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Schauder estimates for drifted fractional operators in the supercritical case

Paul-Éric Chaudru de Raynal*, Stéphane Menozzi† and Enrico Priola‡

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Abstract

We consider a non-local operator L_α which is the sum of a fractional Laplacian $\Delta^{\alpha/2}$, $\alpha \in (0, 1)$, plus a first order term which is measurable in the time variable and locally β -Hölder continuous in the space variables. Importantly, the fractional Laplacian $\Delta^{\alpha/2}$ does not dominate the first order term. We show that global parabolic Schauder estimates hold even in this case under the natural condition $\alpha + \beta > 1$. Thus, the constant appearing in the Schauder estimates is in fact independent of the L^∞ -norm of the first order term. In our approach we do not use the so-called extension property and we can replace $\Delta^{\alpha/2}$ with other operators of α -stable type which are somehow close, including the relativistic α -stable operator. Moreover, when $\alpha \in (1/2, 1)$, we can prove Schauder estimates for more general α -stable type operators like the singular cylindrical one, i.e., when $\Delta^{\alpha/2}$ is replaced by a sum of one dimensional fractional Laplacians $\sum_{k=1}^d (\partial_{x_k x_k}^2)^{\alpha/2}$.

1 Statement of the problem and main results

We are interested in establishing global Schauder estimates for the following parabolic integro-partial differential equation (IPDE):

$$\begin{aligned} \partial_t u(t, x) + L_\alpha u(t, x) + F(t, x) \cdot D_x u(t, x) &= -f(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\ u(T, x) &= g(x), \quad \text{on } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $T > 0$ is a fixed final time horizon.

The operator L_α can be the fractional Laplacian $\Delta^{\alpha/2}$, i.e., for regular functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\Delta^{\alpha/2} \varphi(x) = \text{p.v.} \int_{\mathbb{R}^d} [\varphi(x+y) - \varphi(x)] \nu_\alpha(dy), \quad \text{where } \nu_\alpha(dy) = C_{\alpha,d} \frac{dy}{|y|^{d+\alpha}}, \tag{1.2}$$

or a more general *symmetric* non-local α -stable operator with symbol comparable to $-|\lambda|^\alpha$ but associated with a wider class of Lévy measures ν_α (see Section 1.1 for our precise assumptions). We can also consider some *non-symmetric* stable operators like the relativistic stable one with symbol $-(|\lambda|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} + m$, $m > 0$.

In (1.1), the source $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ and terminal condition $g : \mathbb{R}^d \rightarrow \mathbb{R}$ are assumed to belong to some suitable Hölder spaces and to be bounded.

The drift term F can be unbounded. It is only assumed to be (locally) β -Hölder continuous, $\beta \in (0, 1)$, i.e. $F : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel measurable, locally bounded and there exists $K_0 > 0$ such that

$$|F(t, x) - F(t, y)| \leq K_0 |x - y|^\beta, \quad t \in [0, T], \quad x, y \in \mathbb{R}^d \text{ s.t. } |x - y| \leq 1. \tag{1.3}$$

In particular, we concentrate on the so-called *super-critical* case, i.e., $\alpha \in (0, 1)$, although our estimates can be extended to the simpler case $\alpha \in [1, 2)$. The difficulty is quite clear: in the Fourier space, L_α is of order α and *does not dominate*, when $\alpha \in (0, 1)$, the drift term which is roughly speaking of order one (see also Remark 3.5 in [Pri12] for related issues).

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In order to compensate the *low* smoothing effect of L_α , it is natural to ask *more* on the Hölder exponent β of the drift. Namely, we need that the gradient $D_x u(t, x)$ in (1.1) exists in the classical sense. To this end, since the smoothing effect of L_α on the β -Hölder source f is *expected* to be of order $(\alpha + \beta)$ in space, it is natural to consider $\alpha + \beta > 1$. This condition also appears from a probabilistic viewpoint; it had indeed already been observed in the scalar case by Tanaka *et al.* [TTW74] that uniqueness might fail for the corresponding SDE when $\alpha + \beta \leq 1$.

In the previously described framework, we obtain estimates like

$$\|u\|_{L^\infty([0,T], C_b^{\alpha+\beta})} \leq C(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0,T], C_b^\beta)}), \quad (1.4)$$

with usual notations for Hölder spaces, where C is independent of u , f and g .

An interesting example covered by our assumptions is the non-local Ornstein-Uhlenbeck operator

$$\Delta^{\alpha/2} \varphi(x) + Ax \cdot D_x \varphi(x), \quad (1.5)$$

when $F(t, x) = Ax$ and A is any $d \times d$ real matrix (in this case assumption (1.3) holds for any $\beta \in (0, 1)$). If $\alpha = 2$ Schauder estimates were first proved by Da Prato and Lunardi [DPL95]. After that paper the OU operator has been much investigated as a prototype of operator with unbounded coefficients.

