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To cite this version:

Raphaele Herbin, Jean-Claude Latché. A class of staggered schemes for the compressible Euler equations. NMA, 2018, Borovets, Bulgaria. hal-02019553

HAL Id: hal-02019553

https://hal.science/hal-02019553v1

Submitted on 14 Feb 2019

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A class of staggered schemes for the compressible Euler equations

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Abstract. We present a class of numerical schemes for the solution of the Euler equations; these schemes are based on staggered discretizations and work either on structured meshes or on general simplicial or tetrahedral/hexahedral meshes. The time discretization is performed by fractional-step algorithms, either based on semi-implicit pressure correction techniques or segregated in such a way that only explicit steps are involved (referred to hereafter as "explicit" variants). These schemes solve the internal energy balance, with corrective terms to ensure the correct capture of shocks, and, more generally, the consistency in the Lax-Wendroff sense. To keep the density, the internal energy and the pressure positive, positivity-preserving convection operators for the mass and internal energy balance equations are designed, using upwinding with respect of the material velocity only. The construction of the fluxes thus does not need any Riemann or approximate Riemann solver, and yields particularly efficient algorithms. The stability is obtained without restriction on the time step for the pressure correction time-stepping and under a CFL-like condition for explicit variants: preservation of the integral of the total energy over the computational domain, entropy estimates and positivity of the density and the internal energy.

Keywords: Euler equations · staggered schemes.

1 Introduction

We address in this paper the solution of the Euler equations for an ideal gas, which read:

$$\partial_t \rho + \mathrm{div}(\rho\, \boldsymbol{u}) = 0, \tag{1a}$$

$$\partial_t (\rho\, \boldsymbol{u}) + \mathrm{div}(\rho\, \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \tag{1b}$$

$$\partial_t (\rho\, E) + \mathrm{div}(\rho\, E\, \boldsymbol{u}) + \mathrm{div}(p\, \boldsymbol{u}) = 0, \tag{1c}$$

$$p = (\gamma - 1)\, \rho\, e, \qquad E = \frac{1}{2}|\boldsymbol{u}|^2 + e, \tag{1d}$$

where t stands for the time, ρ, $\boldsymbol{u}$, p, E and e are the density, velocity, pressure, total energy and internal energy respectively, and $\gamma > 1$ is a coefficient specific

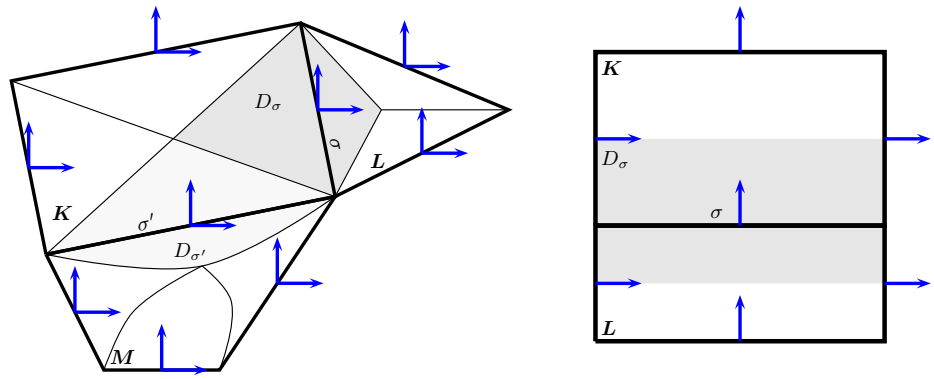

Fig. 1. Meshes and unknowns – Left: unstructured discretizations (the present sketch illustrates the possibility, implemented in our software CALIF^{3}S [2], of mixing simplicial and quadrangular cells); scalars variables are associated to the primal cells (here K, L and M) while velocity vectors are associated to the faces (here, σ and σ') or, equivalently, to dual cells (here, D_σ and $D_{\sigma'}$). – Right: MAC discretization; scalars variables are associated to the primal cells and each face is associated to the component of the velocity normal to the face.

to the considered fluid. The problem is supposed to be posed over $\Omega \times (0, T)$, where Ω is an open bounded connected subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, and $(0, T)$ is a finite time interval. System (1) is complemented by initial conditions for ρ, e and $\boldsymbol{u}$, let us say ρ_0, e_0 and $\boldsymbol{u}_0$ respectively, with $\rho_0 > 0$ and $e_0 > 0$, and by suitable boundary conditions (not specified for short).

