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# Stabilization of the Korteweg-de Vries equation with internal time-delay feedback

Julie Valein\*

February 15, 2019

## Abstract

The aim of this work is to study the exponential stability of the nonlinear Korteweg-de Vries equation in the presence of a delayed internal feedback. We first consider the case where the weight of the feedback with delay is smaller than the weight of the feedback without delay and prove the local exponential stability result by two methods: the first one by a Lyapunov method (which holds for restrictive length of the domain but allow to have an estimation on the decay rate) and the second one by an observability inequality for any length (without estimation of the decay rate). We also prove a semiglobal stabilization result for any length. Secondly we study the case where the support of the feedback without delay is not included in the feedback with delay and give a local exponential stability result if the weight of the delayed feedback is small enough. Some numerical simulations are given to illustrate these results.

**Keyword:** KdV equation, stabilization, delay

## 1 Introduction and main results

The Korteweg-de Vries equation (KdV) equation is the nonlinear dispersive partial differential equation  $y_t + y_x + y_{xxx} + yy_x = 0$ , which models, in the absence of damping, the (unidirectional) propagation of a water wave of small amplitude in a bounded channel. The domains of applications of this equation are various: collision of hydromagnetic waves, ion acoustic waves in a plasma, acoustic waves on a crystal lattice or even subparts of the cardiovascular system... The KdV equation has been the subject of intensive research (see for instance [BW83], [RZ09],...).

The first work concerning the exponential stabilization of the nonlinear KdV equation (without delay) on a bounded domain is [Zha94], with a boundary damping and where the length of the

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spatial domain is  $L = 1$ . It is well-known that the length of the domain plays an important role in the controllability or the stability questions of the KdV equation. For instance, if  $L = 2\pi$ , there exists a solution ( $y(x, t) = 1 - \cos x$ ) of the linearized system around 0 which has a constant energy. More generally, for a length belonging to the set of critical lengths

$$\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N}^* \right\},$$

we can construct an initial data whose the corresponding solution of the linear KdV equation has a constant energy. Nevertheless, in the case of non critical length (i.e.  $L \notin \mathcal{N}$ ), it is not necessary (see [PMVZ02]) to introduce a boundary feedback law as in [Zha94] to have the local exponential stability of the nonlinear KdV equation. Moreover, it is proved in [PMVZ02] and [Paz05] that for any critical length, adding a localized damping in the nonlinear KdV equation allows to have a local exponential stability result, and even a semi-global stability result by working directly with the nonlinear system.

We also refer to [CCS15] and [TCSC16] in which the asymptotic stability for the nonlinear KdV equation for the first critical lengths ( $2k\pi$ ,  $k \in \mathbb{N}^*$ ) and a special second one ( $2\pi\sqrt{\frac{7}{3}}$ ) have been proven without any feedback law. The rapid stabilization (or how to construct a feedback law which stabilizes the system at a prescribed decay rate) has been studied in [KS08] and [CC13] by the backstepping method, and in [CL14] by an integral transform. Finally a related interesting question is the global stabilization of a nonlinear KdV equation with a saturating distributed control studied recently in [MCPA17].

The main goal of this paper is to study the stabilization of the following nonlinear KdV equation with an internal feedback delayed term

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + a(x)y(x, t) + b(x)y(x, t - h) = 0, \\ \quad x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0), \end{cases} \quad (1.1)$$

where  $h > 0$  is the (constant) delay,  $L > 0$  is the length of the spacial domain,  $y(x, t)$  is the amplitude of the water wave at position  $x$  at time  $t$ , and  $a = a(x)$  and  $b = b(x)$  are nonnegative functions belonging to  $L^\infty(0, L)$ . We will also assume that  $\text{supp } b = \omega$ , where  $\text{supp } b$  is the support of the function  $b$ , and

$$b(x) \geq b_0 > 0 \text{ a.e. in } \omega \quad (1.2)$$

where  $\omega$  is an open, nonempty subset of  $(0, L)$ .

In the case without delay (i.e.  $b = 0$ ) it is well-known (see for instance [PMVZ02]) that for every  $T > 0$ ,  $L > 0$  and  $y_0 \in L^2(0, L)$ , the system (1.1) is locally well-posed in

$$\mathcal{B} := C([0, T], L^2(0, L)) \cap L^2(0, T, H_0^1(0, L)).$$

We will give in Section 2 the proof of well-posedness for the case with delayed internal condition (i.e.  $b \neq 0$ ).

The challenge on the specific topic of our contribution, beyond the difficulty of dealing with a nonlinear equation, is to prove that under appropriate conditions, a delay in the internal feedback of this equation will not destabilize the system [Dat88]. Very recently, the robustness with respect to the delay of the boundary stability of the nonlinear KdV equation has been studied in [BCV18], where the boundary condition is  $y_x(L, t) = \alpha y_x(0, t) + \beta y_x(0, t - h)$ . The authors obtain, under appropriate condition on the weights of the feedbacks with and without delay (i.e.  $|\alpha| + |\beta| < 1$ ), the locally exponentially stability result for non critical length. Note that no condition about the size of the delayed weight  $\alpha$  with respect to the non delayed weight  $\beta$  is required. The aim of this present work is to extend these results to internal damping with delay for any length and to study if we need a restrictive assumption on  $a$  and  $b$ .

We first assume that the coefficients  $a$  and  $b$  comply to the following limitation:

$$\exists c_0 > 0, \quad b(x) + c_0 \leq a(x) \text{ in } \omega. \quad (1.3)$$

Note that (1.2) and (1.3) imply that  $\omega = \text{supp } b \subset \text{supp } a$  and

$$a(x) \geq b_0 + c_0 > 0 \text{ in } \omega. \quad (1.4)$$

We define the Hilbert space of the initial and delayed data  $\mathcal{H} := L^2(0, L) \times L^2((0, L) \times (-h, 0))$ , endowed with the norm defined for all  $(y, z) \in \mathcal{H}$  by

$$\|(y, z)\|_{\mathcal{H}}^2 = \int_0^L y^2(x) dx + \int_0^L \int_{-h}^0 \xi(x) z^2(x, s) dx ds,$$

where  $\xi$  is a nonnegative function in  $L^\infty(0, L)$  chosen such that  $\text{supp } \xi = \text{supp } b = \omega$  and

$$b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0 \text{ in } \omega. \quad (1.5)$$

Note that this choice of  $\xi$  is possible due to (1.3).

Let us now give the following definition of the energy of system (1.1), chosen because it corresponds to the norm of  $(y(\cdot, t), y(\cdot, t + \cdot))$  on  $\mathcal{H}$ :

$$E(t) = \int_0^L y^2(x, t) dx + h \int_\omega \int_0^1 \xi(x) y^2(x, t - h\rho) d\rho dx, \quad (1.6)$$

where  $\xi \in L^\infty(0, L)$  is defined by (1.5). The first part of the energy  $E$  corresponds to the natural energy of the KdV equation, and the second part is classical when considering internal delayed terms, as in [NP06] for the wave partial differential equations.

Note also that, in [NP06], dealing with the wave equation, there are some restrictions about the weights of the feedbacks with or without delay similarly to (1.3), i.e. the weight of the feedback with delay is smaller than the weight of the feedback without delay. Actually, it is also the case for hyperbolic and parabolic partial differential equations in [NV10] and even for the Schrödinger equation (which is a dispersive equation, just like KdV) in [NR11]. This kind of assumption is necessary in these cases and if they are not satisfied, it can be shown that instabilities may appear (see for instance [Dat88], [DLP86] with  $a = 0$ , or [NP06] in the more general case for the wave equation). However it is not the case for the delayed boundary stabilization of the non linear KdV equation (see [BCV18]).

Our first main result is obtained for a restricted assumption on the length  $L$  but yields local exponential stability of the solution of system (1.1) with an estimation of the decay rate stated below.

**Theorem 1.** *Assume that  $a$  and  $b$  are nonnegative functions belonging to  $L^\infty(0, L)$  satisfying (1.2) and (1.3), and assume that the length  $L$  fulfills*

$$L < \pi\sqrt{3}. \quad (1.7)$$

*Then, there exists  $r > 0$  sufficiently small, such that for every  $(y_0, z_0) \in \mathcal{H}$  satisfying*

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

*the energy of system (1.1), denoted  $E$  and defined by (1.6), decays exponentially. More precisely, there exist two positive constants  $\gamma$  and  $\kappa$  such that*

$$E(t) \leq \kappa E(0) e^{-2\gamma t}, \quad t > 0,$$

*where for  $\mu_1, \mu_2$  sufficiently small*

$$\gamma \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2h(\mu_2 + \|\xi\|_{L^\infty(0,L)})} \right\}, \quad (1.8)$$

$$\kappa \leq \left( 1 + \max \left\{ L\mu_1, \frac{\mu_2}{b_0} \right\} \right).$$

This theorem will be proved in a constructive manner, allowing an estimation of the decay rate  $\gamma$ . The proof (similar to [BCV18] for a delayed boundary feedback) uses an appropriate Lyapunov functional build with coefficients  $\mu_1$  and  $\mu_2$  and detailed in Section 3. Moreover, note that when the delay  $h$  becomes larger, then the decay rate  $\gamma$  is smaller.

**Remark 1.** *The coefficients  $\mu_1$  and  $\mu_2$  depend on the Lyapunov functional we will use in the proof of the stability result. Nevertheless, one can have an estimation of both  $r$ ,  $\mu_1$  and  $\mu_2$  in the proof of Theorem 1 below.*

On the other hand, our second main result is obtained simply for any lengths (contrary to [BCV18] which holds only for non critical lengths) and gives generic local exponential stability of the solution of system (1.1).

**Theorem 2.** *Assume that  $L > 0$  and that  $a$  and  $b$  are nonnegative functions belonging to  $L^\infty(0, L)$  satisfying (1.2) and (1.3). Then, there exists  $r > 0$  such that for every  $(y_0, z_0) \in \mathcal{H}$  satisfying*

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

*the energy of system (1.1), denoted  $E$  and defined by (1.6), decays exponentially. More precisely, there exist two positive constants  $\nu$  and  $\kappa$  such that*

$$E(t) \leq \kappa E(0)e^{-\nu t}, \quad t > 0.$$

The proof of this theorem relies on an observability inequality and the use of a contradiction argument. Thus, the value of the decay rate can not be estimated precisely in this approach.

Moreover, contrary to [BCV18], we prove a semi-global stabilization result for any length, working directly with the nonlinear system (1.1), without passing by the linear system.

**Theorem 3.** *Assume that  $a$  and  $b$  are nonnegative functions belonging to  $L^\infty(0, L)$  satisfying (1.2) and (1.3). Let  $L > 0$  and  $R > 0$ . There exists  $C = C(R) > 0$  and  $\mu = \mu(R) > 0$  such that*

$$E(t) \leq CE(0)e^{-\mu t}, \quad t > 0,$$

*for any solution of (1.1) with  $\|(y_0, z_0)\|_{\mathcal{H}} \leq R$ .*

The semi-global character of this result comes from the fact that even if we are able to chose any radius  $R$  for the initial data, the decay rate  $\mu$  depends on  $R$ .

Finally, we give a local stabilization result in the case where  $\text{supp } b \not\subset \text{supp } a$ . To do that, following Nicaise and Pignotti in [NP14], we consider a "close" auxiliary problem whose the energy is decreasing and we use a classical perturbation result of Pazy [Paz12].

**Theorem 4.** *Assume that  $a$  and  $b$  are nonnegative functions belonging to  $L^\infty(0, L)$  satisfying (1.2) and assume that the length  $L$  fulfills (1.7). Let  $\xi > 1$ . Then there exist  $\delta > 0$  (depending on  $\xi, L, h$ ) and  $r > 0$  sufficiently small such that if*

$$\|b\|_{L^\infty(0, L)} \leq \delta,$$

for every  $(y_0, z_0) \in \mathcal{H}$  satisfying

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

the energy of system (1.1), denoted  $E$  and defined by (1.6) with  $\xi(x) = \xi b(x)$ , decays exponentially.

It is interesting to note that we can take  $a = 0$  in Theorem 4. Note also that Theorem 3 and Theorem 4 are new with respect to [BCV18].

Section 2 is devoted to the preliminary step dealing with the well-posedness and regularity of the solutions of our specific system coupling the KdV equation and a delayed internal feedback. Section 3 will develop the proof of a first quantified exponential stabilization result stated in Theorem 1 while the proof of our second stabilization result, stated in Theorem 2, will be detailed in Section 4. The semi-global stabilization result stated in Theorem 3 is proved in Section 5. The study of the case  $\text{supp } b \not\subset \text{supp } a$  of Theorem 4 is done in Section 6. Some remarks and numerical simulations are presented in Section 7.

