to consider the cdf given by

$$F(x; \lambda, \delta, \phi) = 1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right). \quad (2)$$

Thus, in comparison to the RB-G cdf given by (1), we have replaced the logarithmic term $-\log[G(x; \phi)] = \log[G(x; \phi)^{-1}]$ by the affine power transformation $[G(x; \phi)^{-\lambda} - 1]/\lambda$, which corresponds to the so-called Box-Cox transformation of $G(x; \phi)^{-1}$ with parameter λ . Indeed, introducing the Box-Cox transformation defined by $b_\lambda(y) = (y^\lambda - 1)/\lambda$, we have $[G(x; \phi)^{-\lambda} - 1]/\lambda = b_\lambda[G(x; \phi)^{-1}]$. The advantages to consider the Box-Cox transformation in our context are the following: (i) When $\lambda \rightarrow 0$, we have $b_\lambda[G(x; \phi)^{-1}] = -\log[G(x; \phi)]$ and the cdf $F(x; \lambda, \delta, \phi)$ given by (2) becomes the RB-G cdf given by (1), (ii) When $\lambda = 1$, we have $b_\lambda[G(x; \phi)^{-1}] = G(x; \phi)^{-1} - 1 = (1 - G(x; \phi))/G(x; \phi)$, which corresponds to the odd transformation of $G(x; \phi)$, widely used this last decades to define new flexible families of distributions, with physical interpretations (in the context of Gamma-G, see Hosseini *et al.* (2018), and the references therein) (iii) In full generality, the consideration of the power transform of $G(x; \phi)$ increases its flexibility and the flexibility of the related family (see, for instance, Gupta *et al.* (1998) for the former power family of distributions). The family of distributions characterized by the cdf (2) will be called the Box-Cox Gamma-G family of distributions, with BCG-G as an abbreviation of Box-Cox Gamma-G.

In this paper, we aim to study the BCG-G family (of distributions) in detail, by examining both the theoretical and practical aspects, with discussions. For the practice, the half-Cauchy cdf is considered for $G(x; \phi)$, offering a new solution for modelling data presenting highly skewed distribution to the right. In particular, we show that our model outperforms, in some sense, well established competitors, highlighting the importance of the BCG-G family.

This paper is organized as follows. In Section 2, we present the two other crucial functions of the BCG-G family: the probability density function and the hazard rate function. Then some members of the BCG-G family with a potential of interest are listed. In Section 3, we derive the general mathematical properties of the BCG-G family such as the asymptotic behavior of the crucial functions, their shapes, some immediate characterizations of the family, the quantile function, a result on stochastic ordering, the useful series expansions of the crucial functions, the moments, the incomplete moments with some derivations, the moment generating function, the Rényi entropy, the probability density function of order statistics

with moments and some generalities on the maximum likelihood estimates. Section 4 is devoted to a member of the BCG-G family defined with the half-Cauchy distribution as baseline. The most interesting features of this new distribution are then presented with illustration by numerical results and graphics. Section 5 is devoted to the statistical inference of this distribution via the use of the maximum likelihood method. Analysis of two practical data sets are also performed. The paper is concluded in Section 6.

2. Description of the BCG-G family

First of all, let us recall that the BCG-G family is characterized by the cdf $F(x; \lambda, \delta, \phi)$ given by (2). By derivation, the associated probability density function (pdf) is given by

$$f(x; \lambda, \delta, \phi) = \frac{1}{\Gamma(\delta)} g(x; \phi) G(x; \phi)^{-\lambda-1} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{\delta-1} e^{-\left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)}, \quad x \in \mathbb{R}. \quad (3)$$

The associated hazard rate function (hrf) is given by

$$\begin{aligned} h(x; \lambda, \delta, \phi) &= \frac{f(x; \lambda, \delta, \phi)}{1 - F(x; \lambda, \delta, \phi)} \\ &= \frac{1}{\gamma\left(\delta, \frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)} g(x; \phi) G(x; \phi)^{-\lambda-1} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{\delta-1} e^{-\left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)}, \quad x \in \mathbb{R}. \quad (4) \end{aligned}$$

Some members of the BCG-G family are presented in Table 1, taking standard distributions for the baseline cdf $G(x; \phi)$, with various supports and number of parameters. To the best of our knowledge, none of them has been studied in the literature.

Table 1: Some members of the BCG-G family described by their cdfs.

Distribution G	Support	cdf of the BCG-G	Parameters
Uniform	$(0, \theta)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{(x/\theta)^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta)$
Exponential	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{(1 - e^{-\theta x})^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta)$
Weibull	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{(1 - e^{-\theta x^\alpha})^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta, \alpha)$
Fréchet	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{e^{\lambda \theta x^{-\alpha}} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta, \alpha)$
Pareto	$(\theta, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[1 - (\theta/x)^k]^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta, k)$
Burr XII	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{\{1 - [1 + (x/s)^c]^{-k}\}^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, s, k, c)$
Gamma	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[\gamma(\theta, x)/\Gamma(\theta)]^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta)$
half-Cauchy	$(0, +\infty)$	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[(2/\pi) \arctan(x/\theta)]^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \theta)$
Logistic	\mathbb{R}	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[1 + e^{-(x-\mu)/s}]^\lambda - 1}{\lambda}\right)$	$(\lambda, \delta, \mu, s)$
Gumbel	\mathbb{R}	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{\exp(\lambda e^{-(x-\mu)/\sigma}) - 1}{\lambda}\right)$	$(\lambda, \delta, \mu, \sigma)$
Normal	\mathbb{R}	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{(\Phi((x-\mu)/\sigma))^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \mu, \sigma)$
Cauchy	\mathbb{R}	$1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[(1/\pi) \arctan((x-\mu)/\theta) + 1/2]^{-\lambda} - 1}{\lambda}\right)$	$(\lambda, \delta, \mu, \theta)$

Let us mention that the BCG-G member defined with the half-Cauchy cdf as baseline will be in the center of the applications in Section 4 (for reasons explained later).

3. General mathematical properties

This section is devoted to general mathematical properties of the BCG-G family.

3.1. Asymptotic behavior

We now investigate the asymptotic behavior of the BCG-G cdf $F(x; \lambda, \delta, \phi)$, pdf $f(x; \lambda, \delta, \phi)$ and hrf $h(x; \lambda, \delta, \phi)$ given by (2), (3) and (4) respectively. When $G(x; \phi) \rightarrow 0$, using the following approximation result: $\int_u^{+\infty} t^{\delta-1} e^{-t} dt \sim u^{\delta-1} e^{-u}$ when $u \rightarrow \infty$, we have

$$F(x; \lambda, \delta, \phi) \sim \frac{1}{\Gamma(\delta)\lambda^{\delta-1}} G(x; \phi)^{-\lambda(\delta-1)} e^{-\frac{G(x; \phi)^{-\lambda}}{\lambda}},$$

$$f(x; \lambda, \delta, \phi) \sim \frac{1}{\Gamma(\delta)\lambda^{\delta-1}} g(x; \phi) G(x; \phi)^{-\lambda\delta-1} e^{-\frac{G(x; \phi)^{-\lambda}}{\lambda}}$$

and

$$h(x; \lambda, \delta, \phi) \sim f(x; \lambda, \delta, \phi) \sim \frac{1}{\Gamma(\delta)\lambda^{\delta-1}} g(x; \phi) G(x; \phi)^{-\lambda\delta-1} e^{-\frac{G(x; \phi)^{-\lambda}}{\lambda}}.$$

On the other side, when $G(x; \phi) \rightarrow 1$, using the following approximation results: $\gamma(\delta, u) \sim u^\delta/\delta$ and $(1-u)^{-\lambda} \sim 1 + \lambda u$ when $u \rightarrow 0$, we have

$$F(x; \lambda, \delta, \phi) \sim 1 - \frac{1}{\Gamma(\delta)\delta} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^\delta \sim 1 - \frac{1}{\Gamma(\delta)\delta} (1 - G(x; \phi))^\delta,$$

$$f(x; \lambda, \delta, \phi) \sim \frac{1}{\Gamma(\delta)} g(x; \phi) \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{\delta-1} \sim \frac{1}{\Gamma(\delta)} g(x; \phi) (1 - G(x; \phi))^{\delta-1}$$

and

$$h(x; \lambda, \delta, \phi) \sim \delta \frac{g(x; \phi)}{1 - G(x; \phi)}.$$

Thus the asymptotic behavior of $h(x; \lambda, \delta, \phi)$ is proportional to the hrf associated to $G(x; \phi)$, with δ as coefficient of proportionality. Furthermore, we can notice that, for the three functions, the impact of the parameter λ is strong when $G(x; \phi) \rightarrow 0$ and inexistant when $G(x; \phi) \rightarrow \infty$. So it plays an important role to modulate the left skewness of the distribution.

