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THE SURVIVAL PROBABILITY OF CRITICAL AND SUBCRITICAL BRANCHING PROCESSES IN FINITE STATE SPACE MARKOVIAN ENVIRONMENT

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Abstract. . Let $(Z_n)_{n\geqslant 0}$ be a branching process in a random environment defined by a Markov chain $(X_n)_{n\geqslant 0}$ with values in a finite state space $\mathbb{X}$. Let $\mathbb{P}_i$ be the probability generated by the trajectories of $(X_n)_{n\geqslant 0}$ starting at $X_0 = i \in \mathbb{X}$. We study the asymptotic behaviour of the joint survival probability $\mathbb{P}_i(Z_n > 0\,,\, X_n = j)$, $j \in \mathbb{X}$ as $n \to +\infty$ in the critical and strongly, intermediate and weakly subcritical cases.

1. Introduction and main results

The Galton-Watson branching process is one of the most used models in the dynamic of populations with numerous applications in different areas such as biology, medicine, physics, economics etc; for an introduction we refer to Harris [17], Athreya and Ney [5] and to the references therein. The random environment in the context of a branching process, say $(Z_n)_{n\geqslant 0}$, has first been introduced in Smith and Wilkinson [22] and Athreya and Karlin [4, 3]. In a remarkable series of papers Afanasyev [1], Dekking [6], Kozlov [19], Liu [21], D'Souza and Hambly [7], Geiger and Kersting [9], Guivarc'h and Liu [16] and Geiger, Kersting and Vatutin [10] have determined the asymptotic behaviour of the survival probability of a branching process with random environment under various assumptions. Based on the recent advances in the study of conditioned limit theorems for sums of functions defined on Markov chains from [11, 12, 13, 14], the goal of the present paper is to prove exact asymptotic results for the survival probability when the environment is a Markov chain.

Let $(X_n)_{n\geqslant 0}$ be a homogeneous Markov chain defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in the finite state space $\mathbb{X}$. Let $\mathscr{C}$ be the set of functions from $\mathbb{X}$ to $\mathbb{C}$. Denote by $\mathbf{P}$ the transition operator of the chain $(X_n)_{n\geqslant 0}$: $\mathbf{P}g(i) = \mathbb{E}_i\left(g(X_1)\right)$, for any $g \in \mathscr{C}$ and $i \in \mathbb{X}$. Set $\mathbf{P}(i,j) = \mathbf{P}(\delta_j)(i)$, where $\delta_j(i) = 1$ if $i = j$ and $\delta_j(i) = 0$ else. Note that $\mathbf{P}^n g(i) = \mathbb{E}_i\left(g(X_n)\right)$. Let $\mathbb{P}_i$ be the probability on $(\Omega, \mathscr{F})$ generated by the finite dimensional distributions of the Markov chain

2010 *Mathematics Subject Classification.* Primary 60J80. Secondary 60J10.

Key words and phrases. Critical and subcritical branching process, random environment, Markov chain, survival probability.

$(X_n)_{n\geqslant 0}$ starting at $X_0 = i$. Denote by $\mathbb{E}$ and $\mathbb{E}_i$ the corresponding expectation associated to $\mathbb{P}$ and $\mathbb{P}_i$.

Assume that $(X_n)_{n\geqslant 0}$ is irreducible and aperiodic which is equivalent to:

Condition 1. *The matrix* $\mathbf{P}$ *is primitive, which means that there exists* $k_0 \geqslant 1$ *such that, for any non-negative and non-identically zero function* $g \in \mathscr{C}$ *and* $i \in \mathbb{X}$ *it holds* $\mathbf{P}^{k_0} g(i) > 0$.

By the Perron-Frobenius theorem, under Condition 1, there exist positive constants c_1 and c_2, a unique positive $\mathbf{P}$-invariant probability $\boldsymbol{\nu}$ on $\mathbb{X}$ and an operator Q on $\mathscr{C}$ such that for any $g \in \mathscr{C}$ and $n \geqslant 1$,

$$\mathbf{P}g(i) = \boldsymbol{\nu}(g) + Q(g)(i) \qquad \text{and} \qquad \|Q^n(g)\|_\infty \leqslant c_1 \, \mathrm{e}^{-c_2 n} \, \|g\|_\infty ,$$

where $\boldsymbol{\nu}(g) := \sum_{i\in\mathbb{X}} g(i)\boldsymbol{\nu}(i)$, $Q(1) = \boldsymbol{\nu}(Q(g)) = 0$, $\|g\|_\infty = \max_{i\in\mathbb{X}} |g(i)|$. In particular, for any $(i, j) \in \mathbb{X}^2$, we have

$$|\mathbf{P}^n(i, j) - \boldsymbol{\nu}(j)| \leqslant c_1 \, \mathrm{e}^{-c_2 n} . \tag{1.1}$$

The branching process in the Markov environment $(X_n)_{n\geqslant 0}$ is defined with the help of a collection of generating functions

$$f_i(s) := \mathbb{E}\left(s^{\xi_i}\right), \quad \forall i \in \mathbb{X}, \ s \in [0, 1], \tag{1.2}$$

where the random variable ξ_i takes its values in $\mathbb{N}$ and means the total offspring of one individual when the environment is $i \in \mathbb{X}$. For any $i \in \mathbb{X}$, let $(\xi_i^{n,j})_{j,n\geqslant 1}$ be independent and identically distributed random variables with the same generating function f_i defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that the sequence $(\xi_i^{n,j})_{j,n\geqslant 1}$ is independent of the Markov chain $(X_n)_{n\geqslant 0}$.

Condition 2. *For any* $i \in \mathbb{X}$*, the random variable* ξ_i *is non-identically zero and has a finite second moment:* $\mathbb{E}(\xi_i) > 0$ *and* $\mathbb{E}(\xi_i^2) < +\infty$.

Condition 2 implies that, $0 < f_i'(1) < +\infty$ and $f_i''(1) < +\infty$, $i \in \mathbb{X}$.

Define the branching process $(Z_n)_{n\geqslant 0}$ iteratively: for each $n = 1, 2, \ldots$, given the environment $X_n = i$, the total offspring of each individual $j \in \{1, \ldots Z_{n-1}\}$ is given by the random variable $\xi_i^{n,j}$, so that the total population is

$$Z_0 = 1 \qquad \text{and} \qquad Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{X_n}^{n,j}, \qquad \forall n \geqslant 1.$$

We shall consider branching processes $(Z_n)_{n\geqslant 0}$ in one of the following two regimes: critical or subcritical (see below for the precise definition). In both cases the probability that the population survives until the n-th generation tends to zero, $\mathbb{P}_i(Z_n > 0) \to 0$, for any $i \in \mathbb{X}$ as $n \to +\infty$, see Smith and Wilkinson [23]. The key point in determining the speed of this convergence is a close relation between the branching process and the associated Markov walk $(S_n)_{n\geqslant 0}$ defined as follows. Let

$$\rho(i) = \ln f_i'(1), \qquad \forall i \in \mathbb{X}.$$

Set $S_0 := 0$ and

$$S_n := \ln\left(f'_{X_1}(1)\cdots f'_{X_n}(1)\right) = \sum_{k=1}^{n} \rho\left(X_k\right), \quad \forall n \geqslant 1. \tag{1.3}$$

Assume that the Markov walk $(S_n)_{n\geqslant 0}$ is non-lattice:

Condition 3. *For any $(\theta, a) \in \mathbb{R}^2$, there exista a path $x_0, \ldots, x_n$ in $\mathbb{X}$ such that $\mathbf{P}(x_0, x_1)\cdots\mathbf{P}(x_{n-1}, x_n)\mathbf{P}(x_n, x_0) > 0$ and $\rho(x_0) + \cdots + \rho(x_n) - (n+1)\theta \notin a\mathbb{Z}$.*

It is shown in Section 2.4 that under Conditions 1 and 3, for any $\lambda \in \mathbb{R}$ and any $i \in \mathbb{X}$, the following limit exists and does not depend on the initial state of the Markov chain $X_0 = i$:

$$k(\lambda) := \lim_{n\to+\infty} \mathbb{E}_i^{1/n}\left(\mathrm{e}^{\lambda S_n}\right).$$

The function k, up to a logarithmic transform, is similar to the function Λ in [7]. It is related to the so-called transfer operator $\mathbf{P}_\lambda$:

$$\mathbf{P}_\lambda g(i) := \mathbf{P}\left(\mathrm{e}^{\lambda\rho}\, g\right)(i) = \mathbb{E}_i\left(\mathrm{e}^{\lambda S_1}\, g(X_1)\right), \quad \text{for} \quad g \in \mathscr{C}, i \in \mathbb{X}. \tag{1.4}$$

In particular, $k(\lambda)$ is an eigenvalue of the operator $\mathbf{P}_\lambda$ corresponding to an eigenvector v_λ and is equal to its spectral radius. Moreover, the function $k(\lambda)$ is analytic on $\mathbb{R}$, see Lemma 2.15. Note also that the transfer operator $\mathbf{P}_\lambda$ is not Markovian, but it can be easily normalized so that the operator $\tilde{\mathbf{P}}_\lambda g = \frac{\mathbf{P}_\lambda(g v_\lambda)}{k(\lambda) v_\lambda}$ is Markovian. We shall denote by $\tilde{\boldsymbol{\nu}}_\lambda$ its unique invariant probability measure.

The branching process in Markovian environment is said to be *subcritical* if $k'(0) < 0$, *critical* if $k'(0) = 0$ and *supercritical* if $k'(0) > 0$. To clarify the relation to the classification in the case of branching processes with i.i.d. environment note that, by Lemma 2.15,

$$k'(0) = \boldsymbol{\nu}(\rho) = \mathbb{E}_{\boldsymbol{\nu}}\left(\rho(X_1)\right) = \mathbb{E}_{\boldsymbol{\nu}}\left(\ln f'_{X_1}(1)\right) = \varphi'(0), \tag{1.5}$$

where $\mathbb{E}_{\boldsymbol{\nu}}$ is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ in the stationary regime and $\varphi(\lambda) = \mathbb{E}_{\boldsymbol{\nu}}(\exp\{\lambda \ln f'_{X_1}(1)\})$, $\lambda \in \mathbb{R}$. When the random variables $(X_n)_{n\geqslant 1}$ are i.i.d. with common law $\boldsymbol{\nu}$, from (1.5) the two classifications coincide.

We proceed to formulate our main result in the critical case.

Theorem 1.1 (Critical case)**.** *Assume Conditions 1-3 and $k'(0) = 0$. Then, there exists a positive function u on $\mathbb{X}$ such that for any $(i, j) \in \mathbb{X}^2$,*

$$\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} \frac{\boldsymbol{\nu}(j) u(i)}{\sqrt{n}}.$$

The critical case has been considered in Le Page and Ye [20] in a more general setting. Nevertheless, the conditions in their paper do not cover the present situation and the employed method is different from ours. For an i.i.d. environment,

it has been established earlier in [9] that $\mathbb{P}(Z_n > 0) \sim \frac{c}{\sqrt{n}}$ as $n \to +\infty$ under weaker assumptions than the finiteness of the state space $\mathbb{X}$.

Now we consider the subcritical case. We say that the branching process in Markovian environment is *strongly subcritical* if $k'(0) < 0, k'(1) < 0$, *intermediately subcritical* if $k'(0) < 0, k'(1) = 0$ and *weakly subcritical* if $k'(0) < 0, k'(1) > 0$. Again by Lemma 2.15,

$$k'(1)/k(1) = \tilde{\boldsymbol{\nu}}_1(\rho) = \mathbb{E}_{\tilde{\boldsymbol{\nu}}_1}\left(\rho(X_1)\right) = \mathbb{E}_{\tilde{\boldsymbol{\nu}}_1}\left(\ln f'_{X_1}(1)\right), \tag{1.6}$$

where $\mathbb{E}_{\tilde{\boldsymbol{\nu}}_\lambda}$ is the expectation generated by the finite dimensional distributions of the Markov chain with transition probabilities $\tilde{\mathbf{P}}_\lambda$ in the stationary regime. When the environment $(X_n)_{n\geqslant 0}$ is an i.i.d. sequence of common law $\boldsymbol{\nu}$ we have in addition

$$\mathbb{E}_{\tilde{\boldsymbol{\nu}}_1}\left(\ln f'_{X_1}(1)\right) = \mathbb{E}_{\boldsymbol{\nu}}\left(f'_{X_1}(1)\ln f'_{X_1}(1)\right) = \varphi'(1). \tag{1.7}$$

This shows that for branching processes with i.i.d. environments both classifications (the one according to $k'(\cdot)$ and the other according to $\varphi'(\cdot)$) are equivalent. In general, (1.7) is not fulfilled for a Markovian environment and therefore the function $\varphi(\cdot)$ is not the appropriate one for the classification. For a Markovian environment the classification equally can be done using the function $K'(\lambda)$, where $K(\lambda) = \ln k(\lambda)$, $\lambda \in \mathbb{R}$.

Note that by Lemma 2.15 the function $\lambda \mapsto K(\lambda)$ is strictly convex. In the strongly and intermediately subcritical cases, this implies that $0 < k(1) < 1$.

Theorem 1.2 (Strongly subcritical case)**.** *Assume Conditions 1-3 and $k'(0) < 0$, $k'(1) < 0$. Then, there exists a positive function u on $\mathbb{X}$ such that for any $(i, j) \in \mathbb{X}^2$,*

$$\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} k(1)^n v_1(i)u(j).$$

Recall that v_1 is the eigenfunction of the transfer operator $\mathbb{P}_1$ (see also Section 2.4 eq. (2.29) for details). Note also that we can drop the assumption $k'(0) < 0$, since it is implied by the assumption $k'(1) < 0$, in view of the strict convexity of $K(\lambda)$. For comparison, the corresponding result in the case when the environment is i.i.d. has been established in [16]: $\mathbb{P}(Z_n > 0) \sim c\varphi(1)^n$, as $n \to +\infty$, where $0 < \varphi(1) = \mathbb{E}f'_{X_1}(1) < 1$.

Theorem 1.3 (Intermediate subcritical case)**.** *Assume Conditions 1-3 and $k'(0) < 0$, $k'(1) = 0$. Then, there exists a positive function u on $\mathbb{X}$ such that for any $(i, j) \in \mathbb{X}^2$,*

$$\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} k(1)^n \frac{v_1(i)u(j)}{\sqrt{n}}.$$

As in Theorem 1.2, $k'(1) = 0$ implies the assumption $k'(0) < 0$.

In the weakly subcritical case, an easy consequence of the strict convexity of K is the existence and the unicity of $\lambda \in (0, 1)$ satisfying $k'(\lambda) = 0$ and $0 < k(\lambda) < 1$ which is used the next result.

Theorem 1.4 (Weakly subcritical case)**.** *Assume Conditions 1-3 and* $k'(0) < 0$, $k'(1) > 0$. *Then, there exist a unique* $\lambda \in (0,1)$ *satisfying* $k'(\lambda) = 0$, $0 < k(\lambda) < 1$ *and a positive function* u *on* $\mathbb{X}^2$ *such that for any* $(i,j) \in \mathbb{X}^2$,

$$\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} k(\lambda)^n \frac{u(i,j)}{n^{3/2}}.$$

Recall the original results in [10] which have been established for an i.i.d. environment. In the intermediate and weakly subcritical cases, respectively: $\mathbb{P}(Z_n > 0) \sim cn^{-1/2}\varphi(1)^n$ and $\mathbb{P}(Z_n > 0) \sim cn^{-3/2}\varphi(\lambda)^n$, where λ is the unique critical point of φ: $\varphi'(\lambda) = 0$ such that $0 < \varphi(1) < 1$.

For stationary and ergodic environments D'Souza and Hambly [7] have studied the large deviation principle for the survival probability. Theorems 1.1-1.4 improve on the results in [7] giving exact asymptotics. In addition, the random environment in our model is not assumed to be stationary.

The proofs of the main results are based on the following relation between the survival probability $\mathbb{P}_i(Z_n > 0)$ and the associated random walk $(S_n)_{n\geqslant 0}$ which goes back to Agresti [2]: for any initial state $X_0 = i$,

$$\mathbb{P}_i(Z_n > 0) = \mathbb{E}_i(q_n), \quad \text{where} \quad q_n^{-1} = \mathrm{e}^{-S_n} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n} \tag{1.8}$$

and under the assumptions of the paper the random variables $\eta_{k+1,n}$ are bounded. To handle the expectation $\mathbb{E}_i(q_n)$ in the right-hand side of (1.8) we make use of three tools: conditioned limit and local limit theorems for Markov chains which have been obtained recently in [13] and [12], the exponential change of measure which is defined with the help of the transfer operator $\mathbf{P}_\lambda$, see Guivarc'h and Hardy [15], and the duality for Markov chains which we develop in Section 2.2.

The outline of the paper is as follows. In Section 2 we introduce the associated Markov chain and relate it to the survival probability. We also introduce the dual Markov chain and state some useful assertions for walks on Markov chains conditioned to stay positive and on the transfer operator. The proofs in the critical, strongly subcritical, intermediate subcritical and weakly subcritical cases are deferred to Sections 3, 4, 5 and 6, respectively.

Let us end this section by fixing some notations. The symbol c, possibly enabled with subscripts, will denote positive constants depending on all previously introduced constants. All these constants are likely to change their values every occurrence. The indicator of an event A is denoted by $\mathbb{1}_A$. For any bounded measurable function f on $\mathbb{X}$, random variable X in some measurable space $\mathbb{X}$ and event A, the integral $\int_{\mathbb{X}} f(x)\mathbb{P}(X \in \mathrm{d}x, A)$ means the expectation $\mathbb{E}\left(f(X); A\right) = \mathbb{E}\left(f(X)\mathbb{1}_A\right)$.

2. Preliminary results on the associated Markov walk

The aim of this section is to provide necessary assertions on the Markov chain and on the associated Markov walk (1.3) and to relate them to the survival probability of the branching process.

2.1. **The link between the branching process and the associated Markov walk.** The proof of the following lemma being elementary is left to the reader.

Lemma 2.1 (Conditioned generating function)**.** *For any $s \in [0,1]$ and $n \geqslant 1$,*

$$\mathbb{E}_i\left(s^{Z_n} \,\middle|\, X_1, \dots, X_n\right) = f_{X_1} \circ \cdots \circ f_{X_n}(s).$$

For any $n \geqslant 1$ and $s \in [0,1]$ set

$$q_n(s) := 1 - f_{X_1} \circ \cdots \circ f_{X_n}(s) \qquad \text{and} \qquad q_n := q_n(0). \tag{2.1}$$

Lemma 2.1 implies that

$$\mathbb{P}_i\left(Z_n > 0 \,|\, X_1, \dots, X_n\right) = q_n. \tag{2.2}$$

Taking the expectation in (2.2), we obtain the well-known equality, which will be the starting point for our study:

$$\mathbb{P}_i\left(Z_n > 0\right) = \mathbb{E}_i\left(q_n\right). \tag{2.3}$$

Under Condition 2, for any $i \in \mathbb{X}$ and $s \in [0,1)$, we have $f_i(s) \in [0,1)$. Therefore $f_{X_1} \circ \cdots \circ f_{X_n}(s) \in [0,1)$ and in particular

$$q_n \in (0,1], \qquad \forall n \geqslant 1. \tag{2.4}$$

Introduce some additional notations, which will be used all over the paper: for any $n \geqslant 1$, $k \in \{1, \dots, n\}$, $i \in \mathbb{X}$ and $s \in [0,1)$,

$$f_{k,n} := f_{X_k} \circ \cdots \circ f_{X_n}, \qquad f_{n+1,n} := \mathrm{id}, \tag{2.5}$$

$$g_i(s) := \frac{1}{1 - f_i(s)} - \frac{1}{f_i'(1)(1-s)}, \tag{2.6}$$

$$\eta_{k,n}(s) := g_{X_k}\left(f_{k+1,n}(s)\right), \qquad \eta_{k,n} := \eta_{k,n}(0). \tag{2.7}$$

The following key point assertion relies the random variable $q_n(s)$ to the associated Markov walk. Its proof being similar to corresponding statements in [2] and [9] is left to the reader.

Lemma 2.2. *For any $s \in [0,1)$ and $n \geqslant 1$,*

$$q_n(s)^{-1} = \frac{\mathrm{e}^{-S_n}}{1-s} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}(s).$$

Taking $s = 0$ in Lemma 2.2 we obtain the following identity which will play the central role in the proofs:

$$q_n^{-1} = \mathrm{e}^{-S_n} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}, \qquad \forall n \geqslant 1. \tag{2.8}$$

Since f_i is convex on $[0,1]$ for all $i \in \mathbb{X}$, the function g_i is non-negative,

$$g_i(s) = \frac{f_i'(1)(1-s) - (1 - f_i(s))}{(1 - f_i(s))\, f_i'(1)(1-s)} \geqslant 0, \qquad \forall s \in [0,1), \tag{2.9}$$

which, in turn, implies that the random variables $\eta_{k+1,n}$ are non-negative for any $n \geqslant 1$ and $k \in \{0, \ldots, n-1\}$.

Lemma 2.3. *Assume Condition 2. For any $n \geqslant 2$, $(i_1, \ldots, i_n) \in \mathbb{X}^n$ and $s \in [0,1)$, we have*

$$0 \leqslant g_{i_1}\left(f_{i_2} \circ \cdots \circ f_{i_n}(s)\right) \leqslant \eta := \max_{i \in \mathbb{X}} \frac{f_i''(1)}{f_i'(1)^2} < +\infty.$$

Moreover, for any $(i_n)_{n \geqslant 1} \in \mathbb{X}^{\mathbb{N}^*}$ and any $k \geqslant 1$,

$$\lim_{n \to +\infty} g_{i_k}\left(f_{i_{k+1}} \circ \cdots \circ f_{i_n}(0)\right) \in [0, \eta]. \tag{2.10}$$

Proof. Fix $(i_n)_{n \geqslant 1} \in \mathbb{X}^{\mathbb{N}^*}$. For any $i \in \mathbb{X}$ and $s \in [0,1)$, we have $f_i(s) \in [0,1)$. So $f_{i_2} \circ \cdots \circ f_{i_n}(s) \in [0,1)$. In addition, by (2.9), g_i is non-negative on $[0,1)$ for any $i \in \mathbb{X}$, therefore $g_{i_1}\left(f_{i_2} \circ \cdots \circ f_{i_n}(s)\right) \geqslant 0$. Moreover by the lemma 2.1 of [9], for any $i \in \mathbb{X}$ and any $s \in [0,1)$,

$$g_i(s) \leqslant \frac{f_i''(1)}{f_i'(1)^2}. \tag{2.11}$$

By Condition 2, $\eta < +\infty$ and so $g_{i_1}\left(f_{i_2} \circ \cdots \circ f_{i_n}(s)\right) \in [0, \eta]$, for any $s \in [0,1)$.