## 2 Well-posedness and regularity results

### 2.1 Study of the linear equation

We begin by proving the well-posedness of the KdV equation linearized around 0, that writes

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) + b(x)y(x, t - h) = 0, & x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0). \end{cases} \quad (2.9)$$

Following Nicaise and Pignotti [NP06], we set  $z(x, \rho, t) = y|_{\omega}(x, t - \rho h)$  for any  $x \in \omega$ ,  $\rho \in (0, 1)$  and  $t > 0$ . Then  $z$  satisfies the transport equation

$$\begin{cases} h z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, & x \in \omega, \rho \in (0, 1), t > 0, \\ z(x, 0, t) = y|_{\omega}(x, t), & x \in \omega, t > 0, \\ z(x, \rho, 0) = z_0|_{\omega}(x, -\rho h), & x \in \omega, \rho \in (0, 1). \end{cases} \quad (2.10)$$

We equipped the Hilbert space  $H = L^2(0, L) \times L^2(\omega \times (0, 1))$  with the inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y \tilde{y} dx + h \int_{\omega} \int_0^1 \xi(x) z \tilde{z} d\rho dx,$$

for any  $(y, z), (\tilde{y}, \tilde{z}) \in H$ , where  $\xi$  is a nonnegative function in  $L^{\infty}(0, L)$  such that  $\text{supp } \xi = \text{supp } b = \omega$  and (1.5) holds. We denote by  $\|\cdot\|_H$  the associated norm and this new norm is clearly equivalent to the usual norm on  $H$  since  $\xi(x) > b(x) \geq b_0 > 0$  on  $\omega$  (see (1.5)).

We then rewrite (2.9) and (2.10) as a first order system:

$$\begin{cases} U_t(t) = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0 \in H, \end{cases} \quad (2.11)$$

where  $U = \begin{pmatrix} y \\ z \end{pmatrix}$ ,  $U_0 = \begin{pmatrix} y_0 \\ z_{0|\omega}(\cdot, -h \cdot) \end{pmatrix}$ , and where the operator  $\mathcal{A}$  is defined by

$$\mathcal{A}U = \begin{pmatrix} -y_{xxx} - y_x - ay - b\tilde{z}(\cdot, 1) \\ -\frac{1}{h} z_\rho \end{pmatrix},$$

where  $\tilde{z}(\cdot, 1) \in L^2(0, L)$  is the extension of  $z(\cdot, 1)$  by zero outside  $\omega$ , with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (y, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)) \mid y(0) = y(L) = y_x(L) = 0, z(x, 0) = y_{|\omega}(x) \text{ in } \omega \right\}.$$

**Theorem 5.** *Assume that  $a$  and  $b$  are nonnegative functions belonging to  $L^\infty(0, L)$  satisfying (1.2) and (1.3), and that  $U_0 \in H$ . Then there exists a unique mild solution  $U \in C([0, +\infty), H)$  for system (2.11). Moreover if  $U_0 \in \mathcal{D}(\mathcal{A})$ , then the solution is classical and satisfies*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), H).$$

*Proof.* We first prove that the operator  $\mathcal{A}$  is dissipative. Let  $U = (y, z) \in \mathcal{D}(\mathcal{A})$ . Then we have

$$\begin{aligned} \langle \mathcal{A}U, U \rangle &= - \int_0^L y_{xxx}y \, dx - \int_0^L y_x y \, dx - \int_0^L a(x)y^2 \, dx - \int_\omega b(x)z(x, 1)y(x) \, dx \\ &\quad - \int_\omega \int_0^1 \xi(x)z_\rho(x, \rho)z(x, \rho) \, dx d\rho \\ &= \int_0^L y_{xx}y_x \, dx - [y_{xx}y]_0^L - \frac{1}{2}[y^2]_0^L - \int_0^L a(x)y^2 \, dx - \int_\omega b(x)z(x, 1)y(x) \, dx - \frac{1}{2} \int_\omega \xi(x)[z^2]_0^1 \, dx \\ &= \frac{1}{2}[y_x^2]_0^L - \int_0^L a(x)y^2 \, dx - \int_\omega b(x)z(x, 1)y(x) \, dx - \frac{1}{2} \int_\omega \xi(x)z^2(x, 1) \, dx + \frac{1}{2} \int_\omega \xi(x)y^2(x) \, dx \\ &\leq -y_x^2(0) + \int_\omega \left( -a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} \right) y^2(x) \, dx - \int_{(0, L) \setminus \omega} a(x)y^2(x) \, dx \\ &\quad + \int_\omega \left( \frac{b(x)}{2} - \frac{\xi(x)}{2} \right) z^2(x, 1) \, dx. \end{aligned}$$

If we take  $\xi$  such that (1.5) holds (which is possible due to (1.3)), then  $-a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} < 0$  and  $\frac{b(x)}{2} - \frac{\xi(x)}{2} < 0$  in  $\omega$ , and  $-a(x) \leq 0$  in  $(0, L) \setminus \omega$ . Consequently  $\langle \mathcal{A}U, U \rangle \leq 0$ , which means that the operator  $\mathcal{A}$  is dissipative.

Secondly we show that the adjoint of  $\mathcal{A}$ , denoted by  $\mathcal{A}^*$ , is also dissipative. It is not difficult to prove that the adjoint is defined by

$$\mathcal{A}^*U = \begin{pmatrix} y_{xxx} + y_x - ay + \xi(x)\tilde{z}(\cdot, 0) \\ \frac{1}{h} z_\rho \end{pmatrix}, \quad U = \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*),$$



with domain

$$\mathcal{D}(\mathcal{A}^*) = \left\{ (y, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)) \mid y(0) = y(L) = y_x(0) = 0, z(x, 1) = -\frac{1}{\xi(x)} b(x) y|_\omega(x) \text{ in } \omega \right\}.$$

Then for all  $U = (y, z) \in \mathcal{D}(\mathcal{A}^*)$ , we have

$$\begin{aligned} \langle \mathcal{A}^* U, U \rangle &= \int_0^L y_{xxx} y \, dx + \int_0^L y_x y \, dx - \int_0^L a(x) y^2 \, dx + \int_\omega \xi(x) z(x, 0) y(x) \, dx + \int_0^1 \int_\omega \xi(x) z_\rho z \, dx d\rho \\ &= -\frac{1}{2} [y_x^2]_0^L - \int_0^L a(x) y^2 \, dx + \int_\omega \xi(x) z(x, 0) y(x) \, dx + \frac{1}{2} \int_\omega \frac{1}{\xi(x)} b^2(x) y^2(x) \, dx - \frac{1}{2} \int_\omega \xi(x) z^2(x, 0) \, dx \\ &\leq -\frac{1}{2} y_x^2(L) + \int_\omega \left( -a(x) + \frac{1}{2\xi(x)} b^2(x) + \frac{\xi(x)}{2} \right) y^2(x) \, dx - \int_{(0, L) \setminus \omega} a(x) y^2(x) \, dx \leq 0, \end{aligned}$$

since, due to (1.5), we have in  $\omega$

$$-a(x) + \frac{1}{2\xi(x)} b^2(x) + \frac{\xi(x)}{2} < -a(x) + \frac{b(x)}{2} + \frac{\xi(x)}{2} \leq 0.$$

Finally, since  $\mathcal{A}$  is a densely defined closed linear operator, and both  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative, then  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$  semigroup of contractions on  $H$  (see for instance [Paz12]), which finishes the proof.  $\square$

We denote by  $\{S(t), t \geq 0\}$  the semigroup of contractions associated with  $\mathcal{A}$ . In the following, by abusing the notation, we identify  $z_0|_\omega$  and  $z_0$ , and the real  $C$  is a positive constant that can depend on  $T, h, \|a\|_{L^\infty(0, L)}$  and  $\|b\|_{L^\infty(0, L)}$ . Let us now detail a few a priori estimates and regularity estimates of the solutions of systems (2.9) and (2.10).

**Proposition 1.** *Assume that (1.2) and (1.3) are satisfied. Then, the map*

$$(y_0, z_0(\cdot, -h \cdot)) \mapsto S(\cdot)(y_0, z_0(\cdot, -h \cdot)) \quad (2.12)$$

*is continuous from  $H$  to  $\mathcal{B} \times C([0, T], L^2(\omega \times (0, 1)))$ , and for  $(y_0, z_0(\cdot, -h \cdot)) \in H$ , the following estimates hold*

$$\begin{aligned} \int_0^T \int_0^L a(x) y^2(x, t) \, dx dt + \int_0^T \int_\omega z^2(x, 1, t) \, dx dt \\ \leq C \left( \|y_0\|_{L^2(0, L)}^2 + \|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2 \right), \end{aligned} \quad (2.13)$$

$$\|y_0\|_{L^2(0, L)}^2 \leq C \left( \|y\|_{L^2(0, T, L^2(0, L))}^2 + \|y_x(0, \cdot)\|_{L^2(0, T)}^2 + \|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2 \right), \quad (2.14)$$

$$\|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2 \leq \|z(\cdot, \cdot, T)\|_{L^2(\omega \times (0, 1))}^2 + \frac{1}{h} \|z(\cdot, 1, \cdot)\|_{L^2(\omega \times (0, T))}^2. \quad (2.15)$$

*Proof.* • First of all, for any  $(y_0, z_0(\cdot, -h \cdot)) \in H$ , Theorem 5 brings  $S(\cdot)(y_0, z_0(\cdot, -h \cdot)) = (y, z) \in C([0, T], H)$  and as the operator  $\mathcal{A}$  generates a  $C_0$  semigroup of contractions we get for all  $t \in [0, T]$ ,

$$\int_0^L y^2(x, t) \, dx + h \int_\omega \int_0^1 \xi(x) z^2(x, \rho, t) \, dx d\rho \leq \int_0^L y_0^2(x) \, dx + h \int_\omega \int_0^1 \xi(x) z_0^2(x, -\rho h) \, dx d\rho. \quad (2.16)$$

Let  $p \in C^\infty([0, 1] \times [0, T])$ ,  $q \in C^\infty([0, L] \times [0, T])$  and  $(y, z) \in \mathcal{D}(\mathcal{A})$ . Then multiplying (2.10) by  $pz$  and (2.9) by  $qy$ , and using some integrations by parts we get

$$\begin{aligned} & \int_0^1 \int_\omega (p(\rho, T)z^2(x, \rho, T) - p(\rho, 0)z_0^2(x, -\rho h)) dx d\rho - \frac{1}{h} \int_0^T \int_0^1 \int_\omega (hp_t + p_\rho)z^2 dx d\rho dt \\ & + \frac{1}{h} \int_0^T \int_\omega (p(1, t)z^2(x, 1, t) - p(0, t)y^2(x, t)) dx dt = 0 \quad (2.17) \end{aligned}$$

$$\begin{aligned} & \int_0^L (q(x, T)y^2(x, T) - q(x, 0)y_0^2(x)) dx - \int_0^T \int_0^L (q_t + q_x + q_{xxx})y^2 dx dt + 3 \int_0^T \int_0^L q_x y_x^2 dx dt \\ & + \int_0^T q(0, t)y_x^2(0, t) dt + 2 \int_0^T \int_0^L a(x)qy^2 dx dt + 2 \int_0^T \int_0^L b(x)qy(x, t)y(x, t-h) dx dt = 0. \end{aligned} \quad (2.18)$$

• Let us first choose  $p(\rho, t) \equiv \rho$  in (2.17). Then we obtain

$$\int_0^1 \int_\omega \rho (z^2(x, \rho, T) - z_0^2(x, -\rho h)) dx d\rho - \frac{1}{h} \int_0^T \int_\omega \int_0^1 z^2 d\rho dx dt + \frac{1}{h} \int_0^T \int_\omega z^2(x, 1, t) dx dt = 0$$

and thanks to (2.16) we get

$$\int_0^T \int_\omega z^2(x, 1, t) dx dt \leq C(\|y_0\|_{L^2(0, L)}^2 + \|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2). \quad (2.19)$$

Secondly, if we choose  $q(x, t) \equiv 1$  in (2.18), then we get,

$$\int_0^L (y^2(x, T) - y_0^2(x)) dx + \int_0^T y_x^2(0, t) dt + 2 \int_0^T \int_0^L a(x)y^2 dx dt + 2 \int_0^T \int_0^L b(x)y(x, t)y(x, t-h) dx dt = 0,$$

which implies

$$2 \int_0^T \int_0^L a(x)y^2 dx dt \leq \|y_0\|_{L^2(0, L)}^2 + 2 \int_0^T \int_0^L b(x)|y(x, t)||y(x, t-h)| dx dt.$$

Therefore, since

$$\begin{aligned} \int_0^T \int_0^L b(x)|y(x, t)||y(x, t-h)| dx dt & \leq \frac{1}{2} \int_0^T \int_0^L b(x)y^2(x, t) dx dt + \frac{1}{2} \int_0^T \int_0^L b(x)y^2(x, t-h) dx dt \\ & \leq \int_0^T \int_0^L b(x)y^2(x, t) dx dt + \frac{1}{2} \int_0^T \int_{-h}^0 b(x)y^2(x, t) dx dt \\ & = \int_0^T \int_0^L b(x)y^2(x, t) dx dt + \frac{1}{2} \int_\omega \int_{-h}^0 b(x)z_0^2(x, \rho) d\rho dx \end{aligned}$$

we get, using (2.16), that

$$\int_0^T \int_0^L a(x)y^2 dx dt \leq C \left( \|y_0\|_{L^2(0, L)}^2 + \|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2 \right),$$

that concludes the proof of (2.13).