Finite volume schemes for the solution of System (1), and, more generally speaking, of hyperbolic problems, generally use a collocated arrangement of the unknowns, all of them being associated to the cell centers, and apply a Godunov-like technique for the computation of the fluxes at the cells faces: the face is seen as a discontinuity line for the beginning-of-time-step numerical solution, supposed to be constant in the two adjacent cells; a solution, either exact or approximate, of the so-posed Riemann problem is constructed and the numerical solution is advanced in time by projection of this construction on piecewise constant functions (see *e.g.* [17, 1] for the development of such solvers). Thanks to the properties of the projection, at least for exact Riemann solvers, application of this process to the Euler equations yields consistant schemes which preserve the non-negativity of the density and the internal energy and, for first-order variants, satisfy an entropy inequality. The price to pay is the computational cost of the evaluation of the fluxes, and the fact that this issue is intricate enough to put almost out of reach implicit-in-time formulations, which would allow to relax CFL time step constraints. In addition, preserving the accuracy for low Mach number flows is a difficult task (see *e.g.* [9] and references herein).

The aim of this paper is to review recent developements performed to derive a class of schemes following a different route. The space discretization is staggered: scalar variables are associated to cell centers while the velocity is associated to the faces, or, equivalently, to staggered mesh(es). Two different space discretizations may be considered: either the so-called Marker-And-Cell (MAC) scheme for structured grids [11] or, for general meshes, a space discretization using degrees of freedom similar to low-order Rannacher-Turek [16] or Crouzeix-Raviart [3] finite elements (see Figure 1). With this space discretization, the use of Riemann solvers seems difficult (scalar unknowns and velocities may still be considered as piecewise constant functions, but not associated to the same partition of the computational domain). The positivity of the internal energy is thus ensured by a non-standard argument: the internal energy balance is discretized instead of the actual (total) energy balance (1c) by a positivity-preserving scheme. This leads to consistency problems, which are the main difficulty faced here. We develop two time discretizations: a pressure correction technique and a fractional step scheme involving only explicit steps. We finally obtain a class of schemes which offer many interesting properties: both the density and internal energy positivity are preserved, unconditionnally for the pressure correction scheme and under CFL-like conditions for the (quasi) explicit variant, and the integral of the total energy on the computational domain is conserved (which yields a stability result); the construction of the fluxes is very simple (essentially based on standard upwinding techniques of the convection operators with respect to the material velocity); finally, the space approximation, the fluxes and the choice of the internal energy balance are consistent with usual discretizations of quasi-incompressible flows, so the pressure correction scheme is asymptotic preserving by construction in the limit of vanishing low Mach number flows. In addition, an entropy estimate is obtained for the pressure correction scheme, while only a weak entropy estimate seems to hold for the explicit variant. The development of this class of schemes started ten years ago, and we review here the essential arguments; details may be found in [12, 8, 13].

The use of staggered discretization for compressible flows began with the very first papers on the MAC scheme [10], and has been the subject of a wide litterature (see [18] for a textbook and references in [12, 8, 13]). However, the use of the internal energy equation associated to a consistency correction seems to be restricted to the context of Lagrangian approaches, up to a very recent work implementing a Lagrange-remap technique on staggered meshes [4].