3.2. Shapes of the BCG-G pdf and the hrf

In what follows, we describe analytically the shapes of the BCG-G pdf and hrf. The critical points of the BCG-G pdf are the root of the equation $\partial \log[f(x; \lambda, \delta, \phi)]/\partial x = 0$, where

$$\begin{aligned} \frac{\partial}{\partial x} \log[f(x; \lambda, \delta, \phi)] &= \frac{\partial g(x; \phi)/\partial x}{g(x; \phi)} - (\lambda + 1) \frac{g(x; \phi)}{G(x; \phi)} - \lambda(\delta - 1) \frac{G(x; \phi)^{-\lambda-1} g(x; \phi)}{G(x; \phi)^{-\lambda} - 1} \\ &+ G(x; \phi)^{-\lambda-1} g(x; \phi). \end{aligned}$$

We have no guaranty for the uniqueness a critical point for any $G(x; \phi)$; more than one root can exist. Let $\xi(x) = \partial^2 \log[f(x; \lambda, \delta, \phi)]/\partial x^2$. Then we have

$$\begin{aligned} \xi(x) &= \frac{(\partial^2 g(x; \phi)/\partial x^2)g(x; \phi) - (\partial g(x; \phi)/\partial x)^2}{g(x; \phi)^2} - (\lambda + 1) \frac{(\partial g(x; \phi)/\partial x)G(x; \phi) - g(x; \phi)^2}{G(x; \phi)^2} \\ &\quad - \lambda(\delta - 1) \frac{(\partial g(x; \phi)/\partial x)G(x; \phi)(1 - G(x; \phi)^\lambda) - g(x; \phi)^2(1 - (\lambda + 1)G(x; \phi)^\lambda)}{G(x; \phi)^2(1 - G(x; \phi)^\lambda)^2} \\ &\quad - (\lambda + 1)G(x; \phi)^{-\lambda-2}g(x; \phi)^2 + G(x; \phi)^{-\lambda-1}\partial g(x; \phi)/\partial x. \end{aligned}$$

If $x = x_0$ is a critical point, then it corresponds to a local maximum if $\xi(x_0) < 0$, a local minimum if $\xi(x_0) > 0$ and a point of inflection if $\xi(x_0) = 0$.

In the same way, the critical points of the BCG-G hrf are the root of the equation $\partial \log[h(x; \lambda, \delta, \phi)]/\partial x = 0$ where

$$\begin{aligned} \frac{\partial}{\partial x} \log[h(x; \lambda, \delta, \phi)] &= \\ \frac{\partial g(x; \phi)/\partial x}{g(x; \phi)} - (\lambda + 1) \frac{g(x; \phi)}{G(x; \phi)} - \lambda(\delta - 1) \frac{G(x; \phi)^{-\lambda-1}g(x; \phi)}{G(x; \phi)^{-\lambda} - 1} + G(x; \phi)^{-\lambda-1}g(x; \phi) \\ &\quad + \frac{1}{\gamma\left(\delta, \frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)} g(x; \phi)G(x; \phi)^{-\lambda-1} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{\delta-1} e^{-\left(\frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)}. \end{aligned}$$

Again, more than one root can be obtained. Let $\zeta(x) = \partial^2 \log[h(x; \lambda, \delta, \phi)]/\partial x^2$. Then we have

$$\begin{aligned} \zeta(x) &= \frac{(\partial^2 g(x; \phi)/\partial x^2)g(x; \phi) - (\partial g(x; \phi)/\partial x)^2}{g(x; \phi)^2} - (\lambda + 1) \frac{(\partial g(x; \phi)/\partial x)G(x; \phi) - g(x; \phi)^2}{G(x; \phi)^2} \\ &\quad - \lambda(\delta - 1) \frac{(\partial g(x; \phi)/\partial x)G(x; \phi)(1 - G(x; \phi)^\lambda) - g(x; \phi)^2(1 - (\lambda + 1)G(x; \phi)^\lambda)}{G(x; \phi)^2(1 - G(x; \phi)^\lambda)^2} \\ &\quad - (\lambda + 1)G(x; \phi)^{-\lambda-2}g(x; \phi)^2 + G(x; \phi)^{-\lambda-1}\partial g(x; \phi)/\partial x \\ &\quad + \frac{1}{\gamma\left(\delta, \frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)} \left[(\partial g(x; \phi)/\partial x)G(x; \phi)^{-\lambda-1} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{\delta-1} \right. \\ &\quad - (\lambda + 1)g(x; \phi)^2G(x; \phi)^{-\lambda-2} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{\delta-1} \\ &\quad - (\delta - 1)g(x; \phi)^2G(x; \phi)^{-2(\lambda+1)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{\delta-2} \\ &\quad \left. + g(x; \phi)^2G(x; \phi)^{-2(\lambda+1)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{\delta-1} \right] e^{-\left(\frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)} \\ &\quad + \frac{1}{\gamma\left(\delta, \frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)^2} g(x; \phi)^2G(x; \phi)^{-2(\lambda+1)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda}\right)^{2(\delta-1)} e^{-2\left(\frac{G(x; \phi)^{-\lambda-1}}{\lambda}\right)}. \end{aligned}$$

If $x = x_0$ is a critical point, then it corresponds to a local maximum if $\zeta(x_0) < 0$, a local minimum if $\zeta(x_0) > 0$ and a point of inflection if $\zeta(x_0) = 0$. Finally, let us mention that the critical points can be determined by using symbolic computation software (Mathematica, Maple...).

3.3. Characterization

Let Y be a random variable following the Gamma distribution with parameters 1 and δ , i.e. with pdf $r(x) = (1/\Gamma(\delta))x^{\delta-1}e^{-x}$, $x > 0$ and $Q_G(y; \phi)$ be the quantile function associated to $G(x; \phi)$, i.e. $G[Q_G(y; \phi); \phi] = Q_G[G(y; \phi); \phi] = y$, $y \in (0, 1)$. Then the random variable

$$X = Q_G \left([1 + \lambda Y]^{-1/\lambda}; \phi \right)$$

has the BCG-G pdf given by (3). Furthermore, if X is a random variable having the BCG-G pdf, then the random variable

$$Y = \frac{G(X; \phi)^{-\lambda} - 1}{\lambda}$$

follows the Gamma distribution with parameters 1 and δ .

3.4. Quantile function, skewness and kurtosis

The quantile function of the BCG family of distributions is defined by

$$Q(y; \lambda, \delta, \phi) = Q_G \left([1 + \lambda \gamma^{-1}(\delta, (1-y)\Gamma(\delta))]^{-1/\lambda}; \phi \right), \quad y \in (0, 1),$$

where $\gamma^{-1}(\delta, u)$ denotes the inverse of $\gamma(\delta, u)$. The first quartile is given by $Q(1/4; \lambda, \delta, \phi)$, the median is given by $M = Q(1/2; \lambda, \delta, \phi)$ and the third quartile is defined by $Q(3/4; \lambda, \delta, \phi)$.

Also, from $Q(y; \lambda, \delta, \phi)$, one can defined several robust measures of skewness and kurtosis as the Galton skewness S introduced by Galton (1883) and the Moors kurtosis K introduced by Moors (1988). The skewness S measures the degree of the long tail, while the kurtosis K measures the degree of tail heaviness. They are respectively defined by

$$S = \frac{Q(6/8; \lambda, \delta, \phi) - 2Q(4/8; \lambda, \delta, \phi) + Q(2/8; \lambda, \delta, \phi)}{Q(6/8; \lambda, \delta, \phi) - Q(2/8; \lambda, \delta, \phi)} \quad (5)$$

and

$$K = \frac{Q(7/8; \lambda, \delta, \phi) - Q(5/8; \lambda, \delta, \phi) + Q(3/8; \lambda, \delta, \phi) - Q(1/8; \lambda, \delta, \phi)}{Q(6/8; \lambda, \delta, \phi) - Q(2/8; \lambda, \delta, \phi)}. \quad (6)$$

Their main advantages are to be robust to eventual outliers and to always exist (whatever the existence of moments).

3.5. Stochastic ordering

A stochastic ordering result on the BCG-G family is now presented. Let $\delta_1 > 0$, $\delta_2 > 0$, and X_1 and X_2 be two random variables with X_1 having the BCG-G pdf $f(x; \lambda, \delta_1, \phi)$ and δ_1 and X_2 having the BCG-G pdf $f(x; \lambda, \delta_2, \phi)$. Then, if $\delta_2 \leq \delta_1$, the ratio function $f(x; \lambda, \delta_1, \phi)/f(x; \lambda, \delta_2, \phi)$ is decreasing. Indeed, we have

$$\frac{f(x; \lambda, \delta_1, \phi)}{f(x; \lambda, \delta_2, \phi)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{\delta_1 - \delta_2},$$

and

$$\frac{\partial}{\partial x} \frac{f(x; \lambda, \delta_1, \phi)}{f(x; \lambda, \delta_2, \phi)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} (\delta_2 - \delta_1) \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{\delta_1 - \delta_2 - 1} G(x; \phi)^{-\lambda - 1} g(x; \phi) \leq 0.$$

This implies that X_2 is stochastically greater than X_1 with respect to the likelihood ratio order, implying others stochastic ordering informations (see, for instance, Shaked and Shanthikumar (2007)).