Since f_i is increasing on $[0,1)$ for any $i \in \mathbb{X}$, it follows that for any $k \geqslant 1$ and any $n \geqslant k+1$,

$$0 \leqslant f_{i_{k+1}} \circ \cdots \circ f_{i_n}(0) \leqslant f_{i_{k+1}} \circ \cdots \circ f_{i_n} \circ f_{i_{n+1}}(0) \leqslant 1,$$

and the sequence $\left(f_{i_{k+1}} \circ \cdots \circ f_{i_n}(0)\right)_{n \geqslant k+1}$ converges to a limit, say $l \in [0,1]$. For any $i \in \mathbb{X}$, the function g_i is continuous on $[0,1)$ and we have

$$\begin{aligned}
\lim_{\substack{s \to 1 \\ s<1}} g_i(s) &= \lim_{\substack{s \to 1 \\ s<1}} \frac{f_i'(1)(1-s) - (1 - f_i(s))}{f'(1)\,(1 - f_i(s))\,(1-s)} \\
&= \lim_{\substack{s \to 1 \\ s<1}} \frac{1}{f_i'(1)} \frac{f_i(s) - 1 - f_i'(1)(s-1)}{(s-1)^2} \frac{1-s}{1 - f_i(s)} \\
&= \frac{1}{f_i'(1)} \frac{f_i''(1)}{2} \frac{1}{f_i'(1)} = \frac{f_i''(1)}{2 f_i'(1)^2} < +\infty. \qquad (2.12)
\end{aligned}$$

Denoting $g_i(l) = \frac{f_i''(1)}{2 f_i'(1)^2}$ if $l = 1$, we conclude that $g_{i_k}\left(f_{i_{k+1}} \circ \cdots \circ f_{i_n}(0)\right)$ converges to $g_{i_k}(l)$ as $n \to +\infty$. By (2.9) and (2.11), we obtain that $g_{i_k}(l) \in [0, \eta]$. □

2.2. **The dual Markov walk.** We will introduce the dual Markov chain $(X_n^*)_{n \geqslant 0}$ and the associated dual Markov walk $(S_n^*)_{n \geqslant 0}$, and state some of their properties.

Since $\boldsymbol{\nu}$ is positive on $\mathbb{X}$, the following dual Markov kernel $\mathbf{P}^*$ is well defined:

$$\mathbf{P}^*(i,j) = \frac{\boldsymbol{\nu}(j)}{\boldsymbol{\nu}(i)} \mathbf{P}(j,i), \quad \forall (i,j) \in \mathbb{X}^2. \tag{2.13}$$

Without loss of generality we assume that the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is rich enough to define on it a Markov chain $(X_n^*)_{n\geqslant 0}$, called dual, with values in $\mathbb{X}$ and with transition probability $\mathbf{P}^*$. Clearly, it can be chosen to be independent of the chain $(X_n)_{n\geqslant 0}$. We define the dual Markov walk by

$$S_0^* = 0 \qquad \text{and} \qquad S_n^* = -\sum_{k=1}^{n} \rho\left(X_k^*\right), \quad \forall n \geqslant 1. \tag{2.14}$$

For any $z \in \mathbb{R}$, let τ_z^* be the associated exit time:

$$\tau_z^* := \inf\left\{k \geqslant 1 : z + S_k^* \leqslant 0\right\}. \tag{2.15}$$

For any $i \in \mathbb{X}$, denote by $\mathbb{P}_i^*$ and $\mathbb{E}_i^*$ the probability, respectively the expectation generated by the finite dimensional distributions of the Markov chain $(X_n^*)_{n\geqslant 0}$ starting at $X_0^* = i$.

It is easy to see that $\boldsymbol{\nu}$ is also $\mathbf{P}^*$-invariant and for any $n \geqslant 1$, $(i,j) \in \mathbb{X}^2$,

$$\left(\mathbf{P}^*\right)^n (i,j) = \mathbf{P}^n(j,i)\frac{\boldsymbol{\nu}(j)}{\boldsymbol{\nu}(i)}.$$

In particular, the last formula implies the following result.

Lemma 2.4. *Assume Conditions 1 and 3 for the Markov kernel* $\mathbf{P}$*. Then Conditions 1 and 3 hold also for dual kernel* $\mathbf{P}^*$*.*

Similarly to (1.1), we have for any $(i,j) \in \mathbb{X}^2$,

$$\left|\left(\mathbf{P}^*\right)^n (i,j) - \boldsymbol{\nu}(j)\right| \leqslant c\,\mathrm{e}^{-cn}. \tag{2.16}$$

Note that the operator $\mathbf{P}^*$ is the adjoint of $\mathbf{P}$ in the space $\mathrm{L}^2(\boldsymbol{\nu})$: for any functions f and g on $\mathbb{X}$,

$$\boldsymbol{\nu}\left(f\left(\mathbf{P}^*\right)^n g\right) = \boldsymbol{\nu}\left(g\mathbf{P}^n f\right).$$

For any measure $\mathfrak{m}$ on $\mathbb{X}$, let $\mathbb{E}_{\mathfrak{m}}$ (respectively $\mathbb{E}_{\mathfrak{m}}^*$) be the expectation associated to the probability generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geqslant 0}$ (respectively $(X_n^*)_{n\geqslant 0}$) with the initial law $\mathfrak{m}$.

Lemma 2.5 (Duality)**.** *For any probability measure* $\mathfrak{m}$ *on* $\mathbb{X}$*, any* $n \geqslant 1$ *and any function* g*:* $\mathbb{X}^n \to \mathbb{C}$*,*

$$\mathbb{E}_{\mathfrak{m}}\left(g\left(X_1, \ldots, X_n\right)\right) = \mathbb{E}_{\boldsymbol{\nu}}^*\left(g\left(X_n^*, \ldots, X_1^*\right)\frac{\mathfrak{m}\left(X_{n+1}^*\right)}{\boldsymbol{\nu}\left(X_{n+1}^*\right)}\right).$$

Moreover, for any $n \geqslant 1$ *and any function* g*:* $\mathbb{X}^n \to \mathbb{C}$*,*

$$\mathbb{E}_i\left(g\left(X_1, \ldots, X_n\right) ;\, X_{n+1} = j\right) = \mathbb{E}_j^*\left(g\left(X_n^*, \ldots, X_1^*\right) ;\, X_{n+1}^* = i\right)\frac{\boldsymbol{\nu}(j)}{\boldsymbol{\nu}(i)}.$$

Proof. The first equality is proved in Lemma 3.2 of [12]. The second can be deduced from the first as follows. Taking $\mathfrak{m} = \boldsymbol{\delta}_i$ and $\tilde{g}(i_1, \cdots, i_n, i_{n+1}) = g(i_1, \cdots, i_n)\mathbb{1}_{\{i_{n+1}=j\}}$, from the first equality of the lemma, we see that

$$\mathbb{E}_i\left(g\left(X_1, \ldots, X_n\right) ; X_{n+1} = j\right) = \mathbb{E}^*_{\boldsymbol{\nu}}\left(\tilde{g}\left(X^*_{n+1}, \ldots, X^*_1\right) ; X^*_{n+2} = i\right)\frac{1}{\boldsymbol{\nu}(i)}$$
$$= \mathbb{E}^*_{\boldsymbol{\nu}}\left(g\left(X^*_{n+1}, \ldots, X^*_2\right) ; X^*_1 = j\,, X^*_{n+2} = i\right)\frac{1}{\boldsymbol{\nu}(i)}.$$

Since $\boldsymbol{\nu}$ is $\mathbf{P}^*$-invariant, we obtain

$$\mathbb{E}_i\left(g\left(X_1, \ldots, X_n\right) ; X_{n+1} = j\right)$$
$$= \sum_{i_1 \in \mathbb{X}} \mathbb{E}^*_{i_1}\left(g\left(X^*_n, \ldots, X^*_1\right) ; X^*_{n+1} = i\right)\frac{1}{\boldsymbol{\nu}(i)}\mathbb{1}_{\{i_1=j\}}\boldsymbol{\nu}(i_1)$$
$$= \mathbb{E}^*_j\left(g\left(X^*_n, \ldots, X^*_1\right) ; X^*_{n+1} = i\right)\frac{\boldsymbol{\nu}(j)}{\boldsymbol{\nu}(i)}.$$

□

2.3. **Markov walks conditioned to stay positive.** In this section we recall the main results from [13] and [12] for Markov walks conditioned to stay positive. We complement them by some new assertions which will be used in the proofs of the main results of the paper.

For any $y \in \mathbb{R}$ define the first time when the Markov walk $(S_n)_{n\geqslant 0}$ becomes non-positive by setting

$$\tau_y := \inf\left\{k \geqslant 1 : y + S_k \leqslant 0\right\}.$$

Under Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$ the stopping time τ_y is well defined and finite $\mathbb{P}_i$-almost surely for any $i \in \mathbb{X}$.

The following three assertions deal with the existence of the harmonic function, the limit behaviour of the probability of the exit time and of the law of the random walk $y + S_n$, conditioned to stay positive and are taken from [13].

Proposition 2.6. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$. There exists a non-negative function V on $\mathbb{X} \times \mathbb{R}$ such that*

1. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geqslant 1$,

$$\mathbb{E}_i\left(V\left(X_n, y + S_n\right) ; \tau_y > n\right) = V(i, y).$$

2. For any $i \in \mathbb{X}$, the function $V(i, \cdot)$ is non-decreasing and for any $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$V(i, y) \leqslant c\left(1 + \max(y, 0)\right).$$

3. For any $i \in \mathbb{X}$, $y > 0$ and $\delta \in (0, 1)$,

$$(1 - \delta)\, y - c_\delta \leqslant V(i, y) \leqslant (1 + \delta)\, y + c_\delta.$$

We define

$$\sigma^2 := \boldsymbol{\nu}\left(\rho^2\right) - \boldsymbol{\nu}\left(\rho\right)^2 + 2\sum_{n=1}^{+\infty}\left[\boldsymbol{\nu}\left(\rho \mathbf{P}^n \rho\right) - \boldsymbol{\nu}\left(\rho\right)^2\right]. \tag{2.17}$$

It is known that under Conditions 1 and 3 we have $\sigma^2 > 0$, see Lemma 10.3 in [12].

Proposition 2.7. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$.*

1. *For any $(i, y) \in \mathbb{X} \times \mathbb{R}$,*

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_i\left(\tau_y > n\right) = \frac{2V(i,y)}{\sqrt{2\pi}\sigma},$$

   where σ is defined by (2.17).
2. *For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geqslant 1$,*

$$\mathbb{P}_i\left(\tau_y > n\right) \leqslant c\frac{1 + \max(y, 0)}{\sqrt{n}}.$$

We denote by $\mathrm{supp}(V) = \{(i, y) \in \mathbb{X} \times \mathbb{R} : \ V(i, y) > 0\}$ the support of the function V. Note that from property 3 of Proposition 2.6, for any fixed $i \in \mathbb{X}$, the function $y \mapsto V(i, y)$ is positive for large y. For more details on the properties of $\mathrm{supp}(V)$ see [13].

Proposition 2.8. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$.*

1. *For any $(i, y) \in \mathrm{supp}(V)$ and $t \geqslant 0$,*

$$\mathbb{P}_i\left(\frac{y + S_n}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_y > n\right) \underset{n\to+\infty}{\longrightarrow} \mathbf{\Phi}^+(t),$$

   where $\mathbf{\Phi}^+(t) = 1 - \mathrm{e}^{-\frac{t^2}{2}}$ is the Rayleigh distribution function.
2. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geqslant 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(i, y) \in \mathbb{X} \times \mathbb{R}$,*

$$\left|\mathbb{P}_i\left(y + S_n \leqslant t\sqrt{n}\sigma\,,\, \tau_y > n\right) - \frac{2V(i,y)}{\sqrt{2\pi n}\sigma}\mathbf{\Phi}^+(t)\right| \leqslant c_{\varepsilon,t_0}\frac{(1 + \max(y, 0)^2)}{n^{1/2+\varepsilon}}.$$

The next assertions are two local limit theorems for the associated Markov walk $y + S_n$ from [12].

Proposition 2.9. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$.*

1. *For any $i \in \mathbb{X}$, $a > 0$, $y \in \mathbb{R}$, $z \geqslant 0$ and any non-negative function ψ: $\mathbb{X} \to \mathbb{R}_+$,*

$$\lim_{n\to+\infty} n^{3/2}\mathbb{E}_i\left(\psi(X_n)\,;\, y + S_n \in [z, z + a]\,,\, \tau_y > n\right)$$
$$= \frac{2V(i,y)}{\sqrt{2\pi}\sigma^3}\int_z^{z+a} \mathbb{E}^*_{\boldsymbol{\nu}}\left(\psi(X_1^*)V^*\left(X_1^*, z' + S_1^*\right)\,;\, \tau^*_{z'} > 1\right)\mathrm{d}z'.$$

2. Moreover, for any $a > 0$, $y \in \mathbb{R}$, $z \geqslant 0$, $n \geqslant 1$ and any non-negative function ψ: $\mathbb{X} \to \mathbb{R}_+$,

$$\sup_{i\in\mathbb{X}} \mathbb{E}_i \left(\psi(X_n)\,;\, y + S_n \in [z, z+a]\,,\, \tau_y > n\right) \leqslant \frac{c\,(1+a^3)}{n^{3/2}} \left\|\psi\right\|_\infty (1+z)\,(1+\max(y,0))\,.$$

Recall that the dual chain $(X_n^*)_{n\geqslant 0}$ has been constructed independently of the chain $(X_n)_{n\geqslant 0}$. For any $(i,j) \in \mathbb{X}^2$, the probability generated by the finite dimensional distributions of the two dimensional Markov chain $(X_n, X_n^*)_{n\geqslant 0}$ starting at $(X_0, X_0^*) = (i,j)$ is given by $\mathbb{P}_{i,j} = \mathbb{P}_i \times \mathbb{P}_j$. Let $\mathbb{E}_{i,j}$ be the corresponding expectation. For any $l \geqslant 1$ we define $\mathscr{C}^+\left(\mathbb{X}^l \times \mathbb{R}_+\right)$ the set of non-negative function g: $\mathbb{X}^l \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

- for any $(i_1, \ldots, i_l) \in \mathbb{X}^l$, the function $z \mapsto g(i_1, \ldots, i_l, z)$ is continuous,
- $\max_{i_1,\ldots i_l \in \mathbb{X}} \sup_{z\geqslant 0} g(i_1, \ldots, i_l, z)(1+z)^{2+\varepsilon} < +\infty$ for some $\varepsilon > 0$.

Proposition 2.10. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$. For any $i \in \mathbb{X}$, $y \in \mathbb{R}$, $l \geqslant 1$, $m \geqslant 1$ and $g \in \mathscr{C}^+\left(\mathbb{X}^{l+m} \times \mathbb{R}_+\right)$,*

$$\begin{aligned}\lim_{n\to+\infty} n^{3/2}\mathbb{E}_i \left(g\left(X_1, \ldots, X_l, X_{n-m+1}, \ldots, X_n, y + S_n\right)\,;\, \tau_y > n\right) \\ = \frac{2}{\sqrt{2\pi}\sigma^3} \int_0^{+\infty} \sum_{j\in\mathbb{X}} \mathbb{E}_{i,j} \left(g\left(X_1, \ldots, X_l, X_m^*, \ldots, X_1^*, z\right)\right. \\ \left. \times V\left(X_l, y + S_l\right) V^*\left(X_m^*, z + S_m^*\right)\,;\, \tau_y > l\,,\, \tau_z^* > m\right) \boldsymbol{\nu}(j)\,\mathrm{d}z.\end{aligned}$$

We complete these results by determining the asymptotic behaviour of the law of the Markov chain $(X_n)_{n\geqslant 1}$ jointly with $\{\tau_y > n\}$.

Lemma 2.11. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$. Then, for any $(i,y) \in \mathbb{X}\times\mathbb{R}$ and $j \in \mathbb{X}$, we have*

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_i \left(X_n = j\,,\, \tau_y > n\right) = \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma}.$$

Proof. Fix $(i,y) \in \mathbb{X} \times \mathbb{R}$ and $j \in \mathbb{X}$. We will prove that

$$\begin{aligned}\frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma} &\leqslant \liminf_{n\to+\infty} \sqrt{n}\mathbb{P}_i \left(X_n = j\,,\, \tau_y > n\right) \\ &\leqslant \limsup_{n\to+\infty} \sqrt{n}\mathbb{P}_i \left(X_n = j\,,\, \tau_y > n\right) \leqslant \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma}.\end{aligned}$$

The upper bound. By the Markov property, for any $n \geqslant 1$ and $k = \left\lfloor n^{1/4} \right\rfloor$ we have

$$\begin{aligned}\mathbb{P}_i \left(X_n = j\,,\, \tau_y > n\right) &\leqslant \mathbb{P}_i \left(X_n = j\,,\, \tau_y > n - k\right) \\ &= \mathbb{E}_i \left(\mathbf{P}^k \left(X_{n-k}, j\right)\,;\, \tau_y > n - k\right).\end{aligned}$$

Using (1.1), we obtain that

$$\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n\right) \leqslant \left(\boldsymbol{\nu}(j) + c\,\mathrm{e}^{-ck}\right)\mathbb{P}_i\left(\tau_y > n-k\right).$$

Using the point 1 of Proposition 2.7 and the fact that $k = \left\lfloor n^{1/4}\right\rfloor$,

$$\limsup_{n\to+\infty}\sqrt{n}\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n\right) \leqslant \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma}. \tag{2.18}$$

The lower bound. Again, let $n \geqslant 1$ and $k = \left\lfloor n^{1/4}\right\rfloor$. We have

$$\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n\right) \geqslant \mathbb{P}_i\left(X_n = j\,,\,\tau_y > n-k\right) - \mathbb{P}_i\left(n-k < \tau_y \leqslant n\right). \tag{2.19}$$

As for the upper bound, by the Markov property and (1.1),

$$\begin{aligned}\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n-k\right) &= \mathbb{E}_i\left(\mathbf{P}^k\left(X_{n-k}, j\right)\,;\,\tau_y > n-k\right)\\ &\geqslant \left(\boldsymbol{\nu}(j) - c\,\mathrm{e}^{-ck}\right)\mathbb{P}_i\left(\tau_y > n-k\right).\end{aligned}$$

Using the point 1 of Proposition 2.7 and the fact that $k = \left\lfloor n^{1/4}\right\rfloor$,

$$\liminf_{n\to+\infty}\sqrt{n}\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n-k\right) \geqslant \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma}. \tag{2.20}$$

Furthermore, on the event $\{n-k < \tau_y \leqslant n\}$, we have

$$0 \geqslant \min_{n-k<i\leqslant n} y + S_i \geqslant y + S_{n-k} - k\left\|\rho\right\|_\infty,$$

where $\left\|\rho\right\|_\infty$ is the maximum of $|\rho|$ on $\mathbb{X}$. Consequently,

$$\begin{aligned}\mathbb{P}_i\left(n-k < \tau_y \leqslant n\right) &\leqslant \mathbb{P}_i\left(y + S_{n-k} \leqslant ck\,,\,\tau_y > n-k\right)\\ &= \mathbb{P}_i\left(y + S_{n-k} \leqslant \frac{ck}{\sqrt{n-k}}\sqrt{n-k}\,,\,\tau_y > n-k\right).\end{aligned}$$

Now, using the point 2 of Proposition 2.8 with $t_0 = \max_{n\geqslant 1}\frac{ck}{\sqrt{n-k}}$, we obtain that, for $\varepsilon > 0$ small enough,

$$\mathbb{P}_i\left(n-k < \tau_y \leqslant n\right) \leqslant \frac{2V(i,y)}{\sqrt{2\pi(n-k)}\sigma}\left(1 - \mathrm{e}^{-\frac{ck^2}{2(n-k)}}\right) + c_\varepsilon\frac{(1+y^2)}{(n-k)^{1/2+\varepsilon}}.$$

Therefore, since $k = \left\lfloor n^{1/4}\right\rfloor$,

$$\lim_{n\to+\infty}\sqrt{n}\mathbb{P}_i\left(n-k < \tau_y \leqslant n\right) = 0. \tag{2.21}$$

Putting together (2.19), (2.20) and (2.21), we conclude that

$$\liminf_{n\to+\infty}\sqrt{n}\mathbb{P}_i\left(X_n = j\,,\,\tau_y > n\right) \geqslant \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma},$$

which together with (2.18) concludes the proof of the lemma. □

Now, with the help of the function V from Proposition 2.6, for any $(i,y) \in \mathrm{supp}(V)$, we define a new probability $\mathbb{P}^+_{i,y}$ on $\sigma\left(X_n, n \geqslant 1\right)$ and the corresponding expectation $\mathbb{E}^+_{i,y}$, which are characterized by the following property: for any $n \geqslant 1$ and any g: $\mathbb{X}^n \to \mathbb{C}$,

(2.22)

$$\mathbb{E}^+_{i,y}\left(g\left(X_1,\ldots,X_n\right)\right) := \frac{1}{V(i,y)}\mathbb{E}_i\left(g\left(X_1,\ldots,X_n\right)V\left(X_n, y+S_n\right)\,;\,\tau_y > n\right).$$

The fact that $\mathbb{P}^+_{i,y}$ is a probability measure and that it does not depend on n follows easily from the point 1 of Proposition 2.6. The probability $\mathbb{P}^+_{i,y}$ is extended obviously to the hole probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The corresponding expectation is again denoted by $\mathbb{E}^+_{i,y}$.