2 A pressure correction scheme

2.1 A basic lemma

Let ρ and $\boldsymbol{u}$ be regular scalar and vector-valued functions, respectively, such that

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0.$$

Let z be a regular scalar function. Then we have:

$$\begin{aligned}\mathcal{C}(z) = \partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u}) &= \rho\big(\partial_t z + \boldsymbol{u}\cdot\boldsymbol{\nabla} z\big) + z\big(\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u})\big)\\ &= \rho\big(\partial_t z + \boldsymbol{u}\cdot\boldsymbol{\nabla} z\big).\end{aligned} \tag{2}$$

Let φ be a regular real function. Then:

$$\varphi'(z)\ \mathcal{C}(z) = \varphi'(z)\,\rho\,\big(\partial_t z + \boldsymbol{u}\cdot\boldsymbol{\nabla} z\big) = \rho\,\big(\partial_t\varphi(z) + \boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi(z)\big).$$

Now, reversing the computation performed in Relation (2) with $\varphi(z)$ instead of z, we get:

$$\varphi'(z)\ \mathcal{C}(z) = \partial_t\big(\rho\varphi(z)\big) + \mathrm{div}\big(\rho\varphi(z)\boldsymbol{u}\big). \tag{3}$$

The following lemma states a time semi-discrete version of this computation.

Lemma 1. *Let ρ^n, ρ^{n+1}, z^n and z^{n+1} be regular scalar functions, let $\boldsymbol{u}$ be a regular vector-valued function and let φ be a twice-differentiable real function. Let us suppose that*

$$\frac{1}{\delta t}\ (\rho^{n+1} - \rho^n) + \mathrm{div}(\rho^{n+1}\boldsymbol{u}) = 0, \tag{4}$$

with δt a positive real number. Then

$$\begin{aligned}&\varphi'(z^{n+1})\ \Big[\frac{1}{\delta t}\ (\rho^{n+1}z^{n+1} - \rho^n z^n) + \mathrm{div}(\rho^{n+1}z^{n+1}\boldsymbol{u})\Big]\\ &\qquad = \frac{1}{\delta t}\ \Big(\rho^{n+1}\varphi(z^{n+1}) - \rho^n\varphi(z^n)\Big) + \mathrm{div}\big(\rho^{n+1}\varphi(z^{n+1})\boldsymbol{u}\big) + \mathcal{R}^n,\end{aligned} \tag{5}$$

with

$$\mathcal{R}^n = \frac{1}{2\,\delta t}\rho^n\varphi''(\bar z)\,(z^{n+1} - z^n)^2, \quad \bar z = \theta z^n + (1-\theta)z^{n+1}, \quad \theta\in[0,1].$$

Proof. We first begin by deriving a discrete analogue to Identity (2):

$$\begin{aligned}&\frac{1}{\delta t}\ (\rho^{n+1}z^{n+1} - \rho^n z^n) + \mathrm{div}(\rho^{n+1}z^{n+1}\boldsymbol{u})\\ &= \frac{1}{\delta t}\ \rho^n\ (z^{n+1} - z^n) + \rho^{n+1}\boldsymbol{u}\cdot\boldsymbol{\nabla} z^{n+1} + z^{n+1}\Big[\frac{1}{\delta t}\ (\rho^{n+1} - \rho^n) + \mathrm{div}(\rho^{n+1}\boldsymbol{u})\Big]\\ &= \frac{1}{\delta t}\ \rho^n\ (z^{n+1} - z^n) + \rho^{n+1}\boldsymbol{u}\cdot\boldsymbol{\nabla} z^{n+1}.\end{aligned} \tag{6}$$

Then the result follows by multiplying this relation by $\varphi'(z^{n+1})$, using a Taylor expansion for the first term and the same combination of partial derivative as in the continuous case for the second one, and finally, still as in the continuous cas, by performing this computation in the reverse sense with $\varphi(z^n)$ and $\varphi(z^{n+1})$ instead of z^n and z^{n+1}.

2.2 The time semi-discrete scheme

We begin with a formal reformulation of the energy equation. Let us suppose that the solution is regular, and let E_k be the kinetic energy, defined by $E_k = \frac{1}{2}|\boldsymbol{u}|^2$. Taking the inner product of (1b) by $\boldsymbol{u}$ yields, after the formal compositions of partial derivatives described in the previous section:

$$\partial_t(\rho E_k) + \mathrm{div}\big(\rho\, E_k\, \boldsymbol{u}\big) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0. \tag{7}$$

This relation is referred to as the kinetic energy balance. Subtracting this relation to the total energy balance (1c), we obtain the so-called internal energy balance equation:

$$\partial_t(\rho e) + \mathrm{div}(\rho e \boldsymbol{u}) + p\,\mathrm{div}\boldsymbol{u} = 0. \tag{8}$$