3.6. Useful expansions

Firstly, let us determine a series expansion for the BCG-G cdf. Using the exponential series expansion, we have

$$\gamma(\delta, u) = \int_0^u t^{\delta-1} e^{-t} dt = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \int_0^u t^{k+\delta-1} dt = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+\delta)} u^{k+\delta}.$$

Therefore

$$\begin{aligned} F(x; \lambda, \delta, \phi) &= 1 - \frac{1}{\Gamma(\delta)} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+\delta)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{k+\delta} \\ &= 1 - \frac{1}{\Gamma(\delta)\lambda^\delta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+\delta)\lambda^k} \left(1 - G(x; \phi)^\lambda \right)^{k+\delta} G(x; \phi)^{-\lambda(k+\delta)}. \end{aligned}$$

It follows from the general binomial theorem that

$$\left(1 - G(x; \phi)^\lambda \right)^{k+\delta} = \sum_{\ell=0}^{+\infty} \binom{k+\delta}{\ell} (-1)^\ell G(x; \phi)^{\lambda\ell}.$$

Applying the general binomial theorem two times, we obtain

$$\begin{aligned} G(x; \phi)^{-\lambda(k+\delta-\ell)} &= \sum_{m=0}^{+\infty} \binom{-\lambda(k+\delta-\ell)}{m} (-1)^m (1 - G(x; \phi))^m \\ &= \sum_{m=0}^{+\infty} \sum_{u=0}^m \binom{-\lambda(k+\delta-\ell)}{m} \binom{m}{u} (-1)^{m+u} G(x; \phi)^u \\ &= \sum_{u=0}^{+\infty} \sum_{m=u}^{+\infty} \binom{-\lambda(k+\delta-\ell)}{m} \binom{m}{u} (-1)^{m+u} G(x; \phi)^u. \end{aligned}$$

Hence we can write

$$F(x; \lambda, \delta, \phi) = 1 - \sum_{u=0}^{+\infty} c_u W_u(x; \phi),$$

where

$$c_u = \frac{(-1)^u}{\Gamma(\delta)\lambda^\delta} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \sum_{m=u}^{+\infty} \frac{(-1)^{k+\ell+m}}{k!(k+\delta)\lambda^k} \binom{k+\delta}{\ell} \binom{-\lambda(k+\delta-\ell)}{m} \binom{m}{u}$$

and $W_u(x; \phi) = G(x; \phi)^u$. Let us notice that $W_u(x; \phi)$ is the well-known exp-G cdf with power parameter u . Further details on this family of distributions can be found in Gupta *et al.* (1998).

An alternative expression is given by

$$F(x; \lambda, \delta, \phi) = \sum_{u=0}^{+\infty} d_u W_u(x; \phi), \quad (7)$$

where $d_0 = 1 - c_0$ and $d_u = -c_u$ for $u \geq 1$. With this configuration, the BCG-G cdf can be expressed as an infinite linear combination of exp-G cdfs.

Let us now introduce the pdf of the exp-G distribution with power parameter $u + 1$, i.e. $w_{u+1}(x; \phi) = (u + 1)G(x; \phi)^u g(x; \phi)$. Then, by derivation of (7), we deduce the series expansion for $f(x; \lambda, \delta, \phi)$ given by

$$f(x; \lambda, \delta, \phi) = \sum_{u=0}^{+\infty} d_{u+1} w_{u+1}(x; \phi). \quad (8)$$

This expansion can be useful to determine important mathematical properties of the BCG-G family, as moments of different nature. Some of them are presented below.

3.7. Moments

Important note: Hereafter, let X be a random variable having the BCG-G pdf $f(x; \lambda, \delta, \theta)$ given by (3) and, for any integer u , let Y_u be a random variable having the exp-G pdf given by $w_{u+1}(x; \phi) = (u + 1)G(x; \phi)^u g(x; \phi)$. Also, when a quantity is introduced, it is assumed that it exists, which is not necessarily the case depending on the choice for $G(x; \phi)$.

The r -th moment of X is given by

$$\mu'_r = \mathbb{E}(X^r) = \int_{-\infty}^{+\infty} x^r f(x; \lambda, \delta, \phi) dx.$$

Using the series expansion given by (8), we also have

$$\mu'_r = \sum_{u=0}^{+\infty} d_{u+1} \mathbb{E}(Y_u^r), \quad (9)$$

where

$$\mathbb{E}(Y_u^r) = \int_{-\infty}^{+\infty} x^r w_{u+1}(x; \phi) dx = (u + 1) \int_0^1 [Q_G(y; \phi)]^r y^u dy.$$

If the closed-form expression is not available, the integral term can be evaluated numerically for a given cdf $G(x; \phi)$. The mean of X is given by $\mathbb{E}(X) = \mu'_1$ and the variance of X is given by $\mathbb{V}(X) = \mathbb{E}((X - \mu'_1)^2) = \mu'_2 - (\mu'_1)^2$. In order to complete this part, let us mention that r -th central moment of X can be determined by using μ'_1, \dots, μ'_r as follows:

$$\mu_r = \mathbb{E}[(X - \mu'_1)^r] = \sum_{k=0}^r \binom{r}{k} (-1)^k (\mu'_1)^k \mu'_{r-k}.$$

3.8. Cumulants, skewness and kurtosis

As usual, the r -th cumulants of X can be obtained by the following recursion formula:

$$\kappa_r = \mu'_r - \sum_{k=1}^{r-1} \binom{r-1}{k-1} \mu'_{r-k} \kappa_k,$$

with the initial value: $\kappa_1 = \mu'_1$. From the cumulants of X , we can define the skewness and the kurtosis of X , which are respectively defined by

$$\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2}. \quad (10)$$

Both of them can be computed numerically for a given cdf $G(x; \phi)$.

3.9. Incomplete moments

The r -th incomplete moment of X is given by

$$\mu_r^*(t) = \mathbb{E}(X^r \mathbf{1}_{\{X \leq t\}}) = \int_{-\infty}^t x^r f(x; \lambda, \delta, \phi) dx, \quad t > 0,$$

where $\mathbf{1}_A$ denotes the indicator function over the event A . Using the series expansion given by (8), we also have

$$\mu_r^*(t) = \sum_{u=0}^{+\infty} d_{u+1} \mathbb{E}(Y_u^r \mathbf{1}_{\{Y_u \leq t\}}),$$

where

$$\mathbb{E}(Y_u^r \mathbf{1}_{\{Y_u \leq t\}}) = \int_{-\infty}^t x^r w_{u+1}(x; \phi) dx = (u+1) \int_0^{G(y; \phi)} [Q_G(y; \phi)]^r y^u dy.$$

From the incomplete moments, several important mathematical quantities related to the BCG-G family can be expressed. Some of them are presented below.

The mean deviation about the mean given by

$$\delta_1 = \mathbb{E}(|X - \mu'_1|) = 2\mu'_1 F(\mu'_1; \lambda, \delta, \phi) - 2\mu_1^*(\mu'_1).$$

The mean deviation about the median is given by

$$\delta_2 = \mathbb{E}(|X - M|) = \mu'_1 - 2\mu_1^*(M).$$

These two mean deviations can be used as measures of the degree of scatter of X .

The Bonferroni curve is given by

$$B(x) = \frac{\mu_1^*[Q(x; \lambda, \delta, \phi)]}{\mu'_1 Q(x; \lambda, \delta, \phi)}, \quad x \in (0, 1).$$

The Lorenz curve is given by

$$L(x) = \frac{\mu_1^*[Q(x; \lambda, \delta, \phi)]}{\mu'_1}, \quad x \in (0, 1).$$

These curves find applications in various areas such as econometrics, finance, medicine, insurance, demography and insurance. We refer to Sarabia (2008).

Finally, one can also mention the r -th moment of the reversed residual life for an integer r . Further details and applications on this mathematical object can be found in Nanda *et al.* (2003). It is defined by

$$V(t) = \mathbb{E}((t - X)^r | X \leq t) = \frac{1}{F(t; \lambda, \delta, \phi)} \int_{-\infty}^t (t - x)^r f(x; \lambda, \delta, \phi) dx, \quad t > 0.$$

Using the binomial theorem, one can express it as

$$V(t) = \frac{1}{F(t; \lambda, \delta, \phi)} \sum_{k=0}^r \binom{r}{k} (-1)^k t^{r-k} \mu_r^*(t).$$

3.10. Moment generating function

The moment generating function of X is given by

$$M(t; \lambda, \delta, \phi) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x; \lambda, \delta, \phi) dx, \quad t \in \mathbb{R}.$$

Using the series expansion given by (8), we also have

$$M(t; \lambda, \delta, \phi) = \sum_{u=0}^{+\infty} d_{u+1} \mathbb{E}(e^{tY_u}),$$

where

$$\mathbb{E}(e^{tY_u}) = \int_{-\infty}^{+\infty} e^{tx} w_{u+1}(x; \phi) dx = (u+1) \int_0^1 e^{tQ_G(y; \phi)} y^u dy.$$

Again, the integral term can be evaluated numerically for a given cdf $G(x; \phi)$. We can refine the r -th moment of X from $M(t; \lambda, \delta, \phi)$ by using the formula $\mu'_r = \partial^r M(t; \lambda, \delta, \phi) / \partial t^r |_{t=0}$.