Lemma 2.12. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$. Let $m \geqslant 1$. For any $n \geqslant 1$, bounded measurable function g: $\mathbb{X}^m \to \mathbb{C}$, $(i,y) \in \mathrm{supp}(V)$ and $j \in \mathbb{X}$,*

$$\lim_{n\to+\infty} \mathbb{E}_i\left(g\left(X_1,\ldots,X_m\right)\,;\,X_n = j \,|\, \tau_y > n\right) = \mathbb{E}^+_{i,y}\left(g\left(X_1,\ldots,X_m\right)\right)\boldsymbol{\nu}(j).$$

Proof. For the sake of brevity, for any $(i,j) \in \mathbb{X}^2$, $y \in \mathbb{R}$ and $n \geqslant 1$, set

$$J_n(i,j,y) := \mathbb{P}_i\left(X_n = j\,,\,\tau_y > n\right).$$

Fix $m \geqslant 1$ and let g be a function $\mathbb{X}^m \to \mathbb{C}$. By the point 1 of Proposition 2.7, it is clear that for any $(i,y) \in \mathrm{supp}(V)$ and n large enough, $\mathbb{P}_i\left(\tau_y > n\right) > 0$. By the Markov property, for any $j \in \mathbb{X}$ and $n \geqslant m+1$ large enough,

$$\begin{aligned} I_0 &:= \mathbb{E}_i\left(g\left(X_1,\ldots,X_m\right)\,;\,X_n = j \,|\, \tau_y > n\right) \\ &= \mathbb{E}_i\left(g\left(X_1,\ldots,X_m\right)\frac{J_{n-m}\left(X_m, j, y+S_m\right)}{\mathbb{P}_i\left(\tau_y > n\right)}\,;\,\tau_y > m\right). \end{aligned}$$

Using Lemma 2.11 and the point 1 of Proposition 2.7, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to+\infty} I_0 &= \mathbb{E}_i\left(g\left(X_1,\ldots,X_m\right)\frac{V\left(X_m, y+S_m\right)}{V(i,y)}\,;\,\tau_y > m\right)\boldsymbol{\nu}(j) \\ &= \mathbb{E}^+_{i,y}\left(g\left(X_1,\ldots,X_m\right)\right)\boldsymbol{\nu}(j). \end{aligned}$$

□

Lemma 2.13. *Assume Conditions 1, 3 and $\boldsymbol{\nu}(\rho) = 0$. For any $(i,y) \in \mathrm{supp}(V)$, we have, for any $k \geqslant 1$,*

$$\mathbb{E}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) \leqslant \frac{c\left(1+\max(y,0)\right)\mathrm{e}^y}{k^{3/2}V(i,y)}.$$

In particular,

$$\mathbb{E}^+_{i,y}\left(\sum_{k=0}^{+\infty}\mathrm{e}^{-S_k}\right) \leqslant \frac{c\left(1+\max(y,0)\right)\mathrm{e}^y}{V(i,y)}.$$

Proof. By (2.22), for any $k \geqslant 1$,

$$\mathbb{E}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) = \mathbb{E}_i\left(\mathrm{e}^{-S_k}\,\frac{V\left(X_k, y+S_k\right)}{V(i,y)}\,;\,\tau_y > k\right).$$

Using the point 2 of Proposition 2.6,

$$\begin{aligned}
\mathbb{E}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) &\leqslant \mathrm{e}^y\,\mathbb{E}_i\left(\mathrm{e}^{-(y+S_k)}\,\frac{c\left(1+\max\left(0,y+S_k\right)\right)}{V(i,y)}\,;\,\tau_y > k\right)\\
&= \mathrm{e}^y \sum_{p=0}^{+\infty} \mathbb{E}_i\left(\mathrm{e}^{-(y+S_k)}\,\frac{c\left(1+\max\left(0,y+S_k\right)\right)}{V(i,y)}\,;\,y+S_k \in (p,p+1]\,,\,\tau_y > k\right)\\
&\leqslant \mathrm{e}^y \sum_{p=0}^{+\infty} \mathrm{e}^{-p}\,\frac{c(1+p)}{V(i,y)}\mathbb{P}_i\left(y+S_k \in [p,p+1]\,,\,\tau_y > k\right).
\end{aligned}$$

By the point 2 of Proposition 2.9,

$$\begin{aligned}
\mathbb{E}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) &\leqslant \frac{c}{k^{3/2}} \sum_{p=0}^{+\infty} \mathrm{e}^{-p}(1+p)^2 \frac{\mathrm{e}^y\left(1+\max(0,y)\right)}{V(i,y)}\\
&= \frac{c\left(1+\max(0,y)\right)\mathrm{e}^y}{k^{3/2}V(i,y)}.
\end{aligned}$$

This proves the first inequality of the lemma. Summing both sides in k and using the Lebesgue monotone convergence theorem, it proves also the second inequality of the lemma. □

2.4. **The change of measure related to the Markov walk.** In this section we shall establish some useful properties of the Markov chain under the exponential change of the probability measure, which will be crucial in the proofs of the results of the paper.

For any $\lambda \in \mathbb{R}$, let $\mathbf{P}_\lambda$ be the transfer operator defined on $\mathscr{C}$ by, for any $g \in \mathscr{C}$ and $i \in \mathbb{X}$,

$$\mathbf{P}_\lambda g(i) := \mathbf{P}\left(\mathrm{e}^{\lambda\rho}\,g\right)(i) = \mathbb{E}_i\left(\mathrm{e}^{\lambda S_1}\,g(X_1)\right). \tag{2.23}$$

From the Markov property, it follows easily that, for any $g \in \mathscr{C}$, $i \in \mathbb{X}$ and $n \geqslant 0$,

$$\mathbf{P}^n_\lambda g(i) = \mathbb{E}_i\left(\mathrm{e}^{\lambda S_n}\,g(X_n)\right). \tag{2.24}$$

For any non-negative function $g \geqslant 0$, $\lambda \in \mathbb{R}$, $i \in \mathbb{X}$ and $n \geqslant 1$, we have

$$\mathbf{P}^n_\lambda g(i) \geqslant \min_{x_1,\ldots,x_n \in \mathbb{X}^n} \mathrm{e}^{\lambda(\rho(x_1)+\cdots+\rho(x_n))}\,\mathbf{P}^n g(i). \tag{2.25}$$

Therefore the matrix $\mathbf{P}_\lambda$ is primitive *i.e.* satisfies the Condition 1. By the Perron-Frobenius theorem, there exists a positive number $k(\lambda) > 0$, a positive function

$v_\lambda : \mathbb{X} \to \mathbb{R}_+^*$, a positive linear form $\boldsymbol{\nu}_\lambda$: $\mathscr{C} \to \mathbb{C}$ and a linear operator Q_λ on $\mathscr{C}$ such that for any $g \in \mathscr{C}$, and $i \in \mathbb{X}$,

$$\mathbf{P}_\lambda g(i) = k(\lambda)\boldsymbol{\nu}_\lambda(g)v_\lambda(i) + Q_\lambda(g)(i), \tag{2.26}$$

$$\boldsymbol{\nu}_\lambda(v_\lambda) = 1 \qquad \text{and} \qquad Q_\lambda(v_\lambda) = \boldsymbol{\nu}_\lambda(Q_\lambda(g)) = 0, \tag{2.27}$$

where the spectral radius of Q_λ is strictly less than $k(\lambda)$:

$$\frac{\|Q_\lambda^n(g)\|_\infty}{k(\lambda)^n} \leqslant c_\lambda \, \mathrm{e}^{-c_\lambda n} \, \|g\|_\infty . \tag{2.28}$$

Note that, in particular, $k(\lambda)$ is equal to the spectral radius of $\mathbf{P}_\lambda$, and, moreover, $k(\lambda)$ is an eigenvalue associated to the eigenvector v_λ:

$$\mathbf{P}_\lambda v_\lambda(i) = k(\lambda)v_\lambda(i). \tag{2.29}$$

From (2.26) and (2.27), we have for any $n \geqslant 1$,

$$\mathbf{P}_\lambda^n g(i) = k(\lambda)^n \boldsymbol{\nu}_\lambda(g)v_\lambda(i) + Q_\lambda^n(g)(i). \tag{2.30}$$

By (2.28), for any $g \in \mathscr{C}$ and $i \in \mathbb{X}$,

$$\lim_{n\to+\infty} \frac{\mathbf{P}_\lambda^n g(i)}{k(\lambda)^n} = \boldsymbol{\nu}_\lambda(g)v_\lambda(i)$$

and so for any non-negative and non-identically zero function $g \in \mathscr{C}$ and $i \in \mathbb{X}$,

$$k(\lambda) = \lim_{n\to+\infty} (\mathbf{P}_\lambda^n g(i))^{1/n} = \lim_{n\to+\infty} \mathbb{E}_i^{1/n} \left(\mathrm{e}^{\lambda S_n} \, g(X_n) \right). \tag{2.31}$$

Note that when $\lambda = 0$, we have $k(0) = 1$, $v_0(i) = 1$ and $\boldsymbol{\nu}_0(i) = \boldsymbol{\nu}(i)$, for any $i \in \mathbb{X}$. However, in general case, the operator $\mathbf{P}_\lambda$ is no longer a Markov operator and we define $\tilde{\mathbf{P}}_\lambda$ for any $\lambda \in \mathbb{R}$ by

$$\tilde{\mathbf{P}}_\lambda g(i) = \frac{\mathbf{P}_\lambda(gv_\lambda)(i)}{k(\lambda)v_\lambda(i)} = \frac{\mathbf{P}\left(\mathrm{e}^{\lambda\rho} \, gv_\lambda\right)(i)}{k(\lambda)v_\lambda(i)} = \frac{\mathbb{E}_i\left(\mathrm{e}^{\lambda S_1} \, g(X_1)v_\lambda(X_1)\right)}{k(\lambda)v_\lambda(i)}, \tag{2.32}$$

for any $g \in \mathscr{C}$ and $i \in \mathbb{X}$. It is clear that $\tilde{\mathbf{P}}_\lambda$ is a Markov operator: by (2.29),

$$\tilde{\mathbf{P}}_\lambda v_0(i) = \frac{\mathbf{P}_\lambda(v_\lambda)(i)}{k(\lambda)v_\lambda(i)} = 1,$$

where for any $i \in \mathbb{X}$, $v_0(i) = 1$. Iterating (2.32) and using (2.24), we see that for any $n \geqslant 1$, $g \in \mathscr{C}$ and $i \in \mathbb{X}$.

$$\tilde{\mathbf{P}}_\lambda^n g(i) = \frac{\mathbf{P}_\lambda^n(gv_\lambda)(i)}{k(\lambda)^n v_\lambda(i)} = \frac{\mathbb{E}_i\left(\mathrm{e}^{\lambda S_n} \, g(X_n)v_\lambda(X_n)\right)}{k(\lambda)^n v_\lambda(i)}. \tag{2.33}$$

In particular, as in (2.25),

$$\tilde{\mathbf{P}}_\lambda^n g(i) \geqslant \min_{x_1,\ldots,x_n \in \mathbb{X}^n} \mathrm{e}^{\lambda(\rho(x_1)+\cdots+\rho(x_n))} \, v_\lambda(x_n) \frac{\mathbf{P}^n g(i)}{k(\lambda)^n v_\lambda(i)}.$$

The following lemma is an easy consequence of this last inequality.

Lemma 2.14. *Assume Conditions 1 and 3 for the Markov kernel* $\mathbf{P}$*. Then for any* $\lambda \in \mathbb{R}$*, Conditions 1 and 3 hold also for the operator* $\tilde{\mathbf{P}}_\lambda$*.*

Using (2.30) and (2.33), the spectral decomposition of $\tilde{\mathbf{P}}_\lambda$ is given by

$$\tilde{\mathbf{P}}_\lambda^n g(i) = \boldsymbol{\nu}_\lambda\left(g v_\lambda\right) v_0(i) + \frac{Q_\lambda^n(g v_\lambda)(i)}{k(\lambda)^n v_\lambda(i)} = \tilde{\boldsymbol{\nu}}_\lambda(g) v_0(i) + \tilde{Q}_\lambda^n(g)(i),$$

with, for any $\lambda \in \mathbb{R}$, $g \in \mathscr{C}$ and $i \in \mathbb{X}$,

$$\tilde{\boldsymbol{\nu}}_\lambda(g) := \boldsymbol{\nu}_\lambda\left(g v_\lambda\right) \quad \text{and} \quad \tilde{Q}_\lambda(g)(i) := \frac{Q_\lambda(g v_\lambda)(i)}{k(\lambda) v_\lambda(i)}. \tag{2.34}$$

By (2.27),

$$\tilde{\boldsymbol{\nu}}_\lambda\left(\tilde{Q}_\lambda(g)\right) = \boldsymbol{\nu}_\lambda\left(\frac{Q_\lambda(g v_\lambda)}{k(\lambda)}\right) = 0 \quad \text{and} \quad \tilde{Q}_\lambda(v_0) = \frac{Q_\lambda(v_\lambda)(i)}{k(\lambda) v_\lambda(i)} = 0.$$

Consequently, $\tilde{\boldsymbol{\nu}}_\lambda$ is the positive invariant measure of $\tilde{\mathbf{P}}_\lambda$ and since by (2.28),

$$\left\|\tilde{Q}_\lambda^n(g)\right\|_\infty \leqslant \frac{\left\|Q_\lambda^n(g v_\lambda)\right\|_\infty}{k(\lambda)^n \min_{i \in \mathbb{X}} v_\lambda} \leqslant c_\lambda \, \mathrm{e}^{-c_\lambda n} \left\|g\right\|_\infty,$$

we can conclude that for any $(i, j) \in \mathbb{X}^2$,

$$\left|\tilde{\mathbf{P}}_\lambda^n(i, j) - \tilde{\boldsymbol{\nu}}_\lambda(j)\right| \leqslant c_\lambda \, \mathrm{e}^{-c_\lambda n}.$$

Fix $\lambda \in \mathbb{R}$ and let $\tilde{\mathbb{P}}_i$ and $\tilde{\mathbb{E}}_i$ be the probability, respectively the expectation, generated by the finite dimensional distributions of the Markov chain $(X_n)_{n \geqslant 0}$ with transition operator $\tilde{\mathbf{P}}_\lambda$ and starting at $X_0 = i$. For any $n \geqslant 1$, $g\colon \mathbb{X}^n \to \mathbb{C}$ and $i \in \mathbb{X}$,

$$\tilde{\mathbb{E}}_i\left(g(X_1, \ldots, X_n)\right) := \frac{\mathbb{E}_i\left(\mathrm{e}^{\lambda S_n} \, g(X_1, \ldots, X_n) v_\lambda(X_n)\right)}{k(\lambda)^n v_\lambda(i)}. \tag{2.35}$$

We now proceed to formulate some properties of the function $\lambda \mapsto k(\lambda)$ which are important to distinguish between the critical and three different subcritical cases.

Lemma 2.15. *Assume Conditions 1 and 3. The function* $\lambda \mapsto k(\lambda)$ *is analytic on* $\mathbb{R}$*. Moreover the function* $K\colon \lambda \mapsto \ln\left(k(\lambda)\right)$ *is strictly convex and satisfies for any* $\lambda \in \mathbb{R}$*,*

$$K'(\lambda) = \frac{k'(\lambda)}{k(\lambda)} = \tilde{\boldsymbol{\nu}}_\lambda(\rho) \tag{2.36}$$

and

$$K''(\lambda) = \tilde{\boldsymbol{\nu}}_\lambda\left(\rho^2\right) - \tilde{\boldsymbol{\nu}}_\lambda\left(\rho\right)^2 + 2 \sum_{n=1}^{+\infty} \left[\tilde{\boldsymbol{\nu}}_\lambda\left(\rho \tilde{\mathbf{P}}_\lambda^n \rho\right) - \tilde{\boldsymbol{\nu}}_\lambda\left(\rho\right)^2\right] =: \tilde{\sigma}_\lambda^2 > 0. \tag{2.37}$$

Proof. It is clear that $\lambda \mapsto \mathbf{P}_\lambda$ is analytic on $\mathbb{R}$ and consequently, by the perturbation theory for linear operators (see for example [18] or [8]) $\lambda \to k(\lambda)$, $\lambda \to v_\lambda$ and $\lambda \mapsto \boldsymbol{\nu}_\lambda$ are also analytic on $\mathbb{R}$. In particular we write for any $h \in \mathbb{R}$,

$$\mathbf{P}_{\lambda+h} = \mathbf{P}_\lambda + h\mathbf{P}'_\lambda + \frac{h^2}{2}\mathbf{P}''_\lambda + o(h^2),$$
$$v_{\lambda+h} = v_\lambda + hv'_\lambda + \frac{h^2}{2}v''_\lambda + o(h^2),$$
$$k(\lambda+h) = k(\lambda) + hk'(\lambda) + \frac{h^2}{2}k''(\lambda) + o(h^2),$$

where for any $h \in \mathbb{R}$, $o(h^2)$ refers to an operator, a function or a real such that $o(h^2)/h^2 \to 0$ as $h \to 0$. Since $v_{\lambda+h}$ is an eigenvector of $\mathbf{P}_{\lambda+h}$ we have $\mathbf{P}_{\lambda+h}v_{\lambda+h} = k(\lambda+h)v_{\lambda+h}$ and its development gives

$$\mathbf{P}_\lambda v_\lambda = k(\lambda)v_\lambda,$$
$$\mathbf{P}_\lambda v'_\lambda + \mathbf{P}'_\lambda v_\lambda = k(\lambda)v'_\lambda + k'(\lambda)v_\lambda, \tag{2.38}$$
$$\frac{1}{2}\mathbf{P}_\lambda v''_\lambda + \mathbf{P}'_\lambda v'_\lambda + \frac{1}{2}\mathbf{P}''_\lambda v_\lambda = \frac{1}{2}k(\lambda)v''_\lambda + k'(\lambda)v'_\lambda + \frac{1}{2}k''(\lambda)v_\lambda. \tag{2.39}$$

Since $\boldsymbol{\nu}_\lambda$ is an invariant measure, $\boldsymbol{\nu}_\lambda(\mathbf{P}_\lambda g) = k(\lambda)\boldsymbol{\nu}_\lambda(g)$ and (2.38) implies that

$$k(\lambda)\boldsymbol{\nu}_\lambda(v'_\lambda) + \boldsymbol{\nu}_\lambda(\mathbf{P}'_\lambda v_\lambda) = k(\lambda)\boldsymbol{\nu}_\lambda(v'_\lambda) + k'(\lambda).$$

In addition, by (2.23), $\mathbf{P}'_\lambda v_\lambda = \mathbf{P}_\lambda(\rho v_\lambda)$. Therefore,

$$k(\lambda)\boldsymbol{\nu}_\lambda(\rho v_\lambda) = k'(\lambda),$$

which, with the definition of $\tilde{\boldsymbol{\nu}}_\lambda$ in (2.34), proves (2.36).

From (2.39) and the fact that $\boldsymbol{\nu}_\lambda(\mathbf{P}_\lambda g) = k(\lambda)\boldsymbol{\nu}_\lambda(g)$, we have

$$\frac{k(\lambda)}{2}\boldsymbol{\nu}_\lambda(v''_\lambda) + k(\lambda)\boldsymbol{\nu}_\lambda(\rho v'_\lambda) + \frac{k(\lambda)}{2}\boldsymbol{\nu}_\lambda\left(\rho^2 v_\lambda\right)$$
$$= \frac{1}{2}k(\lambda)\boldsymbol{\nu}_\lambda(v''_\lambda) + k'(\lambda)\boldsymbol{\nu}_\lambda(v'_\lambda) + \frac{1}{2}k''(\lambda).$$

So,

$$\frac{k''(\lambda)}{k(\lambda)} = \boldsymbol{\nu}_\lambda\left(\rho^2 v_\lambda\right) + 2\left[\boldsymbol{\nu}_\lambda(\rho v'_\lambda) - \frac{k'(\lambda)}{k(\lambda)}\boldsymbol{\nu}_\lambda(v'_\lambda)\right].$$

By (2.36), we obtain that

$$K''(\lambda) = \frac{k''(\lambda)}{k(\lambda)} - \left(\frac{k'(\lambda)}{k(\lambda)}\right)^2$$
$$= \boldsymbol{\nu}_\lambda\left(\rho^2 v_\lambda\right) - \boldsymbol{\nu}_\lambda^2(\rho v_\lambda) + 2\left[\boldsymbol{\nu}_\lambda(\rho v'_\lambda) - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\boldsymbol{\nu}_\lambda(v'_\lambda)\right]. \tag{2.40}$$

It remains to determine v'_λ. By (2.38), we have

$$v'_\lambda - \frac{\mathbf{P}_\lambda v'_\lambda}{k(\lambda)} = \frac{\mathbf{P}_\lambda(\rho v_\lambda)}{k(\lambda)} - \frac{k'(\lambda)}{k(\lambda)}v_\lambda$$

and for any $n \geqslant 0$, using (2.36),

$$\frac{\mathbf{P}_\lambda^n v'_\lambda}{k(\lambda)^n} - \frac{\mathbf{P}_\lambda^{n+1} v'_\lambda}{k(\lambda)^{n+1}} = \frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\, v_\lambda. \tag{2.41}$$

Note that

$$\frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\, v_\lambda = \frac{Q_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}}.$$

By (2.28),

$$\left\| \frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\, v_\lambda \right\|_\infty \leqslant c_\lambda\, \mathrm{e}^{-c_\lambda(n+1)} \left\|\rho v_\lambda\right\|_\infty = c_\lambda\, \mathrm{e}^{-c_\lambda(n+1)}.$$

Consequently, by (2.41), the series $\sum_{n\geqslant 0}\left[\frac{\mathbf{P}_\lambda^n v'_\lambda}{k(\lambda)^n} - \frac{\mathbf{P}_\lambda^{n+1} v'_\lambda}{k(\lambda)^{n+1}}\right]$ converges absolutely and we deduce that

$$v'_\lambda = \sum_{n=0}^{+\infty} \left[\frac{\mathbf{P}_\lambda^{n+1}(\rho v_\lambda)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\, v_\lambda\right].$$

In particular,

$$\boldsymbol{\nu}_\lambda(v'_\lambda) = \sum_{n=0}^{+\infty} \left[\boldsymbol{\nu}_\lambda(\rho v_\lambda) - \boldsymbol{\nu}_\lambda(\rho v_\lambda)\right] = 0,$$

and

$$\boldsymbol{\nu}_\lambda(\rho v'_\lambda) = \sum_{n=0}^{+\infty} \left[\frac{\boldsymbol{\nu}_\lambda\left(\rho \mathbf{P}_\lambda^{n+1}(\rho v_\lambda)\right)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)^2\right].$$

Therefore (2.40) becomes

$$K''(\lambda) = \boldsymbol{\nu}_\lambda\left(\rho^2 v_\lambda\right) - \boldsymbol{\nu}_\lambda^2(\rho v_\lambda) + 2\sum_{n=0}^{+\infty} \left[\frac{\boldsymbol{\nu}_\lambda\left(\rho \mathbf{P}_\lambda^{n+1}(\rho v_\lambda)\right)}{k(\lambda)^{n+1}} - \boldsymbol{\nu}_\lambda(\rho v_\lambda)^2\right].$$

To conclude the proof of the lemma, we establish that $K''(\lambda) > 0$, from which the strict convexity of K follows. By (2.34),

$$K''(\lambda) = \tilde{\boldsymbol{\nu}}_\lambda\left(\tilde{\rho}_\lambda^2\right) + 2\sum_{n=1}^{+\infty} \left[\tilde{\boldsymbol{\nu}}_\lambda\left(\tilde{\rho}_\lambda \tilde{\mathbf{P}}_\lambda^n \tilde{\rho}_\lambda\right)\right], \tag{2.42}$$

where for any $\lambda \in \mathbb{R}$, $\tilde{\rho}_\lambda = \rho - \tilde{\boldsymbol{\nu}}_\lambda(\rho) v_0$. Moreover, Conditions 1 and 3 and Lemma 2.14 imply that the normalized transfer operator $\tilde{\mathbf{P}}_\lambda$ together with the function $\tilde{\rho}_\lambda$ satisfies Conditions 1 and 3. In conjunction with (2.42) and Lemma 10.3 of [12], this proves that (2.42) is positive and so (2.37) holds. □

3. Proofs in the critical case

In this section we prove Theorem 1.1. By equations (2.3) and (2.8), the survival probability of the branching process is related to the study of the sum $q_n^{-1} = \mathrm{e}^{-S_n} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}$ where $(S_n)_{n\geqslant 0}$ is a Markov walk defined by (1.3). Very roughly speaking, the sum q_n^{-1} converges mainly when the walk stays positive: $S_k > 0$ for any $k \geqslant 1$ and we will see that (at least in the critical case) only positive trajectories of the Markov walk $(S_n)_{n\geqslant 0}$ count for the survival of the branching process.