Since,

- as seen in the previous section, thanks to the mass balance equation, the first two terms in the left-hand side of (8) may be recast as a transport operator,
- and, from the equation of state, the pressure vanishes when $e = 0$,

this equation implies that, if $e \geq 0$ at $t = 0$ and with suitable boundary conditions, then e remains non-negative at all time. The same result would hold if (8) featured a non-negative right-hand side, as for the compressible Navier-Stokes equations. Solving (8) instead of the total energy balance is thus appealing, to preserve this positivity property by construction of the scheme. In addition, it avoids introducing a space discretization for the total energy which, for a staggered discretization, combines cell-centered (the internal energy and the density) and face-centered (the velocity) variables. However, a raw discretization of a non-conservative equation derived from a conservative system (formally, *i.e.* supposing unrealistic regularity properties of the solution) may be non-consistent (and the numerical test presented in Section 4 shows that, for the problem at hand, a such a scheme would be unable to capture shock solutions). To deal with this problem, we implement the following strategy:

- First, we derive a discrete kinetic energy balance, by mimicking at the discrete level the computation used to obtain Equation (7). This relation allows to identify the terms which are likely to lead to non-consistency: the numerical diffusion in the momentum balance equation yields dissipation terms in the kinetic energy balance which are observed to behave, when the space and time step tend to zero, as measure born by the shocks which modify the jump conditions.
- These terms are thus compensated in the internal energy balance.

At the fully discrete level, for staggered discretizations, the kinetic and internal energy balances are not posed on the same mesh (the dual and primal mesh respectively) and cannot be combined to provide a local discrete total energy balance, even though the dissipation and correction terms have opposite integrals over the computational domain, so that the integral of the total energy over the domain is conserved. However, we are able to show that the scheme

is consistent, in the Lax-Wendroff sense, to the weak form of the total energy balance: indeed, for a given sequence of discrete solutions (obtained with a sequence of discretizations where the space and time steps tend to zero) controlled and converging to a limit in suitable norms (namely, controlled in BV norms and converging in L^p norms, for $p \in [1, +\infty)$), we show that the limit is a weak solution of the Euler equations. As far as the total energy balance is concerned, the key trick to this purpose is to use two interpolates of the test function, on the dual and primal mesh for the kinetic and energy balance respectively, and to pass to the limit in the equation obtained by summing the two corresponding relations.

We now derive the time semi-discrete formulation of a pressure correction scheme following these guidelines. This scheme takes the following general form:

$$\frac{1}{\delta t}(\rho^n\,\tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1}\,\boldsymbol{u}^n) + \mathrm{div}(\rho^n\,\boldsymbol{u}^n \otimes \tilde{\boldsymbol{u}}^{n+1}) + \zeta^n \boldsymbol{\nabla} p^n = 0, \tag{9a}$$

$$\frac{1}{\delta t}\rho^n\,(\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}) + \boldsymbol{\nabla} p^{n+1} - \zeta^n \boldsymbol{\nabla} p^n = 0, \tag{9b}$$

$$\frac{1}{\delta t}(\rho^{n+1} - \rho^n) + \mathrm{div}(\rho^{n+1}\,\boldsymbol{u}^{n+1}) = 0, \tag{9c}$$

$$\frac{1}{\delta t}(\rho^{n+1}\,e^{n+1} - \rho^n\,e^n) + \mathrm{div}(\rho^{n+1}\,e^{n+1}\,\boldsymbol{u}^{n+1}) + p^{n+1}\mathrm{div}\boldsymbol{u}^{n+1} = S^{n+1}, \tag{9d}$$

