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A Comprehensive Introduction to sub-Riemannian Geometry

Andrei Agrachev, Davide Barilari, Ugo Boscain

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A Comprehensive Introduction to sub-Riemannian Geometry

from Hamiltonian viewpoint

ANDREI AGRACHEV
DAVIDE BARILARI
UGO BOSCAIN

February 6, 2019

Draft version

Preface

This book presents material taught by the authors in graduated courses at Trieste (SISSA), Paris (Institut Henri Poincaré, Orsay, Paris Diderot), and several summer schools, in the period 2008 – 2018.

It contains material for an introductory course in sub-Riemannian geometry at master or PhD level, as well as material for a more advanced course.

The book attempts to be as elementary as possible but, although the main concepts are recalled, it requires a certain ability in managing object in differential geometry (vector fields, differential forms, symplectic manifolds, etc.). We try to avoid as much as possible the use of functional analysis (some is required starting from Chapter 6).

We do not require any knowledge in Riemannian geometry. Actually from the book one can extract an introductory course in Riemannian geometry as a special case of sub-Riemannian one, starting from the geometry of surfaces in Chapter 1.

There are few other books of sub-Riemannian geometry available. Besides the pioneering book edited by A. Bellaïche and J.-J. Risler [BR96], a nowadays classical reference is the book of R. Montgomery [Mon02], that inspired several of our chapters. More recent books, written in a language similar to the one we use, are those of F. Jean [Jea14] and L. Rifford [Rif14]; see also the collection of lectures notes [BBS16a, BBS16b]. Other related books, although with a different approach, are the monographs [BLU07] and [CDPT07].

Example of an introductory course of sub-Riemannian geometry.

Chapters 2, 3 (without the appendices), 4, 7 (without 7.1), 9, 13, 21.

Example of an advanced course of sub-Riemannian geometry.

Chapters 2, 3 (with the appendices), 4, 6, 7 (together with 7.1), 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, the appendix by Zelenko.

Example of a course of Riemannian geometry.

Chapters 1, 2, 3 (without the appendices), 4, 5, 7, 8, 11, 14 (without 14.4-14.5-14.6), 15, 16, 21 (only 21.1).

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Contents

Introduction	12
1 Geometry of surfaces in \mathbb{R}^3	21
1.1 Geodesics and optimality	21
1.1.1 Existence and minimizing properties of geodesics	25
1.1.2 Absolutely continuous curves	27
1.2 Parallel transport	27
1.2.1 Parallel transport and Levi-Civita connection	29
1.3 Gauss-Bonnet theorems	31
1.3.1 Gauss-Bonnet theorem: local version	31
1.3.2 Gauss-Bonnet theorem: global version	35
1.3.3 Consequences of the Gauss-Bonnet theorems	38
1.3.4 The Gauss map	39
1.4 Surfaces in \mathbb{R}^3 with the Minkowski inner product	42
1.5 Model spaces of constant curvature	45
1.5.1 Zero curvature: the Euclidean plane	45
1.5.2 Positive curvature: the sphere	45
1.5.3 Negative curvature: the hyperbolic plane	47
1.6 Bibliographical note	48
2 Vector fields	49
2.1 Differential equations on smooth manifolds	49
2.1.1 Tangent vectors and vector fields	49
2.1.2 Flow of a vector field	51
2.1.3 Vector fields as operators on functions	51
2.1.4 Nonautonomous vector fields	52
2.2 Differential of a smooth map	54
2.3 Lie brackets	56
2.4 Frobenius theorem	59
2.4.1 An application of Frobenius theorem	61
2.5 Cotangent space	62
2.6 Vector bundles	63
2.7 Submersions and level sets of smooth maps	65
2.8 Bibliographical note	66

3	Sub-Riemannian structures	67
3.1	Basic definitions	67
3.1.1	The minimal control and the length of an admissible curve	70
3.1.2	Equivalence of sub-Riemannian structures	73
3.1.3	Examples	74
3.1.4	Every sub-Riemannian structure is equivalent to a free one	76
3.2	Sub-Riemannian distance and Rashevskii-Chow theorem	77
3.2.1	Proof of Rashevskii-Chow theorem	78
3.2.2	Non bracket-generating structures	82
3.3	Existence of length-minimizers	83
3.3.1	On the completeness of the sub-Riemannian distance	85
3.3.2	Lipschitz curves with respect to d vs admissible curves	87
3.3.3	Lipschitz equivalence of sub-Riemannian distances	88
3.3.4	Continuity of d with respect to the sub-Riemannian structure	89
3.4	Pontryagin extremals	91
3.4.1	The energy functional	93
3.4.2	Proof of Theorem 3.59	94
3.5	Appendix: Measurability of the minimal control	97
3.5.1	A measurability lemma	97
3.5.2	Proof of Lemma 3.12	99
3.6	Appendix: Lipschitz vs absolutely continuous admissible curves	99
3.7	Bibliographical note	100
4	Pontryagin extremals: characterization and local minimality	101
4.1	Geometric characterization of Pontryagin extremals	101
4.1.1	Lifting a vector field from M to T^*M	102
4.1.2	The Poisson bracket	103
4.1.3	Hamiltonian vector fields	105
4.2	The symplectic structure	107
4.2.1	Symplectic form vs Poisson bracket	108
4.3	Characterization of normal and abnormal Pontryagin extremals	109
4.3.1	Normal extremals	110
4.3.2	Abnormal extremals	113
4.3.3	Codimension one and contact distributions	115
4.4	Examples	116
4.4.1	2D Riemannian geometry	116
4.4.2	Isoperimetric problem	119
4.4.3	Heisenberg group	122
4.5	Lie derivative	123
4.6	Symplectic manifolds	125
4.7	Local minimality of normal extremal trajectories	127
4.7.1	The Poincaré-Cartan one-form	127
4.7.2	Normal Pontryagin extremal trajectories are geodesics	129
4.8	Bibliographical note	133