Recall that the hypotheses of Theorem 1.1 are Conditions 1-3 and $k'(0) = \boldsymbol{\nu}(\rho) = 0$. Under these assumptions the conclusions of all the theorems of Section 2.3 hold for the probability $\mathbb{P}_i$, for any $i \in \mathbb{X}$. Recall also that $\mathbb{E}^+_{i,y}$ is the expectation corresponding to the probability measure (2.22). We carry out the proof through a series of lemmata.

Lemma 3.1. *Assume conditions of Theorem 1.1. For any $m \geqslant 1$, $(i,y) \in \mathrm{supp}(V)$, and $j \in \mathbb{X}$, we have*

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,;\, X_n = j \,|\, \tau_y > n\right) = \mathbb{E}^+_{i,y}\left(q_m\right) \boldsymbol{\nu}(j).$$

Proof. Fix $m \geqslant 1$, $(i,y) \in \mathrm{supp}(V)$, and $j \in \mathbb{X}$. By (2.2), for any $n \geqslant m+1$,

$$\begin{aligned}
\mathbb{P}_i\left(Z_m > 0\,,\, X_n = j\,,\, \tau_y > n\right) &= \mathbb{E}_i\left(\mathbb{P}_i\left(Z_m > 0 \,|\, X_1, \dots, X_n\right)\,;\, X_n = j\,,\, \tau_y > n\right) \\
&= \mathbb{E}_i\left(\mathbb{E}_i\left(q_m \,|\, X_1, \dots, X_n\right)\,;\, X_n = j\,,\, \tau_y > n\right) \\
&= \mathbb{E}_i\left(q_m\,;\, X_n = j\,,\, \tau_y > n\right).
\end{aligned}$$

Using Lemma 2.12, we conclude that

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,;\, X_n = j \,|\, \tau_y > n\right) &= \lim_{n\to+\infty} \mathbb{E}_i\left(q_m\,;\, X_n = j \,|\, \tau_y > n\right) \\
&= \mathbb{E}^+_{i,y}\left(q_m\right) \boldsymbol{\nu}(j).
\end{aligned}$$

□

By Lemma 2.3, we have for any $(i,y) \in \mathrm{supp}(V)$, $k \geqslant 1$ and $n \geqslant k+1$,

$$0 \leqslant \eta_{k,n} \leqslant \eta := \max_{x\in\mathbb{X}} \frac{f''_x(1)}{f'_x(1)^2} < +\infty \qquad \mathbb{P}^+_{i,y}\text{-a.s.} \tag{3.1}$$

By (2.9) and (2.11), this equation holds also when $n = k$. Moreover, by Lemma 2.3,

$$\eta_{k,\infty} := \lim_{n\to+\infty} \eta_{k,n} \in [0,\eta] \qquad \mathbb{P}^+_{i,y}\text{-a.s.} \tag{3.2}$$

Let q_∞ be the following random variable:

$$q_\infty := \left[\sum_{k=0}^{+\infty} \mathrm{e}^{-S_k}\, \eta_{k+1,\infty}\right]^{-1} \in [0,+\infty]. \tag{3.3}$$

The random variable q_∞^{-1} is $\mathbb{P}_{i,y}^+$-integrable for any $(i,y) \in \mathrm{supp}(V)$: indeed by (3.2),

$$q_\infty^{-1} \leqslant \sum_{k=0}^{+\infty} \mathrm{e}^{-S_k}\, \eta.$$

Using Lemma 2.13, for any $(i,y) \in \mathrm{supp}(V)$

$$\mathbb{E}_{i,y}^+\left(q_\infty^{-1}\right) \leqslant \eta \mathbb{E}_{i,y}^+\left(\sum_{k=0}^{+\infty} \mathrm{e}^{-S_k}\right) \leqslant \eta \frac{c\,(1+\max(y,0))\,\mathrm{e}^y}{V(i,y)} < +\infty. \tag{3.4}$$

Lemma 3.2. *Assume conditions of Theorem 1.1. For any $(i,y) \in \mathrm{supp}(V)$,*

$$\lim_{m\to+\infty} \mathbb{E}_{i,y}^+\left(\left|q_m^{-1} - q_\infty^{-1}\right|\right) = 0, \tag{3.5}$$

and

$$\lim_{m\to+\infty} \mathbb{E}_{i,y}^+\left(|q_m - q_\infty|\right) = 0. \tag{3.6}$$

Proof. Let $(i,y) \in \mathrm{supp}(V)$ and fix $l \geqslant 1$. By (2.8) and (3.3), we have for all $m \geqslant l+2$,

$$\begin{aligned}
\mathbb{E}_{i,y}^+\left(\left|q_m^{-1} - q_\infty^{-1}\right|\right) &= \mathbb{E}_{i,y}^+\left(\left|\mathrm{e}^{-S_m} + \sum_{k=0}^{m-1} \mathrm{e}^{-S_k}\,\eta_{k+1,m} - \sum_{k=0}^{+\infty} \mathrm{e}^{-S_k}\,\eta_{k+1,\infty}\right|\right) \\
&\leqslant \mathbb{E}_{i,y}^+\left(\mathrm{e}^{-S_m}\right) + \mathbb{E}_{i,y}^+\left(\sum_{k=0}^{l} \mathrm{e}^{-S_k}\,|\eta_{k+1,m} - \eta_{k+1,\infty}|\right) \\
&\quad + \mathbb{E}_{i,y}^+\left(\sum_{k=l+1}^{m-1} \mathrm{e}^{-S_k}\,|\eta_{k+1,m} - \eta_{k+1,\infty}|\right) + \mathbb{E}_{i,y}^+\left(\sum_{k=m}^{+\infty} \mathrm{e}^{-S_k}\,\eta_{k+1,\infty}\right).
\end{aligned}$$

By (3.1) and (3.2),

$$\begin{aligned}
\mathbb{E}_{i,y}^+\left(\left|q_m^{-1} - q_\infty^{-1}\right|\right) &\leqslant \mathbb{E}_{i,y}^+\left(\mathrm{e}^{-S_m}\right) + \mathbb{E}_{i,y}^+\left(\sum_{k=0}^{l} \mathrm{e}^{-S_k}\,|\eta_{k+1,m} - \eta_{k+1,\infty}|\right) \\
&\quad + \eta \mathbb{E}_{i,y}^+\left(\sum_{k=l+1}^{+\infty} \mathrm{e}^{-S_k}\right).
\end{aligned}$$

Using Lemma 2.13 and the Lebesgue monotone convergence theorem,

$$\begin{aligned}
\mathbb{E}_{i,y}^+\left(\left|q_m^{-1} - q_\infty^{-1}\right|\right) &\leqslant \frac{c\,(1+\max(y,0))\,\mathrm{e}^y}{V(i,y)}\left(\frac{1}{m^{3/2}} + \eta \sum_{k=l+1}^{+\infty} \frac{1}{k^{3/2}}\right) \\
&\quad + \mathbb{E}_{i,y}^+\left(\sum_{k=0}^{l} \mathrm{e}^{-S_k}\,|\eta_{k+1,m} - \eta_{k+1,\infty}|\right) \\
&\leqslant \frac{c\,(1+\max(y,0))\,\mathrm{e}^y}{V(i,y)}\left(\frac{1}{m^{3/2}} + \frac{\eta}{\sqrt{l}}\right) + \mathbb{E}_{i,y}^+\left(\sum_{k=0}^{l} \mathrm{e}^{-S_k}\,|\eta_{k+1,m} - \eta_{k+1,\infty}|\right).
\end{aligned}$$

Using (3.1) and (3.2), we have $\sum_{k=0}^{l} \mathrm{e}^{-S_k} \left|\eta_{k+1,m} - \eta_{k+1,\infty}\right| \leqslant \eta \sum_{k=0}^{+\infty} \mathrm{e}^{-S_k}$ which is $\mathbb{P}^+_{i,y}$-integrable by Lemma 2.13. Consequently, by the Lebesgue dominated convergence theorem and (3.2), when $m \to +\infty$, we obtain that for any $l \geqslant 1$,

$$\limsup_{m\to+\infty} \mathbb{E}^+_{i,y}\left(\left|q_m^{-1} - q_\infty^{-1}\right|\right) \leqslant \frac{c\eta\,(1+\max(y,0))\,\mathrm{e}^y}{V(i,y)\sqrt{l}}.$$

Letting $l \to +\infty$ it proves (3.5).

Now, it follows easily from (2.4) that $q_\infty \leqslant 1$: for any $\varepsilon > 0$ and $m \geqslant 1$, we write that $\mathbb{P}^+_{i,y}\left(q_\infty^{-1} < 1-\varepsilon\right) \leqslant \mathbb{P}^+_{i,y}\left(q_\infty^{-1} - q_m^{-1} < -\varepsilon\right)$. Since by (3.5), q_m^{-1} converges in $\mathbb{P}^+_{i,y}$-probability to q_∞^{-1}, for any $\varepsilon > 0$ we have $\mathbb{P}^+_{i,y}\left(q_\infty^{-1} < 1-\varepsilon\right) = 0$ and so

$$q_\infty \leqslant 1 \qquad \mathbb{P}^+_{i,y}\text{-a.s.} \tag{3.7}$$

Consequently, $|q_m - q_\infty| = q_m q_\infty \left|q_m^{-1} - q_\infty^{-1}\right| \leqslant \left|q_m^{-1} - q_\infty^{-1}\right|$ and by (3.5) we obtain (3.6). □

Let U be a function defined on $\mathrm{supp}(V)$ by

$$U(i,y) = \mathbb{E}^+_{i,y}\left(q_\infty\right).$$

Note that for any $(i,y) \in \mathrm{supp}(V)$, by (3.4), $q_\infty > 0$ $\mathbb{P}^+_{i,y}$-a.s. and so

$$U(i,y) > 0. \tag{3.8}$$

By (3.7), we have also $U(i,y) \leqslant 1$.

Lemma 3.3. *Assume conditions of Theorem 1.1. For any $(i,y) \in \mathrm{supp}(V)$ and $j \in \mathbb{X}$, we have*

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,;\, X_n = j \,\middle|\, \tau_y > n\right) = \boldsymbol{\nu}(j) U(i,y).$$

Proof. By Lemma 3.1, for any $(i,y) \in \mathrm{supp}(V)$, $j \in \mathbb{X}$ and $m \geqslant 1$, we have

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,;\, X_n = j \,\middle|\, \tau_y > n\right) = \boldsymbol{\nu}(j)\mathbb{E}^+_{i,y}\left(q_m\right).$$

By (3.6), we obtain the desired equality. □

Lemma 3.4. *Assume conditions of Theorem 1.1. For any $(i,y) \in \mathrm{supp}(V)$ and $\theta \in (0,1)$,*

$$\lim_{m\to+\infty} \limsup_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,,\, Z_{\lfloor\theta n\rfloor} = 0 \,\middle|\, \tau_y > n\right) = 0.$$

Proof. Fix $(i,y) \in \mathrm{supp}(V)$ and $\theta \in (0,1)$. For any $m \geqslant 1$ and any $n \geqslant 1$ such that $\lfloor\theta n\rfloor \geqslant m+1$ we define $\theta_n = \lfloor\theta n\rfloor$ and we write

$$\begin{aligned} I_0 &:= \mathbb{P}_i\left(Z_m > 0\,,\, Z_{\theta_n} = 0\,,\, \tau_y > n\right) \\ &= \mathbb{P}_i\left(Z_m > 0\,,\, \tau_y > n\right) - \mathbb{P}_i\left(Z_{\theta_n} > 0\,,\, \tau_y > n\right) \\ &= \mathbb{E}_i\left(\mathbb{P}_i\left(Z_m > 0 \,\middle|\, X_1, \ldots, X_m\right)\,;\, \tau_y > n\right) \\ &\quad - \mathbb{E}_i\left(\mathbb{P}_i\left(Z_{\theta_n} > 0 \,\middle|\, X_1, \ldots, X_{\theta_n}\right)\,;\, \tau_y > n\right). \end{aligned}$$

By (2.2),

$$I_0 = \mathbb{E}_i \left(|q_m - q_{\theta_n}| \, ; \, \tau_y > n \right).$$

We define $J_p(i,y) := \mathbb{P}_i \left(\tau_y > p \right)$ for any $(i,y) \in \mathbb{X} \times \mathbb{R}$ and $p \geqslant 0$ and consider

$$I_1 := \mathbb{P}_i \left(Z_m > 0 \, , \, Z_{\theta_n} = 0 \, | \, \tau_y > n \right)$$

for any $(i,y) \in \mathrm{supp}(V)$. By the Markov property, for any $(i,y) \in \mathrm{supp}(V)$,

$$I_1 = \frac{I_0}{J_n(i,y)} = \mathbb{E}_i \left(|q_m - q_{\theta_n}| \, \frac{J_{n-\theta_n} \left(X_{\theta_n}, y + S_{\theta_n} \right)}{J_n(i,y)} \, ; \, \tau_y > \theta_n \right).$$

By the point 2 of Proposition 2.7,

$$I_1 \leqslant \frac{c}{\sqrt{(1-\theta)n} J_n(i,y)} \mathbb{E}_i \left(|q_m - q_{\theta_n}| \left(1 + y + S_{\theta_n} \right) \, ; \, \tau_y > \theta_n \right).$$

Using also the point 3 of Proposition 2.6, we have

$$I_1 \leqslant \frac{c}{\sqrt{(1-\theta)n} J_n(i,y)} \mathbb{E}_i \left(|q_m - q_{\theta_n}| \left(1 + V \left(X_{\theta_n}, y + S_{\theta_n} \right) \right) \, ; \, \tau_y > \theta_n \right).$$

Using (2.4) and (2.22), we obtain that

$$I_1 \leqslant \frac{c}{\sqrt{(1-\theta)n} J_n(i,y)} \left(\mathbb{P}_i \left(\tau_y > \theta_n \right) + V(i,y) \mathbb{E}^+_{i,y} \left(|q_m - q_{\theta_n}| \right) \right).$$

Using the point 1 of Proposition 2.7, for any $(i,y) \in \mathrm{supp}(V)$,

$$\frac{1}{\sqrt{(1-\theta)n} J_n(i,y)} = \frac{1}{\sqrt{(1-\theta)n} \mathbb{P}_i \left(\tau_y > n \right)} \underset{n \to +\infty}{\sim} \frac{\sqrt{2\pi}\sigma}{2\sqrt{1-\theta} V(i,y)}.$$

Moreover using again the point 1 of Proposition 2.7 and using (3.6),

$$\mathbb{P}_i \left(\tau_y > \theta_n \right) + V(i,y) \mathbb{E}^+_{i,y} \left(|q_m - q_{\theta_n}| \right) \underset{n \to +\infty}{\longrightarrow} V(i,y) \mathbb{E}^+_{i,y} \left(|q_m - q_\infty| \right).$$

Therefore, we obtain that, for any $m \geqslant 1$ and $\theta \in (0,1)$,

$$\limsup_{n \to +\infty} I_1 \leqslant \frac{c}{\sqrt{1-\theta}} \mathbb{E}^+_{i,y} \left(|q_m - q_\infty| \right).$$

Letting m go to $+\infty$ and using (3.6), we obtain $\lim_{m \to +\infty} \limsup_{n \to +\infty} I_1 = 0$, which proves the assertion of the lemma. □

Lemma 3.5. *Assume conditions of Theorem 1.1. For any $(i,y) \in \mathrm{supp}(V)$, $j \in \mathbb{X}$, and $\theta \in (0,1)$,*

$$\lim_{n \to +\infty} \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0 \, , \, X_n = j \, \middle| \, \tau_y > n \right) = \boldsymbol{\nu}(j) U(i,y).$$

In particular,

$$\lim_{n \to +\infty} \mathbb{P}_i \left(Z_{\lfloor \theta n \rfloor} > 0 \, \middle| \, \tau_y > n \right) = U(i,y). \tag{3.9}$$

Proof. Fix $(i,y) \in \mathrm{supp}(V)$ and $j \in \mathbb{X}$. Let $\theta_n := \lfloor \theta n \rfloor$ for any $\theta \in (0,1)$ and $n \geqslant 1$. For any $m \geqslant 1$ and $n \geqslant 1$ such that $\theta_n \geqslant m+1$, we write

$$\begin{aligned}
&\mathbb{P}_i\left(Z_{\theta_n} > 0\,,\, X_n = j \,|\, \tau_y > n\right) \\
&= \mathbb{P}_i\left(Z_m > 0\,,\, Z_{\theta_n} > 0\,,\, X_n = j \,|\, \tau_y > n\right) \\
&= \mathbb{P}_i\left(Z_m > 0\,,\, X_n = j \,|\, \tau_y > n\right) - \mathbb{P}_i\left(Z_m > 0\,,\, Z_{\theta_n} = 0\,,\, X_n = j \,|\, \tau_y > n\right).
\end{aligned}$$

By Lemma 3.4,

$$\begin{aligned}
\lim_{m\to+\infty} \limsup_{n\to+\infty} &\mathbb{P}_i\left(Z_m > 0\,,\, Z_{\theta_n} = 0\,,\, X_n = j \,|\, \tau_y > n\right) \\
&\leqslant \lim_{m\to+\infty} \limsup_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0\,,\, Z_{\theta_n} = 0 \,|\, \tau_y > n\right) = 0.
\end{aligned}$$

Therefore, using Lemma 3.3, the first assertion of the lemma follows. □

Lemma 3.6. *Assume conditions of Theorem 1.1. For any* $(i,y) \in \mathrm{supp}(V)$,

$$\lim_{p\to+\infty} \mathbb{P}_i\left(Z_p > 0 \,|\, \tau_y > p\right) = U(i,y).$$

Proof. Fix $(i,y) \in \mathrm{supp}(V)$. For any $p \geqslant 1$ and $\theta \in (0,1)$, we have

$$\mathbb{P}_i\left(Z_p > 0 \,|\, \tau_y > p\right) = \frac{\mathbb{P}_i\left(Z_p > 0\,,\, \tau_y > \frac{p}{\theta} + 1\right) + \mathbb{P}_i\left(Z_p > 0\,,\, p < \tau_y \leqslant \frac{p}{\theta} + 1\right)}{\mathbb{P}_i\left(\tau_y > p\right)}.$$

Let $n = \left\lfloor \frac{p}{\theta} \right\rfloor + 1$ and note that $\lfloor \theta n \rfloor = p$. So, by (3.9),

$$\begin{aligned}
\lim_{p\to+\infty} \mathbb{P}_i\left(Z_p > 0 \,|\, \tau_y > p\right) = U(i,y) &\lim_{p\to+\infty} \frac{\mathbb{P}_i\left(\tau_y > n\right)}{\mathbb{P}_i\left(\tau_y > p\right)} \\
&+ \lim_{p\to+\infty} \frac{\mathbb{P}_i\left(Z_p > 0\,,\, p < \tau_y \leqslant n\right)}{\mathbb{P}_i\left(\tau_y > p\right)}.
\end{aligned}$$

By the point 1 of Proposition 2.7, we obtain that

$$\lim_{p\to+\infty} \mathbb{P}_i\left(Z_p > 0 \,|\, \tau_y > p\right) = U(i,y)\sqrt{\theta} + \lim_{p\to+\infty} \frac{\mathbb{P}_i\left(Z_p > 0\,,\, p < \tau_y \leqslant n\right)}{\mathbb{P}_i\left(\tau_y > p\right)}.$$

Moreover, using again the point 1 of Proposition 2.7, for any $\theta \in (0,1)$,

$$\frac{\mathbb{P}_i\left(Z_p > 0\,,\, p < \tau_y \leqslant n\right)}{\mathbb{P}_i\left(\tau_y > p\right)} \leqslant \frac{\mathbb{P}_i\left(\tau_y > p\right) - \mathbb{P}_i\left(\tau_y > n\right)}{\mathbb{P}_i\left(\tau_y > p\right)} \underset{p\to+\infty}{\longrightarrow} 1 - \sqrt{\theta}.$$

Letting $\theta \to 1$, we conclude that

$$\lim_{p\to+\infty} \mathbb{P}_i\left(Z_p > 0 \,|\, \tau_y > p\right) = U(i,y).$$

□

Lemma 3.7. *Assume conditions of Theorem 1.1. For any* $(i,y) \in \mathrm{supp}(V)$ *and* $\theta \in (0,1)$,

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_{\lfloor \theta n \rfloor} > 0\,,\, Z_n = 0 \,\middle|\, \tau_y > n\right) = 0.$$

Proof. For any $(i,y) \in \mathrm{supp}(V)$, $\theta \in (0,1)$ and $n \geqslant 1$,

$$\mathbb{P}_i\left(Z_{\lfloor \theta n \rfloor} > 0\,,\, Z_n = 0 \,\middle|\, \tau_y > n\right) = \mathbb{P}_i\left(Z_{\lfloor \theta n \rfloor} > 0 \,\middle|\, \tau_y > n\right) - \mathbb{P}_i\left(Z_n > 0 \,|\, \tau_y > n\right).$$

From (3.9) and Lemma 3.6, the last difference converges to 0. □

Lemma 3.8. *Assume conditions of Theorem 1.1. For any $(i,y) \in \mathrm{supp}(V)$ and $j \in \mathbb{X}$,*