$$p^{n+1} = (\gamma - 1)\,\rho^{n+1}\,e^{n+1}. \tag{9e}$$

The first equation allows for the computation of a tentative velocity $\tilde{\boldsymbol{u}}^{n+1}$; it is decoupled from the other equations of the system, and referred to as the velocity prediction step. Equations (9b)-(9e) constitute the correction step, and are solved simultaneously; note however that using the equation of state to recast $\rho^{n+1}\,e^{n+1}$ as a function of the pressure only in (9d) and eliminating $\boldsymbol{u}^{n+1}$ in this relation thanks to the divergence of (9b) divided by ρ^n yields a nonlinear and nonconservative elliptic problem for the pressure only. This process must be performed at the fully discrete level to preserve the properties of the scheme. The coefficient ζ^n in Equation (9a) and the correction term S^{n+1} in (9d) are computed in the derivation of the scheme so as to ensure stability and consistency. The first step of this process is to obtain a discrete kinetic energy balance. To this purpose, let us multiply (9a) by $\tilde{\boldsymbol{u}}^{n+1}$ and apply Lemma 1 component by component, with $\varphi(s) = \frac{1}{2}s^2$. We get:

$$\frac{1}{2\,\delta t}\left(\rho^n\,|\tilde{\boldsymbol{u}}^{n+1}|^2 - \rho^{n-1}\,|\boldsymbol{u}^n|^2\right) + \frac{1}{2}\mathrm{div}\left(\rho^n\,|\tilde{\boldsymbol{u}}^{n+1}|^2\boldsymbol{u}^n\right) + \zeta^n \boldsymbol{\nabla} p^n \cdot \tilde{\boldsymbol{u}}^{n+1} + R_1^n = 0, \tag{10}$$

with

$$R_1^n = \frac{1}{2\,\delta t}|\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n|^2.$$

Note that the mass balance equation (9c), which is a fundamental assumption of Lemma 1, only holds at this stage of the algorithm with the previous time

step values, hence the shift of the time level of the density in (9a). Let us now recast Equation (9b) as

$$\alpha^n \boldsymbol{u}^{n+1} + \frac{1}{\alpha^n}\boldsymbol{\nabla} p^{n+1} = \alpha^n \tilde{\boldsymbol{u}}^{n+1} + \frac{\zeta^n}{\alpha^n}\boldsymbol{\nabla} p^n, \quad \alpha^n = \big[\frac{\rho^n}{\delta t}\big]^{1/2}$$

and square this relation, to get

$$\frac{1}{2\,\delta t}\,\rho^n\,|\boldsymbol{u}^{n+1}|^2 + \boldsymbol{\nabla} p^{n+1}\cdot\boldsymbol{u}^{n+1} + R_2^n = \frac{1}{2\,\delta t}\,\rho^n\,|\tilde{\boldsymbol{u}}^{n+1}|^2 + \zeta^n\boldsymbol{\nabla} p^n\cdot\tilde{\boldsymbol{u}}^{n+1}, \quad (11)$$

with

$$R_2^n = \frac{\delta t}{\rho^n}|\boldsymbol{\nabla} p^{n+1}|^2 - (\zeta^n)^2\frac{\delta t}{\rho^n}|\boldsymbol{\nabla} p^n|^2.$$

Summing (10) and (11) yields the kinetic energy balance that we are seeking:

$$\frac{1}{2\,\delta t}\,\big(\rho^n\,|\boldsymbol{u}^{n+1}|^2 - \rho^{n-1}\,|\boldsymbol{u}^n|^2\big) + \frac{1}{2}\mathrm{div}\big(\rho^n\,|\tilde{\boldsymbol{u}}^{n+1}|^2\boldsymbol{u}^n\big) + \boldsymbol{\nabla} p^{n+1}\cdot\boldsymbol{u}^{n+1} + R_1^n + R_2^n = 0.$$

We now choose the coefficient ζ^n in such a way that the remainder term R_2^n becomes the difference of two consecutive time levels of the same quantity, which is realized by choosing

$$\zeta^n = \big[\frac{\rho^n}{\rho^{n-1}}\big]^{1/2}.$$

Supposing the control in $L^1(0,T,BV)$ of the pressure and in L^∞ of the pressure and of the inverse of the density, the term R_2^n may thus be seen to tend with zero with the discretization parameters in a distributional sense. We just need to compensate R_1^n in the internal energy balance, which is done by choosing $S^{n+1} = R_1^n$, which thus ensures $S^{n+1} \geq 0$. The definition of the time-discrete scheme is now complete.