5	First integrals and integrable systems	135
5.1	Reduction of Hamiltonian systems with symmetries	135
5.1.1	An example of symplectic reduction: the space of affine lines in \mathbb{R}^n	137
5.2	Riemannian geodesic flow on hypersurfaces	138
5.2.1	Geodesics on hypersurfaces	138
5.2.2	Riemannian geodesic flow and symplectic reduction	139
5.3	Sub-Riemannian structures with symmetries	141
5.4	Completely integrable systems	143
5.5	Arnold-Liouville theorem	146
5.6	Geodesic flows on quadrics	148
5.7	Bibliographical note	152
6	Chronological calculus	153
6.1	Motivation	153
6.2	Duality	154
6.2.1	On the notation	155
6.3	Topology on the set of smooth functions	155
6.3.1	Family of functionals and operators	156
6.4	Operator ODEs and Volterra expansions	157
6.4.1	Volterra expansion	158
6.4.2	Adjoint representation	160
6.5	Variations formulae	162
6.6	Appendix: Estimates and Volterra expansion	163
6.7	Appendix: Remainder term of the Volterra expansion	166
6.8	Bibliographical note	168
7	Lie groups and left-invariant sub-Riemannian structures	169
7.1	Subgroups of $\text{Diff}(M)$ generated by a finite-dimensional Lie algebra of vector fields	169
7.1.1	A finite-dimensional approximation	170
7.1.2	Passage to infinite dimension	173
7.1.3	Proof of Proposition 7.2	174
7.2	Lie groups and Lie algebras	175
7.2.1	Lie groups as groups of diffeomorphisms	176
7.2.2	Matrix Lie groups and the matrix notation	178
7.2.3	Bi-invariant pseudo-metrics and Haar measures	180
7.2.4	The Levi-Malcev decomposition	182
7.3	Trivialization of TG and T^*G	183
7.4	Left-invariant sub-Riemannian structures	183
7.5	Example: Carnot groups of step 2	184
7.5.1	Normal Pontryagin extremals for Carnot groups of step 2	186
7.6	Left-invariant Hamiltonian systems on Lie groups	189
7.6.1	Vertical coordinates in TG and T^*G	189
7.6.2	Left-invariant Hamiltonians	191
7.7	Normal extremals for left-invariant sub-Riemannian structures	193
7.7.1	Explicit expression of normal Pontryagin extremals in the $\mathfrak{d} \oplus \mathfrak{s}$ case	194
7.7.2	Example: The $\mathfrak{d} \oplus \mathfrak{s}$ problem on $SO(3)$	195

7.7.3	Further comments on the $\mathfrak{d} \oplus \mathfrak{s}$ problem: $SO(3)$ and $SO_+(2,1)$	196
7.7.4	Explicit expression of normal Pontryagin extremals in the $\mathfrak{k} \oplus \mathfrak{z}$ case	199
7.8	Rolling spheres	202
7.8.1	Rolling with twisting	202
7.8.2	Rolling without twisting	205
7.8.3	Euler’s “ <i>cvrvae elasticae</i> ”	210
7.8.4	Rolling spheres: further comments	212
7.9	Bibliographical note	213
8	End-point map and exponential map	215
8.1	The end-point map	215
8.1.1	Regularity of the end-point map: proof of Proposition 8.5.	216
8.2	Lagrange multipliers rule	219
8.3	Pontryagin extremals via Lagrange multipliers	219
8.4	Critical points and second order conditions	220
8.4.1	The manifold of Lagrange multipliers	223
8.5	Sub-Riemannian case	228
8.6	Exponential map and Gauss’ Lemma	231
8.7	Conjugate points	234
8.8	Minimizing properties of extremal trajectories	238
8.8.1	Local length-minimality in the $W^{1,2}$ topology. Proof of Theorem 8.52.	239
8.8.2	Local length-minimality in the C^0 topology	241
8.9	Compactness of length-minimizers	245
8.10	Cut locus and global length-minimizers	247
8.11	An example: the first conjugate locus on perturbed sphere	250
8.12	Bibliographical note	252
9	2D Almost-Riemannian Structures	255
9.1	Basic definitions and properties	255
9.1.1	How big is the singular set?	260
9.1.2	Genuinely 2D almost-Riemannian structures have always infinite area	261
9.1.3	Pontryagin extremals	263
9.2	The Grushin plane	264
9.2.1	Geodesics on the Grushin plane	265
9.3	Riemannian, Grushin and Martinet points	266
9.3.1	Normal forms	270
9.4	Generic 2D almost-Riemannian structures	272
9.4.1	Proof of the genericity result	272
9.5	A Gauss-Bonnet theorem	273
9.5.1	Integration of the curvature	274
9.5.2	The Euler number	275
9.5.3	Gauss-Bonnet theorem	276
9.5.4	Every compact orientable 2D manifold can be endowed with a free almost-Riemannian structure with only Riemannian and Grushin points	282
9.6	Bibliographical note	282