Related results. Schauder estimates in the α -stable non-local framework have been addressed by several authors, always assuming that the drift term is globally bounded and mainly assuming that $\alpha \geq 1$. In some papers, the Lévy measure ν_α may also depend on t and x . We mention for instance the so-called stable-like setting, corresponding to time-inhomogeneous operators of the form

$$L_t \varphi(x) = \int_{\mathbb{R}^d} [\varphi(x+y) - \varphi(x) - \mathbb{I}_{\alpha \in [1,2)} D_x \varphi(x) \cdot y] m(t, x, y) \frac{dy}{|y|^{d+\alpha}} + F(t, x) \cdot D_x u(t, x) \mathbb{I}_{\alpha \in [1,2)}, \quad (1.6)$$

where the *diffusion coefficient* m is bounded from above and below, Hölder continuous in the spatial variable, and even in the y variable for $\alpha = 1$. In that framework, Mikulevicius and Pragarauskas [MP14] obtained parabolic Schauder type bounds on the whole space and derived from those estimates the well-posedness of the corresponding martingale problem. Observe that, in (1.6), in the super-critical case $\alpha \in (0, 1)$, the drift term is set to 0. Again, this is mainly due to the fact that, in that case, the drift cannot be viewed anymore as a lower order perturbation of the fractional operator.

In the driftless framework, the (elliptic)-Schauder type estimates for stable-like operators were first derived by Bass [Bas09]. We can refer as well to the recent work of Imbert *et al.* [IJS16] concerning Schauder estimates for a driftless stable like operator of type (1.6) for $\alpha = 1$ and some non-standard diffusion coefficients m with applications to a non-local Burgers type equation. Eventually, still for $F = 0$, in the general non-degenerate symmetric α -stable setting, for which ν_α writes in polar coordinates $y = \rho s$, $(\rho, s) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ as

$$\nu_\alpha(dy) = \rho^{-1-\alpha} d\rho \tilde{\mu}(ds), \quad (1.7)$$

where $\tilde{\mu}$ is a non-degenerate symmetric measure on the sphere \mathbb{S}^{d-1} , we can also mention the works of Ros-Oton and Serra [ROS16] for interior and boundary elliptic-regularity and Fernandez-Real and Ros-Oton [FRRO17] for parabolic equations. We can also refer to Kim and Kim [KK15] for results on the whole space involving more general, but rotationally invariant, Lévy measures, or to Dong and Kim [DK13] for stable like measures that might be non-symmetric and non-regular w.r.t the jump parameter.

In the elliptic setting, when $\alpha \in [1, 2)$ and L_α is a non-degenerate symmetric α -stable operator and for bounded Hölder drifts, global Schauder estimates were obtained by Priola, see e.g. Section 3 in [Pri12] and [Pri18] with respective applications to the strong well-posedness and Davie's uniqueness for the corresponding SDE. Also, when $\alpha \in [1, 2)$, elliptic Schauder estimates can be proved for more general Lévy-type generators invariant for translations, see Section 6 in [Pri18] and Remark 5.

For a non trivial, and potentially *rough*, drift, there is a rather large literature concerning the regularity of (1.1) when the drift (possibly depending on the solution) is divergence free, i.e., $\nabla \cdot F(t, x) = 0$, in connection with the quasi-geostrophic equation even for $\alpha \in (0, 1)$. We can mention the seminal work of Caffarelli and Vasseur [CV10] and the work of Silvestre *et al.* [SVZ13] which exhibits counter-examples to regularity of (1.1) when the terminal condition lacks good integrability properties or when the condition $\alpha + \beta > 1$ is not met (see Theorem 1.1 therein). Conditions on divergence free drifts F in Morrey-Campanato or Besov spaces giving the Hölder continuity of (1.1) are discussed in [CM16] and [CM18].

When, $\alpha \in (0, 1)$ and F is Hölder continuous and bounded (but not necessarily divergence free), Silvestre obtained in [Sil12] sharp Schauder estimates on balls for the fractional Laplacian. His approach heavily relies on the so-called extension property, see [CS07] or [MO69] for a more probabilistic approach, and therefore seems rather delicate to extend to more general operators of stable type or with varying coefficients. Also, it seems that our result when $L_\alpha = \Delta^{\alpha/2}$ and F satisfying (1.3) cannot be obtained from the estimates by Silvestre using a standard covering argument; indeed the Schauder constant in [Sil12] also depends on the global boundedness of F . We can mention as well the recent work of Zhang and Zhao [ZZ18] who address through probabilistic arguments the parabolic Dirichlet problem in the super-critical case for stable-like operators of the form (1.6) with a non trivial bounded drift, i.e., getting rid of the indicator function for the drift. They also obtain interior Schauder estimates and some boundary decay estimates (see e.g. Theorem 1.5 therein).