2.3 The fully discrete scheme

The fully discrete scheme is obtained from System (9) by applying the following guidelines:

- The mass and internal energy balances (*i.e.* Equations (9c) and (9d) respectively) are discretized on the primal mesh, while the velocity prediction (9a) and correction (9b) are discretized on the dual mesh(es). The equation of state only involves cell quantities, and its expression is obtained by writing (9e) for these latter.
- The space arrangement of the unknowns (density discretized at the cell and velocity at the faces) yields a natural expression of the mass fluxes in the mass balance, performed by a first-order upwind scheme (with respect to the velocity). By construction, the density is thus non-negative (in fact, positive, at the discrete level, if the initial density is positive). The discrete mass

balance equation on the cell K of measure $|K|$ and faces $\sigma \in \mathcal{E}(K)$ takes the form:

$$\frac{|K|}{\delta t}\left(\rho_K^{n+1} - \rho_K^n\right) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \tag{12}$$

where $F_{K,\sigma}$ is the mass flux across σ outward K.

- The form of the time-derivative and convection operator from the internal energy (let us denote $\mathcal{C}_K(e^{n+1})$ the sum of these two terms) in Equation (9d) follows from this relation:

$$\mathcal{C}_K(e^{n+1}) = \frac{|K|}{\delta t}\left(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n\right) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} e_\sigma^{n+1},$$

  where e_σ^{n+1} is the upwind approximation of e^{n+1} at σ with respect to $F_{K,\sigma}^{n+1}$ (or, equivalently, since the density is positive, with respect to the velocity). This was shown in [14] to be a sufficient condition to obtain a positivity-preserving operator, and is also a necessary condition for a fully discrete version of Lemma 1 to hold; this is of course linked since both results rely on the possibility to recast $\mathcal{C}_K$ as a transport operator, and the positivity-preserving property of $\mathcal{C}_K$ may be proved by applying Lemma 1 with $\varphi(s) = \min(s, 0)^2$. Once again, thanks to the arrangement of the unknowns, a natural discretization for $\mathrm{div}\boldsymbol{u}^{n+1}$ is available. Since p^{n+1} is a function of e^{n+1} given by the equation of state and invoking the corrective term is non-negative, we are able to show that the discrete internal energy is kept positive by the scheme.

- For the derivation of a discrete kinetic energy balance, the same structure is needed for the time-derivative and convection operator in the velocity prediction step (9a). This raises a difficulty since this equation is posed on the dual mesh, and thus we need an analogue of the mass balance (12) to also hold on this mesh. The way to build the face density and the mass fluxes across the faces of the dual mesh for such a relation to hold, while still ensuring the scheme consistency, is a central ingredient of the scheme; it is detailed in [5] for the MAC discretization and in [15] for unstructured discretizations.
  Once the face density is defined, the discretization of the coefficient ζ^n is straightforward. In order to combine the discrete equivalents of $\boldsymbol{u}\cdot\boldsymbol{\nabla}p$ (kinetic energy balance) and $p\,\mathrm{div}\boldsymbol{u}$ (internal energy balance), the discrete gradient is defined as the transposed of the divergence operator with respect to the L^2 inner product (if $\boldsymbol{u}\cdot\boldsymbol{\nabla}p + p\,\mathrm{div}\boldsymbol{u} = \mathrm{div}(p\,\boldsymbol{u})$, the integral of this quantity over the computational domain vanishes when the normal velocity is prescribed to zero at the boundary). Note that this definition is consistent with the usual treatment in the incompressible case, and is a key ingredient for the scheme to be asymptotic preserving in the limit of vanishing Mach number flows. As in the incompressible case, it also allows to control the L^2 norm of the pressure by a weak norm of its gradient, which is central for convergence studies; with this respect, a discrete *inf-sup* condition is required in some sense, which is true for staggered discretizations.