10 Nonholonomic tangent space	285
10.1 Flag of the distribution and Carnot groups	285
10.2 Jet spaces	287
10.2.1 Jets of curves	287
10.2.2 Jets of vector fields	289
10.3 Admissible variations and nonholonomic tangent space	290
10.3.1 Admissible variations	291
10.3.2 Nonholonomic tangent space	292
10.4 Nonholonomic tangent space and privileged coordinates	294
10.4.1 Privileged coordinates	294
10.4.2 Description of the nonholonomic tangent space in privileged coordinates . . .	297
10.4.3 Existence of privileged coordinates: proof of Theorem 10.32.	303
10.4.4 Nonholonomic tangent spaces in low dimension	307
10.5 Metric meaning	309
10.5.1 Convergence of the sub-Riemannian distance and the Ball-Box theorem . . .	310
10.6 Algebraic meaning	314
10.6.1 Nonholonomic tangent space: the equiregular case	316
10.7 Carnot groups: normal forms in low dimension	317
10.8 Bibliographical note	321
11 Regularity of the sub-Riemannian distance	323
11.1 Regularity of the sub-Riemannian squared distance	323
11.2 Locally Lipschitz functions and maps	330
11.2.1 Locally Lipschitz map and Lipschitz submanifolds	334
11.2.2 A non-smooth version of Sard Lemma	336
11.3 Regularity of sub-Riemannian spheres	339
11.4 Geodesic completeness and Hopf-Rinow theorem	342
11.5 Bibliographical note	342
12 Abnormal extremals and second variation	345
12.1 Second variation	345
12.2 Abnormal extremals and regularity of the distance	346
12.3 Goh and generalized Legendre conditions	352
12.3.1 Proof of Goh condition - (i) of Theorem 12.13	354
12.3.2 Proof of generalized Legendre condition - (ii) of Theorem 12.13	360
12.3.3 More on Goh and generalized Legendre conditions	361
12.4 Rank 2 distributions and nice abnormal extremals	363
12.5 Minimality of nice abnormal in rank 2 structures	366
12.5.1 Proof of Theorem 12.34	367
12.6 Conjugate points along abnormal	374
12.6.1 Abnormals in dimension 3	376
12.6.2 Higher dimension	380
12.7 Equivalence of local minimality with respect to $W^{1,2}$ and C^0 topology	381
12.8 Non-minimality of corners	383
12.9 Bibliographical note	388

13	Some model spaces	391
13.1	Carnot groups of step 2	392
13.2	Multi-dimensional Heisenberg groups	394
13.2.1	Pontryagin extremals in the contact case	395
13.2.2	Optimal synthesis	397
13.3	Free Carnot groups of step 2	399
13.3.1	Intersection of the cut locus with the vertical subspace	402
13.3.2	The cut locus for the free step-two Carnot group of rank three	403
13.4	An extended Hadamard technique to compute the cut locus	404
13.5	The Grushin structure	409
13.5.1	Optimal synthesis starting from a Riemannian point	409
13.5.2	Optimal synthesis starting from a singular point	413
13.6	The standard sub-Riemannian structure on $SU(2)$	415
13.7	Optimal synthesis on the groups $SO(3)$ and $SO_+(2,1)$	419
13.8	Synthesis for the group of Euclidean transformations of the plane $SE(2)$	423
13.8.1	Mechanical interpretation	424
13.8.2	Geodesics	424
13.9	The Martinet flat sub-Riemannian structure	431
13.9.1	Abnormal extremals	431
13.9.2	Normal extremals	432
13.10	Bibliographical note	436
14	Curves in the Lagrange Grassmannian	441
14.1	The geometry of the Lagrange Grassmannian	441
14.1.1	The Lagrange Grassmannian	444
14.2	Regular curves in Lagrange Grassmannian	446
14.3	Curvature of a regular curve	449
14.4	Reduction of non-regular curves in Lagrange Grassmannian	452
14.5	Ample curves	453
14.6	From ample to regular	455
14.7	Conjugate points in $L(\Sigma)$	459
14.8	Comparison theorems for regular curves	460
14.9	Bibliographical note	463
15	Jacobi curves	465
15.1	From Jacobi fields to Jacobi curves	465
15.1.1	Jacobi curves	466
15.2	Conjugate points and optimality	468
15.3	Reduction of the Jacobi curves by homogeneity	469
15.4	Bibliographical note	471
16	Riemannian curvature	473
16.1	Ehresmann connection	473
16.1.1	Curvature of an Ehresmann connection	474
16.1.2	Linear Ehresmann connections	475
16.1.3	Covariant derivative and torsion for linear connections	476