3.11. Rényi entropy

The Rényi entropy introduced by Rényi (1961) is a useful measure of variation of the uncertainty used in many areas as engineering, quantum information and ecology. This subsection is devoted to the Rényi entropy of the BCG-G family. Let $v > 0$ with $v \neq 1$. Then the Rényi entropy of the BCG-G family is given by

$$I(v) = \frac{1}{1-v} \log \left[\int_{-\infty}^{+\infty} f(x; \lambda, \delta, \phi)^v dx \right]. \quad (11)$$

We have

$$f(x; \lambda, \delta, \phi)^v = \frac{1}{\Gamma(\delta)^v} g(x; \phi)^v G(x; \phi)^{-v(\lambda+1)} \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)^{v(\delta-1)} e^{-v \left(\frac{G(x; \phi)^{-\lambda} - 1}{\lambda} \right)}.$$

Using the exponential series decomposition, we obtain

$$\begin{aligned} f(x; \lambda, \delta, \phi)^v &= \frac{1}{\Gamma(\delta)^v \lambda^{v(\delta-1)}} g(x; \phi)^v G(x; \phi)^{-v(\lambda+1)} \sum_{k=0}^{+\infty} \frac{(-1)^k v^k}{k! \lambda^k} \left(G(x; \phi)^{-\lambda} - 1 \right)^{k+v(\delta-1)} \\ &= \frac{1}{\Gamma(\delta)^v \lambda^{v(\delta-1)}} g(x; \phi)^v \sum_{k=0}^{+\infty} \frac{(-1)^k v^k}{k! \lambda^k} \left(1 - G(x; \phi)^\lambda \right)^{k+v(\delta-1)} G(x; \phi)^{-\lambda k - v(\lambda\delta+1)}. \end{aligned}$$

By the general binomial theorem, we obtain

$$f(x; \lambda, \delta, \phi)^v = \frac{1}{\Gamma(\delta)^v \lambda^{v(\delta-1)}} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} e_{k,\ell} G(x; \phi)^{-\lambda(k-\ell) - v(\lambda\delta+1)} g(x; \phi)^v,$$

where

$$e_{k,\ell} = \frac{(-1)^{k+\ell} v^k}{k! \lambda^k} \binom{k+v(\delta-1)}{\ell}.$$

Putting this series expansion in (11), we can express $I(v)$ as

$$I(v) = \frac{1}{1-v} \left\{ -v \log[\Gamma(\delta)] - v(\delta-1) \log(\lambda) \right. \\ \left. + \log \left[\sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} e_{k,\ell} \int_{-\infty}^{+\infty} G(x; \phi)^{-\lambda(k-\ell)-v(\lambda\delta+1)} g(x; \phi)^v dx \right] \right\}.$$

The last integral term can be expressed as

$$\int_{-\infty}^{+\infty} G(x; \phi)^{-\lambda(k-\ell)-v(\lambda\delta+1)} g(x; \phi)^v dx = \int_0^1 y^{-\lambda(k-\ell)-v(\lambda\delta+1)} g(Q_G(y; \phi); \phi)^{v-1} dy.$$

Numerical evaluation of this integral is feasible.

3.12. Order statistics

The order statistics are useful in statistics and probability theory. Here we aim to give tractable expressions for the pdfs of the order statistics, as well as their moments, in the context of the BCG-G family. Let X_1, \dots, X_n be n random variable having the BCG-G pdf. Then the pdf of the i -th order statistic of X_1, \dots, X_n is given by

$$f_{i:n}(x; \lambda, \delta, \phi) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F(x; \lambda, \delta, \phi)^{j+i-1} f(x; \lambda, \delta, \phi).$$

It follows from the series expansions for $F(x; \lambda, \delta, \phi)$ and $f(x; \lambda, \delta, \phi)$ given by (7) and (8) respectively that

$$f_{i:n}(x; \lambda, \delta, \phi) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[\sum_{u=0}^{+\infty} d_u W_u(x; \phi) \right]^{j+i-1} \sum_{v=0}^{+\infty} d_{v+1} w_{v+1}(x; \phi). \quad (12)$$

Since $W_u(x; \phi) = G(x; \phi)^u$, by virtue of a result by Gradshteyn and Ryzhik (2000), we have the following equality:

$$\left[\sum_{u=0}^{+\infty} d_u W_u(x; \phi) \right]^{j+i-1} = \sum_{k=0}^{+\infty} \xi_k W_k(x; \phi),$$

with ξ_k defined by the following recursive formula: $\xi_0 = d_0^{j+i-1}$ and, for $k \geq 1$,

$$\xi_k = \frac{1}{k d_0} \sum_{\ell=1}^k [\ell(j+i) - k] d_\ell \xi_{k-\ell}.$$

Now equation (12) becomes

$$f_{i:n}(x; \lambda, \delta, \phi) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{k=0}^{+\infty} \sum_{v=0}^{+\infty} \xi_k d_{v+1} W_k(x; \phi) w_{v+1}(x; \phi). \quad (13)$$

Let $X_{i:n}$ be the i -th order statistic of X_1, \dots, X_n . Then, using (13), the r -th moment of $X_{i:n}$ is given by

$$\begin{aligned} \mathbb{E}(X_{i:n}^r) &= \int_{-\infty}^{+\infty} x^r f_{i:n}(x; \lambda, \delta, \phi) dx \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{k=0}^{+\infty} \sum_{v=0}^{+\infty} \xi_k d_{v+1} \int_{-\infty}^{+\infty} x^r W_k(x; \phi) w_{v+1}(x; \phi) dx, \end{aligned}$$

with

$$\begin{aligned} \int_{-\infty}^{+\infty} x^r W_k(x; \phi) w_{v+1}(x; \phi) dx &= (v+1) \int_{-\infty}^{+\infty} x^r G(x; \phi)^{k+v} g(x; \phi) dx \\ &= (v+1) \int_0^1 [Q_G(y; \phi)]^r y^{k+v} dy. \end{aligned}$$

The last integral term can be computed numerically for most of the considered cdf $G(x; \phi)$.

Proceeding as the subsection above, one can also express other mathematical quantities, as the incomplete moments and the moment generating function of $X_{i:n}$.

3.13. Maximum likelihood: general formula

In this section, we investigate the estimation of the parameters of the BCG-G model by the method of the maximum likelihood. Let x_1, \dots, x_n be observations of n independent and identically distributed random variables having the BCG-G pdf. The log-likelihood function is given by

$$\begin{aligned} \ell(\lambda, \delta, \phi) &= \sum_{i=1}^n \log [f(x_i, \lambda, \delta, \phi)] \\ &= -n \log [\Gamma(\delta)] + \sum_{i=1}^n \log [g(x_i; \phi)] - (\lambda + 1) \sum_{i=1}^n \log [G(x_i; \phi)] \\ &\quad + (\delta - 1) \sum_{i=1}^n \log [G(x_i; \phi)^{-\lambda} - 1] - n(\delta - 1) \log(\lambda) - \sum_{i=1}^n \frac{G(x_i; \phi)^{-\lambda}}{\lambda} + \frac{n}{\lambda}. \end{aligned}$$

The maximum likelihood estimates (MLEs) for λ , δ and ϕ are the real numbers $\hat{\lambda}$, $\hat{\delta}$ and $\hat{\phi}$ such that $\ell(\hat{\lambda}, \hat{\delta}, \hat{\phi})$ is maximal. They are simultaneous solutions of the three nonlinear equations: $\partial \ell(\lambda, \delta, \phi) / \partial \lambda = 0$, $\partial \ell(\lambda, \delta, \phi) / \partial \delta = 0$ and $\partial \ell(\lambda, \delta, \phi) / \partial \phi = 0$, with

$$\begin{aligned} \frac{\partial}{\partial \lambda} \ell(\lambda, \delta, \phi) &= - \sum_{i=1}^n \log [G(x_i; \phi)] - (\delta - 1) \sum_{i=1}^n \frac{\log [G(x_i; \phi)]}{1 - G(x_i; \phi)^\lambda} - n(\delta - 1) \frac{1}{\lambda} \\ &\quad + \sum_{i=1}^n \frac{G(x_i; \phi)^{-\lambda} \log [G(x_i; \phi)]}{\lambda} + \sum_{i=1}^n \frac{G(x_i; \phi)^{-\lambda}}{\lambda^2} - \frac{n}{\lambda^2}, \end{aligned} \quad (14)$$

$$\frac{\partial}{\partial \delta} \ell(\lambda, \delta, \phi) = -n \frac{\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^n \log [G(x_i; \phi)^{-\lambda} - 1] - n \log(\lambda) \quad (15)$$

and

$$\begin{aligned} \frac{\partial}{\partial \phi} \ell(\lambda, \delta, \phi) &= \sum_{i=1}^n \frac{\partial g(x_i; \phi) / \partial \phi}{g(x_i; \phi)} - (\lambda + 1) \sum_{i=1}^n \frac{\partial G(x_i; \phi) / \partial \phi}{G(x_i; \phi)} \\ &\quad - \lambda(\delta - 1) \sum_{i=1}^n \frac{(\partial G(x_i; \phi) / \partial \phi) G(x_i; \phi)^{-\lambda-1}}{G(x_i; \phi)^{-\lambda} - 1} + \sum_{i=1}^n (\partial G(x_i; \phi) / \partial \phi) G(x_i; \phi)^{-\lambda-1}. \end{aligned} \quad (16)$$

Naturally, $\hat{\lambda}$, $\hat{\delta}$ and $\hat{\phi}$ can be determined numerically via a statistical software as R. Under specific regularity conditions, their random versions have the features to be asymptotic unbiased and asymptotic normal. This allows to construct confidence intervals (Wald interval...), hypothesis testing (Likelihood-ratio test...) and various measures of goodness of fit. This aspect will be developed for a special cdf $G(x; \phi)$ in Section 5.