3 A "quasi-explicit" variant

A variant of the proposed scheme which consists only in explicit steps (in the sense that these steps do not require the solution of any linear or non-linear algebraic system) reads, in the time semi-discrete setting:

$$\frac{1}{\delta t}(\rho^{n+1} - \rho^n) + \mathrm{div}(\rho^n \, \boldsymbol{u}^n) = 0, \tag{13a}$$

$$\frac{1}{\delta t}(\rho^{n+1} \, e^{n+1} - \rho^n \, e^n) + \mathrm{div}(\rho^n \, e^n \, \boldsymbol{u}^n) + p^n \mathrm{div} \boldsymbol{u}^n = S^n, \tag{13b}$$

$$p^{n+1} = (\gamma - 1) \, \rho^{n+1} \, e^{n+1}, \tag{13c}$$

$$\frac{1}{\delta t}(\rho^{n+1} \, \boldsymbol{u}^{n+1} - \rho^n \, \boldsymbol{u}^n) + \mathrm{div}(\rho^n \, \boldsymbol{u}^n \otimes \boldsymbol{u}^n) + \boldsymbol{\nabla} p^{n+1} = 0. \tag{13d}$$

The update of the pressure before the solution of the momentum balance equation is crucial in our derivation of entropy estimates (see Section 5 below). This issue seems to be supported by numerical experiments: ommitting it, we observe the appearance of non-entropic discontinuities in rarefaction waves.

The space discretization differs from the pressure correction scheme in two points:

- the discretization of the convection operator in the momentum balance equation (13d) is performed by the first order upwind scheme (still with respect to the material velocity $\boldsymbol{u}^n$),
- the corrective term S^n is still obtained by deriving a kinetic energy balance multiplying Equation (13d) by $\boldsymbol{u}^{n+1}$, but its expression is quite different, due to the time-level used in the convection operator. The time-discretization is now anti-diffusive but, as usual for explicit schemes, this anti-diffusion is counterbalanced by the diffusion in the approximation of the convection (hence the upwinding) and S^n is non-negative only under a CFL condition.

4 A numerical test

In this section, we assess the behaviour of the scheme on a one dimensional Riemann problem. We choose initial conditions such that the structure of the solution consists in two shock waves, separated by the contact discontinuity, with sufficiently strong shocks to allow an easy discrimination of correct numerical solutions. These initial conditions are those proposed in [17, chapter 4], for the test referred to as Test 5. The computations are performed with the open-source software CALIF^{3}S [2].

The density fields obtained with $h = 1/2000$ (or a number of cells $n = 2000$) at $t = 0.035$, with and without assembling the corrective source term in the internal energy balance, together with the analytical solution, are shown on Figure 2. We observe that both schemes seem to converge, but the corrective

term is necessary to obtain the right solution. Without a corrective term, one can check that the obtained solution is not a weak solution to the Euler system (Rankine-Hugoniot conditions are not verified). We also observe that the scheme is rather diffusive especially for contact discontinuities for which the beneficial compressive effect of the shocks does not apply; this may be cured in the explicit variant by implementing MUSCL-like algorithms [7].

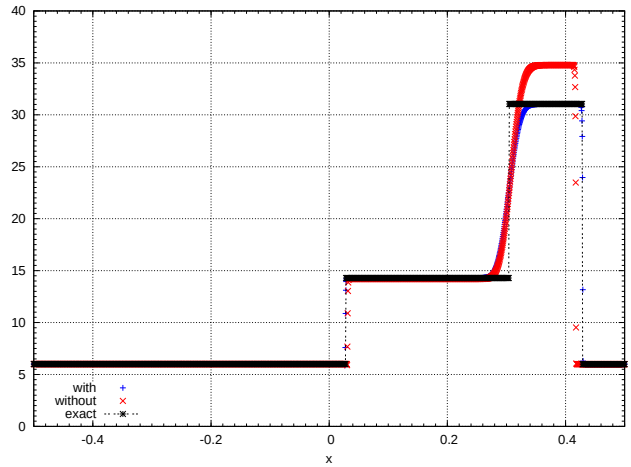

Fig. 2. Test 5 of [17, chapter 4] - Density obtained with $n = 2000$ cells, with and without corrective source terms, and analytical solution.