16.2	Riemannian connection	478
16.3	Relation with Hamiltonian curvature	483
16.4	Comparison theorems for conjugate points	485
16.5	Locally flat spaces	486
16.6	Curvature of 2D Riemannian manifolds	487
16.7	Bibliographical note	488
17	Curvature in 3D contact sub-Riemannian geometry	489
17.1	A worked-out example: the 2D Riemannian case	489
17.2	3D contact sub-Riemannian manifolds	493
17.3	Canonical frames	496
17.4	Curvature of a 3D contact structure	498
17.4.1	Geometric interpretation	503
17.5	Local classification of 3D left-invariant structures	504
17.5.1	A description of the classification	506
17.5.2	A sub-Riemannian isometry between non isomorphic Lie groups	508
17.5.3	Canonical frames and classification. Proof of Theorem 17.29	510
17.5.4	An explicit isometry. Proof of Theorem 17.32	513
17.6	Appendix: Remarks on curvature coefficients	517
17.7	Bibliographical note	518
18	Integrability of the sub-Riemannian geodesic flow on 3D Lie groups	519
18.1	Poisson manifolds and symplectic leaves	519
18.2	Integrability of Hamiltonian systems on Lie groups	523
18.2.1	The Poisson manifold \mathfrak{g}^*	523
18.2.2	The Casimir first integral	524
18.2.3	First integrals associated with a right-invariant vector field	525
18.2.4	Complete integrability on Lie groups	526
18.3	Left-invariant Hamiltonian systems on 3D Lie groups	526
18.3.1	Rank 2 sub-Riemannian structures on 3D Lie groups	530
18.3.2	Classification of symplectic leaves on 3D Lie groups	532
18.4	Bibliographical note	538
19	Asymptotic expansion of the 3D contact exponential map	541
19.1	The exponential map	541
19.1.1	The nilpotent case	542
19.2	General case: second order asymptotic expansion	544
19.2.1	Proof of Proposition 19.2: second order asymptotics	544
19.3	General case: higher order asymptotic expansion	548
19.3.1	Proof of Theorem 19.6: asymptotics of the exponential map	550
19.3.2	Asymptotics of the conjugate locus	553
19.3.3	Asymptotics of the conjugate length	555
19.3.4	Stability of the conjugate locus	556
19.4	Bibliographical note	557

20	The volume in sub-Riemannian geometry	559
20.1	Equiregular sub-Riemannian manifolds	559
20.2	The Popp volume	560
20.3	A formula for Popp volume in terms of adapted frames	562
20.4	Popp volume and smooth isometries	565
20.5	Hausdorff dimension and Hausdorff volume	566
20.6	Hausdorff volume on sub-Riemannian manifolds	567
20.6.1	Hausdorff dimension	568
20.6.2	On the metric convergence	570
20.6.3	Induced volumes and estimates	571
20.7	Density of the spherical Hausdorff volume with respect to a smooth volume	573
20.8	Bibliographical note	574
21	The sub-Riemannian heat equation	575
21.1	The heat equation	575
21.1.1	The heat equation in the Riemannian context	575
21.1.2	The heat equation in the sub-Riemannian context	578
21.1.3	The Hörmander theorem and the existence of the heat kernel	580
21.1.4	The heat equation in the non bracket-generating case	582
21.2	The heat-kernel on the Heisenberg group	582
21.2.1	The Heisenberg group as a group of matrices	583
21.2.2	The heat equation on the Heisenberg group	584
21.2.3	Construction of the Gaveau-Hulanicki fundamental solution	585
21.2.4	Small-time asymptotics for the Gaveau-Hulanicki fundamental solution	592
21.3	Bibliographical note	593
A	Geometry of parametrized curves in Lagrangian Grassmannians (by Igor Zelanko)	595
A.1	Preliminaries	595
A.2	Algebraic theory of curves in Grassmannians and flag varieties	601
A.3	Application to differential geometry of monotonic parametrized curves in Lagrangian Grassmannians	611
	Bibliography	620
	Index	637

Introduction

This book concerns a fresh development of the eternal idea of the distance as the length of a shortest path. In Euclidean geometry, shortest paths are segments of straight lines that satisfy all classical axioms. In the Riemannian world, Euclidean geometry is just one of a huge amount of possibilities. However, each of these possibilities is well approximated by Euclidean geometry at very small scale. In other words, Euclidean geometry is treated as geometry of initial velocities of the paths starting from a fixed point of the Riemannian space rather than the geometry of the space itself.