4. Box-Cox Gamma-half-Cauchy distribution

Among all the distributions belonging to the BCG-G family, we now focus on the one defined with the half-Cauchy distribution as baseline. The reasons are threefold: (i) The half-Cauchy distribution has the feature to be a simple distribution with heavy tailed, highly skewed to the right (ii) The few existing generalizations of the half-Cauchy distribution give models that demonstrates nice goodness of fit properties (see Alzaatreh *et al.* (2016), and the references therein) (iii) Since the parameter λ has a great influence on the BCG pdf and hrf on the neighborhood of $x = 0$, one can expect to add more flexibility on the left tail of the half-Cauchy distribution (with success as we shall see in Section 5).

Let $\theta > 0$. The cdf of the half-Cauchy distribution with parameter θ is given by

$$G(x; \theta) = \frac{2}{\pi} \arctan\left(\frac{x}{\theta}\right), \quad x > 0.$$

The associated pdf is given by

$$g(x; \theta) = \frac{2}{\pi\theta} \frac{1}{1 + (x/\theta)^2}, \quad x > 0.$$

Putting these expressions into the BCG-G cdf given by (2), we obtain the cdf given by

$$F(x; \lambda, \delta, \theta) = 1 - \frac{1}{\Gamma(\delta)} \gamma\left(\delta, \frac{[(2/\pi) \arctan(x/\theta)]^{-\lambda} - 1}{\lambda}\right), \quad x > 0. \quad (17)$$

The related distribution is called the Box-Cox Gamma-half-Cauchy distribution (BCG-HC for short). Then all the general mathematical properties presented in Section 2 can be applied to this special case (with $\phi = \theta$ and a quantile function $Q_G(y; \theta)$ that will be presented later). In the following, we present and discuss the most useful mathematical properties of this new distribution.

The associated pdf is given by

$$\begin{aligned} f(x; \lambda, \delta, \theta) &= \frac{1}{\Gamma(\delta)} \frac{2}{\pi\theta} \frac{1}{1 + (x/\theta)^2} \left[\frac{2}{\pi} \arctan\left(\frac{x}{\theta}\right) \right]^{-\lambda-1} \left(\frac{[(2/\pi) \arctan(x/\theta)]^{-\lambda} - 1}{\lambda} \right)^{\delta-1} \\ &\quad \times e^{-\left(\frac{[(2/\pi) \arctan(x/\theta)]^{-\lambda} - 1}{\lambda} \right)}, \quad x > 0. \end{aligned} \quad (18)$$

The associated hrf is given by

$$h(x; \lambda, \delta, \theta) = \frac{2}{\pi\theta} \frac{1}{\gamma\left(\delta, \frac{[(2/\pi)\arctan(x/\theta)]^{-\lambda}-1}{\lambda}\right)} \frac{1}{1+(x/\theta)^2} \left[\frac{2}{\pi} \arctan\left(\frac{x}{\theta}\right)\right]^{-\lambda-1} \\ \times \left(\frac{[(2/\pi)\arctan(x/\theta)]^{-\lambda}-1}{\lambda}\right)^{\delta-1} e^{-\left(\frac{[(2/\pi)\arctan(x/\theta)]^{-\lambda}-1}{\lambda}\right)}, \quad x > 0.$$

Let us now investigate the asymptotic behavior of the BCG-HC pdf only. When $x \rightarrow 0$, we have $g(x; \theta) \sim 2/(\pi\theta)$ and $G(x; \theta) \sim (2/(\pi\theta))x$. Then, when $x \rightarrow 0$, we have

$$f(x; \lambda, \delta, \theta) \sim \frac{1}{\Gamma(\delta)\lambda^{\delta-1}} \left(\frac{2}{\pi\theta}\right)^{-\lambda\delta} x^{-\lambda\delta-1} e^{-\frac{(\pi\theta)^\lambda x^{-\lambda}}{\lambda 2^\lambda}} \rightarrow 0.$$

When $x \rightarrow +\infty$, we have $g(x; \theta) \sim (2\theta/\pi)(1/x^2)$ and $G(x; \theta) = 1 - (2/\pi)\arctan(\theta/x) \sim 1 - (2\theta/\pi)(1/x)$. Hence, when $x \rightarrow +\infty$, we have

$$f(x; \lambda, \delta, \theta) \sim \frac{1}{\Gamma(\delta)} \left(\frac{2\theta}{\pi}\right)^\delta \frac{1}{x^{\delta+1}} \rightarrow 0.$$

In Figure 1, we have plotted the BCG-HC pdfs and hrfs for selected values of λ , δ and θ . We see various shapes with different level of bell shaped and right-skewed. In some cases, a light left tail can be observed. These features are welcome to construct flexible models for a wide variety of lifetime data.

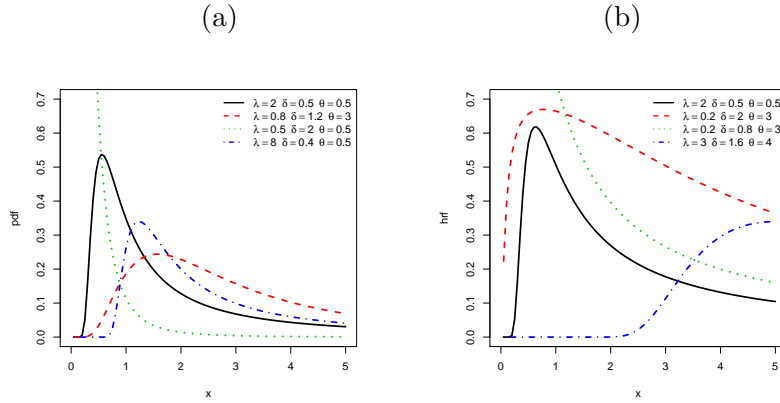


Figure 1: Plots of some BCG-HC (a) pdfs and (b) hrfs.

Let us notice that the quantile function of the half-Cauchy distribution with parameter θ is given by

$$Q_G(y, \theta) = \theta \tan\left(\frac{\pi}{2}y\right).$$

Thus we have the following immediate characterization. Let Y be a random variable following the Gamma distribution with parameters 1 and δ . Then the random variable

$$X = \theta \tan\left(\frac{\pi}{2}[1 + \lambda Y]^{-1/\lambda}\right)$$

follows the BCG-HC distribution.

Moreover, the BCG-HC quantile function is given by

$$Q(y; \lambda, \delta, \theta) = \theta \tan \left(\frac{\pi}{2} \left[1 + \lambda \gamma^{-1} (\delta, (1-y)\Gamma(\delta)) \right]^{-1/\lambda} \right), \quad y \in (0, 1).$$

From this quantile function, we can express the Galton skewness S defined by (5) and the Moors kurtosis K defined by (6). Figure 2 presents the graphics of these two measures for $\theta = 2.5$, $\alpha \in (1, 5)$ and $\delta \in (1, 5)$. We see that the skewness S increases when δ increases, with a various magnitude according to λ . Varying shapes are observed for the kurtosis K .

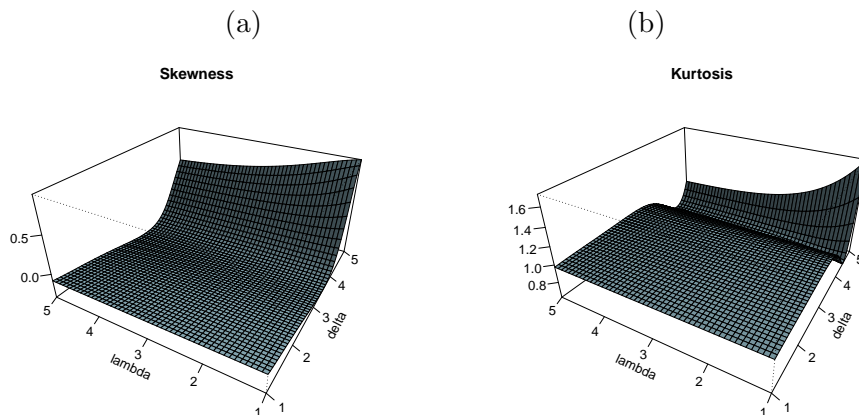


Figure 2: Plots of (a) the Galton skewness S and (b) the Moor kurtosis K for the BCG-HC distribution with parameters $\theta = 2.5$, $\alpha \in (1, 5)$ and $\delta \in (1, 5)$.

A random variable X following the BCG-HC distribution has no moment of all order. More precisely, μ'_r exists if and only if we have $\delta > r$. Indeed, when $x \rightarrow +\infty$, we have

$$x^r f(x; \lambda, \delta, \theta) \sim \frac{1}{\Gamma(\delta)} \left(\frac{2\theta}{\pi} \right)^\delta \frac{1}{x^{\delta-r+1}}$$

which converge as Riemann integral if and only if $\delta > r$ (there is no problem of convergence for $x^r f(x; \lambda, \delta, \theta)$ at $x = 0$). Under this condition, the r -th moments are given by (9). As consequences, if $\delta > 2$, the variance exists and, if $\delta > 4$, the skewness γ_1 and the kurtosis γ_2 given by (10) exist. Table 2 provides a numerical evaluation of these quantities for selected values for λ , δ and θ .