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_n > 0\,,\, X_n = j \,|\, \tau_y > n\right) = \boldsymbol{\nu}(j) U(i,y).$$

Proof. For any $(i,y) \in \mathrm{supp}(V)$, $j \in \mathbb{X}$, $\theta \in (0,1)$ and $n \geqslant 1$,

$$\begin{aligned}\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j \,|\, \tau_y > n\right) &= \mathbb{P}_i\left(Z_{\lfloor \theta n \rfloor} > 0\,,\, X_n = j \,\middle|\, \tau_y > n\right) \\ &\quad - \mathbb{P}_i\left(Z_{\lfloor \theta n \rfloor} > 0\,,\, Z_n = 0\,,\, X_n = j \,\middle|\, \tau_y > n\right).\end{aligned}$$

Using Lemmas 3.5 and 3.7, the result follows. □

Proof of Theorem 1.1. Fix $(i,j) \in \mathbb{X}^2$. For any $y \in \mathbb{R}$, we have

$$\begin{aligned}0 &\leqslant \mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) - \mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\,,\, \tau_y > n\right) \\ &\leqslant \mathbb{P}_i\left(Z_n > 0\,,\, \tau_y \leqslant n\right). \qquad (3.10)\end{aligned}$$

Using (2.2),

$$\mathbb{P}_i\left(Z_n > 0\,,\, \tau_y \leqslant n\right) = \mathbb{E}_i\left(q_n\,;\, \tau_y \leqslant n\right).$$

Moreover, by the definition of q_n in (2.1), for any $k \geqslant 1$,

$$q_k \leqslant f'_{X_k}(1) \times \cdots \times f'_{X_1}(1) = \mathrm{e}^{S_k}.$$

Since $(q_k)_{k\geqslant 1}$ is non-increasing, $q_n = \min_{1\leqslant k\leqslant n} q_k \leqslant \mathrm{e}^{\min_{1\leqslant k\leqslant n} S_k}$. Therefore

$$\begin{aligned}\mathbb{P}_i\left(Z_n > 0\,,\, \tau_y \leqslant n\right) &\leqslant \mathbb{E}_i\left(\mathrm{e}^{\min_{1\leqslant k\leqslant n} S_k}\,;\, \tau_y \leqslant n\right) \\ &= \mathrm{e}^{-y} \sum_{p=0}^{+\infty} \mathbb{E}_i\Bigg(\mathrm{e}^{\min_{1\leqslant k\leqslant n}\{y+S_k\}}\,; \\ &\qquad -(p+1) < \min_{1\leqslant k\leqslant n}\{y+S_k\} \leqslant -p\,,\, \tau_y \leqslant n\Bigg) \\ &\leqslant \mathrm{e}^{-y} \sum_{p=0}^{+\infty} \mathrm{e}^{-p}\, \mathbb{P}_i\left(\tau_{y+p+1} > n\right). \qquad (3.11)\end{aligned}$$

By the point 2 of Proposition 2.7,

$$\begin{aligned}\mathbb{P}_i\left(Z_n > 0\,,\, \tau_y \leqslant n\right) &= \frac{c\,\mathrm{e}^{-y}}{\sqrt{n}} \sum_{p=0}^{+\infty} \mathrm{e}^{-p}\left(1 + p + 1 + \max(y,0)\right) \\ &\leqslant \frac{c\,\mathrm{e}^{-y}\left(1+\max(y,0)\right)}{\sqrt{n}}. \qquad (3.12)\end{aligned}$$

Note that from the point 3 of Proposition 2.6, it is clear that there exits $y_0 = y_0(i) < +\infty$ such that for any $y \geqslant y_0$, we have $V(i,y) > 0$ i.e. $(i,y) \in \mathrm{supp}(V)$

(for more information on supp(V) see [13]). Using Lemma 3.8 and the point 1 of Proposition 2.7, for any $y \geqslant y_0$,

$$\sqrt{n}\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\,,\, \tau_y > n\right) \underset{n\to+\infty}{\longrightarrow} \frac{2\boldsymbol{\nu}(j)U(i,y)V(i,y)}{\sqrt{2\pi}\sigma}. \tag{3.13}$$

Let

$$I(i,j) = \liminf_{n\to+\infty} \sqrt{n}\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right)$$

and

$$J(i,j) = \limsup_{n\to+\infty} \sqrt{n}\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right).$$

Using (3.10), (3.12) and (3.13), we obtain that, for any $y \geqslant y_0(i)$,

$$\begin{aligned} &\frac{2\boldsymbol{\nu}(j)U(i,y)V(i,y)}{\sqrt{2\pi}\sigma} \leqslant I(i,j) \\ &\leqslant J(i,j) \leqslant \frac{2\boldsymbol{\nu}(j)U(i,y)V(i,y)}{\sqrt{2\pi}\sigma} + c\,\mathrm{e}^{-y}\left(1+\max(y,0)\right) < +\infty. \end{aligned} \tag{3.14}$$

From (3.13), it is clear that $y \mapsto \frac{2U(i,y)V(i,y)}{\sqrt{2\pi}\sigma}$ is non-decreasing and from (3.14) the function is bounded by $I(i,j)/\boldsymbol{\nu}(j) < +\infty$. Therefore

$$u(i) := \lim_{y\to+\infty} \frac{2U(i,y)V(i,y)}{\sqrt{2\pi}\sigma}$$

exists. Moreover by (3.8), for any $y \geqslant y_0(i)$,

$$u(i) \geqslant \frac{2U(i,y)V(i,y)}{\sqrt{2\pi}\sigma} > 0.$$

Taking the limit as $y \to +\infty$ in (3.14), we conclude that

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) = \boldsymbol{\nu}(j)u(i),$$

which finishes the proof of Theorem 1.1.

4. Proofs in the strongly subcritical case

Assume the hypotheses of Theorem 1.2 that is Conditions 1-3 and $k'(1) < 0$. We fix $\lambda = 1$ and define the probability $\tilde{\mathbb{P}}_i$ and the corresponding expectation $\tilde{\mathbb{E}}_i$ by (2.35), such that, for any $n \geqslant 1$ and any $g\colon \mathbb{X}^n \to \mathbb{C}$,

$$\tilde{\mathbb{E}}_i\left(g(X_1,\ldots,X_n)\right) = \frac{\mathbb{E}_i\left(\mathrm{e}^{S_n}\, g(X_1,\ldots,X_n)v_1(X_n)\right)}{k(1)^n v_1(i)}. \tag{4.1}$$

By (2.2), we have, for any $(i,j) \in \mathbb{X}^2$ and $n \geqslant 1$,

$$\begin{aligned}\mathbb{P}_i\left(Z_{n+1} > 0\,,\, X_{n+1} = j\right) &= \mathbb{E}_i\left(q_{n+1}\,,\, X_{n+1} = j\right)\\ &= \tilde{\mathbb{E}}_i\left(\frac{\mathrm{e}^{-S_{n+1}}}{v_1\left(X_{n+1}\right)} q_{n+1}\,;\, X_{n+1} = j\right) k(1)^{n+1} v_1(i)\\ &= \tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_n}\, q_n\left(f_j(0)\right)\,;\, X_{n+1} = j\right) k(1)^{n+1} \frac{v_1(i)\,\mathrm{e}^{-\rho(j)}}{v_1(j)},\end{aligned}$$

where $q_n(s)$ is defined for any $s \in [0,1]$ by (2.1). From Lemma 2.2, we write

$$\begin{aligned}\mathrm{e}^{-S_n}\, q_n\left(f_j(0)\right) &= \left[\frac{1}{1 - f_j(0)} + \sum_{k=0}^{n-1} \mathrm{e}^{S_n - S_k}\, \eta_{k+1,n}\left(f_j(0)\right)\right]^{-1}\\ &= \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{n} \mathrm{e}^{S_n - S_{n-k}}\, \eta_{n-k+1,n}\left(f_j(0)\right)\right]^{-1}. \qquad (4.2)\end{aligned}$$

As in Section 2.2, we define the dual Markov chain $(X_n^*)_{n\geqslant 0}$, where the dual Markov kernel is given, for any $(i,j) \in \mathbb{X}^2$, by

$$\tilde{\mathbf{P}}_1^*(i,j) = \tilde{\mathbf{P}}_1(j,i) \frac{\tilde{\boldsymbol{\nu}}_1(j)}{\tilde{\boldsymbol{\nu}}_1(i)} = \mathbf{P}(j,i) \frac{\mathrm{e}^{\rho(i)}\, \boldsymbol{\nu}_1(j)}{k(1) \boldsymbol{\nu}_1(i)}.$$

Let $(S_n^*)_{n\geqslant 0}$ be the associated Markov walk defined by (2.14) and

$$q_n^*(j) := \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{n} \mathrm{e}^{-S_k^*}\, \eta_k^*(j)\right]^{-1}, \qquad (4.3)$$

where

$$\eta_k^*(j) := g_{X_k^*}\left(f_{X_{k-1}^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)\right) \quad \text{and} \quad \eta_1^*(j) := g_{X_1^*}\left(f_j(0)\right). \qquad (4.4)$$

Following the proof of Lemma 2.2, we obtain

$$q_n^*(j) = \mathrm{e}^{S_n^*}\left(1 - f_{X_n^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)\right). \qquad (4.5)$$

We are going to apply duality Lemma 2.5. The following correspondences designed by the two-sided arrow $\longleftrightarrow$ are included for the ease of the reader:

$$\begin{aligned} X_k^* &\longleftrightarrow X_{n-k+1},\\ S_k^* &\longleftrightarrow S_{n-k} - S_n,\\ \eta_k^*(j) &\longleftrightarrow \eta_{n-k+1,n}\left(f_j(0)\right),\\ q_n^*(j) &\longleftrightarrow \mathrm{e}^{-S_n}\, q_n\left(f_j(0)\right).\end{aligned}$$

Now Lemma 2.5 implies,

$$\begin{aligned}&\mathbb{P}_i\left(Z_{n+1} > 0\,,\, X_{n+1} = j\right)\\ &= \tilde{\mathbb{E}}_j^*\left(q_n^*(j)\,;\, X_{n+1}^* = i\right) k(1)^{n+1} \frac{\tilde{\boldsymbol{\nu}}_1(j) v_1(i)\,\mathrm{e}^{-\rho(j)}}{\tilde{\boldsymbol{\nu}}_1(i) v_1(j)}, \qquad (4.6)\end{aligned}$$

where $\tilde{\mathbb{E}}_j^*$ is the expectation generated by the trajectories of the Markov chain $(X_n^*)_{n\geqslant 0}$ starting at $X_0^* = j$.

Note that, under Condition 2, by Lemma 2.3 we have, for any $j \in \mathbb{X}$ and $k \geqslant 1$,

$$0 \leqslant \eta_k^*(j) \leqslant \eta = \max_{i\in\mathbb{X}} \frac{f_i''(1)}{f_i'(1)^2} < +\infty \qquad \tilde{\mathbb{P}}_j^*\text{-a.s.} \tag{4.7}$$

In particular, by (4.3),

$$q_n^*(j) \in (0,1], \quad \forall n \geqslant 1. \tag{4.8}$$

For any $j \in \mathbb{X}$, consider the random variable

$$q_\infty^*(j) := \left[\frac{1}{1-f_j(0)} + \sum_{k=1}^{\infty} \mathrm{e}^{-S_k^*}\, \eta_k^*(j)\right]^{-1} \in [0,1]. \tag{4.9}$$

Lemma 4.1. *Assume that the conditions of Theorem 1.2 are satisfied. For any $j \in \mathbb{X}$,*

$$\lim_{n\to+\infty} q_n^*(j) = q_\infty^*(j) \in (0,1], \qquad \tilde{\mathbb{P}}_j^*\text{-a.s.} \tag{4.10}$$

and

$$\lim_{n\to+\infty} \tilde{\mathbb{E}}_j^* \left(|q_n^*(j) - q_\infty^*(j)|\right) = 0. \tag{4.11}$$

Proof. Fix $j \in \mathbb{X}$. By the law of large numbers for finite Markov chains,

$$\frac{S_k^*}{k} \underset{k\to+\infty}{\longrightarrow} \tilde{\boldsymbol{\nu}}_1(-\rho), \qquad \tilde{\mathbb{P}}_j^*\text{-a.s.}$$

This means that there exists a set N of null probability $\tilde{\mathbb{P}}_j^*(N) = 0$, such that for any $\omega \in \Omega \setminus N$ and any $\varepsilon > 0$, there exists $k_0(\omega,\varepsilon)$ such that for any $k \geqslant k_0(\omega,\varepsilon)$,

$$\mathrm{e}^{-S_k^*(\omega)}\, \eta_k^*(j)(\omega) \leqslant \mathrm{e}^{k\tilde{\boldsymbol{\nu}}_1(\rho)+k\varepsilon}\, \eta,$$

where for the last inequality we used the bound (4.7). By Lemma 2.15, we have $\tilde{\boldsymbol{\nu}}_1(\rho) = k'(1)/k(1) < 0$. Taking $\varepsilon = -\tilde{\boldsymbol{\nu}}_1(\rho)/2$ we obtain that, for any $k \geqslant k_0(\omega)$,

$$0 \leqslant \mathrm{e}^{-S_k^*(\omega)}\, \eta_k^*(j)(\omega) \leqslant \mathrm{e}^{k\frac{\tilde{\boldsymbol{\nu}}_1(\rho)}{2}}\, \eta.$$

Consequently, the series $(q_n^*(j))^{-1}$ converges a.s. to $(q_\infty^*(j))^{-1} \in [1,+\infty)$ which proves (4.10). By (4.8), $(q_n^*(j))_{n\geqslant 1}$ is bounded a.s., so by the Lebesgue dominated convergence theorem, $\lim_{n\to+\infty} \tilde{\mathbb{E}}_j^* \left(|q_n^*(j) - q_\infty^*(j)|\right) = 0$. □

Lemma 4.2. *Assume that the conditions of Theorem 1.2 are satisfied. For any $(i,j) \in \mathbb{X}^2$,*

$$\lim_{n\to+\infty} \tilde{\mathbb{E}}_j^* \left(q_n^*(j)\,;\, X_{n+1}^* = i\right) = \tilde{\boldsymbol{\nu}}_1(i)\tilde{\mathbb{E}}_j^* \left(q_\infty^*(j)\right).$$

Proof. Let $m \geqslant 1$. For any $(i,j) \in \mathbb{X}^2$, and $n \geqslant m$,

$$\begin{aligned}&\tilde{\mathbb{E}}_j^* \left(q_n^*(j) \, ; \, X_{n+1}^* = i \right) \\ &\qquad = \tilde{\mathbb{E}}_j^* \left(q_m^*(j) \, ; \, X_{n+1}^* = i \right) + \tilde{\mathbb{E}}_j^* \left(q_n^*(j) - q_m^*(j) \, ; \, X_{n+1}^* = i \right). \end{aligned} \tag{4.12}$$

By the Markov property,

$$\tilde{\mathbb{E}}_j^* \left(q_m^*(j) \, ; \, X_{n+1}^* = i \right) = \tilde{\mathbb{E}}_j^* \left(q_m^*(j) \left(\tilde{\mathbf{P}}_1^* \right)^{n-m+1} (X_m^*, i) \right).$$

Using (2.16) (which holds also for $\tilde{\mathbf{P}}_1^*$ by Lemmas 2.14 and 2.4) and (4.11), we have

$$\begin{aligned}&\lim_{m \to +\infty} \lim_{n \to +\infty} \tilde{\mathbb{E}}_j^* \left(q_m^*(j) \, ; \, X_{n+1}^* = i \right) \\ &\qquad = \lim_{m \to +\infty} \tilde{\mathbb{E}}_j^* \left(q_m^*(j) \right) \tilde{\boldsymbol{\nu}}_1(i) = \tilde{\mathbb{E}}_j^* \left(q_\infty^*(j) \right) \tilde{\boldsymbol{\nu}}_1(i). \end{aligned} \tag{4.13}$$

Moreover, again by (4.11),

$$\begin{aligned}&\lim_{m \to +\infty} \lim_{n \to +\infty} \left| \tilde{\mathbb{E}}_j^* \left(q_n^*(j) - q_m^*(j) \, ; \, X_{n+1}^* = i \right) \right| \\ &\quad \leqslant \lim_{m \to +\infty} \lim_{n \to +\infty} \tilde{\mathbb{E}}_j^* \left(|q_n^*(j) - q_m^*(j)| \right) = \lim_{m \to +\infty} \tilde{\mathbb{E}}_j^* \left(|q_\infty^*(j) - q_m^*(j)| \right) = 0. \end{aligned}$$

Together with (4.12) and (4.13), this concludes the lemma. □

Proof of Theorem 1.2. By (4.10), the function

$$u(j) = \frac{\tilde{\boldsymbol{\nu}}_1(j) \, \mathrm{e}^{-\rho(j)} \, \tilde{\mathbb{E}}_j^* \left(q_\infty^*(j) \right)}{v_1(j)}$$

is positive. The result of the theorem follows from Lemma 4.2 and the identity (4.6).

5. Proofs in the intermediate subcritical case

We assume the conditions of Theorem 1.3, that is Conditions 1-3 and $k'(1) = 0$. As in the critical case the proof is carried out through a series of lemmata.

The beginning of the reasoning is the same as in the strongly subcritical case. Keeping the same notation as in Section 4 (see (4.1)-(4.6)), we have

$$\mathbb{P}_i \left(Z_{n+1} > 0 \, , \, X_{n+1} = j \right) = \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \, ; \, X_{n+1}^* = i \right) k(1)^{n+1} \frac{\tilde{\boldsymbol{\nu}}_1(j) v_1(i) \, \mathrm{e}^{-\rho(j)}}{\tilde{\boldsymbol{\nu}}_1(i) v_1(j)}. \tag{5.1}$$

Under the hypotheses of Theorem 1.3, the Markov walk $(S_n^*)_{n \geqslant 0}$ is centred under the probability $\tilde{\mathbb{P}}_j^*$ for any $j \in \mathbb{X}$: indeed $\tilde{\boldsymbol{\nu}}_1(-\rho) = -k'(1)/k(1) = 0$ (see Lemma 2.15) and by Lemma 2.14, Conditions 1 and 3 hold for $\tilde{\mathbf{P}}_1$. In this case, by Lemma 2.4, Conditions 1 and 3 hold also for $\tilde{\mathbf{P}}_1^*$. Therefore all the results of Section 2.3 hold for the probability $\tilde{\mathbb{P}}^*$. Let τ_z^* be the exit time of the Markov walk $(z + S_n^*)_{n \geqslant 0}$:

$$\tau_z^* := \inf \left\{ k \geqslant 1 : z + S_k^* \leqslant 0 \right\}.$$

Denote by $\tilde{V}_1^*$ the harmonic function defined by Proposition 2.6 with respect to the probability $\tilde{\mathbb{P}}^*$. As in (2.22), for any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$, define a new probability $\tilde{\mathbb{P}}_{j,z}^{*+}$ and its associated expectation $\tilde{\mathbb{E}}_{j,z}^{*+}$ on $\sigma\left(X_n^*, n \geqslant 1\right)$ by

$$\tilde{\mathbb{E}}_{j,z}^{*+}\left(g\left(X_1^*,\dots,X_n^*\right)\right) := \frac{1}{\tilde{V}_1^*(j,z)}\tilde{\mathbb{E}}_j^*\left(g\left(X_1^*,\dots,X_n^*\right)\tilde{V}_1^*\left(X_n^*, z+S_n^*\right)\,;\,\tau_z^* > n\right),$$

for any $n \geqslant 1$ and any $g\colon \mathbb{X}^n \to \mathbb{C}$.

Lemma 5.1. *Assume that the conditions of Theorem 1.3 are satisfied. For any $m \geqslant 1$, $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$, and $i \in \mathbb{X}$, we have*

$$\lim_{n\to+\infty} \tilde{\mathbb{E}}_j^*\left(q_m^*(j)\,;\, X_{n+1}^* = i \,\middle|\, \tau_z^* > n+1\right) = \tilde{\mathbb{E}}_{j,z}^{*+}\left(q_m^*(j)\right)\tilde{\boldsymbol{\nu}}_1(i).$$

Proof. The equation (4.5) gives an expression for $q_m^*(j)$ in terms of $(X_1^*,\dots,X_m^*)$. Therefore, the assertion of the lemma is a straightforward consequence of Lemma 2.12. □

As in Section 4, using Lemma 2.3 we have for any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$ and $k \geqslant 1$,

$$0 \leqslant \eta_k^*(j) \leqslant \eta = \max_{i\in\mathbb{X}} \frac{f_i''(1)}{f_i'(1)^2} < +\infty \qquad \text{and} \qquad q_n^*(j) \in (0,1], \qquad \tilde{\mathbb{P}}_{j,z}^{*+}\text{-a.s.} \tag{5.2}$$

Consider the random variable

$$q_\infty^*(j) := \left[\frac{1}{1-f_j(0)} + \sum_{k=1}^{+\infty} \mathrm{e}^{-S_k^*}\,\eta_k^*(j)\right]^{-1} \in [0,1]. \tag{5.3}$$

Lemma 5.2. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$*

$$\lim_{m\to+\infty} \tilde{\mathbb{E}}_{j,z}^{*+}\left(\left|(q_m^*(j))^{-1} - (q_\infty^*(j))^{-1}\right|\right) = 0, \tag{5.4}$$

and

$$\lim_{m\to+\infty} \tilde{\mathbb{E}}_{j,z}^{*+}\left(|q_m^*(j) - q_\infty^*(j)|\right) = 0. \tag{5.5}$$

Proof. Fix $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$. By (4.3), (5.3) and (5.2), for any $m \geqslant 1$,

$$\tilde{\mathbb{E}}_{j,z}^{*+}\left(\left|(q_m^*(j))^{-1} - (q_\infty^*(j))^{-1}\right|\right) \leqslant \eta\tilde{\mathbb{E}}_{j,z}^{*+}\left(\sum_{k=m+1}^{+\infty} \mathrm{e}^{-S_k^*}\right).$$

From this bound, by Lemma 2.13 and the dominated convergence theorem when $m \to +\infty$, we obtain (5.4).