The Riemannian construction was based on the previous study of smooth surfaces in the Euclidean space undertaken by Gauss. The distance between two points on the surface is the length of a shortest path on the surface connecting the points. Initial velocities of smooth curves starting from a fixed point on the surface form a tangent plane to the surface, that is an Euclidean plane. Tangent planes at two different points are isometric, but neighborhoods of the points on the surface are not locally isometric in general; certainly not if the Gaussian curvature of the surface is different at the two points.

Riemann generalized Gauss' construction to higher dimensions and realized that it can be done in an intrinsic way; you do not need an ambient Euclidean space to measure the length of curves. Indeed, to measure the length of a curve it is sufficient to know the Euclidean length of its velocities. A Riemannian space is a smooth manifold whose tangent spaces are endowed with Euclidean structures; each tangent space is equipped with its own Euclidean structure that smoothly depends on the point where the tangent space is attached.

For a habitant sitting at a point of the Riemannian space, tangent vectors give directions where to move or, more generally, to send and receive information. He measures lengths of vectors, and angles between vectors attached at the same point, according to the Euclidean rules, and this is essentially all what he can do. It is important that our habitant can, in principle, completely recover the geometry of the space by performing these simple measurements along different curves.

In the sub-Riemannian space we cannot move, receive and send information in all directions. There are restrictions (imposed by the God, the moral imperative, the government, or simply a physical law). A sub-Riemannian space is a smooth manifold with a fixed admissible subspace in any tangent space where admissible subspaces are equipped with Euclidean structures. Admissible paths are those curves whose velocities are admissible. The distance between two points is the infimum of the length of admissible paths connecting the points. It is assumed that any pair of points in the same connected component of the manifold can be connected by at least an admissible path. The last assumption might look strange at a first glance, but it is not. The admissible subspace depends on the point where it is attached, and our assumption is satisfied for a more or less general smooth dependence on the point; better to say that it is not satisfied only for very special families of admissible subspaces.

Let us describe a simple model. Let our manifold be \mathbb{R}^3 with coordinates x, y, z . We consider

the differential 1-form $\omega = -dz + \frac{1}{2}(xdy - ydx)$. Then $d\omega = dx \wedge dy$ is the pullback on \mathbb{R}^3 of the area form on the xy -plane. In this model the subspace of admissible velocities at the point (x, y, z) is assumed to be the kernel of the form ω . In other words, a curve $t \mapsto (x(t), y(t), z(t))$ is an admissible path if and only if $\dot{z}(t) = \frac{1}{2}(x(t)\dot{y}(t) - y(t)\dot{x}(t))$ or equivalently if

$$z(t) = z(0) + \frac{1}{2} \int_0^t (x(s)\dot{y}(s) - y(s)\dot{x}(s)) ds.$$

If $x(0) = y(0) = z(0) = 0$, then $z(t)$ is the signed area of the domain bounded by the curve and the segment connecting $(0, 0)$ with $(x(t), y(t))$.

In this geometry, the length of an admissible tangent vector $(\dot{x}, \dot{y}, \dot{z})$ is defined to be $(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}}$, that is the length of the projection of the vector to the xy -plane. By construction, the sub-Riemannian length of the admissible curve in \mathbb{R}^3 is equal to the Euclidean length of its projection to the plane.

In this geometry, to compute the shortest paths connecting the origin $(0, 0, 0)$ to a fixed point (x_1, y_1, z_1) we are then reduced to solve the classical Dido isoperimetric problem: find a shortest planar curve among those connecting $(0, 0)$ with (x_1, y_1) and such that the signed area of the domain bounded by the curve and the segment joining $(0, 0)$ and (x_1, y_1) is equal to z_1 (see Figure 1).

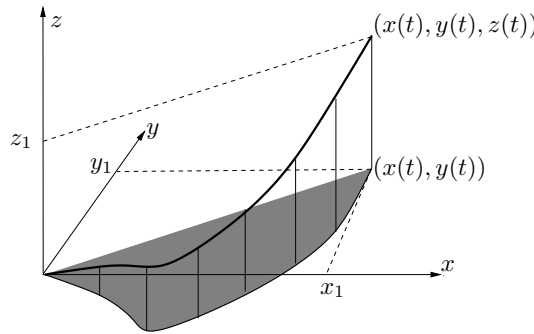


Figure 1: The Dido problem

Solutions of the Dido problem are arcs of circles and their lifts to \mathbb{R}^3 are spirals where $z(t)$ is the area of the piece of disc cut by the chord connecting $(0, 0)$ with $(x(t), y(t))$ (see Figure 2).

A piece of such a spiral is a shortest admissible path between its endpoints while the planar projection of this piece is an arc of the circle. The spiral ceases to be a shortest path when its planar projection starts to run the circle for the second time, i.e., when the spiral starts its second turn. Sub-Riemannian balls centered at the origin for this model look like apples with singularities at the poles (see Figure 3).

Singularities are points on the sphere connected with the center by more than one shortest path. The dilation $(x, y, z) \mapsto (rx, ry, r^2z)$ transforms the ball of radius 1 into the ball of radius r . In particular, arbitrary small balls have singularities. This is always the case when admissible subspaces are proper subspaces.