- Taking now  $q(x, t) \equiv x$  in (2.18), we can write

$$\begin{aligned} \int_0^L x (y^2(x, T) - y_0^2(x)) dx - \int_0^T \int_0^L y^2 dx dt + 3 \int_0^T \int_0^L y_x^2 dx dt \\ + 2 \int_0^T \int_0^L xa(x)y^2 dx dt + 2 \int_0^T \int_0^L xb(x)y(x, t)y(x, t-h) dx dt = 0 \end{aligned}$$

and we have

$$\begin{aligned} 3 \int_0^T \int_0^L y_x^2(x, t) dx dt \leq L \|y_0\|_{L^2(0, L)}^2 + (1 + 2L \|b\|_{L^\infty(0, L)}) \int_0^T \int_0^L y^2 dx dt \\ + L \|b\|_{L^\infty(0, L)} \int_{-h}^0 \int_\omega z_0^2(x, t) dx dt. \end{aligned}$$

Using (2.16), we obtain that there exists  $C > 0$  such that

$$\|y_x\|_{L^2(0, T, L^2(0, L))}^2 \leq C \left( \|y_0\|_{L^2(0, L)}^2 + \|z_0(\cdot, -h \cdot)\|_{L^2(\omega \times (0, 1))}^2 \right)$$

that brings, together with (2.16), the continuity of the map (2.12).

- Choosing  $q(x, t) \equiv T - t$  in (2.18) yields easily inequality (2.14) since it writes

$$\begin{aligned} - \int_0^L T y_0^2(x) dx + \int_0^T \int_0^L y^2 dx dt + \int_0^T (T - t) y_x^2(0, t) dt \\ + 2 \int_0^T \int_0^L (T - t) a(x) y^2 dx dt + 2 \int_0^T \int_0^L (T - t) b(x) y(x, t) y(x, t - h) dx dt = 0, \end{aligned}$$

and since we use the fact that

$$\begin{aligned} 2 \int_0^T \int_0^L (T - t) b(x) y(x, t) y(x, t - h) dx dt &\leq T \int_0^T \int_0^L b(x) y^2(x, t) dx dt \\ &\quad + T \int_0^T \int_0^L b(x) y^2(x, t - h) dx dt \\ &\leq 2T \int_0^T \int_0^L b(x) y^2(x, t) dx dt \\ &\quad + T \|b\|_{L^\infty(0, L)} \int_{-h}^0 \int_\omega z_0^2(x, t) dx dt. \end{aligned}$$

- Finally, taking  $p(\rho, t) = 1$  in (2.17) brings inequality (2.15) since it writes

$$\int_0^1 \int_\omega (z^2(x, \rho, T) - z_0^2(x, -\rho h)) dx d\rho + \frac{1}{h} \int_0^T \int_\omega (z^2(x, 1, t) - y^2(x, t)) dx dt = 0.$$

By density of  $\mathcal{D}(\mathcal{A})$  in  $H$ , the results extend to arbitrary  $(y_0, z_0(\cdot, -h \cdot)) \in H$ .  $\square$

## 2.2 KdV linear equation with a source term

Consider now the KdV linear equation with a right hand side:

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) + b(x)y(x, t - h) = f(x, t), & x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0). \end{cases} \quad (2.20)$$

**Proposition 2.** *Assume that (1.2) and (1.3) hold. For any  $(y_0, z_0(\cdot, -h \cdot)) \in H$  and  $f \in L^1(0, T, L^2(0, L))$ , there exists a unique mild solution  $(y, y(\cdot, t-h \cdot)) \in \mathcal{B} \times C([0, T], L^2(\omega \times (0, 1)))$  to (2.20). Moreover, there exists  $C > 0$  independent of  $T$  such that*

$$\|(y, z)\|_{C([0, T], H)}^2 \leq C \left( \|(y_0, z_0(\cdot, -h \cdot))\|_H^2 + \|f\|_{L^1(0, T, L^2(0, L))}^2 \right), \quad (2.21)$$

$$\|y_x\|_{L^2(0, T, L^2(0, L))}^2 \leq C(1 + T) \left( \|(y_0, z_0(\cdot, -h \cdot))\|_H^2 + \|f\|_{L^1(0, T, L^2(0, L))}^2 \right). \quad (2.22)$$

*Proof.* The well-posedness of system (2.20) in  $C([0, T], H)$ , when we rewrite it as a first order system (see (2.11)) with source term  $(f(\cdot, t), 0)$ , and the proof of (2.21), stem from  $\mathcal{A}$  being the infinitesimal generator of a  $C_0$ -semigroup of contractions on  $H$  (see [Paz12]).

The proof of (2.22) follows exactly the steps of the proof of Proposition 1 (see the third step). One has to pay attention to the right hand side terms that are not homogeneous anymore (but involve the source  $f$ ) and to note that

$$\begin{aligned} \left| \int_0^T \int_0^L f y dx dt \right| &\leq \int_0^T \|f\|_{L^2(0, L)} \|y\|_{L^2(0, L)} dt \leq \max_{t \in [0, T]} \|y(t)\|_{L^2(0, L)} \int_0^T \|f\|_{L^2(0, L)} dt \\ &\leq \frac{1}{2} \max_{t \in [0, T]} \|y(t)\|_{L^2(0, L)}^2 + \frac{1}{2} \|f\|_{L^1(0, T, L^2(0, L))}^2. \end{aligned}$$

□

### 2.3 Global existence of the solution of the nonlinear system

We endow the space  $\mathcal{B}$  with the norm

$$\|y\|_{\mathcal{B}} = \max_{t \in [0, T]} \|y(\cdot, t)\|_{L^2(0, L)} + \left( \int_0^T \|y(\cdot, t)\|_{H_0^1(0, L)}^2 dt \right)^{1/2}.$$

To prove the well-posedness result of the nonlinear system (1.1), we exactly follow [PMVZ02] (see also [CC04], [Cer14]).

The first step is to show that the nonlinearity term  $yy_x$  can be considered as a source term of the linear equation (2.20):

**Proposition 3.** *Let  $y \in \mathcal{B}$ . Then  $yy_x \in L^1(0, T, L^2(0, L))$  and the map*

$$y \in \mathcal{B} \mapsto yy_x \in L^1(0, T, L^2(0, L))$$

*is continuous. In particular, there exists  $K > 0$  such that, for any  $y, \tilde{y} \in \mathcal{B}$ , we have*

$$\int_0^T \|yy_x - \tilde{y}\tilde{y}_x\|_{L^2(0, L)} \leq KT^{1/4} (\|y\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}}) \|y - \tilde{y}\|_{\mathcal{B}}.$$

*Proof.* The proof can be found in [PMVZ02], which is a variant of Proposition 4.1 of [Ros97] (see also [Cer14]). The proof is given here for the sake of completeness. Applying triangular inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|yy_x - \tilde{y}\tilde{y}_x\|_{L^1(0,T,L^2(0,L))} &\leq \int_0^T \|(y - \tilde{y})y_x\|_{L^2(0,L)} dt + \int_0^T \|(y_x - \tilde{y}_x)\tilde{y}\|_{L^2(0,L)} dt \\
&\leq \int_0^T \|y - \tilde{y}\|_{L^\infty(0,L)} \|y_x\|_{L^2(0,L)} dt \\
&\quad + \int_0^T \|y_x - \tilde{y}_x\|_{L^2(0,L)} \|\tilde{y}\|_{L^\infty(0,L)} dt \\
&\leq \|y - \tilde{y}\|_{L^2(0,T,L^\infty(0,L))} \|y\|_{L^2(0,T,H^1(0,L))} \\
&\quad + \|y - \tilde{y}\|_{L^2(0,T,H^1(0,L))} \|\tilde{y}\|_{L^2(0,T,L^\infty(0,L))}.
\end{aligned}$$

Using Gagliardo-Nirenberg's inequality ( $\|y\|_{L^\infty(0,L)} \leq C \|y\|_{L^2(0,L)}^{\frac{1}{2}} \|y_x\|_{L^2(0,L)}^{\frac{1}{2}}$  for every  $y \in H_0^1(0,L)$ ) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\tilde{y}\|_{L^2(0,T,L^\infty(0,L))} &= \left( \int_0^T \|\tilde{y}\|_{L^\infty(0,L)}^2 dt \right)^{\frac{1}{2}} \leq C \left( \int_0^T \|\tilde{y}\|_{L^2(0,L)} \|\tilde{y}_x\|_{L^2(0,L)} dt \right)^{\frac{1}{2}} \\
&\leq C \|\tilde{y}\|_{L^\infty(0,T,L^2(0,L))}^{\frac{1}{2}} \left( \int_0^T \|\tilde{y}_x\|_{L^2(0,L)} dt \right)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{4}} \|\tilde{y}\|_{L^\infty(0,T,L^2(0,L))}^{\frac{1}{2}} \|\tilde{y}\|_{L^2(0,T,H^1(0,L))}^{\frac{1}{2}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|yy_x - \tilde{y}\tilde{y}_x\|_{L^1(0,T,L^2(0,L))} &\leq CT^{\frac{1}{4}} \|y - \tilde{y}\|_{L^\infty(0,T,L^2(0,L))}^{\frac{1}{2}} \|y - \tilde{y}\|_{L^2(0,T,H^1(0,L))}^{\frac{1}{2}} \|y\|_{L^2(0,T,H^1(0,L))} \\
&\quad + CT^{\frac{1}{4}} \|y - \tilde{y}\|_{L^2(0,T,H^1(0,L))} \|\tilde{y}\|_{L^\infty(0,T,L^2(0,L))}^{\frac{1}{2}} \|\tilde{y}\|_{L^2(0,T,H^1(0,L))}^{\frac{1}{2}},
\end{aligned}$$

which finishes the proof.  $\square$

**Remark 2.** Proposition 4.1 of [Ros97] states that if  $y \in L^2(0,T,H^1(0,L)) =: L^2(H^1)$ , then  $yy_x \in L^1(0,T,L^2(0,L))$  and there exists  $K > 0$  such that, for any  $y, \tilde{y} \in L^2(H^1)$ , we have

$$\int_0^T \|yy_x - \tilde{y}\tilde{y}_x\|_{L^2(0,L)} \leq K \left( \|y\|_{L^2(H^1)} + \|\tilde{y}\|_{L^2(H^1)} \right) \|y - \tilde{y}\|_{L^2(H^1)}.$$

We prove the following proposition:

**Proposition 4.** Let  $L > 0$  and assume that (1.2) and (1.3) hold. Then for every  $(y_0, z_0(\cdot, -h)) \in H$ , there exists a unique  $y \in \mathcal{B}$  solution of system (1.1).

*Proof.* We closely follow [PMVZ02] (see also [Paz05]): we can obtain the global existence of mild solution by proving the local (in time) existence and using the decay of the energy to obtain the global existence of solution.

Indeed, if we prove the local (in time) existence and uniqueness of solution of (1.1), global existence will then be an immediate consequence of the decay of the energy

$$E(t_2) \leq E(t_1) \leq E(0), \quad \forall 0 < t_1 < t_2,$$

provided by the facts that  $E(t) = \|U(t)\|_H^2$  and  $\mathcal{A}$  generates a  $C_0$  semigroup of contractions on  $H$ .

We are then reduced to prove the local (in time) existence and uniqueness of solution of (1.1). Let  $(y_0, z_0(\cdot, -h\cdot)) \in H$ . Given  $y \in \mathcal{B}$ , we consider the map  $\Phi : \mathcal{B} \rightarrow \mathcal{B}$  defined by  $\Phi(y) = \tilde{y}$  where  $\tilde{y}$  is solution of

$$\begin{cases} \tilde{y}_t(x, t) + \tilde{y}_{xxx}(x, t) + \tilde{y}_x(x, t) + a(x)\tilde{y}(x, t) + b(x)\tilde{y}(x, t-h) = -y(x, t)y_x(x, t), & x \in (0, L), t > 0, \\ \tilde{y}(0, t) = \tilde{y}(L, t) = \tilde{y}_x(L, t) = 0, & t > 0, \\ \tilde{y}(x, 0) = y_0(x), & x \in (0, L), \\ \tilde{y}(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0). \end{cases}$$

Clearly  $y \in \mathcal{B}$  is a solution of (1.1) if and only if  $y$  is a fixed point of the map  $\Phi$ .