Now by (5.2) and (5.3) we have for any $m \geqslant 1$,

$$\begin{aligned}\tilde{\mathbb{E}}_{j,z}^{*+}\left(|q_m^*(j) - q_\infty^*(j)|\right) &= \tilde{\mathbb{E}}_{j,z}^{*+}\left(|q_m^*(j)q_\infty^*(j)|\left|(q_m^*(j))^{-1} - (q_\infty^*(j))^{-1}\right|\right)\\ &\leqslant \tilde{\mathbb{E}}_{j,z}^{*+}\left(\left|(q_m^*(j))^{-1} - (q_\infty^*(j))^{-1}\right|\right),\end{aligned}$$

which proves (5.5). □

Let U be the function defined on $\mathrm{supp}(\tilde{V}_1^*)$ by

$$U^*(j,z) = \tilde{\mathbb{E}}^{*+}_{j,z}\left(q^*_\infty(j)\right).$$

Using (5.2) and Lemma 2.13, we have

$$\tilde{\mathbb{E}}^{*+}_{j,z}\left((q^*_\infty(j))^{-1}\right) \leqslant \frac{1}{1-f_j(0)} + \eta\tilde{\mathbb{E}}^{*+}_{j,z}\left(\sum_{k=1}^{+\infty} \mathrm{e}^{-S^*_k}\right) < +\infty. \tag{5.6}$$

Therefore $q^*_\infty > 0$ $\mathbb{P}^+_{i,y}$-a.s. and so $U^*(j,z) > 0$. In addition, by (5.3), $U^*(j,z) \leqslant 1$. For any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$,

$$U^*(j,z) \in (0,1]. \tag{5.7}$$

Lemma 5.3. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$ and $i \in \mathbb{X}$, we have*

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \tilde{\mathbb{E}}^*_j\left(q^*_m(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right) = U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i).$$

Proof. The assertion of the lemma is straightforward consequence of Lemmas 5.1 and 5.2. □

Lemma 5.4. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$ and $\theta \in (0,1)$, we have*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty} \tilde{\mathbb{E}}^*_j\left(\left|q^*_m(j) - q^*_{\lfloor\theta n\rfloor}(j)\right| \,\middle|\, \tau^*_z > n+1\right) = 0.$$

Proof. Fix $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$ and $\theta \in (0,1)$. Let $m \geqslant 1$ and $n \geqslant 1$ be such that $\theta n \geqslant m+1$. Set $\theta_n = \lfloor \theta n \rfloor$. Denote

$$I_0 := \tilde{\mathbb{E}}^*_j\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right| \,\middle|\, \tau^*_z > n+1\right) \qquad \text{and} \qquad J_n(j,z) := \tilde{\mathbb{P}}^*_j\left(\tau^*_z > n\right).$$

Note that by the point 1 of Proposition 2.7, we have $J_n(j,z) > 0$ for any n large enough. By the Markov property and the point 2 of Proposition 2.7,

$$\begin{aligned} I_0 &= \frac{1}{J_{n+1}(j,z)}\tilde{\mathbb{E}}^*_j\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right| J_{n+1-\theta_n}\left(X^*_{\theta_n}, z + S^*_{\theta_n}\right)\,;\, \tau^*_z > \theta_n\right) \\ &\leqslant \frac{c}{J_{n+1}(j,z)\sqrt{n+1-\theta_n}}\tilde{\mathbb{E}}^*_j\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right|\left(1 + z + S^*_{\theta_n}\right)\,;\, \tau^*_z > \theta_n\right). \end{aligned}$$

Using the point 3 of Proposition 2.6 and (5.2),

$$\begin{aligned} I_0 &\leqslant \frac{c}{J_{n+1}(j,z)\sqrt{n(1-\theta)}} \\ &\qquad \times \tilde{\mathbb{E}}^*_j\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right|\left(1 + \tilde{V}^*_1\left(X^*_{\theta_n}, z + S^*_{\theta_n}\right)\right)\,;\, \tau^*_z > \theta_n\right) \\ &\leqslant \frac{c}{J_{n+1}(j,z)\sqrt{n(1-\theta)}}\left(\tilde{\mathbb{P}}^*_j\left(\tau^*_z > \theta_n\right) + \tilde{V}_1(j,z)\tilde{\mathbb{E}}^{*+}_{j,z}\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right|\right)\right). \end{aligned}$$

By the point 1 of Proposition 2.7 and (5.5), we obtain that

$$\begin{aligned}\limsup_{n\to+\infty} I_0 &\leqslant \limsup_{n\to+\infty} \frac{c\sqrt{n+1}}{\sqrt{n(1-\theta)}} \tilde{\mathbb{E}}^{*+}_{j,z}\left(\left|q^*_m(j) - q^*_{\theta_n}(j)\right|\right)\\ &= \frac{c}{\sqrt{(1-\theta)}} \tilde{\mathbb{E}}^{*+}_{j,z}\left(|q^*_m(j) - q^*_\infty(j)|\right).\end{aligned}$$

Taking the limit as $m \to +\infty$ and using (5.5), we conclude the proof. □

Lemma 5.5. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z) \in \mathrm{supp}(\tilde{V}^*_1)$, $i \in \mathbb{X}$ and $\theta \in (0,1)$, we have*

$$\lim_{n\to+\infty} \tilde{\mathbb{E}}^*_j\left(q^*_{\lfloor \theta n\rfloor}(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right) = U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i).$$

Proof. For any $(j,z) \in \mathrm{supp}(\tilde{V}^*_1)$, $i \in \mathbb{X}$, $\theta \in (0,1)$, $m \geqslant 1$ and $n \geqslant m+1$ such that $\lfloor \theta n\rfloor \geqslant m$, we have

$$\begin{aligned}I_0 &:= \tilde{\mathbb{E}}^*_j\left(q^*_{\lfloor \theta n\rfloor}(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right)\\ &= \tilde{\mathbb{E}}^*_j\left(q^*_m(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right)\\ &\qquad + \underbrace{\tilde{\mathbb{E}}^*_j\left(q^*_{\lfloor \theta n\rfloor}(j) - q^*_m(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right)}_{=:I_1}.\end{aligned}$$

By Lemma 5.4, $\limsup_{m\to+\infty} \limsup_{n\to+\infty} |I_1| = 0$. Therefore, using Lemma 5.3, $\lim_{n\to+\infty} I_0 = U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i)$ and the assertion of the lemma follows. □

Lemma 5.6. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z) \in \mathrm{supp}(\tilde{V}^*_1)$, we have*

$$\lim_{p\to+\infty} \tilde{\mathbb{E}}^*_j\left(q^*_p(j) \,\middle|\, \tau^*_z > p+1\right) = U^*(j,z).$$

Proof. Fix $(j,z) \in \mathrm{supp}(\tilde{V}^*_1)$. For any $p \geqslant 1$ and $\theta \in (0,1)$ set $n = \lfloor p/\theta\rfloor + 1$. Note that $p = \lfloor \theta n\rfloor$. We write, for any $p \geqslant 1$,

$$\begin{aligned}&\tilde{\mathbb{E}}^*_j\left(q^*_p(j) \,\middle|\, \tau^*_z > p+1\right)\\ &\qquad = \frac{\tilde{\mathbb{E}}^*_j\left(q^*_p(j)\,;\, \tau^*_z > n+1\right) + \tilde{\mathbb{E}}^*_j\left(q^*_p(j)\,;\, p+1 < \tau^*_z \leqslant n+1\right)}{\tilde{\mathbb{P}}^*_j\left(\tau^*_z > p+1\right)}.\end{aligned}$$

By Lemma 5.5 and the point 1 of Proposition 2.7,

$$\begin{aligned}&\frac{\tilde{\mathbb{E}}^*_j\left(q^*_p(j)\,;\, \tau^*_z > n+1\right)}{\tilde{\mathbb{P}}^*_j\left(\tau^*_z > p+1\right)} =\\ &\sum_{i\in\mathbb{X}} \tilde{\mathbb{E}}^*_j\left(q^*_p(j)\,;\, X^*_{n+1} = i \,\middle|\, \tau^*_z > n+1\right) \frac{\tilde{\mathbb{P}}^*_j\left(\tau^*_z > n+1\right)}{\tilde{\mathbb{P}}^*_j\left(\tau^*_z > p+1\right)} \underset{p\to+\infty}{\longrightarrow} U^*(j,z)\sqrt{\theta}.\end{aligned}$$

Moreover, using (5.2) and the point 1 of Proposition 2.7,

$$\frac{\tilde{\mathbb{E}}_j^*\left(q_p^*(j)\,;\, p+1<\tau_z^*\leqslant n+1\right)}{\tilde{\mathbb{P}}_j^*\left(\tau_z^*>p+1\right)}\leqslant 1-\frac{\tilde{\mathbb{P}}_j^*\left(\tau_z^*>n+1\right)}{\tilde{\mathbb{P}}_j^*\left(\tau_z^*>p+1\right)}\underset{p\to+\infty}{\longrightarrow}1-\sqrt{\theta}.$$

Therefore, for any $\theta\in(0,1)$,

$$\left|\lim_{p\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_p^*(j)\,\middle|\,\tau_z^*>p+1\right)-U^*(j,z)\sqrt{\theta}\right|\leqslant 1-\sqrt{\theta}.$$

Taking the limit as $\theta\to 1$ it concludes the proof. □

Lemma 5.7. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z)\in\operatorname{supp}(\tilde{V}_1^*)$ and $\theta\in(0,1)$, we have*

$$\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(\left|q_{\lfloor\theta n\rfloor}^*(j)-q_n^*(j)\right|\,\middle|\,\tau_z^*>n+1\right)=0.$$

Proof. Using the fact that $\eta_k^*(j)$ are non-negative and the definition of $q_n^*(j)$ in (4.3), we see that $(q_n^*(j))_{n\geqslant 1}$ is non-increasing. Therefore, using Lemmas 5.5 and 5.6,

$$\begin{aligned}I_0&:=\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(\left|q_{\lfloor\theta n\rfloor}^*(j)-q_n^*(j)\right|\,\middle|\,\tau_z^*>n+1\right)\\&=\lim_{n\to+\infty}\sum_{i\in\mathbb{X}}\tilde{\mathbb{E}}_j^*\left(q_{\lfloor\theta n\rfloor}^*(j)\,;\,X_{n+1}^*=i\,\middle|\,\tau_z^*>n+1\right)\\&\qquad-\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_n^*(j)\,|\,\tau_z^*>n+1\right)=U^*(j,z)-U^*(j,z)=0.\end{aligned}$$

□

Lemma 5.8. *Assume that the conditions of Theorem 1.3 are satisfied. For any $(j,z)\in\operatorname{supp}(\tilde{V}_1^*)$ and $i\in\mathbb{X}$, we have*

$$\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_n^*(j)\,;\,X_{n+1}^*=i\,\middle|\,\tau_z^*>n+1\right)=U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i).$$

Proof. By Lemmas 5.5 and 5.7, for any $(j,z)\in\operatorname{supp}(\tilde{V}_1^*)$, $i\in\mathbb{X}$ and $\theta\in(0,1)$,

$$\begin{aligned}I_0&:=\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_n^*(j)\,;\,X_{n+1}^*=i\,\middle|\,\tau_z^*>n+1\right)\\&=\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_{\lfloor\theta n\rfloor}^*(j)\,;\,X_{n+1}^*=i\,\middle|\,\tau_z^*>n+1\right)\\&+\lim_{n\to+\infty}\tilde{\mathbb{E}}_j^*\left(q_n^*(j)-q_{\lfloor\theta n\rfloor}^*(j)\,;\,X_{n+1}^*=i\,\middle|\,\tau_z^*>n+1\right)=U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i).\end{aligned}$$

□

Lemma 5.9. *Assume that the conditions of Theorem 1.3 are satisfied. There exists $\tilde{u}$ a positive function on $\mathbb{X}$ such that, for any $(i,j)\in\mathbb{X}^2$, we have*

$$\tilde{\mathbb{E}}_j^*\left(q_n^*(j)\,;\,X_{n+1}^*=i\right)\underset{n\to+\infty}{\sim}\frac{\tilde{u}(j)\tilde{\boldsymbol{\nu}}_1(i)}{\sqrt{n}}.$$

Proof. Fix $(i,j) \in \mathbb{X}^2$. For any $z \in \mathbb{R}$ and $n \geqslant 1$,

$$\begin{aligned} 0 &\leqslant \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, X_{n+1}^* = i \right) - \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, X_{n+1}^* = i \,,\, \tau_z^* > n+1 \right) \\ &\leqslant \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n+1 \right). \end{aligned} \tag{5.8}$$

Since $q_n^*(j) \leqslant 1$ (see (5.2)), we have

$$\tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n+1 \right) \leqslant \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n \right) + \tilde{\mathbb{P}}_j \left(\tau_z^* = n+1 \right). \tag{5.9}$$

By (4.5), it holds $q_n^*(j) \leqslant \mathrm{e}^{S_n^*}$. Since $(q_n^*(j))_{n \geqslant 1}$ is non-increasing, we have $q_n^*(j) = \min_{1 \leqslant k \leqslant n} q_k^*(j) \leqslant \mathrm{e}^{\min_{1 \leqslant k \leqslant n} S_k^*}$. Consequently,

$$\begin{aligned} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n \right) &\leqslant \mathrm{e}^{-z} \, \tilde{\mathbb{E}}_j^* \left(\mathrm{e}^{\min_{1 \leqslant k \leqslant n} z + S_k^*} \,;\, \tau_z^* \leqslant n \right) \\ &\leqslant \mathrm{e}^{-z} \sum_{p=0}^{+\infty} \mathrm{e}^{-p} \, \tilde{\mathbb{P}}_j^* \left(-(p+1) < \min_{1 \leqslant k \leqslant n} z + S_k^* \leqslant -p \,,\, \tau_z^* \leqslant n \right) \\ &\leqslant \mathrm{e}^{-z} \sum_{p=0}^{+\infty} \mathrm{e}^{-p} \, \tilde{\mathbb{P}}_j^* \left(\tau_{z+p+1}^* > n \right). \end{aligned}$$

Using the point 2 of Proposition 2.7,

$$\tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n \right) \leqslant \frac{c \, \mathrm{e}^{-z} \left(1 + \max(0,z) \right)}{\sqrt{n}}. \tag{5.10}$$

By the point 3 of Proposition 2.6, there exists $z_0 \in \mathbb{R}$ such that for any $z \geqslant z_0$, $\tilde{V}_1^*(j,z) > 0$, which means that $(j,z) \in \mathrm{supp}(\tilde{V}_1^*)$. Therefore, using the point 1 of Proposition 2.7, for any $z \geqslant z_0$,

$$\begin{aligned} &\lim_{n \to +\infty} \sqrt{n} \tilde{\mathbb{P}}_j \left(\tau_z^* = n+1 \right) \\ &\qquad = \lim_{n \to +\infty} \sqrt{n} \tilde{\mathbb{P}}_j \left(\tau_z^* > n \right) - \lim_{n \to +\infty} \sqrt{n} \tilde{\mathbb{P}}_j \left(\tau_z^* > n+1 \right) = 0. \end{aligned} \tag{5.11}$$

Putting together (5.9), (5.10) and (5.11), we obtain that, for any $z \geqslant z_0$,

$$\lim_{n \to +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, \tau_z^* \leqslant n+1 \right) \leqslant c \, \mathrm{e}^{-z} \left(1 + \max(0,z) \right). \tag{5.12}$$

Moreover, using Lemma 5.8 and the point 1 of Proposition 2.7,

$$\lim_{n \to +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, X_{n+1}^* = i \,,\, \tau_z^* > n+1 \right) = \frac{2 \tilde{V}_1^*(j,z)}{\sqrt{2\pi} \tilde{\sigma}_1} U^*(j,z) \tilde{\boldsymbol{\nu}}_1(i), \tag{5.13}$$

where $\tilde{\sigma}_1$ is defined in (2.37). Denoting

$$I(i,j) = \liminf_{n \to +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, X_{n+1}^* = i \right)$$

and

$$J(i,j) = \limsup_{n \to +\infty} \sqrt{n} \tilde{\mathbb{E}}_j^* \left(q_n^*(j) \,;\, X_{n+1}^* = i \right),$$

and using (5.8), (5.12) and (5.13), we obtain that, for any $z \geqslant z_0$,

$$\frac{2\tilde{V}_1^*(j,z)}{\sqrt{2\pi}\tilde{\sigma}_1}U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i) \leqslant I(i,j) \leqslant J(i,j) \leqslant \frac{2\tilde{V}_1^*(j,z)}{\sqrt{2\pi}\tilde{\sigma}_1}U^*(j,z)\tilde{\boldsymbol{\nu}}_1(i) + c\,\mathrm{e}^{-z}\left(1+\max(0,z)\right). \tag{5.14}$$

By (5.13), we observe that $z \mapsto \frac{2\tilde{V}_1^*(j,z)U^*(j,z)}{\sqrt{2\pi}\tilde{\sigma}_1}$ is non-decreasing and by (5.14), this function is bounded by $I(i,j)/\tilde{\boldsymbol{\nu}}_1(i)$. Consequently the limit

$$\tilde{u}(j) := \lim_{z\to+\infty} \frac{2\tilde{V}_1^*(j,z)U^*(j,z)}{\sqrt{2\pi}\tilde{\sigma}_1}$$

exists and for any $z \geqslant z_0$, by (5.7),

$$\tilde{u}(j) \geqslant \frac{2\tilde{V}_1^*(j,z)U^*(j,z)}{\sqrt{2\pi}\tilde{\sigma}_1} > 0. \tag{5.15}$$

Taking the limit as $z \to +\infty$ in (5.14), we conclude that

$$I(i,j) = J(i,j) = \tilde{u}(j)\tilde{\boldsymbol{\nu}}_1(i).$$

□

Proof of Theorem 1.3. By (5.15) the function

$$u(j) = \tilde{u}(j)\frac{\tilde{\boldsymbol{\nu}}_1(j)\,\mathrm{e}^{-\rho(j)}}{v_1(j)}, \qquad \forall j \in \mathbb{X},$$

is positive on $\mathbb{X}$. The assertion of Theorem 1.3 is a consequence of (5.1) and Lemma 5.9.

6. Proofs in the weakly subcritical case

We assume the conditions of Theorem 1.4, that is Conditions 1-3 and $\boldsymbol{\nu}(\rho) = k'(0) < 0$, $k'(1) > 0$. By Lemma 2.15, the function $\lambda \mapsto K'(\lambda)$ is increasing. Consequently, there exists $\lambda \in (0,1)$ such that

$$K'(\lambda) = \frac{k'(\lambda)}{k(\lambda)} = \tilde{\boldsymbol{\nu}}_\lambda(\rho) = 0. \tag{6.1}$$

For this λ and any $i \in \mathbb{X}$, define the changed probability measure $\tilde{\mathbb{P}}_i$ and the corresponding expectation $\tilde{\mathbb{E}}_i$ by (2.35), such that for any $n \geqslant 1$ and any g: $\mathbb{X}^n \to \mathbb{C}$,

$$\tilde{\mathbb{E}}_i\left(g(X_1,\dots,X_n)\right) = \frac{\mathbb{E}_i\left(\mathrm{e}^{\lambda S_n}\, g(X_1,\dots,X_n)v_\lambda(X_n)\right)}{k(\lambda)^n v_\lambda(i)}. \tag{6.2}$$

Our starting point is the following formula which is a consequence of (2.1): for any $(i,j) \in \mathbb{X}^2$ and $n \geqslant 1$,

$$\begin{aligned} &\mathbb{E}_i \left(q_{n+1} \, ; \, X_{n+1} = j \, , \, \tau_y > n \right) \\ &= \tilde{\mathbb{E}}_i \left(\mathrm{e}^{-\lambda S_n} \, q_n \left(f_j(0) \right) \, ; \, X_{n+1} = j \, , \, \tau_y > n \right) k(\lambda)^{n+1} \frac{v_\lambda(i)}{v_\lambda(j)} \, \mathrm{e}^{-\lambda \rho(j)} \, . \end{aligned} \tag{6.3}$$

The transition probabilities of $(X_n)_{n \geqslant 0}$ under the changed measure are given by (2.32):

$$\tilde{\mathbf{P}}_\lambda(i,j) = \frac{\mathrm{e}^{\lambda \rho(j)} \, v_\lambda(j)}{k(\lambda) v_\lambda(i)} \mathbf{P}(i,j).$$

By (6.1), the Markov walk $(S_n)_{n \geqslant 0}$ is centred under $\tilde{\mathbb{P}}_i$. Note that under the hypotheses of Theorem 1.4, by Lemma 2.14, Conditions 1 and 3 hold also for $\tilde{\mathbf{P}}_\lambda$. Therefore all the results of Section 2.3 hold for the Markov walk $(S_n)_{n \geqslant 0}$ under $\tilde{\mathbb{P}}_i$.

Let $(X_n^*)_{n \geqslant 0}$ be the dual Markov chain independent of $(X_n)_{n \geqslant 0}$, with transition probabilities $\tilde{\mathbf{P}}_\lambda^*$ defined by (cp. (2.13))

$$\tilde{\mathbf{P}}_\lambda^*(i,j) = \frac{\tilde{\boldsymbol{\nu}}_\lambda(j)}{\tilde{\boldsymbol{\nu}}_\lambda(i)} \tilde{\mathbf{P}}(j,i) = \frac{\boldsymbol{\nu}_\lambda(j)}{\boldsymbol{\nu}_\lambda(i)} \frac{\mathrm{e}^{\rho(i)}}{k(\lambda)} \mathbf{P}(j,i). \tag{6.4}$$

As in Section 2.2, we define the dual Markov walk $(S_n^*)_{n \geqslant 0}$ by (2.14) and its exit time τ_z^* for any $z \in \mathbb{R}$ by (2.15). Let $\tilde{\mathbb{P}}_{i,j}$ be the probability on $(\Omega, \mathscr{F})$ generated by the finite dimensional distributions of $(X_n, X_n^*)_{n \geqslant 0}$ starting at $(X_0, X_0^*) = (i,j)$. By (6.1), the Markov walk $(S_n^*)_{n \geqslant 1}$ is centred under $\tilde{\mathbb{P}}_{i,j}$:

$$\tilde{\boldsymbol{\nu}}_\lambda(\rho) = \tilde{\boldsymbol{\nu}}_\lambda(-\rho) = 0$$

and by Lemma 2.4, Conditions 1 and 3 hold for $\tilde{\mathbf{P}}_\lambda^*$. Let $\tilde{V}_\lambda$ and $\tilde{V}_\lambda^*$ be the harmonic functions of the Markov walks $(S_n)_{n \geqslant 0}$ and $(S_n^*)_{n \geqslant 0}$, respectively (see Proposition 2.6).

We start by some preliminary bounds. The following assertion is similar to Lemma 2.13.