Outline for Schauder estimates through perturbative approach. In this work we will establish global Schauder estimates for the solution of (1.1) inspired by the perturbative approach first introduced in [CHM18] to derive such estimates in anisotropic Hölder spaces for degenerate Kolmogorov equations.

Roughly speaking, the main steps of our perturbative approach are the following: choose first a suitable *proxy* for the main equation (i.e., an integro-partial differential operator whose associated semi-group and heat kernel are known and close enough to the original one), exhibit then suitable regularization properties associated with the proxy, expand consequently the solution of the IPDE of interest around the proxy (Duhamel type formula or variation of constants formula) and eventually use such a representation to obtain Schauder estimates. Let us emphasize that the derivation of a robust Duhamel representation for the IPDE is crucial in this approach.

More precisely, the perturbative argument takes here the following form: first choose a flow $\theta_{s,\tau}(\xi) = \xi + \int_\tau^s F(v, \theta_{v,\tau}(\xi)) dv$ depending on parameters $\xi \in \mathbb{R}^d$ and $\tau \in [0, T]$ to be chosen carefully and introduce then the time inhomogeneous drift $F(t, \theta_{t,\tau}(\xi))$ *frozen along the considered flow*. Rewrite then (1.1) as

$$\begin{aligned} \partial_t u(t, x) + L_\alpha u(t, x) + F(t, \theta_{t,\tau}(\xi)) \cdot D_x u(t, x) &= -f(t, x) + [F(t, \theta_{t,\tau}(\xi)) - F(t, x)] \cdot D_x u(t, x), \\ u(T, x) &= g(x), \quad \text{on } \mathbb{R}^d. \end{aligned} \tag{1.8}$$

This system reflects more or less the main ingredients needed for our perturbative approach: the integro-partial differential operator in the above l.h.s. will be our proxy which is hence a *frozen* version of the operator in (1.1), where the freezing is done along the chosen flow, and the second term in the r.h.s. is precisely the error made when expanding the solution around the proxy. Roughly speaking, by the Duhamel principle we get a representation formula for the solution u and we can perform estimates by choosing the proxy parameters τ and ξ . In this respect it is useful to look at the proof of Proposition 8 and in particular to the derivation of estimates (2.30) and (2.31). On the other hand, a more general Duhamel formula is needed in Section 2.4.2 to complete the proof of Schauder estimates.

At this stage, let us eventually mention that when dealing with unbounded first order coefficients, the previous associated flow is a rather natural object to consider in order to establish Schauder estimates and was already used by Krylov and Priola [KP10] in the diffusive setting.

In comparison with [CHM18], where the main difficulties encountered consisted in handling the degeneracy of the operator and its associated anisotropic behavior (while the derivation of a Duhamel representation as well as the existence of a solution were the easier parts), we here face different problems, especially when trying to obtain a suitable Duhamel representation or when dealing with the existence part. Such difficulties come from two main features of our framework: the stable operator L_α induces major integrability issues and we consider drift terms that are only locally Hölder continuous (see again (1.3)).

To overcome these particularities, we introduce a *localized* version of (1.8). The point is to multiply u by a suitable *localizing* test function $\eta_{\tau,\xi}$ where (τ, ξ) are freezing parameters and to establish a Duhamel type representation formula for $u\eta_{\tau,\xi}$ (cf. equation (2.25)). We point out that, in our current setting, this localization is not *simply* motivated by the fact to get weaker assumptions on F (i.e., from *global* to *local* Hölder continuity). Indeed, even when $L_\alpha = \Delta^{\frac{\alpha}{2}}$ for $\alpha \in (0, 1/2)$, it is also needed to give a proper meaning to the Duhamel representation of the solution because of the low integrability properties of the underlying heat-kernel (see again Proposition 8 and its proof). Let us also emphasize that, the key to perform our analysis consists in having *good controls* on the heat kernel (or density) p_α associated with L_α and some of its spatial derivatives (cf. **(NDb)** in Section 1.1.1).

This will for instance be the case when the spherical measure $\tilde{\mu}$ in (1.7) has a smooth density w.r.t. the Lebesgue measure of \mathbb{S}^{d-1} , following the work of Kolkoltsov [Kol00]. Roughly speaking, in that framework, the heat-kernel p_α associated with L_α , and its first two derivatives, will behave *similarly* to the rotationally invariant density of $\Delta^{\frac{\alpha}{2}}$ for which we have precise pointwise controls. In this framework we establish Schauder estimates for any $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ s.t. $\beta + \alpha > 1$ for the potentially unbounded drift satisfying (1.3).