From (2.21), (2.22) and Proposition 3, we get

$$\begin{aligned} \|\Phi(y)\|_{\mathcal{B}} &= \|\tilde{y}\|_{\mathcal{B}} \leq C(1 + \sqrt{T}) \left( \|(y_0, z_0(\cdot, -h\cdot))\|_H + \int_0^T \|yy_x(t)\|_{L^2(0,L)} dt \right) \\ &\leq C(1 + \sqrt{T}) \left( \|(y_0, z_0(\cdot, -h\cdot))\|_H + T^{\frac{1}{4}} \|y\|_{\mathcal{B}}^2 \right) \\ &\leq C(1 + \sqrt{T}) \|(y_0, z_0(\cdot, -h\cdot))\|_H + 2CT^{\frac{1}{4}} \|y\|_{\mathcal{B}}^2, \end{aligned}$$

with  $T < 1$ . Moreover, for the same reasons, we have

$$\begin{aligned} \|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} &\leq C(1 + \sqrt{T}) \int_0^T \|-y_1y_{1,x} + y_2y_{2,x}\|_{L^2(0,L)} dt \\ &\leq C(1 + \sqrt{T})T^{\frac{1}{4}} (\|y_1\|_{\mathcal{B}} + \|y_2\|_{\mathcal{B}}) \|y_1 - y_2\|_{\mathcal{B}}. \end{aligned}$$

We consider  $\Phi$  restricted to the closed ball  $\{y \in \mathcal{B}, \|y\|_{\mathcal{B}} \leq R\}$  with  $R > 0$  to be chosen later.

Then

$$\|\Phi(y)\|_{\mathcal{B}} \leq C(1 + \sqrt{T}) \|(y_0, z_0(\cdot, -h\cdot))\|_H + 2CT^{\frac{1}{4}} R^2 \quad \text{and} \quad \|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq 2C(1 + \sqrt{T})T^{\frac{1}{4}} R \|y_1 - y_2\|_{\mathcal{B}}.$$

So if we take  $R = 2C \|(y_0, z_0(\cdot, -h\cdot))\|_H$  and  $T > 0$  satisfying

$$\sqrt{T} + 8C^2 \|(y_0, z_0(\cdot, -h\cdot))\|_H T^{\frac{1}{4}} < 1 \quad \text{and} \quad T < \min \left\{ 1, \frac{1}{(4CR)^4} \right\},$$

then  $\|\Phi(y)\|_{\mathcal{B}} < R$  and  $\|\Phi(y_1) - \Phi(y_2)\|_{\mathcal{B}} \leq C_1 \|y_1 - y_2\|_{\mathcal{B}}$ , with  $C_1 < 1$ . Consequently, we can apply the Banach fixed point theorem and the map  $\Phi$  has a unique fixed point.  $\square$

### 3 Lyapunov approach for a first local stabilization result

The goal of this section is to prove our first main result, presented in Theorem 1. We will basically detail the proof of the exponential stability of the solution of system (1.1), which is based on the appropriate choice of a candidate Lyapunov functional. A first step is the following proposition concerning the energy of the system.

**Proposition 5.** *Let (1.2) and (1.3) be satisfied. Then, for any regular solution of (1.1) the energy  $E$  defined by (1.6) is non-increasing and there exists a positive constant  $C_1$  such that*

$$\frac{d}{dt} E(t) \leq -C_1 \left[ y_x^2(0, t) + \int_0^L a(x) y^2(x, t) dx + \int_{\omega} y^2(x, t-h) dx \right] \leq 0. \quad (3.23)$$

*Proof.* Differentiating (1.6) and using (1.1), we obtain,

$$\begin{aligned} \frac{d}{dt} E(t) &= -2 \int_0^L y(x, t) (y_{xxx} + y_x + yy_x + ay)(x, t) dx - 2 \int_0^L b(x) y(x, t) y(x, t-h) dx \\ &\quad - 2 \int_0^1 \int_{\omega} \xi(x) y(x, t-h\rho) \partial_{\rho} y(x, t-h\rho) d\rho \\ &= -y_x^2(0, t) - 2 \int_0^L a(x) y^2(x, t) dx - 2 \int_0^L b(x) y(x, t) y(x, t-h) dx + \int_{\omega} \xi(x) y^2(x, t) dx \\ &\quad - \int_{\omega} \xi(x) y^2(x, t-h) dx \\ &\leq -y_x^2(0, t) + \int_{\omega} (-2a(x) + b(x) + \xi(x)) y^2(x, t) dx - 2 \int_{(0,L)\setminus\omega} a(x) y^2(x) dx \\ &\quad + \int_{\omega} (b(x) - \xi(x)) y^2(x, t-h) dx. \end{aligned}$$

As in the proof of Theorem 5, assumptions (1.4) and (1.5) (due to (1.2)-(1.3)) end the proof.  $\square$

This result on the energy of the system does not yield the exponential stability we are seeking. Therefore, we choose now the following candidate Lyapunov functional (similar to that one of [BCV18]):

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \quad (3.24)$$

where  $\mu_1$  and  $\mu_2$  are positive constants that will be fixed small enough later on,  $E$  is the energy defined by (1.6),  $V_1$  is defined by

$$V_1(t) = \int_0^L xy^2(x, t) dx, \quad (3.25)$$

and  $V_2$  is defined by

$$V_2(t) = h \int_{\omega} \int_0^1 (1-\rho) y^2(x, t-h\rho) dx d\rho, \quad (3.26)$$

for any regular solution of (1.1).

It is clear that the two energies  $E$  and  $V$  are equivalent, in the sense that

$$E(t) \leq V(t) \leq \left( 1 + \max \left\{ L\mu_1, \frac{\mu_2}{b_0} \right\} \right) E(t) \quad (3.27)$$

(see (1.2) and (1.5)).

*Proof of Theorem 1.* Let  $y$  be a regular solution of (1.1) with  $(y_0, z_0(\cdot, -h\cdot)) \in H$  satisfying  $\|(y_0, z_0(\cdot, -h\cdot))\|_{\mathcal{H}} \leq r$ . Differentiating (3.25) and using (1.1), we obtain by using several integrations by parts

$$\begin{aligned}
\frac{d}{dt} V_1(t) &= -2 \int_0^L xy(x, t)(y_{xxx} + y_x + yy_x + ay)(x, t)dx - 2 \int_0^L xb(x)y(x, t)y(x, t-h)dx \\
&= -2 \int_0^L y_x^2(x, t)dx + 2 [y(x, t)y_x(x, t)]_0^L - \int_0^L y_x^2(x, t)dx + [xy_x^2(x, t)]_0^L \\
&\quad + \int_0^L y^2(x, t)dx + \frac{2}{3} \int_0^L y^3(x, t)dx - 2 \int_0^L xa(x)y^2(x, t)dx - 2 \int_0^L xb(x)y(x, t)y(x, t-h)dx \\
&= -3 \int_0^L y_x^2(x, t)dx + \int_0^L y^2(x, t)dx + \frac{2}{3} \int_0^L y^3(x, t)dx \\
&\quad - 2 \int_0^L xa(x)y^2(x, t)dx - 2 \int_{\omega} xb(x)y(x, t)y(x, t-h)dx.
\end{aligned}$$

Moreover, differentiating (3.26), using an integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt} V_2(t) &= 2h \int_{\omega} \int_0^1 (1-\rho)y(x, t-h\rho)\partial_t y(x, t-h\rho)d\rho dx \\
&= -2 \int_{\omega} \int_0^1 (1-\rho)y(x, t-h\rho)\partial_{\rho}(y(x, t-h\rho))d\rho dx \\
&= - \int_{\omega} [(1-\rho)y^2(x, t-h\rho)]_0^1 dx - \int_{\omega} \int_0^1 y^2(x, t-h\rho)d\rho dx \\
&= \int_{\omega} y^2(x, t)dx - \int_{\omega} \int_0^1 y^2(x, t-h\rho)d\rho dx.
\end{aligned}$$

Consequently, with the proof of Proposition 5, for any  $\gamma > 0$ , we have

$$\begin{aligned}
\frac{d}{dt} V(t) + 2\gamma V(t) &\leq \int_{\omega} (-2a(x) + b(x) + \xi(x) + \mu_1 Lb(x) + \mu_2)y^2(x, t)dx \\
&\quad - 2 \int_{(0, L) \setminus \omega} a(x)y^2(x, t)dx + \int_{\omega} (b(x) - \xi(x) + \mu_1 Lb(x))y^2(x, t-h)dx \\
&\quad + (\mu_1 + 2\gamma + 2\gamma\mu_1 L) \int_0^L y^2(x, t)dx - 3\mu_1 \int_0^L y_x^2(x, t)dx \\
&\quad + \frac{2}{3} \mu_1 \int_0^L y^3(x, t)dx + \int_{\omega} \int_0^1 (2\gamma\xi(x)h + 2\gamma\mu_2 h - \mu_2)y^2(x, t-h\rho)d\rho dx.
\end{aligned}$$

Using Poincaré inequality,  $(\|y\|_{L^2(0, L)} \leq \frac{L}{\pi}\|y_x\|_{L^2(0, L)})$  for  $y \in H_0^1(0, L)$ , we obtain that

$$\begin{aligned}
\frac{d}{dt} V(t) + 2\gamma V(t) &\leq \int_{\omega} (-2a(x) + b(x) + \xi(x) + \mu_1 Lb(x) + \mu_2)y^2(x, t)dx \\
&\quad + \int_{\omega} (b(x) - \xi(x) + \mu_1 Lb(x))y^2(x, t-h)dx + \left( \frac{L^2(\mu_1 + 2\gamma + 2\gamma\mu_1 L)}{\pi^2} - 3\mu_1 \right) \int_0^L y_x^2(x, t)dx \\
&\quad + \frac{2}{3} \mu_1 \int_0^L y^3(x, t)dx + \int_{\omega} \int_0^1 (2\gamma\xi(x)h + 2\gamma\mu_2 h - \mu_2)y^2(x, t-h\rho)d\rho dx.
\end{aligned}$$

Using (1.5), it is sufficient to take  $\mu_1$  and  $\mu_2$  sufficiently small to have  $-2a(x) + b(x) + \xi(x) + \mu_1 Lb(x) + \mu_2 \leq 0$  and  $b(x) - \xi(x) + \mu_1 Lb(x) \leq 0$  for  $x \in \omega$ . More precisely, we can take

$$\mu_1 \leq \inf_{x \in \omega} \left\{ \frac{2a(x) - b(x) - \xi(x)}{Lb(x)}, \frac{\xi(x) - b(x)}{Lb(x)} \right\},$$



$$\mu_2 \leq \inf_{x \in \omega} \{2a(x) - b(x) - \xi(x) - \mu_1 Lb(x)\}.$$

For instance, by (1.5), we can take

$$0 < \mu_1 < \frac{c_0}{L \|b\|_{L^\infty(0,L)}}, \quad 0 < \mu_2 < c_0 - L\mu_1 \|b\|_{L^\infty(0,L)}.$$

Moreover, using Cauchy-Schwarz inequality, Proposition 5 and since  $H_0^1(0, L) \subset L^\infty(0, L)$ , we have:

$$\begin{aligned} \int_0^L y^3(x, t) dx &\leq \|y(\cdot, t)\|_{L^\infty(0,L)}^2 \int_0^L |y(x, t)| dx \\ &\leq L\sqrt{L} \|y_x(\cdot, t)\|_{L^2(0,L)}^2 \|y(\cdot, t)\|_{L^2(0,L)} \\ &\leq L^{3/2} \|(y_0, z_0(\cdot, -h\cdot))\|_H \|y_x(\cdot, t)\|_{L^2(0,L)}^2 \\ &\leq L^{3/2} r \|y_x(\cdot, t)\|_{L^2(0,L)}^2. \end{aligned}$$

Consequently, we have

$$\frac{d}{dt} V(t) + 2\gamma V(t) \leq \Upsilon \|y_x(t)\|_{L^2(0,L)}^2 + \int_\omega \int_0^1 (2h\gamma(\mu_2 + \xi(x)) - \mu_2) y^2(x, t - h\rho) dx d\rho$$

where  $\Upsilon = \frac{L^2(2\gamma(1 + L\mu_1) + \mu_1)}{\pi^2} - 3\mu_1 + \frac{2L^{3/2}r\mu_1}{3}$ .

Since  $L$  satisfies the constraint (1.7), it is possible to choose  $r$  small enough to have  $r < \frac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$ . Then one can choose  $\gamma > 0$  such that (1.8) holds in order to obtain

$$\frac{d}{dt} V(t) + 2\gamma V(t) \leq 0, \quad \forall t > 0.$$

Integrating over  $(0, t)$  and using (3.27), we finally obtain that

$$E(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{b_0}\right\}\right) E(0)e^{-2\gamma t}, \quad \forall t > 0.$$

By density of  $\mathcal{D}(\mathcal{A})$  in  $H$ , the results extend to arbitrary  $(y_0, z_0(\cdot, -h\cdot)) \in H$ .  $\square$

**Remark 3. On the size of the delay.** *As one can deduce from (1.8), when the delay  $h$  increases, the decay rate  $\gamma$  decreases.*

**Remark 4. On the length of the spacial domain.** *The condition  $L < \sqrt{3}\pi$  is a technical one and comes from the choice of the multiplier  $x$  in the expression of  $V_1$ . To find a better multiplier is an open problem as far as we know.*

In the next section we will prove a stabilization result for any lengths but without any bound on the decay rate.