Lemma 6.1. *Assume that the conditions of Theorem 1.4 are satisfied. For any $i \in \mathbb{X}$, $y \in \mathbb{R}$, $k \geqslant 1$ and $n \geqslant k+1$, we have*

$$n^{3/2} \tilde{\mathbb{E}}_i \left(\mathrm{e}^{-S_k} \, \mathrm{e}^{-\lambda S_n} \, ; \, \tau_y > n \right) \leqslant \mathrm{e}^{(1+\lambda)y} (1 + \max(y,0)) \frac{c n^{3/2}}{(n-k)^{3/2} k^{3/2}}.$$

Proof. Fix $i \in \mathbb{X}$, $y \in \mathbb{R}$, $k \geqslant 1$ and $n \geqslant k+1$. By the Markov property,

$$\begin{aligned} I_0 &:= n^{3/2}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_k}\,\mathrm{e}^{-\lambda S_n}\,;\,\tau_y > n\right) \\ &\leqslant \sum_{p=0}^{+\infty} n^{3/2}\,\mathrm{e}^{\lambda y}\,\mathrm{e}^{-\lambda p}\,\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_k}\,;\,y+S_n \in [p,p+1]\,,\,\tau_y > n\right) \\ &= \sum_{p=0}^{+\infty} n^{3/2}\,\mathrm{e}^{\lambda y}\,\mathrm{e}^{-\lambda p}\,\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_k}\,J_{n-k}\left(X_k, y+S_k\right)\,;\,\tau_y > k\right), \end{aligned}$$

where for any $i' \in \mathbb{X}$, $y' \in \mathbb{R}$ and $p \geqslant 1$

$$J_{n-k}(i',y') = \tilde{\mathbb{P}}_{i'}\left(y'+S_{n-k} \in [p,p+1]\,,\,\tau_{y'} > n-k\right).$$

By the point 2 of Proposition 2.9,

$$J_{n-k}(i',y') \leqslant \frac{c}{(n-k)^{3/2}}(1+p)(1+\max(y',0)).$$

Consequently,

$$\begin{aligned} I_0 &\leqslant \mathrm{e}^{\lambda y}\,\frac{cn^{3/2}}{(n-k)^{3/2}}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_k}\,(1+y+S_k)\,;\,\tau_y > k\right)\sum_{p=0}^{+\infty}\mathrm{e}^{-\lambda p}(1+p) \\ &\leqslant \mathrm{e}^{\lambda y}\,\frac{cn^{3/2}}{(n-k)^{3/2}}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-S_k}\,(1+y+S_k)\,;\,\tau_y > k\right) \\ &\leqslant \mathrm{e}^{(1+\lambda)y}\,\frac{cn^{3/2}}{(n-k)^{3/2}}\sum_{p=0}^{+\infty}\mathrm{e}^{-p}(2+p)\tilde{\mathbb{P}}_i\left(y+S_k \in [p,p+1]\,;\,\tau_y > k\right). \end{aligned}$$

Again by the point 2 of Proposition 2.9,

$$I_0 \leqslant \mathrm{e}^{(1+\lambda)y}(1+\max(y,0))\frac{cn^{3/2}}{(n-k)^{3/2}k^{3/2}}\sum_{p=0}^{+\infty}\mathrm{e}^{-p}(2+p)(1+p).$$

This concludes the proof of the lemma. □

For any $l \geqslant 1$ and $n \geqslant l+1$, set

$$q_{l,n}\left(f_j(0)\right) := 1 - f_{l+1,n}\left(f_j(0)\right) = 1 - f_{X_{l+1}} \circ \cdots \circ f_{X_n} \circ f_j(0),$$

In the same way as in Lemma 2.2, we obtain:

$$q_{l,n}\left(f_j(0)\right)^{-1} = \frac{\mathrm{e}^{S_l - S_n}}{1-f_j(0)} + \sum_{k=l}^{n-1}\mathrm{e}^{S_l-S_k}\,\eta_{k+1,n}\left(f_j(0)\right), \tag{6.5}$$

where $\eta_{k+1,n}(s)$ are defined by (2.7). Moreover, similarly to (2.4), we have for any $n \geqslant l+1 \geqslant 2$,

$$q_{l,n}\left(f_j(0)\right) \in (0,1] \qquad \tilde{\mathbb{P}}_i\text{-a.s.} \tag{6.6}$$

In addition, by Lemma 2.3, for any $k \leqslant n-1$,

$$0 \leqslant \eta_{k+1,n}\left(f_j(0)\right) \leqslant \eta \qquad \tilde{\mathbb{P}}_i\text{-a.s.} \tag{6.7}$$

Lemma 6.2. *Assume that the conditions of Theorem 1.4 are satisfied. For any $(i,j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$, we have*

$$\lim_{l,m\to+\infty} \limsup_{n\to+\infty} n^{3/2} \tilde{\mathbb{E}}_i \Big(\Big| \mathrm{e}^{-S_{n-m}} q_{n-m,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_l}\, q_{l,n}\left(f_j(0)\right)^{-1} \Big| \mathrm{e}^{-\lambda S_n}\, ;\, \tau_y > n \Big) = 0.$$

Proof. Fix $(i,j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$. For any $l \geqslant 1$, $m \geqslant 1$ and $n \geqslant l+m+1$, we have

$$\begin{aligned} I_0 &:= n^{3/2} \tilde{\mathbb{E}}_i \left(\left| \mathrm{e}^{-S_{n-m}}\, q_{n-m,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_l}\, q_{l,n}\left(f_j(0)\right)^{-1} \right| \mathrm{e}^{-\lambda S_n}\, ;\, \tau_y > n \right) \\ &= n^{3/2} \tilde{\mathbb{E}}_i \left(\sum_{k=l}^{n-m-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}\left(f_j(0)\right) \mathrm{e}^{-\lambda S_n}\, ;\, \tau_y > n \right). \end{aligned}$$

Using (6.7) and Lemma 6.1,

$$I_0 \leqslant \eta \sum_{k=l}^{n-m-1} \mathrm{e}^{(1+\lambda)y} (1+\max(y,0)) \frac{c n^{3/2}}{(n-k)^{3/2} k^{3/2}}.$$

Let $n_1 := \lfloor n/2 \rfloor$. We note that

$$\begin{aligned} \sum_{k=l}^{n-m-1} \frac{c n^{3/2}}{(n-k)^{3/2} k^{3/2}} &\leqslant \frac{c n^{3/2}}{(n-n_1)^{3/2}} \sum_{k=l}^{n_1} \frac{1}{k^{3/2}} + \frac{c n^{3/2}}{n_1^{3/2}} \sum_{k=n_1+1}^{n-m-1} \frac{1}{(n-k)^{3/2}} \\ &\leqslant c \sum_{k=l}^{+\infty} \frac{1}{k^{3/2}} + c \sum_{k=m}^{+\infty} \frac{1}{k^{3/2}}. \end{aligned}$$

Consequently,

$$\limsup_{n\to+\infty} I_0 \leqslant c\eta\, \mathrm{e}^{(1+\lambda)y} (1+\max(y,0)) \left(\sum_{k=l}^{+\infty} \frac{1}{k^{3/2}} + \sum_{k=m}^{+\infty} \frac{1}{k^{3/2}} \right).$$

Taking the limits as $l \to +\infty$ and $m \to +\infty$, proves the lemma. □

For any $l \geqslant 1$, $m \geqslant 1$ and $n \geqslant l+m+1$, consider the random variables

$$\begin{aligned} r_n^{(l,m)}(j) &:= 1 - f_{1,l} \left(\left[1 - f'_{l+1,n-m}(1) \left(1 - f_{n-m+1,n}\left(f_j(0)\right)\right) \right]^+ \right) \\ &= 1 - f_{X_1} \circ \cdots \circ f_{X_l} \Big(\Big[1 - f'_{X_{l+1}}(1) \times \ldots \\ &\qquad \times f'_{X_{n-m}}(1) \left(1 - f_{X_{n-m+1}} \circ \cdots \circ f_{X_n} \circ f_j(0) \right) \Big]^+ \Big), \end{aligned}$$

where $[t]^+ = \max(t,0)$ for any $t \in \mathbb{R}$. The random variable $r_n^{(l,m)}(j)$ approximates $q_n\left(f_j(0)\right)$ in the following sense:

Lemma 6.3. *Assume that the conditions of Theorem 1.4 are satisfied. For any $(i,j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$,*

$$\lim_{l,m\to+\infty} \limsup_{n\to+\infty} n^{3/2} \tilde{\mathbb{E}}_i \left(\left| q_n\left(f_j(0)\right) - r_n^{(l,m)}(j) \right| \mathrm{e}^{-\lambda S_n} \, ; \, \tau_y > n \right) = 0.$$

Proof. Fix $(i,j) \in \mathbb{X}^2$ and $y \in \mathbb{R}$. Since for any $i' \in \mathbb{X}$, $f_{i'}$ is increasing and convex, the function $f_{l+1,n-m}$ is convex. So, for any $l \geqslant 1$, $m \geqslant 1$ and $n \geqslant l+m+1$,

$$\begin{aligned} f_{l+1,n}\left(f_j(0)\right) &= f_{l+1,n-m}\left(f_{n-m+1,n}\left(f_j(0)\right)\right) \\ &\geqslant \left[1 - f'_{l+1,n-m}(1)\left(1 - f_{n-m+1,n}\left(f_j(0)\right)\right)\right]^+. \end{aligned}$$

Since $f_{1,l}$ is increasing,

$$q_n\left(f_j(0)\right) = 1 - f_{1,n}\left(f_j(0)\right) \leqslant r_n^{(l,m)}(j),$$

or equivalently

$$0 \leqslant r_n^{(l,m)}(j) - q_n\left(f_j(0)\right).$$

Moreover, by the convexity of $f_{1,l}$,

$$\begin{aligned} & r_n^{(l,m)}(j) - q_n\left(f_j(0)\right) \\ &= f_{1,l} \circ f_{l+1,n}\left(f_j(0)\right) - f_{1,l}\left(\left[1 - f'_{l+1,n-m}(1)\left(1 - f_{n-m+1,n}\left(f_j(0)\right)\right)\right]^+\right) \\ &\leqslant f'_{1,l}(1)\left(f_{l+1,n}\left(f_j(0)\right) - \left[1 - f'_{l+1,n-m}(1)\left(1 - f_{n-m+1,n}\left(f_j(0)\right)\right)\right]^+\right) \\ &\leqslant f'_{1,l}(1)\left(f'_{l+1,n-m}(1) q_{n-m,n}\left(f_j(0)\right) - q_{l,n}\left(f_j(0)\right)\right) \\ &= \mathrm{e}^{S_{n-m}} q_{n-m,n}\left(f_j(0)\right) - \mathrm{e}^{S_l} q_{l,n}\left(f_j(0)\right) \\ &= \mathrm{e}^{S_{n-m}} q_{n-m,n}\left(f_j(0)\right) \mathrm{e}^{S_l} q_{l,n}\left(f_j(0)\right) \\ &\qquad \times \left(\mathrm{e}^{-S_l} q_{l,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_{n-m}} q_{n-m,n}\left(f_j(0)\right)^{-1}\right). \end{aligned}$$

By (6.5), we have $q_{l,n}\left(f_j(0)\right) \leqslant \mathrm{e}^{S_n - S_l}$ and so

$$r_n^{(l,m)}(j) - q_n\left(f_j(0)\right) \leqslant \mathrm{e}^{2S_n}\left(\mathrm{e}^{-S_l} q_{l,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_{n-m}} q_{n-m,n}\left(f_j(0)\right)^{-1}\right).$$

In addition, by the definition of $r_n^{(l,m)}(j)$ and $q_n\left(f_j(0)\right)$, we have $r_n^{(l,m)}(j) - q_n\left(f_j(0)\right) \leqslant 1$. Therefore, $\tilde{\mathbb{P}}_i$-a.s. it holds,

$$\begin{aligned} & r_n^{(l,m)}(j) - q_n\left(f_j(0)\right) \\ &\leqslant \min\left(1, \mathrm{e}^{2S_n}\left(\mathrm{e}^{-S_l} q_{l,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_{n-m}} q_{n-m,n}\left(f_j(0)\right)^{-1}\right)\right). \end{aligned}$$

Using the previous bound, it follows that, for any integer $N \geqslant 1$,

$$I_0 := n^{3/2}\tilde{\mathbb{E}}_i\left(\left|q_n\left(f_j(0)\right) - r_n^{(l,m)}(j)\right| \mathrm{e}^{-\lambda S_n}\,;\, \tau_y > n\right) \leqslant \mathrm{e}^{2(N-y)}\, n^{3/2}$$
$$\times\, \tilde{\mathbb{E}}_i\left(\left|\mathrm{e}^{-S_l}\, q_{l,n}\left(f_j(0)\right)^{-1} - \mathrm{e}^{-S_{n-m}}\, q_{n-m,n}\left(f_j(0)\right)^{-1}\right| \mathrm{e}^{-\lambda S_n}\,;\, \tau_y > n\right)$$
$$+\, n^{3/2}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-\lambda S_n}\,;\, y + S_n > N\,,\, \tau_y > n\right).$$

Moreover, using the point 2 of Proposition 2.9,

$$n^{3/2}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-\lambda S_n}\,;\, y + S_n > N\,,\, \tau_y > n\right) \leqslant \sum_{p=N}^{+\infty} \mathrm{e}^{\lambda y}\, \mathrm{e}^{-\lambda p}\, n^{3/2}$$

$$\times\, \tilde{\mathbb{P}}_i\left(y + S_n \in [p, p+1]\,,\, \tau_y > n\right) \leqslant c\, \mathrm{e}^{\lambda y}(1 + \max(y, 0)) \sum_{p=N}^{+\infty} \mathrm{e}^{-\lambda p}(1+p).$$

Consequently, using Lemma 6.2, we obtain that

$$\lim_{l,m\to+\infty} \limsup_{n\to+\infty} I_0 \leqslant c\, \mathrm{e}^{\lambda y}(1 + \max(y, 0)) \sum_{p=N}^{+\infty} \mathrm{e}^{-\lambda p}(1+p).$$

Taking the limit as $N \to +\infty$, proves the lemma. □

We now introduce the following random variable: for any $j \in \mathbb{X}$, $u \in \mathbb{R}$, $l \geqslant 1$ and $m \geqslant 1$

$$r_\infty^{(l,m)}(j, u) := 1 - f_{X_1} \circ \cdots \circ f_{X_l}\left(\left[1 - \mathrm{e}^{-S_l}\, \mathrm{e}^{u}\, q_m^*(j)\right]^+\right) \in [0, 1],$$

where, as in (4.3) and (4.5), for any $m \geqslant 1$,

$$q_m^*(j) := \mathrm{e}^{S_m^*}\left(1 - f_{X_m^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)\right) = \left[\frac{1}{1 - f_j(0)} + \sum_{k=1}^{n} \mathrm{e}^{-S_k^*}\, \eta_k^*(j)\right]^{-1}$$

and as in (4.4), for any $k \geqslant 2$,

$$\eta_k^*(j) := g_{X_k^*}\left(f_{X_{k-1}^*} \circ \cdots \circ f_{X_1^*} \circ f_j(0)\right) \qquad \text{and} \qquad \eta_1^* := g_{X_1^*}\left(f_j(0)\right).$$

For any $(i, y) \in \mathrm{supp}(\tilde{V}_\lambda)$ and $(j, z) \in \mathrm{supp}(\tilde{V}_\lambda^*)$, let $\tilde{\mathbb{P}}_{i,y,j,z}^+$ and $\tilde{\mathbb{E}}_{i,y,j,z}^+$ be, respectively, the probability and its associated expectation defined for any $n \geqslant 1$ and any function g: $\mathbb{X}^{l,m} \to \mathbb{C}$ by

$$\tilde{\mathbb{E}}_{i,y,j,z}^+\left(g\left(X_1, \ldots, X_l, X_m^*, \ldots, X_1^*\right)\right) = \tilde{\mathbb{E}}_{i,j}\Bigg(g\left(X_1, \ldots, X_l, X_m^*, \ldots, X_1^*\right) \times$$

$$\frac{\tilde{V}_\lambda\left(X_l, y + S_l\right)}{\tilde{V}_\lambda(i, y)} \frac{\tilde{V}_\lambda^*\left(X_m^*, z + S_m^*\right)}{\tilde{V}_\lambda^*(j, z)}\,;\, \tau_y > l\,,\, \tau_z^* > m\Bigg). \tag{6.8}$$

For any $j \in \mathbb{X}$ let $z_0(j) \in \mathbb{R}$ be the unique real such that $(j, z) \in \mathrm{supp}\left(\tilde{V}_\lambda^*\right)$ for any $z > z_0$ and $(j, z) \notin \mathrm{supp}\left(\tilde{V}_\lambda^*\right)$ for any $z < z_0$ (see [13] for details on the domain of positivity of the harmonic function). Set $z_0(j)^+ = \max\{z_0(j), 0\}$.

Lemma 6.4. *Assume that the conditions of Theorem 1.4 are satisfied. For any $j \in \mathbb{X}$, $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$, $l \geqslant 1$ and $m \geqslant 1$,*

$$\begin{aligned}&\lim_{n\to+\infty} n^{3/2}\tilde{\mathbb{E}}_i\left(r_n^{(l,m)}(j)\,\mathrm{e}^{-\lambda S_n}\,;\,X_{n+1}=j\,,\,\tau_y>n\right)\\&=\frac{2}{\sqrt{2\pi}\sigma^3}\,\mathrm{e}^{\lambda y}\int_{z_0(j)^+}^{+\infty}\mathrm{e}^{-\lambda z}\,\tilde{\mathbb{E}}^+_{i,y,j,z}\left(r_\infty^{(l,m)}(j,z-y)\right)\tilde{V}_\lambda(i,y)\tilde{V}^*_\lambda(j,z)\,\mathrm{d}z\tilde{\boldsymbol{\nu}}_\lambda(j).\end{aligned}$$

Proof. Fix $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$, $j \in \mathbb{X}$, $l \geqslant 1$ and $m \geqslant 1$ and let g be a function $\mathbb{X}^{l+m} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$\begin{aligned}g(i_1,\dots,i_l,i_{n-m+1},\dots,i_n,z) &= \mathrm{e}^{\lambda y}\,\mathrm{e}^{-\lambda z}\,\mathbb{1}_{\{z\geqslant 0\}}\tilde{\mathbf{P}}_\lambda(i_n,j)\\&\quad\times\Big[1-f_{i_1}\circ\cdots\circ f_{i_l}\Big(\Big[1-\mathrm{e}^{z-y-\rho(i_n)-\cdots-\rho(i_{n-m+1})-\rho(i_l)-\cdots-\rho(i_1)}\\&\qquad\times\left(1-f_{i_{n-m+1}}\circ\cdots\circ f_{i_n}\circ f_j(0)\right)\Big]^+\Big)\Big],\end{aligned}$$

for all $(i_1,\dots,i_l,i_{n-m+1},\dots,i_n,z) \in \mathbb{X}^{l+m}\times\mathbb{R}$ and note that on $\{\tau_y > n\}$,

$$g(X_1,\dots,X_l,X_{n-m+1},\dots,X_n,y+S_n) = r_n^{(l,m)}(j)\,\mathrm{e}^{-\lambda S_n}\,\tilde{\mathbf{P}}_\lambda(i_n,j).$$

Observe also that since $0 \leqslant g(i_1,\dots,i_l,i_{n-m+1},\dots,i_n,z) \leqslant \mathrm{e}^{\lambda y}\,\mathrm{e}^{-\lambda z}\,\mathbb{1}_{\{z\geqslant 0\}}$, the function g belongs to the set, say $\mathscr{C}^+\left(\mathbb{X}^{l+m}\times\mathbb{R}_+\right)$, of non-negative function g: $\mathbb{X}^{l+m}\times\mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following properties:

- for any $(i_1,\dots,i_{l+m}) \in \mathbb{X}^{l+m}$, the function $z \mapsto g(i_1,\dots,i_{l+m},z)$ is continuous,
- $\max_{i_1,\dots i_{l+m}\in\mathbb{X}} \sup_{z\geqslant 0} g(i_1,\dots,i_{l+m},z)(1+z)^{2+\varepsilon} < +\infty$ for some $\varepsilon > 0$.