The same numerical approach can be performed to compute the r -th incomplete moments with a given value for t , the moment generating function (defined with $t < 0$), the Rényi entropy, the moments of the order statistics and the maximum likelihood estimates (as done in Subsection 5.3 for two practical data sets).

5. Statistical inference and data analysis with the BCG-HC model

The BCG-HC distribution with parameters λ , δ and θ , characterized by the cdf (17) (and having the pdf (18)), can be considered as a parametric model. Statistical inference and applications of the BCG-HC model are explored in this section.

Table 2: Some moments, skewness and kurtosis of X for BCG-HC distribution for the following selected parameters values in order $(\lambda, \delta, \theta)$: (i): (5, 5, 2.5), (ii): (0.8, 12, 15), (iii): (1, 8, 8), (iv): (0.2, 7, 10), (v): (9, 10, 0.3) and (vi): (6, 6, 8).

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
$\mathbb{E}(X)$	2.8163	41.7295	20.1226	61.23518	31.71016	9.656562
$\mathbb{E}(X^2)$	8.1589	2575.5120	542.6544	13840.32	6980.461	94.88158
$\mathbb{E}(X^3)$	24.4395	172252.4	16508.19	4368337	2240102	950.4667
$\mathbb{E}(X^4)$	76.3290	12359349	557195.4	1595868914	824905608	9732.13
$\mathbb{V}(X)$	0.2271	834.1558	137.7351	10090.58	5974.927	1.632379
γ_1	1.6795	-0.2009	0.02808	2.254328	3.550561	1.299637
γ_2	10.8262	2.04022	2.8966	7.808801	16.24243	7.022025

5.1. Maximum likelihood method

The MLEs $\hat{\lambda}$, $\hat{\delta}$ and $\hat{\theta}$ of the parameters λ , δ and θ respectively can be obtained by solving the nonlinear equations (14), (15) and (16) with

$$\frac{\partial}{\partial \theta} g(x; \theta) = \frac{2}{\pi} \frac{x^2 - \theta^2}{(x^2 + \theta^2)^2}, \quad \frac{\partial}{\partial \theta} G(x; \theta) = -\frac{2}{\pi} \frac{x}{x^2 + \theta^2}.$$

These estimates will be considered in the rest of study.

5.2. Simulation study

Here we provide a numerical evaluation of the performance of the MLEs $\hat{\lambda}$, $\hat{\delta}$ and $\hat{\theta}$ in the estimation of λ , δ and θ respectively via a graphical (Monte Carlo) simulation study. The R program is used. We generate $N = 3000$ samples of size $n = 5, 10, 20, 40, \dots, 140$ from BCG-HC distribution with the following parameters values: $\lambda = 3.5$, $\delta = 5$ and $\theta = 2$. We, for $h \in \{\lambda, \delta, \theta\}$, we calculate

- the empirical bias of the MLEs defined by $\widehat{Bias}_h = (1/N) \sum_{i=1}^N (\hat{h}_i - h)$
- the empirical mean square error (MSE) of the MLEs defined by

$$\widehat{MSE}_h = (1/N) \sum_{i=1}^N (\hat{h}_i - h)^2.$$

The results of this simulation study can be viewed in Figure 3 for the empirical bias and in Figure 4 for the empirical MSE. From these figures, we observe that, when the sample size increases, the empirical biases and MSEs approach to 0 in all cases, which is consistent with the theoretical properties of the MLEs.

5.3. Data analysis

In this subsection, we prove empirically the flexibility of the BCG-HC distribution by means of two practical data sets. The BCG-HC distribution will be compared with some competitive models listed in Table 3. We compare the fitted distributions by using the following usual goodness of fit measures: $-\hat{\ell}$ (where $\hat{\ell}$ the maximized log-likelihood), AIC (Akaike information criterion), BIC (Bayesian information criterion), CVM (Cramér-Von Mises), AD (Anderson-Darling) and KS (Kolmogorov Smirnov with its p-value (PV)) statistics. These

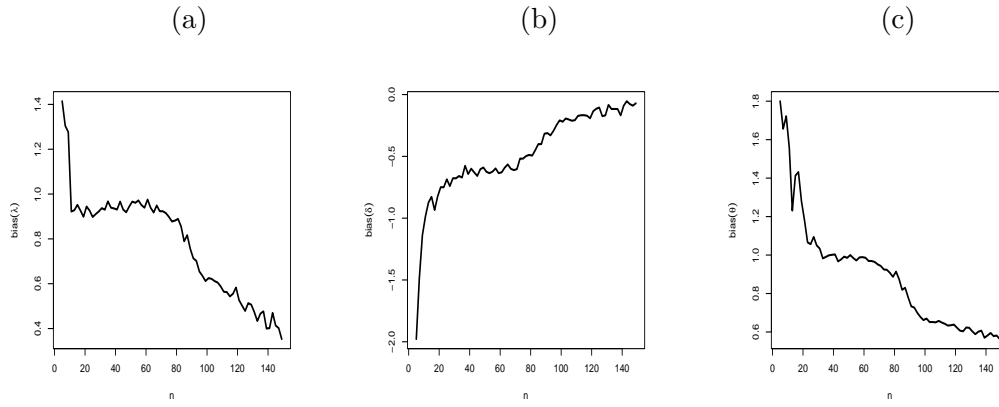


Figure 3: The empirical biases of the MLEs (a) $\hat{\lambda}$, (b) $\hat{\delta}$ and (c) $\hat{\theta}$ for the selected parameter values for BCG-HC distribution.

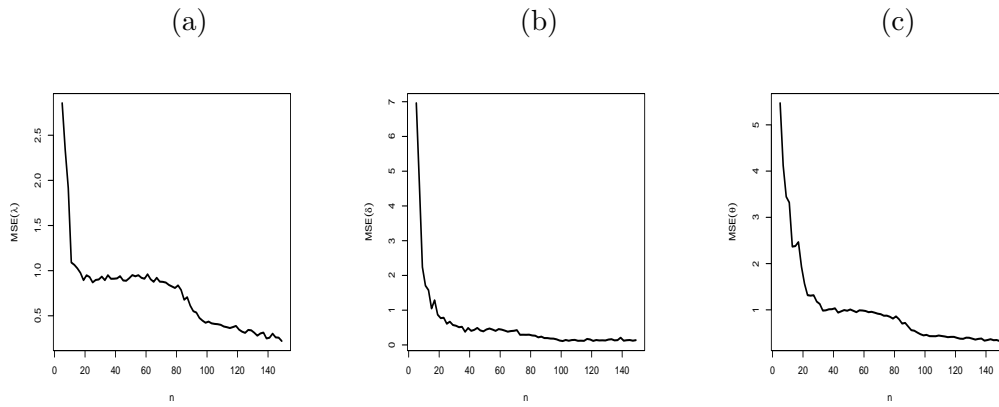


Figure 4: The empirical MSEs of the MLEs (a) $\hat{\lambda}$, (b) $\hat{\delta}$ and (c) $\hat{\theta}$ for the selected parameter values for BCG-HC distribution.

Table 3: The competitive models of the BCG-HC distributions.

Distribution	Author(s)
Half-Cauchy (HC)	(Standard)
Exponential half-Cauchy (EHC)	Gupta <i>et al.</i> (1998)
Marshall Olkin half-Cauchy (MHC)	Jacob and Jayakumar (2012)
The gamma-exponentiated exponential distribution (GEE)	Ristić and Balakrishnan (2012)
Kumaraswamy half-Cauchy (KHC)	Ghosh (2014)
The gamma half-Cauchy distribution (GHC)	Alzaatreh <i>et al.</i> (2016)

measures typically summarize the discrepancy between the data and the expected values under the considered model. Again, we mention that the R program is used.

The two considered data sets are described below.

Data set 1: We consider the actual taxes data set used by Mead (2010). The data consists of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. An immediate histogram plot shows that the distribution is highly skewed to the right. The actual taxes revenue data (in 1000 million Egyptian pounds) are given by: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.8, 6.1, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

Data set 2: The second data set was obtained in Proschan (2000) and corresponds to the time of successive failures of the air conditioning system of jet airplanes. These data were also studied by Dahiya and Gurland (1972), Gupta and Kundu (2001), Kus (2007) and De Andrade *et al.* (2017), among others. The data are given by: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 100, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 57, 33, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 14, 70, 47, 62, 142, 3, 104, 85, 67, 169, 24, 21, 246, 47, 68, 15, 2, 91, 59, 447, 56, 29, 176, 225, 77, 197, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 5, 61, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 156, 11, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 26, 71, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 62, 11, 191, 14, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95.

The descriptive statistics of these two data sets are presented in Table 4.

Firstly, let us analyze Data set 1. Table 5 lists the MLEs and their corresponding standard errors (SEs) (in parentheses) for the BCG-HC model and other fitted models.

To show the uniqueness of the obtained MLEs for the BCG-HC model, we provide the profiles plots of the log-likelihood function of λ , δ and θ in Figure 5.