## 4 Second local stabilization result - Observability approach

This section aims at proving our second main result, stated in Theorem 2, which is obtained simply for any lengths and gives generic exponential stability of the solution of system (1.1). The proof relies on an observability inequality and the use of a contradiction argument. It will need several steps in order to handle the nonlinearity of the KdV equation under consideration.

### 4.1 Proof of the stability of the linear equation

We first prove the following observability result.

**Theorem 6.** *Assume that (1.2) and (1.3) are satisfied. Let  $L > 0$  and  $T > h$ . Then there exists  $C > 0$  such that for all  $(y_0, z_0(\cdot, -h \cdot)) \in H$ , we have the observability inequality*

$$\begin{aligned} \int_0^L y_0^2(x) dx + h \int_\omega \int_0^1 \xi(x) z_0^2(x, -h\rho) dx d\rho \\ \leq C \left( \int_0^T y_x^2(0, t) dt + \int_0^T \int_0^L a(x) y^2(x, t) dx dt + \int_0^T \int_\omega z^2(x, 1, t) dx dt \right) \end{aligned} \quad (4.28)$$

where  $(y, z) = S(\cdot)(y_0, z_0(\cdot, -h \cdot))$ .

*Proof.* We proceed by contradiction as in [Ros97] (see also [BCV18]). Let us suppose that (4.28) is false. Then there exists a sequence  $\left( (y_0^n, z_0^n(\cdot, -h \cdot)) \right)_n \subset H$  such that

$$\int_0^L (y_0^n)^2(x) dx + h \int_\omega \int_0^1 \xi(x) (z_0^n)^2(x, -h\rho) dx d\rho = 1$$

and

$$\int_0^T (y_x^n)^2(0, t) dt + \int_0^T \int_0^L a(x) (y^n)^2(x, t) dx dt + \int_0^T \int_\omega (z^n)^2(x, 1, t) dx dt \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (4.29)$$

where  $(y^n, z^n) = S(y_0^n, z_0^n(\cdot, -h \cdot))$ . Thanks to Proposition 1,  $(y^n)_n$  is a bounded sequence in  $L^2(0, T, H^1(0, L))$ , and then  $y_t^n = -y_x^n - y_{xxx}^n - ay^n - bz^n(\cdot, 1, \cdot)$  is bounded in  $L^2(0, T, H^{-2}(0, L))$ . Due to a result of Simon [Sim87], the set  $\{y^n\}_n$  is relatively compact in  $L^2(0, T, L^2(0, L))$  and we may assume that  $(y^n)_n$  is convergent in  $L^2(0, T, L^2(0, L))$ .

We now prove that if  $T > h$ ,  $(z_0^n(\cdot, -h \cdot))_n$  is a Cauchy sequence in  $L^2(\omega \times (0, 1))$ . Indeed, since  $z^n(x, \rho, T) = y_\omega^n(x, T - \rho h)$ , if  $T > h$ , we have

$$\int_\omega \int_0^1 (z^n(x, \rho, T))^2 dx d\rho = \int_\omega \int_0^1 (y^n(x, T - \rho h))^2 dx d\rho \leq \frac{1}{h} \int_\omega \int_0^T (y^n(x, t))^2 dt dx.$$

Using (2.15), for  $T > h$  we have

$$\int_\omega \int_0^1 (z_0^n)^2(x, -h\rho) d\rho dx \leq \frac{1}{h} \int_0^T \int_\omega (z^n)^2(x, 1, t) dx dt + \frac{1}{h} \int_\omega \int_0^T (y^n)^2(x, t) dx dt.$$

Thus  $(z_0^n(\cdot, -h \cdot))_n$  is a Cauchy sequence in  $L^2(\omega \times (0, 1))$  using also (4.29) and (1.4). Thanks to (2.14) and (4.29), we deduce that  $(y_0^n)_n$  is a Cauchy sequence in  $L^2(0, L)$ .

Let  $(y_0, z_0(\cdot, -h \cdot)) = \lim_{n \rightarrow \infty} (y_0^n, z_0^n(\cdot, -h \cdot))$  in  $H$  and  $(y, z) = S(\cdot)(y_0, z_0(\cdot, -h \cdot))$ . By using Proposition 1,

$$\begin{aligned} \int_0^T \int_0^L a(x)(y^n)^2(x, t) dx dt + \int_0^T \int_\omega b(x)(z^n)^2(x, 1, t) dx dt \\ \rightarrow \int_0^T \int_0^L a(x)y^2(x, t) dx dt + \int_0^T \int_\omega b(x)z^2(x, 1, t) dx dt. \end{aligned}$$

Thus we have that  $\int_0^L y_0^2(x) dx + h \int_\omega \int_0^1 \xi(x) z_0^2(x, -h\rho) dx d\rho = 1$  and  $\int_0^T \int_0^L a(x)y^2(x, t) dx dt + \int_0^T \int_\omega b(x)z^2(x, 1, t) dx dt = 0$ . As  $z(x, 1, t) = y(x, t - h) = 0$  in  $\omega \times (0, T)$  we deduce that  $z_0 = 0$  and  $z = 0$ . Moreover  $y = 0$  on  $\omega \times (0, T)$  (see (1.4)). Consequently  $y$  is solution of the linear equation

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) = 0, & x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \end{cases}$$

with  $y = 0$  in  $\omega \times (0, T)$ , where we recall that  $\omega$  is a nonempty open subset of  $(0, L)$ . Therefore, by the Holmgren's uniqueness theorem,  $y \equiv 0$  in  $(0, L) \times (0, T)$ .

Then we obtain a contradiction, which ends the proof of Theorem 6.  $\square$

From observability inequality (4.28), one can deduce the exponential stability of the KdV linear system (2.9), stated here:

**Theorem 7.** *Let  $L > 0$ . Assume that (1.2) and (1.3) are satisfied. Then, for every  $(y_0, z_0(\cdot, -h \cdot)) \in H$ , the energy of system (2.9), denoted by  $E$  and defined by (1.6), decays exponentially. More precisely, there exist two positive constants  $\nu$  and  $\kappa$  such that  $E(t) \leq \kappa E(0)e^{-\nu t}$ , for all  $t > 0$ .*

*Proof.* We follow the same kind of proof as in [NP06]. Let  $(y_0, z_0(\cdot, -h \cdot)) \in \mathcal{D}(\mathcal{A})$ . Integrating (3.23) between 0 and  $T > h$ , we have

$$E(T) - E(0) \leq -C_1 \left( \int_0^T y_x^2(0, t) dt + \int_0^T \int_0^L a(x)y^2(x, t) dx dt + \int_0^T \int_\omega y^2(x, t - h) dx dt \right),$$

which is equivalent to

$$\int_0^T y_x^2(0, t) dt + \int_0^T \int_0^L a(x)y^2(x, t) dx dt + \int_0^T \int_\omega y^2(x, t - h) dx dt \leq \frac{1}{C_1} (E(0) - E(T)). \quad (4.30)$$

As the energy is non-increasing, we have, using the observability inequality (4.28) and (4.30),

$$E(T) \leq E(0) \leq C \left( \int_0^T y_x^2(0, t) dt + \int_0^T \int_0^L a(x)y^2(x, t) dx dt + \int_0^T \int_\omega y^2(x, t - h) dx dt \right) \leq \frac{C}{C_1} (E(0) - E(T)),$$

which implies that

$$E(T) \leq \gamma E(0), \text{ with } \gamma = \frac{C}{1 + \frac{C}{C_1}} < 1. \quad (4.31)$$

Using this argument on  $[(m-1)T, mT]$  for  $m = 1, 2, \dots$  (which is valid because the system is invariant by translation in time), we will get

$$E(mT) \leq \gamma E((m-1)T) \leq \dots \leq \gamma^m E(0).$$

Therefore, we have  $E(mT) \leq e^{-\nu mT} E(0)$  with  $\nu = \frac{1}{T} \ln \frac{1}{\gamma} = \frac{1}{T} \ln \left(1 + \frac{C_1}{C}\right) > 0$ . For an arbitrary positive  $t$ , there exists  $m \in \mathbb{N}^*$  such that  $(m-1)T < t \leq mT$ , and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0).$$

By density of  $\mathcal{D}(\mathcal{A})$  in  $H$ , we deduce that the exponential decay of the energy  $E$  holds for any initial data in  $H$ .  $\square$

## 4.2 Stability of the nonlinear equation

We consider in this section the more general case than in Theorem 1 for any  $L > 0$ , and prove the exponential decay of small amplitude solutions of the nonlinear KdV equation (1.1):

*Proof of Theorem 2.* The proof follows [Cer14] for the stabilization of the nonlinear KdV equation with internal feedback without delay. Consider initial data  $\|(y_0, z_0(\cdot, -h \cdot))\|_H \leq r$  with  $r$  chosen later. The solution  $y$  of (1.1) can be written as  $y = y^1 + y^2$  where  $y^1$  is solution of

$$\begin{cases} y_t^1(x, t) + y_{xxx}^1(x, t) + y_x^1(x, t) + a(x)y^1(x, t) + b(x)y^1(x, t-h) = 0, & x \in (0, L), t > 0, \\ y^1(0, t) = y^1(L, t) = y_x^1(L, t) = 0, & t > 0, \\ y^1(x, 0) = y_0(x), & x \in (0, L), \\ y^1(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0), \end{cases}$$

and  $y^2$  is solution of

$$\begin{cases} y_t^2(x, t) + y_{xxx}^2(x, t) + y_x^2(x, t) + a(x)y^2(x, t) + b(x)y^2(x, t-h) = -y(x, t)y_x(x, t), & x \in (0, L), t > 0, \\ y^2(0, t) = y^2(L, t) = y_x^2(L, t) = 0, & t > 0, \\ y^2(x, 0) = 0, & x \in (0, L), \\ y^2(x, t) = 0, & x \in (0, L), t \in (-h, 0). \end{cases}$$

More precisely,  $y^1$  is solution of (2.9) with initial data  $(y_0, z_0(\cdot, -h \cdot)) \in H$  and  $y^2$  is solution of (2.20) with initial data  $(0, 0)$  and right-hand side  $f = -yy_x \in L^1(0, T, L^2(0, L))$ . Using (4.31), Proposition 2 and Remark 2, we have

$$\begin{aligned} \|(y(T), z(T))\|_H &\leq \|(y^1(T), z^1(T))\|_H + \|(y^2(T), z^2(T))\|_H \\ &\leq \gamma \|(y_0, z_0(\cdot, -h \cdot))\|_H + C \|yy_x\|_{L^1(0, T, L^2(0, L))} \\ &\leq \gamma \|(y_0, z_0(\cdot, -h \cdot))\|_H + C \|y\|_{L^2(0, T, H^1(0, L))}^2, \end{aligned} \quad (4.32)$$

with  $0 < \gamma < 1$ . The aim is now to deal with the last term of the previous inequality. For that, we multiply the first equation of (1.1) by  $xy$  and integrate to obtain

$$\begin{aligned} & 3 \int_0^T \int_0^L y_x^2(x, t) dx dt + \int_0^L xy^2(x, T) dx + 2 \int_0^T \int_0^L a(x) xy^2(x, t) dx dt \\ &= \int_0^T \int_0^L y^2(x, t) dx dt + \int_0^L xy_0^2(x) dx - 2 \int_0^T \int_0^L b(x) xy(x, t) y(x, t-h) dt + \frac{2}{3} \int_0^T \int_0^L y^3(x, t) dx dt. \end{aligned}$$

Consequently, by the fact that  $E$  is non increasing (see (3.23)), we have

$$\begin{aligned} \int_0^T \int_0^L y_x^2(x, t) dx dt &\leq \frac{T+L}{3} \|(y_0, z_0(\cdot, -h\cdot))\|_H^2 \\ &\quad + \frac{2L}{3} \int_0^T \int_\omega b(x) |y(x, t) y(x, t-h)| dx dt + \frac{2}{9} \int_0^T \int_0^L |y(x, t)|^3 dx dt. \end{aligned}$$

As  $H^1(0, L)$  embeds into  $C([0, L])$  and using Cauchy-Schwarz inequality and (3.23), we have

$$\begin{aligned} \int_0^T \int_0^L |y|^3(x, t) dx dt &\leq \int_0^T \|y\|_{L^\infty(0, L)} \int_0^L y^2(x, t) dx dt \\ &\leq \sqrt{L} \int_0^T \|y\|_{H^1(0, L)} \int_0^L y^2(x, t) dx dt \\ &\leq \sqrt{L} \|y\|_{L^\infty(0, T, L^2(0, L))}^2 \int_0^T \|y\|_{H^1(0, L)} dt \\ &\leq \sqrt{LT} \|(y_0, z_0(\cdot, -h\cdot))\|_H^2 \|y\|_{L^2(0, T, H^1(0, L))}. \end{aligned}$$