Another important symmetry connects balls with different centers. Indeed, the product operation

$$(x, y, z) \cdot (x', y', z') \doteq \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y) \right)$$

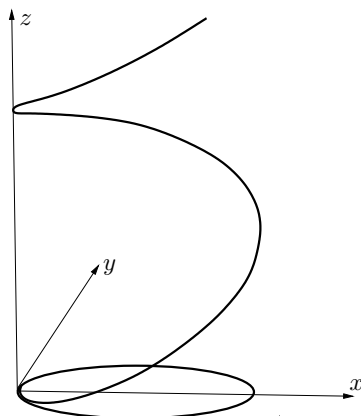


Figure 2: Solutions to the Dido problem

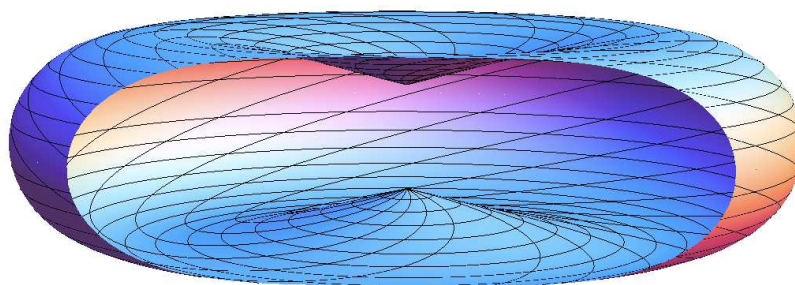


Figure 3: The Heisenberg sub-Riemannian sphere

turns \mathbb{R}^3 into a group, the *Heisenberg group*. The origin in \mathbb{R}^3 is the unit element of this group. It is easy to see that left-translations of the group transform admissible curves into admissible ones and preserve the sub-Riemannian length. Hence left translations transform balls in balls of the same radius. A detailed description of this example and other models of sub-Riemannian spaces is done in Sections 4.4.3, 7.5.1, 13.2.

Actually, even this simplest model tells us something about life in a sub-Riemannian space. Here we deal with planar curves but, in fact, operate in the three-dimensional space. Sub-Riemannian spaces always have a kind of hidden extra dimension. A good and not yet exploited source for mystic speculations but also for theoretical physicists who are always searching new crazy formalizations. In mechanics, this is a natural geometry for systems with nonholonomic constraints like skates, wheels, rolling balls, bearings etc. This kind of geometry could also serve to model social behavior that allows to increase the level of freedom without violation of a restrictive legal system.

Anyway, in this book we perform a purely mathematical study of sub-Riemannian spaces to provide an appropriate formalization ready for all potential applications. Riemannian spaces appear as a very special case. Of course, we are not the first to study the sub-Riemannian stuff. There is a broad literature even if there are not so many experts who could claim that sub-Riemannian geometry is his main field of expertise. Important motivations come from CR geometry, hyperbolic

geometry, analysis of hypoelliptic operators, and some other domains. Our first motivation was control theory: length minimizing is a nice class of optimal control problems.

Indeed, one can find a control theory spirit in our treatment of the subject. First of all, we include admissible paths in admissible flows that are flows generated by vector fields whose values in all points belong to admissible subspaces. The passage from admissible subspaces attached at different points of the manifold to a globally defined space of admissible vector fields makes the structure more flexible and well-adapted to algebraic manipulations. We pick generators f_1, \dots, f_k of the space of admissible fields, and this allows us to describe all admissible paths as solutions to time-varying ordinary differential equations of the form: $\dot{q}(t) = \sum_{i=1}^k u_i(t) f_i(q(t))$. Different admissible paths correspond to the choice of different *control functions* $u_i(\cdot)$ and initial points $q(0)$ while the vector fields f_i are fixed at the very beginning.

We also use a Hamiltonian approach supported by the Pontryagin maximum principle to characterize shortest paths. Few words about the Hamiltonian approach: sub-Riemannian geodesics are admissible paths whose sufficiently small pieces are length-minimizers, i. e. the length of such a piece is equal to the distance between its endpoints. In the Riemannian setting, any geodesic is uniquely determined by its velocity at the initial point q . In the general sub-Riemannian situation we have much more geodesics based at the the point q than admissible velocities at q . Indeed, every point in a neighborhood of q can be connected with q by a length-minimizer, while the dimension of the admissible velocities subspace at q is usually smaller than the dimension of the manifold.

What is a natural parametrization of the space of geodesics? To understand this question, we adapt a classical “trajectory – wave front” duality. Given a length-parameterized geodesic $t \mapsto \gamma(t)$, we expect that the values at a fixed time t of geodesics starting at $\gamma(0)$ and close to γ fill a piece of a smooth hypersurface (see Figure 4). For small t this hypersurface is a piece of the sphere of radius t , while in general it is only a piece of the “wave front”.

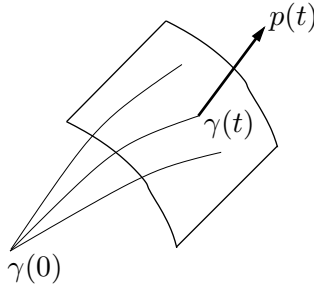


Figure 4: The “wave front” and the “impulse”

Moreover, we expect that $\dot{\gamma}(t)$ is transversal to this hypersurface. It is not always the case but this is true for a generic geodesic.