5 Entropy estimates

Let us consider the following subsystem of the Euler equations:

$$\partial_t \rho + \operatorname{div}(\rho\, \boldsymbol{u}) = 0, \tag{14a}$$

$$\partial_t (\rho\, e) + \operatorname{div}(\rho\, e\, \boldsymbol{u}) + p \operatorname{div}(\boldsymbol{u}) = \mathcal{R} \geq 0, \tag{14b}$$

$$p = (\gamma - 1)\, \rho\, e. \tag{14c}$$

The derivation of an entropy estimate for the continuous Euler system may be deduced from the subsystem (14) in the following way. We seek an entropy function η satisfying the entropy balance:

$$\partial_t \eta(\rho, e) + \operatorname{div}\big[\eta(\rho, e)\, \boldsymbol{u}\big] \leq 0. \tag{15}$$

To this end, let the convex functions φ_ρ and φ_e be defined as follows:

$$\varphi_\rho(z) = z \log(z), \quad \varphi_e(z) = \frac{-1}{\gamma - 1} \log(z), \quad \text{for } z > 0.$$

Let us multiply (14a) by $\varphi'_\rho(\rho)$, which yields:

$$\partial_t\big[\varphi_\rho(\rho)\big] + \mathrm{div}\big[\varphi_\rho(\rho)\,\boldsymbol{u}\big] + \big[\rho\varphi'_\rho(\rho) - \varphi_\rho(\rho)\big]\mathrm{div}(\boldsymbol{u}) = 0. \tag{16}$$

Then, multiplying (14b) by $\varphi'_e(e)$ yields, once again formally, since $\varphi'_e(z) < 0$ for $z > 0$:

$$\partial_t\big[\rho\,\varphi_e(e)\big] + \mathrm{div}\big[\rho\,\varphi_e(e)\,\boldsymbol{u}\big] + \varphi'_e(e)\,p\,\mathrm{div}(\boldsymbol{u}) \leq 0. \tag{17}$$

Summing (16) and (17) and noting that $\rho\varphi'_\rho(\rho) - \varphi_\rho(\rho) + \varphi'_e(e)\,p = 0$, we obtain (15) for $\eta(\rho, e) = \varphi_\rho(\rho) + \rho\varphi_e(e)$.

Depending on the time and space discretization, we obtain two types of results [6]:

- local entropy estimates, *i.e.* discrete analogues of (15), in which case the scheme is called entropy-stable,
- "weak local" entropy inequalities, *i.e.* results of the form:

$$\partial_t\eta(\rho, e) + \mathrm{div}\big[\eta(\rho, e)\,\boldsymbol{u}\big] + \mathcal{R} \leq 0,$$

  with $\mathcal{R}$ tending to zero with respect to the space and time discretization steps, provided that the solution is controlled in reasonable norms (here, L^∞ and BV norms). Such an inequality readily yields a "Lax-consistency" property, in the sense that the limit of a convergent sequence of solutions, bounded in suitable norms, satisfies the following weak entropy inequality:

$$-\int_0^T\int_\Omega \eta(\rho, e)\,\partial_t\varphi + \eta(\rho, e)\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \eta(\rho, e)(\boldsymbol{x}, 0)\;\varphi(\boldsymbol{x}, 0)\,\mathrm{d}\boldsymbol{x} \leq 0,$$
$$\text{for any function } \varphi \in C_c^\infty\big([0, T) \times \bar{\Omega}\big), \varphi \geq 0.$$

The pressure correction scheme - In the pressure correction scheme, the correction step includes a fully implicit discretization of Subsystem (14). A fully discrete analogue of the relation (16) may be found in [12, Lemma A1], while the relation (17) is a direct application of Lemma 1. The pressure correction scheme is thus entropy-stable.

The explicit variant - In this variant, the time discretization of Subsystem (14) is explicit. An adaptation of Lemma 1 still holds under a CFL condition, but it does not seem to be the case for Relation (16). Consequently, we only obtain a "weak local" entropy inequality, under the restrictive assumption that the ratio of the time to the space step tends to zero for the sequence of discretization at hand. Nevertheless, we never observed in numerical experiment any phenomena likely to lead to thinking that the scheme could converge to a non-entropy weak solution.

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