Moreover, using again (3.23), we have

$$\begin{aligned} \int_0^T \int_\omega b(x) |y(x, t) y(x, t-h)| dx dt &\leq \frac{\|b\|_{L^\infty(0, L)}}{2} \left( \int_0^T \int_\omega y^2(x, t) dx dt + \int_0^T \int_\omega y^2(x, t-h) dx dt \right) \\ &\leq \|b\|_{L^\infty(0, L)} \int_0^T \int_0^L y^2(x, t) dx dt \\ &\quad + \frac{\|b\|_{L^\infty(0, L)}}{2} \int_{-h}^0 \int_\omega z_0^2(x, t) dx dt \\ &\leq \|b\|_{L^\infty(0, L)} \left( T + \frac{1}{2} \right) \|(y_0, z_0(\cdot, -h\cdot))\|_H^2. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \int_0^T \int_0^L y_x^2(x, t) dx dt &\leq \left( \frac{T+L}{3} + \frac{L \|b\|_{L^\infty(0, L)} (2T+1)}{3} \right) \|(y_0, z_0(\cdot, -h\cdot))\|_H^2 \\ &\quad + \frac{2\sqrt{LT}}{9} \|(y_0, z_0(\cdot, -h\cdot))\|_H^2 \|y\|_{L^2(0, T, H^1(0, L))}. \end{aligned}$$

Using Young's inequality, there exists  $C > 0$  such that

$$\int_0^T \int_0^L y_x^2(x, t) dx dt \leq C \left( \|(y_0, z_0(\cdot, -h\cdot))\|_H^2 + \|(y_0, z_0(\cdot, -h\cdot))\|_H^4 \right). \quad (4.33)$$

Therefore, gathering (4.32) and (4.33), there exists  $C > 0$  so that

$$\|(y(T), z(T))\|_H \leq \|(y_0, z_0(\cdot, -h\cdot))\|_H \left( \gamma + C \|(y_0, z_0(\cdot, -h\cdot))\|_H + C \|(y_0, z_0(\cdot, -h\cdot))\|_H^3 \right)$$

which implies

$$\|(y(T), z(T))\|_H \leq \|(y_0, z_0(\cdot, -h\cdot))\|_H (\gamma + Cr + Cr^3).$$

Given  $\epsilon > 0$  small enough such that  $\gamma + \epsilon < 1$ , we can take  $r$  small enough such that  $r + r^3 < \frac{\epsilon}{C}$ , in order to have

$$\|(y(T), z(T))\|_H \leq (\gamma + \epsilon) \|(y_0, z_0(\cdot, -h\cdot))\|_H,$$

with  $\gamma + \epsilon < 1$ . The end of the proof follows the same lines as in the linear case (see after (4.31)).  $\square$

## 5 Semi-global stabilization result

The goal of this section is to prove Theorem 3 using directly the nonlinear system (1.1). The two main difficulties to the semi-global stabilization result are the pass to the limit in the nonlinear term and the fact that this nonlinear term do not allow to use Holmgren's theorem. Instead we will use the following unique continuation property for the nonlinear system due to Saut and Scheurer [SS87]:

**Theorem 8.** *Let  $u \in L^2(0, T, H^3(0, L))$  be a solution of*

$$u_t + u_x + u_{xxx} + uu_x = 0$$

*such that*

$$u(x, t) = 0, \quad \forall t \in (t_1, t_2), \forall x \in \omega,$$

*where  $\omega$  is an open nonempty subset of  $(0, L)$ . Then*

$$u(x, t) = 0, \quad \forall t \in (t_1, t_2), \forall x \in (0, L).$$

To prove Theorem 3, in order to use Theorem 8, we have to show that the limit solution in the contradiction argument is in  $L^2(0, T, H^3(0, L))$ .

*Proof of Theorem 3.* We follow [Paz05] (see also [Cer14]). Let  $y$  be the solution of (1.1). We note that, using (3.23), (4.30) holds. Consequently, if we succeed to prove the observability inequality (4.28) for the nonlinear system (1.1), then we will obtain the exponential decay of the energy  $E$  for the solution of the nonlinear system as in the proof of Theorem 7.

We are then reduced to prove the observability inequality (4.28) for the nonlinear system (1.1). First, since  $\int_0^L y^2 y_x dx = \frac{1}{3}[y^3]_0^L = 0$ , we can obtain, similarly to (2.18) with  $q \equiv 1$ ,

$$\begin{aligned} \int_0^L y^2(x, t) dx - \int_0^L y_0^2(x) dx + \int_0^t \int_0^L y_x^2(0, s) ds \\ + 2 \int_0^t \int_0^L a(x) y^2(x, s) dx ds + 2 \int_0^t \int_0^L b(x) y(x, s) y(x, s - h) dx ds = 0. \end{aligned}$$

By integrating this last equation between 0 and  $T$ , we have

$$\begin{aligned} T \int_0^L y_0^2(x) dx &\leq \int_0^T \int_0^L y^2(x, t) dx dt + T \int_0^T y_x^2(0, t) dt \\ &\quad + 2T \int_0^T \int_0^L a(x) y^2(x, t) dx dt + 2T \int_0^T \int_0^L b(x) y(x, t) y(x, t-h) dx dt. \end{aligned}$$

Using the fact that

$$\begin{aligned} \int_0^T \int_0^L b(x) y(x, t) y(x, t-h) dx dt \\ \leq \frac{\|b\|_{L^\infty(0,L)}}{2} \int_0^T \int_\omega y^2(x, t) dx dt + \frac{1}{2} \int_0^T \int_\omega b(x) y^2(x, t-h) dx dt, \end{aligned}$$

and (1.4), there exists  $C > 0$  such that

$$\begin{aligned} T \int_0^L y_0^2(x) dx &\leq \|y\|_{L^2((0,L) \times (0,T))}^2 + T \int_0^T y_x^2(0, t) dt \\ &\quad + C \int_0^T \int_0^L a(x) y^2(x, t) dx dt + C \int_0^T \int_\omega b(x) y^2(x, t-h) dx dt. \quad (5.34) \end{aligned}$$

Moreover, integrating (2.15) between 0 and  $T$ , we have,

$$T \int_0^1 \int_\omega z_0^2(x, -h\rho) d\rho dx \leq \int_0^T \int_0^1 \int_\omega z^2(x, \rho, t) dx d\rho dt + \frac{T}{h} \int_0^T \int_\omega y^2(x, t-h) dx dt,$$

with, if  $T > h$ , using (1.2) and (1.4),

$$\begin{aligned} \int_0^T \int_0^1 \int_\omega z^2(x, \rho, t) dx d\rho dt &= \int_0^T \int_0^1 \int_\omega y^2(x, t - \rho h) dx d\rho dt = \frac{1}{h} \int_0^T \int_{t-h}^t \int_\omega y^2(x, u) dx du dt \\ &\leq \frac{T}{h} \int_{-h}^T \int_\omega y^2(x, u) dx du \\ &= \frac{T}{h} \int_{-h}^{T-h} \int_\omega y^2(x, u) dx du + \frac{T}{h} \int_{T-h}^T \int_\omega y^2(x, u) dx du \\ &\leq \frac{T}{h} \int_0^T \int_\omega (y^2(x, t) + y^2(x, t-h)) dx dt \\ &\leq C \int_0^T \int_0^L a(x) y^2(x, t) dx dt + C \int_0^T \int_\omega b(x) y^2(x, t-h) dx dt. \end{aligned}$$

Gathering these estimates with (5.34), we see that it suffices, in order to prove the observability inequality (4.28) for the nonlinear system (1.1), to prove that for any  $T, R > 0$  there exists  $K > 0$  (which depends on  $R$  and  $T$ ) such that

$$K \|y\|_{L^2((0,L) \times (0,T))}^2 \leq \int_0^T y_x^2(0, t) dt + \int_0^T \int_0^L a(x) y^2(x, t) dx dt + \int_0^T \int_\omega b(x) y^2(x, t-h) dx dt \quad (5.35)$$

for the solutions of the nonlinear system (1.1) with  $\|(y_0, z_0(\cdot, -h\cdot))\|_H \leq R$ . We do that by a contradiction argument. We assume that (5.35) does not hold and we built a sequence  $(y^n)_n \subset \mathcal{B}$  solution of (1.1) with  $\|(y_0^n, z_0^n(\cdot, -h\cdot))\|_H \leq R$  such that

$$\lim_{n \rightarrow +\infty} \frac{\|y^n\|_{L^2((0,L) \times (0,T))}^2}{\|y_x^n(0, \cdot)\|_{L^2(0,T)}^2 + \int_0^T \int_0^L a |y^n|^2 dx dt + \int_0^T \int_\omega b |y^n(x, t-h)|^2 dx dt} = +\infty.$$

We define  $\lambda_n = \|y^n\|_{L^2((0,L)\times(0,T))}$  and  $v_n = \frac{y^n}{\lambda_n}$ . Then,  $v^n$  satisfies

$$\begin{cases} v_t^n(x, t) + v_{xxx}^n(x, t) + v_x^n(x, t) + \lambda_n v^n(x, t) v_x^n(x, t) + a(x) v^n(x, t) \\ \quad + b(x) v^n(x, t-h) = 0, \\ v^n(0, t) = v^n(L, t) = v_x^n(L, t) = 0, \end{cases} \quad (5.36)$$

$$\|v^n\|_{L^2((0,L)\times(0,T))} = 1 \quad (5.37)$$

and

$$\|v_x^n(0, \cdot)\|_{L^2(0,L)}^2 + \int_0^T \int_0^L a |v^n|^2 dxdt + \int_0^T \int_\omega b(x) |v^n(x, t-h)|^2 dxdt \xrightarrow{n \rightarrow +\infty} 0. \quad (5.38)$$

Using the fact that

$$\int_0^T \int_0^L (T-t)(v^n)^2 v_x^n dxdt = 0,$$

we have, as for the linear case (see (2.14)) that

$$\|v^n(\cdot, 0)\|_{L^2(0,L)}^2 \leq C \left( \|v^n\|_{L^2(0,T,L^2(0,L))}^2 + \|v_x^n(0, \cdot)\|_{L^2(0,T)}^2 + \|v^n(\cdot, -h)\|_{L^2(\omega \times (0,1))}^2 \right),$$

with, if  $T > h$  and since (1.2) holds,

$$\begin{aligned} \|v^n(\cdot, -h)\|_{L^2(\omega \times (0,1))}^2 &= h \int_{-h}^0 \int_\omega |v^n(x, t)|^2 dxdt \\ &\leq h \int_{-h}^{T-h} \int_\omega |v^n(x, t)|^2 dxdt \leq h \int_0^T \int_\omega b(x) |v^n(x, t-h)|^2 dxdt. \end{aligned}$$

Gathering this with (5.37) and (5.38), we see that  $(v^n(\cdot, 0))_n$  is bounded in  $L^2(0, L)$ . Moreover, we note that  $(\lambda_n)_n$  is bounded in  $\mathbb{R}$  since, due to (3.23),

$$\lambda_n = \|y^n\|_{L^2((0,L)\times(0,T))} \leq T \|(y_0^n, z_0^n(\cdot, -h))\|_H \leq TR.$$

Consequently, using the same inequality as (4.33) for (5.36),  $(v^n)_n$  is bounded in  $L^2(0, T, H^1(0, L))$ .

We also notice that  $(v^n v_x^n)_n$  is a subset of  $L^2(0, T, L^1(0, L))$ , since by Cauchy-Schwarz inequality, we have

$$\|v^n v_x^n\|_{L^2(0,T,L^1(0,L))} \leq \|v^n\|_{C([0,T],L^2(0,L))} \|v^n\|_{L^2(0,T,H^1(0,L))}.$$

All these are used to show that  $v_t^n = -(v_{xxx}^n + v_x^n + \lambda_n v^n v_x^n + av^n + bv^n(t-h))$  is bounded in  $L^2(0, T, H^{-2}(0, L))$  and consequently using a result of Simon [Sim87], the set  $\{v^n\}_n$  is relatively compact in  $L^2(0, T, L^2(0, L))$  and a subsequence of  $(v^n)_n$ , also denoted by  $(v^n)_n$ , converges strongly in  $L^2(0, T, L^2(0, L))$  to a limit  $v$  verifying  $\|v\|_{L^2(0,T,L^2(0,L))} = 1$ . Furthermore, by weak lower semicontinuity, we have

$$\begin{aligned} &\|v_x(0, \cdot)\|_{L^2(0,L)}^2 + \int_0^T \int_0^L a |v|^2 dxdt + \int_0^T \int_\omega b(x) |v(x, t-h)|^2 dxdt \\ &\leq \liminf_{n \rightarrow \infty} \left( \|v_x^n(0, \cdot)\|_{L^2(0,L)}^2 + \int_0^T \int_0^L a |v^n|^2 dxdt + \int_0^T \int_\omega b(x) |v^n(x, t-h)|^2 dxdt \right) = 0 \end{aligned}$$



and therefore

$$v(x, t) = 0 \quad \text{in } \omega \times (-h, T) \quad \text{and} \quad v_x(0, t) = 0 \quad \text{in } (0, T).$$