The “impulse” $p(t) \in T_{\gamma(t)}^*M$ is the covector orthogonal to the “wave front” and normalized by the condition $\langle p(t), \dot{\gamma}(t) \rangle = 1$. The curve $t \mapsto (p(t), \gamma(t))$ in the cotangent bundle T^*M satisfies a Hamiltonian system. This is exactly what happens in rational mechanics or geometric optics.

The sub-Riemannian Hamiltonian $H : T^*M \rightarrow \mathbb{R}$ is defined by the formula $H(p, q) = \frac{1}{2} \langle p, v \rangle^2$, where $p \in T_q^*M$, and $v \in T_qM$ is an admissible velocity of length 1 that maximizes $\langle p, w \rangle$ among all admissible velocities w of length one at $q \in M$.

Any smooth function on the cotangent bundle defines a Hamiltonian vector field and such a

field generates a Hamiltonian flow. The Hamiltonian flow on T^*M associated to H is the *sub-Riemannian geodesic flow*. The Riemannian geodesic flow is just a special case.

As we mentioned, in general, the construction described above cannot be applied to all geodesics: the so-called abnormal geodesics are missed. An abnormal geodesic $\gamma(t)$ also possesses its “impulse” $p(t) \in T_{\gamma(t)}^*M$ but this impulse belongs to the orthogonal complement to the subspace of admissible velocities and does not satisfy the above Hamiltonian system. Geodesics that are trajectories of the geodesic flow are called *normal*. Actually, abnormal geodesics belong to the closure of the space of the normal ones, and elementary symplectic geometry provides a uniform characterization of the impulses for both classes of geodesics. Such a characterization is, in fact, a very special case of the Pontryagin maximum principle.

Recall that all velocities are admissible in the Riemannian case, and the Euclidean structure on the tangent bundle induces the identification of tangent vectors and covectors, i.e., of the velocities and impulses. We should however remember that this identification depends on the metric. One can think to a sub-Riemannian metric as the limit of a family of Riemannian metrics when the length of forbidden velocities tends to infinity, while the length of admissible velocities remains untouched. It is easy to see that the Riemannian Hamiltonians defined by such a family converge with all derivatives to the sub-Riemannian Hamiltonian. Hence the Riemannian geodesics with a prescribed initial impulse converge to the sub-Riemannian geodesic with the same initial impulse. On the other hand, we cannot expect any reasonable convergence for the family of Riemannian geodesics with a prescribed initial velocity: those with forbidden initial velocities disappear at the limit, while the number of geodesics with admissible initial velocities jumps to infinity.

Outline of the book

We start in Chapter 1 from surfaces in \mathbb{R}^3 that is the beginning of everything in differential geometry, and also a starting point of the story told in this book. There are not yet Hamiltonians here, but a control flavor is already present. The presentation is elementary and self-contained. A student in applied mathematics or analysis who missed the geometry of surfaces at the university or simply is not satisfied by his understanding of these classical ideas, might find it useful to read just this chapter even if he does not plan to study the rest of the book.

In Chapter 2, we recall some basic properties of vector fields and vector bundles. Sub-Riemannian structures are defined in Chapter 3 where we also study three fundamental facts: the finiteness and the continuity of the sub-Riemannian distance, the existence of length-minimizers, and the infinitesimal characterization of geodesics. The first is the classical Rashevskii-Chow theorem, the second and the third one are simplified versions of the Filippov existence theorem and of the Pontryagin maximum principle.

In Chapter 4, we introduce the symplectic language. We define the geodesic Hamiltonian flow, we consider some interesting two- and three-dimensional problems, and we prove a general sufficient condition for length-minimality of normal trajectories. Chapter 5 is devoted to integrable Hamiltonian systems. We explain the construction of the action-angle coordinates and we describe classical examples of integrable geodesic flows, such as the geodesic flow on ellipsoids.

Chapters 1–5 form a first part of the book where we do not use any tool from functional analysis. In fact, even the knowledge of the Lebesgue integration and elementary real analysis are not essential with a unique exception of the existence theorem in Section 3.3. In all other parts of the text, the reader will nevertheless understand the content just replacing the terms “Lipschitz” and “absolutely continuous” with “piecewise C^1 ” and the term “measurable” with “piecewise continuous”.

We start to use some basic functional analysis in Chapter 6. In this chapter, we give elements of an operator calculus that simplifies and clarifies calculations with non-stationary flows, their variations and compositions. In Chapter 7, we give a brief introduction to the Lie group theory. Lie groups are introduced as subgroups of the groups of diffeomorphisms of a manifold M induced by a family of vector fields whose Lie algebra is finite dimensional. Then we study left-invariant sub-Riemannian structures and their geodesics.