Table 4: Descriptive statistics for Data sets 1 and 2.

Statistics	n	Mean	Median	Standard deviation	Skewness	Kurtosis
Data 1	59	13.4900	10.6000	8.0500	1.5700	2.0800
Data 2	213	93.1400	57.0000	106.7600	2.1000	4.8500

Table 5: MLEs and SEs (in parentheses) for Data set 1.

Model	Estimates		
BCG-HC	1.1642	2.5801	19.4444
$(\lambda, \delta, \theta)$	(0.5017)	(1.1276)	(2.5591)
GHC	17.7164	0.1176	2.3306
(α, β, σ)	(19.2298)	(0.0576)	(3.1104)
GEE	0.2457	37.5865	0.5408
$(\alpha, \lambda, \delta)$	(0.1011)	(4.6796)	(0.2810)
KHC	20.8633	4.9716	1.6795
(δ, a, b)	(3.2415)	(1.2284)	(2.9067)
MHC	1.0729	10.9676	–
(α, σ)	(0.5804)	(4.4320)	–
EHC	5.6949	2.6629	–
(α, σ)	(3.5708)	(1.5689)	–
HC	11.52048	–	–
(σ)	(1.6603)	–	–

Table 6 gives the confidence intervals of the parameters of the BCG-HC model for Data set 1 at the levels 95% and 99%.

Table 6: Confidence intervals of the parameters of the BCG-HC model for Data set 1.

CI	λ	δ	θ
95%	[0.1808 1.8144]	[0.3700 4.7901]	[14.4285 24.4602]
99%	[0 2.4585]	[0 5.4893]	[12.8419 26.0468]

Table 7 provides the values of goodness of fit measures for the BCG-HC model and other fitted models. Based on these numerical results, we see that the BCG-HC model provide better fit to Data set 1 than the competitors (smallest value of AIC, BIC, KS...).

The PP, QQ, epdf and ecdf plots of the BCG-HC are shown in Figure 6. The Box plot and the Kaplan Meier survival plot are presented in Figure 7. The nice fits of the different estimated curves indicate that the BCG-HC model yields the best fit to Data set 1.

Let us now focus on the analysis of Data set 2 with the same tools as for Data set 1. Table 8 lists the MLEs and their corresponding standard errors (SEs) (in parentheses) for the BCG-HC model and other fitted models.

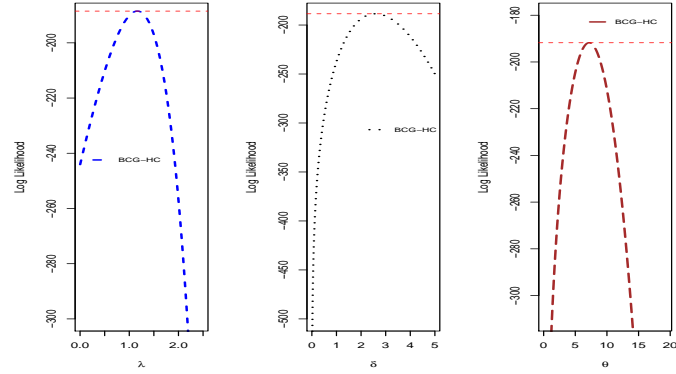


Figure 5: Profiles log-likelihood function of the log-likelihood function of the BCG-HC distribution for Data set 1.

Table 7: Goodness of fit measures for Data set 1.

Model	$-\hat{\ell}$	AIC	BIC	CVM	AD	KS	PV
BCG-HC	188.3004	382.0049	388.2375	0.0465	0.2762	0.0660	0.9589
GHC	188.5255	383.0510	389.2836	0.0597	0.3441	0.0708	0.9282
GEE	197.4691	400.9382	407.1708	0.1826	1.1285	0.1602	0.0966
KHC	188.7694	383.5388	389.7714	0.0683	0.3903	0.0767	0.8781
MHC	219.7464	443.4928	447.6478	0.1057	0.6053	0.2483	0.0013
EHC	209.5714	423.1428	427.2979	0.0498	0.2919	0.2817	0.0001
HC	219.7548	441.5097	443.5872	0.1071	0.6133	0.2504	0.0012

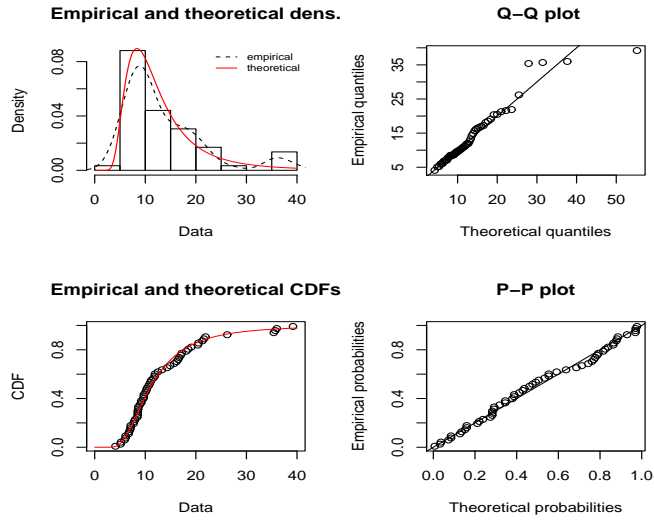


Figure 6: PP, QQ, epdf and ecdf plots of the BCG-HC distribution for Data set 1.

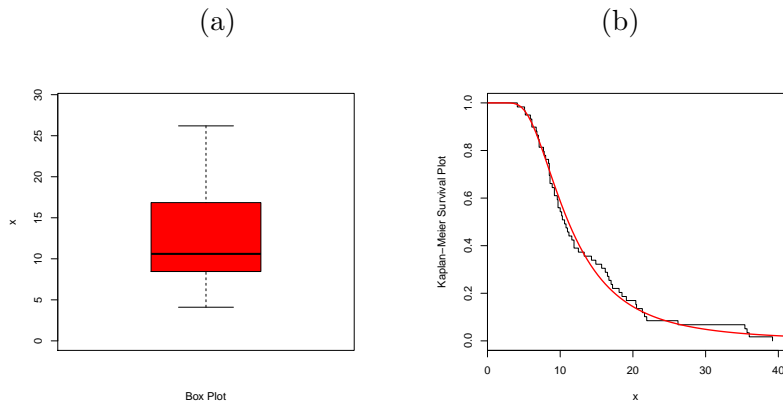


Figure 7: Plots of (a) Box plot and (b) Kaplan Meier survival plot of the HCG-HC model for Data set 1.

Table 8: MLEs and SEs (in parentheses) for Data set 2.

Model	Estimates		
BCG-HC	0.1291	3.1037	375.5790
$(\lambda, \delta, \theta)$	(0.0277)	(0.7969)	(2.5457)
GHC	46.6670	0.1919	0.0100
(α, β, σ)	(7.0641)	(0.0218)	(0.0051)
GEE	0.0026	5.0801	4.7598
$(\alpha, \lambda, \delta)$	(0.0025)	(1.4180)	(0.2454)
KHC	0.9911	2.234	132.2408
(δ, a, b)	(0.1110)	(0.7374)	(3.7441)
MHC	0.4511	100.3208	–
(α, σ)	(0.3240)	(5.1647)	–
EHC	1.1920	43.1570	–
(α, σ)	(0.1658)	(8.0007)	–
HC	52.8000	–	–
(σ)	(4.9808)	–	–

The uniqueness of the obtained MLEs for the BCG-HC model is observed in Figure 8.

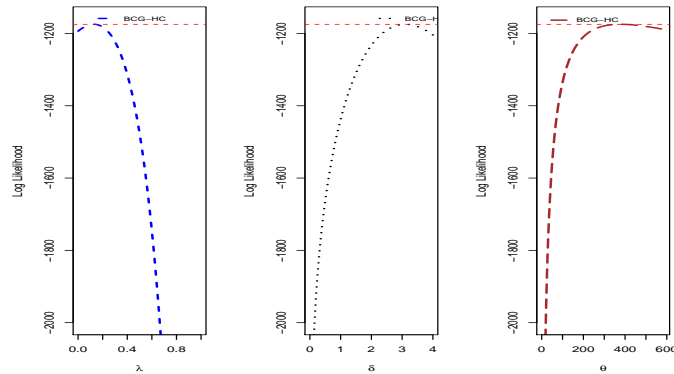


Figure 8: Profiles log-likelihood function of the log-likelihood function of the BCG-HC distribution for Data set 2.

Table 9 gives the confidence intervals of the parameters of the BCG-HC model for Data set 1 at the levels 95% and 99%.

Table 9: Confidence intervals of the parameters of the BCG-HC model for Data set 2.

CI	λ	δ	θ
95%	[0.0748 0.1649]	[1.5417 4.6656]	[370.5894 380.5685]
99%	[0.0576 0.2005]	[1.0476 5.1597]	[369.0110 382.1469]

Table 10 provides the values of goodness of fit measures for the BCG-HC model and other fitted models. Based on these numerical results, we see that the BCG-HC model provide better fit to Data set 2 than the competitors.

Table 10: Goodness of fit measures for Data set 2.