Since  $(\lambda_n)_n$  is bounded, we can also extract a subsequence, still denoted by  $(\lambda_n)_n$ , which converges to  $\lambda \geq 0$ . Consequently, the limit  $v$  satisfies

$$\begin{cases} v_t(x, t) + v_{xxx}(x, t) + v_x(x, t) + \lambda v(x, t)v_x(x, t) = 0, & x \in (0, L), t \in (0, T), \\ v(0, t) = v(L, t) = v_x(L, t) = 0, & t \in (0, T), \\ v(x, t) = 0, & x \in \omega, t \in (0, T), \\ v_x(0, t) = 0, & t \in (0, T), \\ \|v\|_{L^2(0, T, L^2(0, L))} = 1. \end{cases}$$

We then distinguish two cases:

- First case:  $\lambda = 0$ . Then the system satisfied by  $v$  is linear and we can apply Holmgren's theorem to get that the solution  $v$  is trivial, which contradicts  $\|v\|_{L^2(0, T, L^2(0, L))} = 1$ .
- Second case:  $\lambda > 0$ . We then prove that in fact  $v \in L^2(0, T, H^3(0, L))$ . For that, we consider  $u = v_t$ . Then  $u$  satisfies

$$\begin{cases} u_t(x, t) + u_{xxx}(x, t) + u_x(x, t) + \lambda u(x, t)v_x(x, t) + \lambda v(x, t)u_x(x, t) = 0, & x \in (0, L), t \in (0, T), \\ u(0, t) = u(L, t) = u_x(L, t) = 0, & t \in (0, T), \\ u(x, t) = 0, & x \in \omega, t \in (0, T), \\ u_x(0, t) = 0, & t \in (0, T), \\ u(x, 0) = -v'(x, 0) - v'''(x, 0) - \lambda v(x, 0)v'(x, 0) \in H^{-3}(0, L). \end{cases}$$

Using Lemma A.2 of [Cer14], we get that the initial data  $u(\cdot, 0) \in L^2(0, L)$  and so  $u = v_t \in \mathcal{B}$ . We deduce that  $v_{xxx} = -v_t - v_x - \lambda v v_x \in L^2(0, T, L^2(0, L))$  and then that  $v \in L^2(0, T, H^3(0, L))$ . Applying Theorem 8, we obtain that the solution  $v$  is trivial, which contradicts  $\|v\|_{L^2(0, T, L^2(0, L))} = 1$  and ends the proof. □

## 6 Third local stabilization result

In this section, we prove the exponential stability of (1.1) in the case where  $\omega = \text{supp } b \not\subset \text{supp } a$ .

In this case, the derivative of the energy  $E$  defined by (1.6) satisfies

$$\begin{aligned} \frac{d}{dt} E(t) &= -y_x^2(0, t) - 2 \int_{\text{supp } a} a(x)y^2(x, t)dx - 2 \int_{\omega} b(x)y(x, t)y(x, t-h)dx + \int_{\omega} \xi(x)y^2(x, t)dx \\ &\quad - \int_{\omega} \xi(x)y^2(x, t-h)dx \\ &\leq -y_x^2(0, t) - 2 \int_{\text{supp } a} a(x)y^2(x, t)dx + \int_{\omega} (b(x) + \xi(x))y^2(x, t)dx + \int_{\omega} (b(x) - \xi(x))y^2(x, t-h)dx, \end{aligned}$$

and so the energy is not decreasing in general due to the term  $b(x) + \xi(x) > 0$  on  $\omega$ .

Following [NP14], we consider the next auxiliary problem, which is "close" to (1.1) but whose the energy is decreasing:

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + a(x)y(x, t) \\ \quad + b(x)y(x, t - h) + \xi b(x)y(x, t) = 0, & x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0), \end{cases} \quad (6.39)$$

where  $\xi$  is a positive constant. Then we consider the energy defined by (1.6) with  $\xi(x) = \xi b(x)$ , i.e.

$$E(t) = \int_0^L y^2(x, t)dx + h\xi \int_\omega \int_0^1 b(x)y^2(x, t - h\rho)d\rho dx. \quad (6.40)$$

Then the derivative of this energy  $E$  satisfies

$$\begin{aligned} \frac{d}{dt} E(t) &= -y_x^2(0, t) - 2 \int_{\text{supp } a} a(x)y^2(x, t)dx - 2 \int_\omega b(x)y(x, t)y(x, t - h)dx - 2\xi \int_\omega b(x)y^2(x, t)dx \\ &\quad + \xi \int_\omega b(x)y^2(x, t)dx - \xi \int_\omega b(x)y^2(x, t - h)dx \\ &\leq -y_x^2(0, t) - 2 \int_{\text{supp } a} a(x)y^2(x, t)dx + \int_\omega (b(x) - \xi b(x))y^2(x, t)dx \\ &\quad + \int_\omega (b(x) - \xi b(x))y^2(x, t - h)dx \leq 0 \end{aligned}$$

taking  $\xi > 1$ , for any regular solution.

We would like to use the classical perturbation result of Pazy [Paz12]:

**Theorem 9.** *Let  $X$  be a Banach space and let  $A$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$  on  $X$  satisfying  $\|T(t)\| \leq Me^{\omega t}$ . If  $B$  is a bounded linear operator on  $X$ , then  $A + B$  is the infinitesimal generator of a  $C_0$  semigroup  $S(t)$  on  $X$  satisfying  $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$ .*

The strategy to treat the case  $\text{supp } b \not\subset \text{supp } a$  is then the following: we first prove the exponential stability for (6.39) linearized around 0 by the Lyapunov approach for all  $L < \sqrt{3}\pi$  (allowing to have an estimation of the decay rate), then we show the exponential stability of (2.9) using Theorem 9 for all  $L < \sqrt{3}\pi$  and for  $\|b\|_{L^\infty(0, L)}$  small enough. Finally we obtain the local exponential stability of the nonlinear system (1.1) for all  $L < \sqrt{3}\pi$  and for  $\|b\|_{L^\infty(0, L)}$  small enough using the same proof as in Section 4.2.

## 6.1 Exponential stability for a linear auxiliary system by the Lyapunov approach

We first consider the system (6.39) linearized around 0:

$$\begin{cases} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) + b(x)y(x, t - h) + \xi b(x)y(x, t) = 0, \\ x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0). \end{cases} \quad (6.41)$$

As in Section 2, setting  $z(x, \rho, t) = y|_\omega(x, t - \rho h)$  for any  $x \in \omega$ ,  $\rho \in (0, 1)$  and  $t > 0$ ,  $z$  satisfies the transport equation (2.10). We equipped the Hilbert space  $H = L^2(0, L) \times L^2(\omega \times (0, 1))$  with the inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y \tilde{y} dx + h \xi \int_\omega \int_0^1 b(x) z \tilde{z} d\rho dx,$$

for any  $(y, z), (\tilde{y}, \tilde{z}) \in H$  and where  $\xi > 1$ .

We then rewrite (6.41) as a first order system:

$$\begin{cases} U_t(t) = \mathcal{A}_0 U(t), & t > 0, \\ U(0) = U_0 \in H, \end{cases} \quad (6.42)$$

where  $U = \begin{pmatrix} y \\ z \end{pmatrix}$ ,  $U_0 = \begin{pmatrix} y_0 \\ z_{0|\omega}(\cdot, -h \cdot) \end{pmatrix}$ , and where the operator  $\mathcal{A}_0$  is defined by

$$\mathcal{A}_0 U = \begin{pmatrix} -y_{xxx} - y_x - ay - \xi by - b\tilde{z}(\cdot, 1) \\ -\frac{1}{h} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{ (y, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)) \mid y(0) = y(L) = y_x(L) = 0, z(x, 0) = y|_\omega(x) \text{ in } \omega \right\}.$$

**Theorem 10.** *Assume that  $a$  and  $b$  are nonnegative functions in  $L^\infty(0, L)$  satisfying (1.2) and that  $U_0 \in H$ . Let  $\xi > 1$ . Then there exists a unique mild solution  $U \in C([0, +\infty), H)$  for system (6.42). Moreover if  $U_0 \in \mathcal{D}(\mathcal{A}_0)$ , then the solution is classical and satisfies*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}_0)) \cap C^1([0, +\infty), H).$$

*Proof.* We follow the proof of Theorem 5. We first prove that the operator  $\mathcal{A}_0$  is dissipative.

Let  $U = (y, z) \in \mathcal{D}(\mathcal{A}_0)$ . Then we have

$$\begin{aligned} \langle \mathcal{A}_0 U, U \rangle &= \frac{1}{2} [y_x^2]_0^L - \int_0^L a(x) y^2 dx - \int_\omega \xi b(x) y^2 dx - \int_\omega b(x) z(x, 1) y(x) dx - \frac{\xi}{2} \int_\omega b(x) z^2(x, 1) dx \\ &\quad + \frac{\xi}{2} \int_\omega b(x) y^2(x) dx \\ &\leq -y_x^2(0) - \int_{\text{supp } a} a(x) y^2(x) dx + \frac{1}{2} \int_\omega b(x) (1 - \xi) y^2(x) dx + \frac{1}{2} \int_\omega b(x) (1 - \xi) z^2(x, 1) dx. \end{aligned}$$

If we take  $\xi > 1$ , then  $\langle \mathcal{A}_0 U, U \rangle \leq 0$ , which means that the operator  $\mathcal{A}_0$  is dissipative.

Secondly we show that the adjoint of  $\mathcal{A}_0$ , denoted by  $\mathcal{A}_0^*$ , is also dissipative. It is not difficult to prove that the adjoint is defined by

$$\mathcal{A}_0^* U = \begin{pmatrix} y_{xxx} + y_x - ay - \xi by + \xi b\tilde{z}(\cdot, 0) \\ \frac{1}{h} z_\rho \end{pmatrix}, \quad U = \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0^*),$$

with domain

$$\mathcal{D}(\mathcal{A}_0^*) = \left\{ (y, z) \in H^3(0, L) \times L^2(\omega, H^1(0, 1)) \mid y(0) = y(L) = y_x(0) = 0, z(x, 1) = -\frac{1}{\xi} y|_\omega(x) \text{ in } \omega \right\}.$$

Then for all  $U = (y, z) \in \mathcal{D}(\mathcal{A}_0^*)$ ,

$$\begin{aligned} \langle \mathcal{A}_0^* U, U \rangle &= -\frac{1}{2} [y_x^2]_0^L - \int_0^L a(x) y^2 dx - \int_\omega \xi b(x) y^2 dx + \int_\omega \xi b(x) z(x, 0) y(x) dx \\ &\quad + \frac{1}{2\xi} \int_\omega b(x) y^2(x) dx - \frac{\xi}{2} \int_\omega b(x) z^2(x, 0) dx \\ &\leq -\frac{1}{2} y_x^2(L) - \int_{\text{supp } a} a(x) y^2(x) dx + \int_\omega b(x) \left( -\frac{\xi}{2} + \frac{1}{2\xi} \right) y^2(x) dx, \end{aligned}$$

and, since  $\xi > 1$ , we have  $-\frac{\xi}{2} + \frac{1}{2\xi} < 0$ .

Finally, the facts that  $\mathcal{A}_0$  is a densely defined closed linear operator, and both  $\mathcal{A}_0$  and  $\mathcal{A}_0^*$  are dissipative, imply that  $\mathcal{A}_0$  is the infinitesimal generator of a  $C_0$  semigroup of contractions on  $H$ , which finishes the proof.  $\square$

We denote by  $\{T(t), t \geq 0\}$  the semigroup of contractions associated with  $\mathcal{A}_0$ .

To prove the exponential stability of (6.41), we closely follow Section 3. More precisely, we choose the following candidate Lyapunov functional:

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \quad (6.43)$$

where  $\mu_1$  and  $\mu_2$  are positive constants that will be fixed small enough later on,  $E$  is the energy defined by (6.40),  $V_1$  is defined by (3.25) and  $V_2$  is defined by

$$V_2(t) = h \int_\omega \int_0^1 (1 - \rho) b(x) y^2(x, t - h\rho) dx d\rho, \quad (6.44)$$

for any regular solution of (6.41).