Therefore, by the Markov property and Proposition 2.10, we obtain that

$$\begin{aligned}I_0 &:= \lim_{n\to+\infty} n^{3/2}\tilde{\mathbb{E}}_i\left(r_n^{(l,m)}(j)\,\mathrm{e}^{-\lambda S_n}\,;\,X_{n+1}=j\,,\,\tau_y>n\right)\\&= \lim_{n\to+\infty} n^{3/2}\tilde{\mathbb{E}}_i\left(g\left(X_1,\dots,X_l,X_{n-m+1},\dots,X_n,y+S_n\right)\,;\,\tau_y>n\right)\\&= \frac{2}{\sqrt{2\pi}\sigma^3}\int_0^{+\infty}\mathrm{e}^{-\lambda(z-y)}\sum_{j'\in\mathbb{X}}\tilde{\mathbb{E}}_{i,j'}\Big(r_\infty^{(l,m)}(j,z-y)\tilde{\mathbf{P}}_\lambda(X_1^*,j)\\&\qquad\times\tilde{V}_\lambda\left(X_l,y+S_l\right)\tilde{V}^*_\lambda\left(X^*_m,z+S^*_m\right)\,;\,\tau_y>l\,,\,\tau^*_z>m\Big)\tilde{\boldsymbol{\nu}}_\lambda(j')\,\mathrm{d}z.\end{aligned}$$

Since $\tilde{\boldsymbol{\nu}}_\lambda$ is $\tilde{\mathbf{P}}^*_\lambda$-invariant, we write

$$\begin{aligned}I_0 &= \frac{2}{\sqrt{2\pi}\sigma^3}\int_0^{+\infty}\mathrm{e}^{-\lambda(z-y)}\sum_{j_1\in\mathbb{X}}\tilde{\mathbf{P}}_\lambda(j_1,j)\tilde{\boldsymbol{\nu}}_\lambda(j_1)\tilde{\mathbb{E}}_i\Big(r_\infty^{(l,m)}(j,z-y)\\&\qquad\times\tilde{V}_\lambda\left(X_l,y+S_l\right)\tilde{V}^*_\lambda\left(X^*_m,z+S^*_m\right)\,;\,\tau_y>l\,,\,\tau^*_z>m\,\Big|\,X^*_1=j_1\,\Big)\,\mathrm{d}z.\end{aligned}$$

Using the definition of $\tilde{\mathbf{P}}^*_\lambda$ in (6.4), we have

$$I_0 = \frac{2}{\sqrt{2\pi}\sigma^3}\int_0^{+\infty} \mathrm{e}^{-\lambda(z-y)}\,\tilde{\boldsymbol{\nu}}_\lambda(j)\tilde{\mathbb{E}}_{i,j}\Big(r_\infty^{(l,m)}(j,z-y)\tilde{V}_\lambda\left(X_l, y+S_l\right) \times \tilde{V}^*_\lambda\left(X^*_m, z+S^*_m\right)\,;\,\tau_y > l\,,\,\tau^*_z > m\Big)\,\mathrm{d}z.$$

Now, note that when $(j,z) \notin \operatorname{supp}\left(\tilde{V}^*_\lambda\right)$, using the point 1 of Proposition 2.6,

$$\begin{aligned}
&\tilde{\mathbb{E}}_{i,j}\left(r_\infty^{(l,m)}(j,z-y)\tilde{V}_\lambda\left(X_l,y+S_l\right)\tilde{V}^*_\lambda\left(X^*_m,z+S^*_m\right)\,;\,\tau_y>l\,,\,\tau^*_z>m\right)\\
&\quad\leqslant \tilde{\mathbb{E}}_i\left(\tilde{V}_\lambda\left(X_l,y+S_l\right)\,;\,\tau_y>l\right)\tilde{\mathbb{E}}^*_j\left(\tilde{V}^*_\lambda\left(X^*_m,z+S^*_m\right)\,;\,\tau^*_z>m\right)\\
&\quad= \tilde{V}_\lambda(i,y)\tilde{V}^*_\lambda(j,z) = 0.
\end{aligned}$$

Together with (6.8), it proves the lemma. □

Consider for any $l \geqslant 1$, $j \in \mathbb{X}$ and $u \in \mathbb{R}$,

$$r_\infty^{(l,\infty)}(j,u) = 1 - f_{X_1}\circ\cdots\circ f_{X_l}\left(\left[1-\mathrm{e}^{-S_l}\,\mathrm{e}^u\,q^*_\infty(j)\right]^+\right) \in [0,1], \tag{6.9}$$

where as in (4.9),

$$q^*_\infty(j) = \left[\frac{1}{1-f_j(0)} + \sum_{k=1}^{\infty}\mathrm{e}^{-S^*_k}\,\eta^*_k(j)\right]^{-1}.$$

Lemma 6.5. *Assume that the conditions of Theorem 1.4 are satisfied. For any $u \in \mathbb{R}$, $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$, $(j,z) \in \operatorname{supp}\left(\tilde{V}^*_\lambda\right)$ and $l \geqslant 1$,*

$$\lim_{m\to+\infty}\tilde{\mathbb{E}}^+_{i,y,j,z}\left(\left|r_\infty^{(l,m)}(j,u) - r_\infty^{(l,\infty)}(j,u)\right|\right) = 0.$$

Proof. Fix $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$, $(j,z) \in \operatorname{supp}\left(\tilde{V}^*_\lambda\right)$, $l \geqslant 1$ and $u \in \mathbb{R}$. By the convexity of $f_{1,l}$, for any $m \geqslant 1$, we have $\tilde{\mathbb{P}}^+_{i,y,j,z}$ a.s.,

$$\begin{aligned}
&\left|r_\infty^{(l,m)}(j,u) - r_\infty^{(l,\infty)}(j,u)\right|\\
&\leqslant \left(f_{X_1}\circ\cdots\circ f_{X_l}\right)'(1)\left|\left[1-\mathrm{e}^{-S_l}\,\mathrm{e}^u\,q^*_m(j)\right]^+ - \left[1-\mathrm{e}^{-S_l}\,\mathrm{e}^u\,q^*_\infty(j)\right]^+\right|\\
&\leqslant \mathrm{e}^{S_l}\left|\mathrm{e}^{-S_l}\,\mathrm{e}^u\,q^*_m(j) - \mathrm{e}^{-S_l}\,\mathrm{e}^u\,q^*_\infty(j)\right|\\
&= \mathrm{e}^u\left|q^*_m(j)q^*_\infty(j)\right|\left|(q^*_\infty(j))^{-1} - (q^*_m(j))^{-1}\right|.
\end{aligned}$$

Moreover, for any $m \geqslant 1$,

$$q^*_m(j) = \left[\frac{1}{1-f_j(0)} + \sum_{k=1}^{m}\mathrm{e}^{-S^*_k}\,\eta^*_k(j)\right]^{-1} \in (0,1],$$

$$q^*_\infty(j) = \left[\frac{1}{1-f_j(0)} + \sum_{k=1}^{\infty}\mathrm{e}^{-S^*_k}\,\eta^*_k(j)\right]^{-1} \in [0,1]$$

and by Lemma 2.3, for any $k \geqslant 1$,

$$0 \leqslant \eta_k^*(j) \leqslant \eta. \tag{6.10}$$

Therefore,

$$\left| r_\infty^{(l,m)}(j,u) - r_\infty^{(l,\infty)}(j,u) \right| \leqslant \mathrm{e}^u \, \eta \sum_{k=m+1}^{+\infty} \mathrm{e}^{-S_k^*}.$$

Using Lemma 2.13 and the Lebesgue dominated convergence theorem,

$$\tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,m)}(j,u) - r_\infty^{(l,\infty)}(j,u) \right| \right) \leqslant \mathrm{e}^u \, \eta \sum_{k=m+1}^{+\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\mathrm{e}^{-S_k^*} \right).$$

By Lemma 2.13, we obtain the desired assertion. □

For any $l \geqslant 1$, $j \in \mathbb{X}$ and $u \in \mathbb{R}$, set

$$s_l(j,u) = \left[1 - \mathrm{e}^{-S_l} \, \mathrm{e}^u \, q_\infty^*(j) \right]^+. \tag{6.11}$$

Note that, by Lemma 2.13, $(q_\infty^*(j))^{-1}$ is integrable and so finite a.s. (see (5.6)). Therefore $s_l(j,u) \in [0,1)$. In addition, by the convexity of $f_{X_{l+1}}$, we have for any $j \in \mathbb{X}$, $u \in \mathbb{R}$ and $l \geqslant 1$,

$$\begin{aligned} f_{X_{l+1}}(s_{l+1}(j,u)) &\geqslant 1 - f'_{X_{l+1}}(1) \, (1 - s_{l+1}(j,u)) \\ &\geqslant 1 - \mathrm{e}^{\rho(X_{l+1})} \, \mathrm{e}^{-S_{l+1}} \, \mathrm{e}^u \, q_\infty^*(j) = 1 - \mathrm{e}^{-S_l} \, \mathrm{e}^u \, q_\infty^*(j). \end{aligned}$$

Since $f_{X_{l+1}}$ is non-negative on $[0,1]$, we see that $f_{X_{l+1}}(s_{l+1}(j,u)) \geqslant s_l(j,u)$ and so for any $k \geqslant 1$, $(f_{k+1,l}(s_l(j,u)))_{l \geqslant k}$ is non-decreasing and bounded by 1. Using the continuity of g_{X_k} and (2.12), we deduce that $(\eta_{k,l}(s_l(j,u)))_{l \geqslant k}$ converges and we denote for any $k \geqslant 1$,

$$\eta_{k,\infty}(j,u) := \lim_{l \to +\infty} \eta_{k,l}(s_l(j,u)). \tag{6.12}$$

Moreover, by Lemma 2.3, we have for any $k \geqslant 1$, $l \geqslant k$ and $u \in \mathbb{R}$,

$$0 \leqslant \eta_{k,l}(s_l(j,u)) \leqslant \eta \qquad \text{and} \qquad 0 \leqslant \eta_{k,\infty}(j,u) \leqslant \eta. \tag{6.13}$$

For any $j \in \mathbb{X}$ and $u \in \mathbb{R}$, set

$$r_\infty(j,u) := \left[\frac{\mathrm{e}^{-u}}{q_\infty^*(j)} + \sum_{k=0}^{+\infty} \mathrm{e}^{-S_k} \, \eta_{k+1,\infty}(j,u) \right]^{-1}.$$

Lemma 6.6. *Assume that the conditions of Theorem 1.4 are satisfied. For any $u \in \mathbb{R}$, $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$ and $(j,z) \in \operatorname{supp}\left(\tilde{V}_\lambda^*\right)$,*

$$\lim_{l \to +\infty} \tilde{\mathbb{E}}_{i,y,j,z}^+ \left(\left| r_\infty^{(l,\infty)}(j,u) - r_\infty(j,u) \right| \right) = 0.$$

Proof. Fix $(i,y) \in \operatorname{supp}\left(\tilde{V}_\lambda\right)$, $(j,z) \in \operatorname{supp}\left(\tilde{V}^*_\lambda\right)$ and $u \in \mathbb{R}$. By (6.9), Lemma 2.2 and (6.11), we have

$$\left(r_\infty^{(l,\infty)}(j,u)\right)^{-1} = \frac{\mathrm{e}^{-S_l}}{1-s_l(j,u)} + \sum_{k=0}^{l-1} \mathrm{e}^{-S_k}\, \eta_{k+1,l}(s_l(j,u)).$$

So, for any $p \geqslant 1$ and $l \geqslant p$, using (6.13),

$$\begin{aligned}\left|\left(r_\infty^{(l,\infty)}(j,u)\right)^{-1} - r_\infty(j,u)^{-1}\right| &\leqslant \sum_{k=0}^{p} \mathrm{e}^{-S_k}\, |\eta_{k+1,l}(s_l(j,u)) - \eta_{k+1,\infty}(j,u)| \\ &\quad + \left|\frac{\mathrm{e}^{-u}}{q^*_\infty(j)} - \frac{\mathrm{e}^{-S_l}}{1-s_l(j,u)}\right| + 2\eta \sum_{k=p+1}^{+\infty} \mathrm{e}^{-S_k}\,.\end{aligned}$$

Therefore,

$$\begin{aligned} I_0 &:= \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\left|\left(r_\infty^{(l,\infty)}(j,u)\right)^{-1} - r_\infty(j,u)^{-1}\right|\right) \\ &\leqslant \sum_{k=0}^{p} \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\mathrm{e}^{-S_k}\, |\eta_{k+1,l}(s_l(j,u)) - \eta_{k+1,\infty}(j,u)|\right) \\ &\quad + \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\left|\frac{\mathrm{e}^{-u}}{q^*_\infty(j)} - \mathrm{e}^{-S_l}\right| ;\, \mathrm{e}^{-S_l} > \frac{\mathrm{e}^{-u}}{q^*_\infty(j)}\right) + 2\eta \tilde{\mathbb{E}}^+_{i,y}\left(\sum_{k=p+1}^{+\infty} \mathrm{e}^{-S_k}\right), \end{aligned}$$

where $\tilde{\mathbb{P}}^+_{i,y}$ is the marginal law of $\tilde{\mathbb{P}}^+_{i,y,j,z}$ on $\sigma\left(X_n\,,\, n \geqslant 1\right)$. Using Lemma 2.13 and the Lebesgue dominated convergence theorem,

$$\begin{aligned} I_0 &\leqslant \tilde{\mathbb{E}}^+_{i,y}\left(\mathrm{e}^{-S_l}\right) + \sum_{k=0}^{p} \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\mathrm{e}^{-S_k}\, |\eta_{k+1,l}\left(s_l(j,u)\right) - \eta_{k+1,\infty}(j,u)|\right) \\ &\qquad + 2\eta \sum_{k=p+1}^{+\infty} \tilde{\mathbb{E}}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) \\ &\leqslant \frac{c\left(1+\max(y,0)\right)\mathrm{e}^{y}}{V(i,y)}\left(\frac{1}{l^{3/2}} + \sum_{k=p+1}^{+\infty} \frac{\eta}{k^{3/2}}\right) \\ &\qquad + \sum_{k=0}^{p} \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\mathrm{e}^{-S_k}\, |\eta_{k+1,l}(s_l(j,u)) - \eta_{k+1,\infty}(j,u)|\right). \end{aligned}$$

Since $|\eta_{k+1,l}(s_l(j,u)) - \eta_{k+1,\infty}(j,u)| \leqslant 2\eta$, by the Lebesgue dominated convergence theorem and (6.12)

$$\limsup_{l\to+\infty} I_0 \leqslant \frac{c\left(1+\max(y,0)\right)\mathrm{e}^{y}}{V(i,y)} \sum_{k=p+1}^{+\infty} \frac{\eta}{k^{3/2}}.$$

Letting $p \to +\infty$, we obtain that $\lim_{l\to+\infty} I_0 = 0$. Moreover, by (6.9) for any $l \geqslant 1$, $r_\infty^{(l,\infty)}(j,u) \in [0,1]$. In the same manner as we proved (3.7), we have also

$$r_\infty(j,u) \leqslant 1.$$

Consequently,

$$\lim_{l\to+\infty} \tilde{\mathbb{E}}^+_{i,y,j,z}\left(\left|r^{(l,\infty)}_\infty(j,u) - r_\infty(j,u)\right|\right) \leqslant \lim_{l\to+\infty} I_0 = 0.$$

□

We now consider the function

$$U(i,y,j) := \frac{2}{\sqrt{2\pi}\sigma^3}\frac{v_\lambda(i)}{v_\lambda(j)}\,\mathrm{e}^{\lambda(y-\rho(j))} \times \int_{z_0(j)^+}^{+\infty} \mathrm{e}^{-\lambda z}\,\tilde{\mathbb{E}}^+_{i,y,j,z}\left(r_\infty(j,z-y)\right)\tilde{V}_\lambda(i,y)\tilde{V}^*_\lambda(j,z)\,\mathrm{d}z\tilde{\boldsymbol{\nu}}_\lambda(j).$$

Using (6.10), (6.13) and Lemma 2.13, for any $(i,y)\in\operatorname{supp}\left(\tilde{V}_\lambda\right)$, $(j,z)\in\operatorname{supp}\left(\tilde{V}^*_\lambda\right)$ and $u\in\mathbb{R}$,

$$\tilde{\mathbb{E}}^+_{i,y,j,z}\left(r_\infty(j,u)^{-1}\right) \leqslant \mathrm{e}^{-u}\left(\frac{1}{1-f_j(0)} + \eta\tilde{\mathbb{E}}^+_{i,y,j,z}\left(\sum_{k=1}^{+\infty}\mathrm{e}^{-S^*_k}\right)\right) + \eta\tilde{\mathbb{E}}^+_{i,y,j,z}\left(\sum_{k=1}^{+\infty}\mathrm{e}^{-S_k}\right) < +\infty.$$

So, $r_\infty(j,u)>0$ $\tilde{\mathbb{P}}^+_{i,y,j,z}$-a.s. and therefore, for any $(i,y)\in\operatorname{supp}\left(\tilde{V}_\lambda\right)$, $j\in\mathbb{X}$,

$$U(i,y,j) > 0. \tag{6.14}$$

Lemma 6.7. *Assume that the conditions of Theorem 1.4 are satisfied. For any $(i,y)\in\operatorname{supp}\left(\tilde{V}_\lambda\right)$ and $j\in\mathbb{X}$, we have*

$$\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=i\,,\,\tau_y>n\right) \underset{n\to+\infty}{\sim} \frac{U(i,y,j)k(\lambda)^{n+1}}{(n+1)^{3/2}}.$$

Proof. Fix $(i,y)\in\operatorname{supp}\left(\tilde{V}_\lambda\right)$ and $j\in\mathbb{X}$. By (6.3), for any $n\geqslant 1$,

$$I_0 := \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=i\,,\,\tau_y>n\right) = \frac{v_\lambda(i)\,\mathrm{e}^{-\lambda\rho(j)}}{v_\lambda(j)}(n+1)^{3/2}\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-\lambda S_n}\,q_{n+1}\,;\,X_{n+1}=j\,,\,\tau_y>n\right).$$

Using Lemmas 6.3 and 6.4,

$$\begin{aligned}\lim_{n\to+\infty} I_0 &= \lim_{(l,m)\to+\infty}\lim_{n\to+\infty}\frac{v_\lambda(i)\,\mathrm{e}^{-\lambda\rho(j)}}{v_\lambda(j)}(n+1)^{3/2} \times \tilde{\mathbb{E}}_i\left(r^{(l,m)}_n(j)\,\mathrm{e}^{-\lambda S_n}\,;\,X_{n+1}=j\,,\,\tau_y>n\right)\\ &= \lim_{(l,m)\to+\infty}\frac{2v_\lambda(i)}{\sqrt{2\pi}\sigma^3 v_\lambda(j)}\,\mathrm{e}^{\lambda(y-\rho(j))}\int_{z_0(j)^+}^{+\infty}\mathrm{e}^{-\lambda z}\,\tilde{\mathbb{E}}^+_{i,y,j,z}\left(r^{(l,m)}_\infty(j,z-y)\right) \times \tilde{V}_\lambda(i,y)\tilde{V}^*_\lambda(j,z)\,\mathrm{d}z\tilde{\boldsymbol{\nu}}_\lambda(j).\end{aligned}$$

Since for any $l \geqslant 1$, $m \geqslant 1$ and $u \in \mathbb{R}$, $r_\infty^{(l,m)}(j,u) \leqslant 1$, by the Lebesgue dominated convergence theorem and Lemmas 6.5 and 6.6,

$$\lim_{n\to+\infty} I_0 = \frac{2v_\lambda(i)}{\sqrt{2\pi}\sigma^3 v_\lambda(j)}\,\mathrm{e}^{\lambda(y-\rho(j))}\int_{z_0(j)^+}^{+\infty}\mathrm{e}^{-\lambda z}\,\tilde{\mathbb{E}}^+_{i,y,j,z}\left(r_\infty(j,z-y)\right) \times \tilde{V}_\lambda(i,y)\tilde{V}^*_\lambda(j,z)\,\mathrm{d}z\tilde{\boldsymbol{\nu}}_\lambda(j).$$

Denoting the right-hand side by $U(i,y,j)$ ends the proof. □

Proof of Theorem 1.4. We use arguments similar to those of the proof of Lemma 5.9. Fix $(i,j) \in \mathbb{X}^2$. For any $y \in \mathbb{R}$ and $n \geqslant 1$, let

$$I_0 := \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=j\right)$$

and

$$\begin{aligned} I_1 &:= I_0 - \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=j\,,\,\tau_y>n\right) \\ &= \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_n\left(f_j(0)\right)\,;\,X_{n+1}=j\,,\,\tau_y\leqslant n\right). \end{aligned} \tag{6.15}$$

Since $f_j(0) \geqslant 0$, it is easy to see that $q_n\left(f_j(0)\right) \leqslant q_n(0)$. Using the fact that $(q_k(0))_{k\geqslant 1}$ is non-increasing and Lemma 2.2, it holds

$$q_n\left(f_j(0)\right) \leqslant \min_{1\leqslant k\leqslant n} q_k(0) \leqslant \mathrm{e}^{\min_{1\leqslant k\leqslant n} S_k}.$$

Therefore, as in (3.11),

$$\begin{aligned} I_1 &\leqslant \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(\mathrm{e}^{\min_{1\leqslant k\leqslant n} S_k}\,;\,X_{n+1}=j\,,\,\tau_y\leqslant n\right) \\ &\leqslant \frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\,\mathrm{e}^{-y}\sum_{p=0}^{+\infty}\mathrm{e}^{-p}\,\mathbb{P}_i\left(X_{n+1}=j\,,\,\tau_{y+p+1}>n\right). \end{aligned}$$

By (6.2),

$$\begin{aligned} I_1 &\leqslant \frac{(n+1)^{3/2}}{n^{3/2}}\frac{v_\lambda(i)}{v_\lambda(j)}\,\mathrm{e}^{-y-\lambda\rho(j)}\sum_{p=0}^{+\infty}\mathrm{e}^{-p}\,\tilde{\mathbb{E}}_i\left(\mathrm{e}^{-\lambda S_n}\,;\,\tau_{y+p+1}>n\right) \\ &\leqslant c\frac{v_\lambda(i)}{v_\lambda(j)}\,\mathrm{e}^{-y-\lambda\rho(j)}\sum_{p=0}^{+\infty}\mathrm{e}^{-p}\sum_{l=0}^{+\infty}\mathrm{e}^{\lambda(y+p+1)}\,\mathrm{e}^{-\lambda l} \\ &\qquad\qquad \times n^{3/2}\tilde{\mathbb{P}}_i\left(y+p+1+S_n\in[l,l+1]\,;\,\tau_{y+p+1}>n\right). \end{aligned}$$

Using the point 2 of Proposition 2.9,

$$I_1 \leqslant c\frac{v_\lambda(i)\,\mathrm{e}^{-\lambda\rho(j)}}{v_\lambda(j)}\,\mathrm{e}^{-(1-\lambda)y}\sum_{p=0}^{+\infty}\mathrm{e}^{-(1-\lambda)p}\sum_{l=0}^{+\infty}\mathrm{e}^{-\lambda l}(1+\max(y+p+1,0))(1+l)$$
$$\leqslant c\frac{v_\lambda(i)\,\mathrm{e}^{-\lambda\rho(j)}}{v_\lambda(j)}\,\mathrm{e}^{-(1-\lambda)y}(1+\max(y,0)).$$

Moreover, there exists $y_0(i)\in\mathbb{R}$ such that, for any $y\geqslant y_0(i)$ it holds $(i,y)\in\operatorname{supp}\left(\tilde{V}_\lambda\right)$. Using (6.15) and Lemma 6.7, we obtain that, for any $y\geqslant y_0(i)$,

$$U(i,y,j)\leqslant\liminf_{n\to+\infty} I_0\leqslant\limsup_{n\to+\infty} I_0$$
$$\leqslant U(i,y,j)+c\frac{v_\lambda(i)\,\mathrm{e}^{-\lambda\rho(j)}}{v_\lambda(j)}\,\mathrm{e}^{-(1-\lambda)y}(1+\max(y,0)). \tag{6.16}$$

This proves that $\limsup_{n\to+\infty} I_0$ is a finite real which does not depend on y and so $y\mapsto U(i,y,j)$ is a bounded function. Moreover, by Lemma 6.7,

$$U(i,y,j)=\lim_{n\to\infty}\frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=j\,,\,\tau_y>n\right)$$

and so $y\mapsto U(i,y,j)$ is non-decreasing. Let u be its limit:

$$u(i,j):=\lim_{y\to+\infty}U(i,y,j)\in\mathbb{R}.$$

By (6.14), for any $y\geqslant y_0(i)$,

$$u(i,j)\geqslant U(i,y,j)>0.$$

Taking the limit as $y\to+\infty$ in (6.16),

$$\lim_{n\to+\infty} I_0=u(i,j).$$

Finally, by (2.2),

$$\lim_{n\to+\infty}\frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{P}_i\left(Z_{n+1}>0\,,\,X_{n+1}=j\right)$$
$$=\lim_{n\to+\infty}\frac{(n+1)^{3/2}}{k(\lambda)^{n+1}}\mathbb{E}_i\left(q_{n+1}\,;\,X_{n+1}=j\right)=u(i,j).$$

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