Model	$-\widehat{\ell}$	AIC	BIC	CVM	AD	KS	PV
BCG-HC	1174.7000	2355.4000	2365.5000	0.0335	0.2541	0.0380	0.9200
GHC	1185.4000	2376.7000	2386.8000	0.2197	1.5109	0.0712	0.2300
GEE	1177.4000	2360.9000	2371.0000	0.0847	0.5412	0.0571	0.4900
KHC	1178.4000	2362.8000	2372.9000	0.0983	0.6798	0.0501	0.6600
MHC	1185.5000	2375.1000	2381.8000	0.0885	0.6624	0.0562	0.5100
EHC	1185.6000	2375.2000	2381.9000	0.1764	1.1690	0.0600	0.4300
HC	1186.4000	2374.9000	2378.2000	0.1303	0.9009	0.0620	0.3900

The PP, QQ, epdf and ecdf plots of the BCG-HC are shown in Figure 9. The Box plot and the Kaplan Meier survival plot are presented in Figure 10. These plots indicate that the

BCG-HC model yields the best fit to Data set 2.

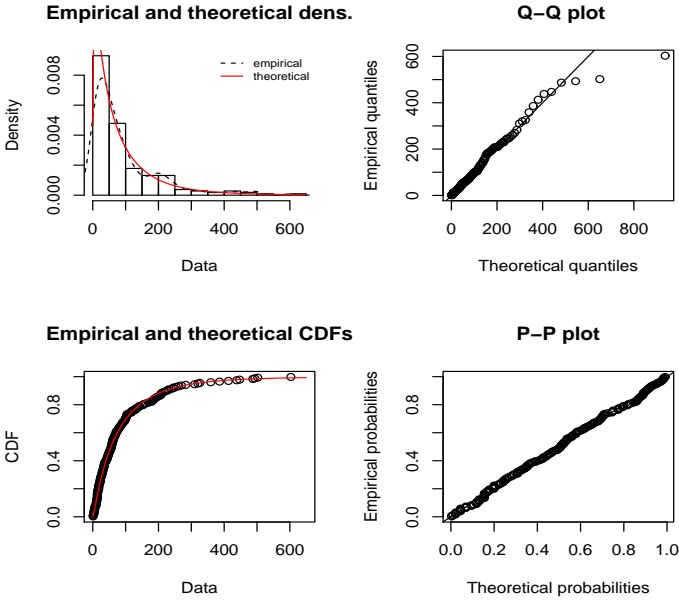


Figure 9: PP, QQ, epdf and ecdf plots of the BCG-HC distribution for Data set 2.

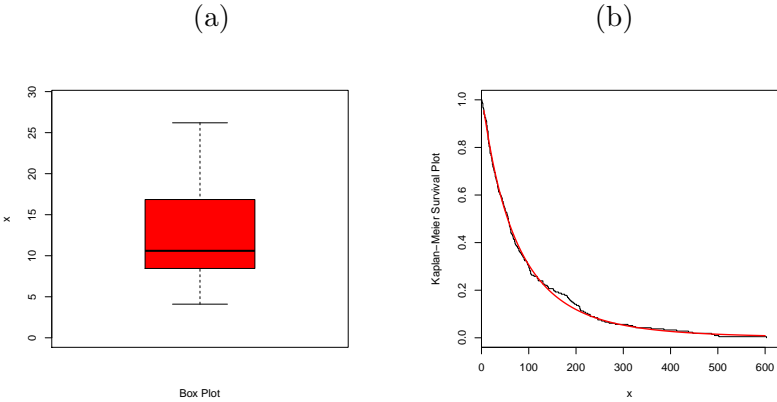


Figure 10: Plots of (a) Box plot and (b) Kaplan Meier survival plot of the HCG-HC model for Data set 2.

6. Concluding remarks

In this paper, a new family of distributions called the BCG-G family is studied. This new family can be viewed as a generalization of the RB-G family of distributions introduced by Ristić and Balakrishnan (2012) by the use of the Box-Cox transformation. The mathematical and practical properties of the BCG-G family are investigated in detail. A member of the BCG-G family using the half-Cauchy distribution as baseline distribution is introduced and considered as statistical model. It is called the BCG-HC distribution. The maximum likelihood estimation of the parameters of the BCG-HC model is discussed. In terms of the statistical significance of the model adequacy, we show that it leads to a better goodness of fit than some serious competitors of the literature.

References

- Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71, 63-79.
- Alzaatreh, A., Mansoor, M., Tahirz, M.H., Zubair, M. and Ghazali, S.A. (2016). The gamma half-Cauchy distribution: Properties and applications. *Hacetatepe Journal of Mathematics and Statistics*, 45, 4, 1143-1159.
- Bakouch, H.S., Chesneau, C. and Khan, M. N. (2019). The extended odd family of probability distributions with practice to a submodel, *Filomat*, (to appear).
- Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science*, 12, 1253-1268.
- Chesneau, C., Bakouch, H.S. and Hussain, T. (2018). A new class of probability distributions via cosine and sine functions with applications, *Communications in Statistics: Simulation and Computation*, (to appear).
- Cordeiro, G.M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, 81, 7, 883-898.
- Cordeiro, G.M., Ortega, E.M.M. and Cunha, D.C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11, 1, 1-27.
- Cordeiro, G.M., Alizadeh, M. and Ortega, E.M.M. (2014). The exponentiated half-logistic family of distributions: Properties and applications. *Journal of Probability and Statistics*, Article ID 864396, 21 pages.
- Cordeiro, G. M. and Bourguignon, M. (2016). New results on the Ristić-Balakrishnan family of distributions. *Communications in Statistics - Theory and Methods*, 45, 23, 6969-6988.
- Dahiya, R.C. and Gurland, J. (1972). Goodness of fit tests for the gamma and exponential distributions. *Technometrics*, 14, 791-801.
- De Andrade, T.A.N., Zea, L.M., Gomes-Silva, S. and Cordeiro, G.M. (2017). The gamma generalized pareto distribution with applications in survival analysis. *International Journal of Statistics and Probability*, 6, 141-156.
- Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*, 31, 4, 497-512.

- Galton, F. (1883). *Inquiries into human faculty and its development*. Macmillan and Company. London, Macmillan.
- Ghosh, I. (2014). The Kumaraswamy-half-Cauchy distribution: properties and applications. *Journal of Statistical Theory and Applications*, 13, 122-134.
- Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Table of Integrals, Series and Products*. Academic Press, New York.
- Gupta, R.C., Gupta, P.I. and Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. *Communications in statistics-Theory and Methods*, 27, 887-904.
- Gupta, R.D. and Kundu, D. (2001). Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. *Biometrical Journal*, 43, 117-130.
- Hosseini, B., Afshari, M. and Alizadeh, M. (2018). The Generalized Odd Gamma-G Family of Distributions: Properties and Applications, *Austrian Journal of Statistics*, 47, 69-89.
- Jacob, E. and Jayakumar, K. (2012). On half-Cauchy distribution and process. *International Journal of Statistika and Matematika*, 3, 77-81.
- Jamal, F., Nasir, M.A., Tahir, M.H. and Montazeri, N.H. (2017). The odd Burr-III family of distributions. *Journal of Statistics Applications and Probability*, 6, 105-122.
- Kus, C. (2007). A new lifetime distribution, *Computational Statistics and Data Analysis*, 51, 9, 4497-4509.
- Marshall, A.N. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika*, 84, 3, 641-652.
- Mead, M. E. (2010). A new generalization of Burr XII distribution. *Journal of Statistics: Advances in Theory and Applications*, 12, 2, 53-73.
- Moors, J.J.A. (1988). A quantile alternative for Kurtosis. *The Statistician*, 37, 25-32.
- Nanda, A.K., Singh, H., Misra, N. and Paul, P. (2003). Reliability properties of reversed residual lifetime. *Communications in Statistics, Theory and Methods*, 32, 2031-2042.
- Proschan, F. (2000). Theoretical explanation of observed decreasing failure rate. *American Statistical Society 1963*, reprinted in *Technometrics*, 42, 1, 7-11.
- Rényi, A. (1961). On measures of entropy and information. In: *4th Berkeley Symposium on Mathematical Statistics and Probability*, 1, 547-561.
- Ristić, M. M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, 82, 8, 1191-1206.
- Sarabia, J. M. (2008). Parametric Lorenz Curves: Models and Applications, in *Modeling Income Distributions and Lorenz Curves*, ed. by D. Chotikapanich, vol. 5 of *Economic Studies in Equality, Social Exclusion and Well-Being*, chapter 9, 167-190. Springer, New-York.
- Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*, Wiley, New York, NY, USA.

- Tahir, M.H., Cordeiro, G.M., Alizadeh, M., Mansoor, M., Zubair, M. and Hamedani, G.G. (2015). The odd generalized exponential family of distributions with applications. *Journal of Statistical Distributions and Applications*, 2, 1, 1-28.
- Torabi, H. and Montazeri, N.H. (2014). The logistic-uniform distribution and its applications. *Communications in Statistics- Simulation and Computation*, 43, 2551-2569.
- Zografos, K. and Balakrishnan, N. (2009). On the families of beta-and gamma-generated generalized distribution and associated inference. *Statistical Methodology*, 6, 4, 344-362.