It is clear that the two energies  $E$  and  $V$  are equivalent, in the sense that

$$E(t) \leq V(t) \leq \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{\xi} \right\}\right) E(t). \quad (6.45)$$

**Proposition 6.** *Assume that  $a$  and  $b$  are nonnegative functions in  $L^\infty(0, L)$  satisfying (1.2) and that the length  $L$  fulfills (1.7). Let  $\xi > 1$ . Then, for every  $(y_0, z_0(\cdot, -h)) \in H$ , the energy of system (6.41), denoted  $E$  and defined by (6.40), decays exponentially. More precisely, there exist two positive constants  $\alpha$  and  $\beta$  such that*

$$E(t) \leq \beta E(0) e^{-2\alpha t}, \quad t > 0,$$

where, for  $\mu_1, \mu_2$  sufficiently small,

$$\alpha \leq \min \left\{ \frac{\mu_2}{2h(\xi + \mu_2)}, \frac{\mu_1(3\pi^2 - L^2)}{2L^2(1 + \mu_1 L)} \right\}, \quad (6.46)$$

$$\beta = \left(1 + \max \left\{ L\mu_1, \frac{\mu_2}{\xi} \right\}\right).$$

*Proof.* Let  $y$  be a regular solution of (6.41). Differentiating (3.25) and using (6.41), we obtain by using several integrations by parts and as in the proof of Theorem 1

$$\begin{aligned} \frac{d}{dt} V_1(t) &= -3 \int_0^L y_x^2(x, t) dx + \int_0^L y^2(x, t) dx - 2 \int_{\text{supp } a} x a(x) y^2(x, t) dx \\ &\quad - 2 \int_\omega \xi x b(x) y^2(x, t) dx - 2 \int_\omega x b(x) y(x, t) y(x, t - h) dx. \end{aligned}$$

Moreover, differentiating (6.44), using an integration by parts, we obtain

$$\frac{d}{dt} V_2(t) = \int_\omega b(x) y^2(x, t) dx - \int_\omega \int_0^1 b(x) y^2(x, t - h\rho) d\rho dx.$$

Consequently, for any  $\alpha > 0$ , we have

$$\begin{aligned} \frac{d}{dt} V(t) + 2\alpha V(t) &\leq -2 \int_0^L a(x) (1 + \mu_1 x) y^2(x, t) dx + \int_\omega b(x) (1 - \xi + \mu_2 + \mu_1 L) y^2(x, t) dx \\ &\quad + \int_\omega b(x) (1 - \xi + \mu_1 L) y^2(x, t - h) dx + (\mu_1 + 2\alpha + 2\alpha\mu_1 L) \int_0^L y^2(x, t) dx \\ &\quad - 3\mu_1 \int_0^L y_x^2(x, t) dx + \int_\omega \int_0^1 b(x) (2\alpha\xi h + 2\alpha\mu_2 h - \mu_2) y^2(x, t - h\rho) d\rho dx. \end{aligned}$$

Using Poincaré inequality, we obtain that

$$\begin{aligned} \frac{d}{dt} V(t) + 2\alpha V(t) &\leq \int_\omega b(x) (1 - \xi + \mu_2 + \mu_1 L) y^2(x, t) dx \\ &\quad + \int_\omega b(x) (1 - \xi + \mu_1 L) y^2(x, t - h) dx + \left( \frac{L^2}{\pi^2} (\mu_1 + 2\alpha + 2\alpha\mu_1 L) - 3\mu_1 \right) \int_0^L y_x^2(x, t) dx \\ &\quad + \int_\omega \int_0^1 b(x) (2\alpha\xi h + 2\alpha\mu_2 h - \mu_2) y^2(x, t - h\rho) d\rho dx. \end{aligned}$$

Then, we choose  $\mu_1 > 0$ ,  $\mu_2 > 0$  and  $\alpha > 0$  such that

$$\mu_1 < \frac{\xi - 1}{L}, \quad \mu_2 < \xi - 1 - L\mu_1 \quad \text{and} \quad \alpha < \min \left\{ \frac{\mu_2}{2h(\xi + \mu_2)}, \frac{\mu_1(3\pi^2 - L^2)}{2L^2(1 + \mu_1 L)} \right\}.$$

This is possible by (1.7). Therefore we obtain

$$\frac{d}{dt} V(t) + 2\alpha V(t) \leq 0, \quad \forall t > 0.$$

Integrating over  $(0, t)$  and using (6.45), we finally obtain that

$$E(t) \leq \left( 1 + \max \left\{ L\mu_1, \frac{\mu_2}{\xi} \right\} \right) E(0)e^{-2\alpha t}, \quad \forall t > 0.$$

By density of  $\mathcal{D}(\mathcal{A})$  in  $H$ , the results extend to arbitrary  $(y_0, z_0(\cdot, -h\cdot)) \in H$ .  $\square$

## 6.2 Exponential stability for the linear system using a perturbation argument

We consider now the linear system (2.9) in the case where  $\text{supp } b = \omega \not\subset \text{supp } a$  that we can rewrite as the first system order (2.11). It is clear that the corresponding operator  $\mathcal{A}$  satisfy

$$\mathcal{A} = \mathcal{A}_0 + B,$$

with domain  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0)$  and where the bounded operator  $B$  is defined by

$$BU = \begin{pmatrix} \xi by \\ 0 \end{pmatrix}, \quad U = \begin{pmatrix} y \\ z \end{pmatrix} \in H.$$

**Proposition 7.** *Assume that  $a$  and  $b$  are nonnegative functions in  $L^\infty(0, L)$  satisfying (1.2) and assume that (1.7) holds. Let  $\xi > 1$ . Then for every  $U_0 \in H$ , there exists a unique mild solution  $U \in C([0, +\infty), H)$  for system (2.9), and for every  $U_0 \in \mathcal{D}(\mathcal{A})$ , the solution is classical and satisfies  $U \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), H)$ . Moreover there exists  $\delta > 0$  (depending on  $\xi, L, h$ ) such that if*

$$\|b\|_{L^\infty(0, L)} \leq \delta,$$

*then, for every  $(y_0, z_0(\cdot, -h\cdot)) \in H$  the energy of system (2.9), denoted  $E$  and defined by (6.40), decays exponentially. More precisely, there exist two positive constants  $\nu$  and  $\beta$  (defined in Proposition 6) such that*

$$E(t) \leq \beta E(0)e^{-2\nu t}, \quad t > 0.$$

*Proof.* It suffices to apply Theorem 9 and to note that, using Proposition 6 and the fact that  $\|B\| = \xi \|b\|_{L^\infty(0, L)}$ , we have

$$-\alpha + \sqrt{\beta}\xi \|b\|_{L^\infty(0, L)} < 0 \Leftrightarrow \|b\|_{L^\infty(0, L)} \leq \frac{\alpha}{\xi\sqrt{\beta}}.$$

$\square$

**Remark 5.** *Note that if  $h$  is large, the choice of  $b$  is such that  $\|b\|_{L^\infty(0, L)}$  is small, due to (6.46).*

### 6.3 Local exponential stability for the nonlinear system (1.1)

We finally obtain the local exponential stability result enounced in Theorem 4 by considering the nonlinear KdV equation (1.1) in the case where  $\text{supp } b = \omega \not\subset \text{supp } a$ .

*Proof of Theorem 4.* It suffices to adapt the proofs of Sections 2.3 and 4.2.  $\square$

**Remark 6.** *In order to apply Theorem 9, we need to have an estimation of the decay rate  $\alpha$  of the linear auxiliary system (6.41). That is why we use a Lyapunov method and we assume that (1.7) holds. If we prove an observability inequality as in Section 4 (which holds without restriction on the length of the domain), we do not have an estimation of the observability constant  $C$  in (4.28) since we use a contradiction argument. The decay rate of the linear auxiliary system is then given by  $\alpha = \frac{1}{T} \ln \left(1 + \frac{C_1}{C}\right)$  (see the proof of Theorem 7) and we should verify that  $-\alpha + \sqrt{\beta}\xi \|b\|_{L^\infty(0,L)} < 0$ . But the observability constant  $C$  may depend on  $\|b\|_{L^\infty(0,L)}$  and so this assumption is difficult to verify. To remove the assumption (1.7) in the case where  $\text{supp } b = \omega \not\subset \text{supp } a$  is, to our knowledge, an open question.*

## 7 Conclusion and remarks

In this paper, we studied the robustness with respect to the delay of the stabilization of the nonlinear KdV equation with internal feedbacks. We first considered the case of the support of the weight of the internal feedback with delay  $b$  is included in the support of the weight of the internal feedback without delay  $a$  and when  $b$  is strictly smaller than  $a$ . We proved the local exponential stability result by two methods: the first one by a Lyapunov approach giving an estimation of the decay rate but with a technical limitation on the length of the domain (i.e.  $L < \sqrt{3}\pi$ ) and a second one by an observability inequality which holds for any length (but without information on the exponential decay rate). We also obtained the semiglobal stabilization result for all length. Secondly we considered the case where the support of the weight of the internal feedback with delay  $b$  is not included in the support of the weight of the internal feedback without delay  $a$ . In this case, if  $b$  is small enough (and even if  $a = 0$ ), we showed that the nonlinear system is locally exponentially stable when  $L < \sqrt{3}\pi$ .

To illustrate these results, we present now some numerical simulations. Adapting the numerical scheme of [CG01] (see also [BCV18]), and using the parameters  $T = 10$ ,  $L = 3$ ,  $h = 2$  and initial conditions  $y_0(x) = 1 - \cos(2\pi x)$  and  $z_0(x, \rho) = (1 - \cos(2\pi x)) \cos(2\pi \rho h)$  with  $\text{supp } a = \text{supp } b = (0, L/5)$  and where  $a$  and  $b$  are constant on their support, we obtain the following figure, that represents  $t \mapsto \ln(E(t))$  for different values of  $a$  and  $b$ . We can see that when there is no feedback ( $a = b = 0$ ), the energy is exponentially decreasing, and if the feedback without delay increases ( $a = 1$  and  $b = 0$ ), the energy is quickly exponentially decreasing. Moreover if the coefficient of

delay  $b$  increases, then the energy is not exponentially decreasing, except if  $b$  is very small (for instance  $b = 0.1$ ) or if  $b$  is smaller than  $a$  ( $a = 4, b = 1$ ). More precisely, with  $a = 0, b = 10$  or  $a = b = 10$ , before that the delay acts ( $t < 2$ ), the energy decays exponentially, which is not longer the case when  $b$  is effective ( $t > 2$ ). Consequently Figure 1 illustrates Theorems 1 and 4.

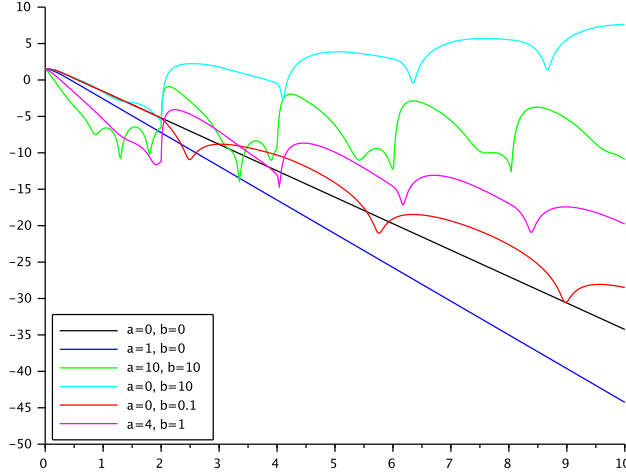


Figure 1: Representation of  $t \mapsto \ln(E(t))$  for different values of  $a$  and  $b$ .

We finish this paper by considered the cases of mixed internal and boundary dampings with delay.

The most simple case is the case where we have an internal feedback without delay and a boundary feedback with delay, i.e.

$$\left\{ \begin{array}{ll} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + a(x)y(x, t) = 0, & x \in (0, L), t > 0, \\ y(0, t) = y(L, t) = 0, & t > 0, \\ y_x(L, t) = \beta y_x(0, t - h), & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y_x(0, t) = z_0(t), & t \in (-h, 0), \end{array} \right.$$

where  $|\beta| < 1$  (see [BCV18] for an explanation of this assumption), and where  $a$  is a nonnegative function in  $L^\infty(0, L)$  such that  $a(x) \geq a_0 > 0$  a.e. in  $\omega$  an open nonempty subset of  $(0, L)$ . In this case, it is sufficient to combine [Paz05] and [BCV18] to obtain the local exponential stability result for every non critical length (i.e.  $L \notin \mathcal{L} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N}^* \right\}$ ).

If we consider now the case of an internal feedback with delay and a boundary feedback without



delay, i.e.

$$\left\{ \begin{array}{ll} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + b(x)y(x, t-h) = 0, & \\ \quad x \in (0, L), t > 0, & \\ y(0, t) = y(L, t) = 0, & t > 0, \\ y_x(L, t) = \alpha y_x(0, t), & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0), \end{array} \right. \quad (7.47)$$

where  $|\alpha| < 1$  (see [Zha94]), and where  $b$  is a nonnegative function in  $L^\infty(0, L)$  such that  $b(x) \geq b_0 > 0$  a.e. in  $\text{supp } b = \omega$  an open nonempty subset of  $(0, L)$  and where  $\|b\|_{L^\infty(0, L)}$  is small enough. Then we can follow Section 6 to obtain the local exponential stability result for every  $L < \sqrt{3}\pi$ . For that, we introduce the following auxiliary exponentially stable system

$$\left\{ \begin{array}{ll} y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + b(x)y(x, t-h) + \xi b(x)y(x, t) = 0, & \\ \quad x \in (0, L), t > 0, & \\ y(0, t) = y(L, t) = 0, & t > 0, \\ y_x(L, t) = \alpha y_x(0, t), & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \\ y(x, t) = z_0(x, t), & x \in (0, L), t \in (-h, 0), \end{array} \right.$$

with the energy defined by (6.40) and  $\xi > 1$ . Note that we can take  $\alpha = 0$  here.

An interesting question to investigate is to remove the technical assumption (1.7) in Theorem 4 or for (7.47). An other subject of future research could be the study of the stability of the KdV equation with a delay in the nonlinear term.

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