In Chapter 8, we interpret the “impulses” as Lagrange multipliers for constrained optimization problems and apply this point of view to the sub-Riemannian case. We also introduce the sub-Riemannian exponential map and we study cut and conjugate points.

In Chapter 9, we consider two-dimensional sub-Riemannian metrics; such a metric coincides with a Riemannian one on an open and dense subset. We describe in details the model space of this geometry, known as the Grushin plane, and we discuss several properties in the generic case, among which a Gauss-Bonnet like theorem.

In Chapter 10, we construct the nonholonomic tangent space at a point q of the manifold: a first quasi-homogeneous approximation of the space if you observe and exploit it from q by means of admissible paths. In general, such a tangent space is a homogeneous space of a nilpotent Lie group equipped with an invariant vector distribution; its structure may depend on the point where the tangent space is attached. At generic points, this is a nilpotent Lie group endowed with a left-invariant vector distribution. The construction of the nonholonomic tangent space does not need a metric; if we take into account the metric, we obtain the Gromov–Hausdorff tangent to the sub-Riemannian metric space. Useful “ball-box” estimates of small balls follow automatically.

In Chapter 11, we study general analytic properties of the sub-Riemannian distance as a function of points of the manifold. It is shown that the distance is smooth on an open dense subset and is Lipschitz out of the points connected by abnormal length-minimizers. Moreover, if these bad points are absent, then almost every sphere is a Lipschitz submanifold.

In Chapter 12, we turn to abnormal geodesics, which provide the deepest singularities of the distance. Abnormal geodesics are critical points of the endpoint map defined on the space of admissible paths, and the main tool for their study is the Hessian of the endpoint map. This study permits to prove also that the cut locus from a point is adjacent to the point itself as soon as the structure is not Riemannian.

Chapter 13 is devoted to the explicit calculation of the sub-Riemannian optimal synthesis for model spaces. After a discussion on Carnot groups, we describe a technique based on the Hadamard theorem that permits, under certain assumptions, to compute the cut locus explicitly. We then apply this technique to several relevant examples.

This is the end of the second part of the book; next few chapters are devoted to the curvature and its applications. Let $\Phi^t : T^*M \rightarrow T^*M$, for $t \in \mathbb{R}$, be a sub-Riemannian geodesic flow. Submanifolds $\Phi^t(T_q^*M)$, $q \in M$, form a fibration of T^*M . Given $\lambda \in T^*M$, let $J_\lambda(t) \subset T_\lambda(T^*M)$ be the tangent space to the leaf of this fibration.

Recall that Φ^t is a Hamiltonian flow and T_q^*M are Lagrangian submanifolds; hence the leaves of our fibrations are Lagrangian submanifolds and $J_\lambda(t)$ is a Lagrangian subspace of the symplectic space $T_\lambda(T^*M)$.

In other words, $J_\lambda(t)$ belongs to the Lagrangian Grassmannian of $T_\lambda(T^*M)$, and $t \mapsto J_\lambda(t)$ is a curve in the Lagrangian Grassmannian, a *Jacobi curve* of the sub-Riemannian structure. The curvature of the sub-Riemannian space at λ is simply the “curvature” of this curve in the Lagrangian Grassmannian.

Chapter 14 is devoted to the elementary differential geometry of curves in the Lagrangian Grassmannian. In Chapter 15 we apply this geometry to Jacobi curves, that are curves in the Lagrange Grassmannian representing Jacobi fields.

The language of Jacobi curves is translated to the traditional language in the Riemannian case in Chapter 16. We recover the Levi-Civita connection and the Riemannian curvature and demonstrate their symplectic meaning. In Chapter 17, we explicitly compute the sub-Riemannian curvature for contact three-dimensional spaces and we show how the curvature invariants appear in the classification of sub-Riemannian left-invariant structures on 3D Lie groups. In Chapter 18, after a brief introduction on Poisson manifolds, we prove the integrability of the sub-Riemannian geodesic flow on 3D Lie groups. As a byproduct, we obtain a classification of coadjoint orbits on 3D Lie algebras. In the next Chapter 19 we study the small distance asymptotics of the exponential map for three-dimensional contact case and see how the structure of the conjugate locus is encoded in the curvature.

In Chapter 20 we address the problem of defining a canonical volume in sub-Riemannian geometry. First we introduce the Popp volume, that is a canonical volume that is smooth for equiregular sub-Riemannian manifold, and we study its basic properties. Then we define the Hausdorff volume and we study its density with respect to Popp's one.

In the last Chapter 21 we define the sub-Riemannian Laplace operator, and we study its properties (hypoellipticity, self-adjointness, etc.). We conclude with a discussion of the sub-Riemannian heat equation and an explicit formula for the heat kernel in the three-dimensional Heisenberg case.

The book is finished by an Appendix on the canonical frames for a wide class of curves in the Lagrangian Grassmannians, written by Igor Zelenko. This is a necessary background for a deeper systematic study of the curvature-type sub-Riemannian invariants, beyond the scope of this book.

We stop here this introduction into the “Comprehensive Introduction”. We hope that the reader won't be bored; comments to the chapters contain references and suggestions for further reading.