On the other hand, in the case of more general, and possibly singular, fractional operators of symmetric stable type, following the approach initiated by Watanabe [Wat07] and also used in Huang *et al.* [HMP19] consisting in treating separately the small and large jumps for the considered characteristic time scale, we have an additional constraint. We are only able to derive that the spatial derivatives of the heat kernel $p_\alpha(t, z)$ (which have the expected additional time singularity associated with the derivation order), can integrate $z \mapsto |z|^\beta$, i.e., $\int_{\mathbb{R}^d} |z|^\beta |D_z^k p_\alpha(t, z)| dz < \infty$, $k \in \{1, 2\}$, provided $\beta < \alpha$, $t > 0$. The constraint $\alpha + \beta > 1$ then gives that we can handle in this general super-critical case, indexes $\alpha \in (1/2, 1)$. The difference between the previous two cases can be intuitively explained as follows: for the fractional Laplacian the derivation of $p_\alpha(t, \cdot)$ induces a concentration gain at infinity, see e.g. Bogdan and Jakubowicz [BJ07], which precisely permits to get rid of the integrability constraints that we have to face; for operators whose symbol is equivalent to $|\xi|^\alpha$ but whose Lévy measure ν_α has a very singular spherical part, we do not have such concentration gain (cf. Remark 4).

Eventually, our approach will also allow to handle stable fractional truncated operators viewing the difference between the truncated and the non-truncated operators as a bounded perturbative term under control.

Organization of this paper. The article is organized as follows. We state our precise framework and give our main results at the end of the current section. Section 2 is then dedicated to the perturbative approach which is the central point to derive our estimates. In particular, we obtain therein some Schauder estimates for drifted operators along the inhomogeneous flow as well as the key Duhamel representation for solutions. We establish in Section 3 existence results. Eventually, Section 4 is devoted to the derivation in the previously described cases, i.e., stable-like and general stable operator for $\alpha \in (1/2, 1)$, of the properties required to obtain our main estimates. The proof of some technical results concerning the stability properties of non-Lipschitz flows are postponed to Appendix A.

1.1 Setting

1.1.1 Operators considered

We consider a Lévy generator L such that for $\phi \in C_0^\infty(\mathbb{R}^d)$, where $C_0^\infty(\mathbb{R}^d)$ stands for the space of real-valued infinitely differentiable functions with compact support one has:

$$L\phi(x) = \int_{\mathbb{R}^d} (\phi(x+y) - \phi(x)) \nu(dy), \quad x \in \mathbb{R}^d, \quad (1.9)$$

where ν is a Borel measure on \mathbb{R}^d such that $\int_{\mathbb{R}^d} (1 \wedge |x|) \nu(dx) < \infty$, and $\nu(\{0\}) = 0$ (ν is an example of Lévy measure). It is well known (see e.g. Sato [Sat99]) that there exists a convolution Markov semigroup (P_t) associated with L :

$$P_t h(x) = \int_{\mathbb{R}^d} h(x+y) \mu_t(dy), \quad h \in B_b(\mathbb{R}^d), \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1.10)$$

$P_0 = I$, where (μ_t) is a family of Borel probability measures on \mathbb{R}^d and $B_b(\mathbb{R}^d)$ stands for the set of real-valued bounded measurable functions. The function $v(t, x) = P_t \phi(x)$ provides the classical solution to the Cauchy problem

$$\partial_t v(t, x) = Lv(t, x) = Lv(t, \cdot)(x), \quad t > 0, \quad v(0, x) = \phi(x) \quad \text{on } \mathbb{R}^d. \quad (1.11)$$

In probabilistic term, μ_t is the distribution at time $t \geq 0$ of a purely jump Lévy process $(Z_t)_{t \geq 0}$.

(NDa) We assume that μ_t has a C^2 -density $p(t, \cdot)$, $t > 0$, and that there exists $\alpha \in (0, 1)$ such that if $0 < \gamma < \alpha$:

$$\int_{\mathbb{R}^d} |y|^\gamma p(t, y) dy \leq c t^{\gamma/\alpha}, \quad t \in [0, 1]. \quad (1.12)$$

for some $c = c(\gamma, \alpha) > 0$.

In the sequel we write $L = L_\alpha$, $\nu = \nu_\alpha$, $P_t = P_t^\alpha$ and $p = p_\alpha$ in order to explicitly emphasize the dependence of these objects w.r.t. the parameter α .

(NDb) To prove Schauder estimates with Hölder index $\beta \in (0, 1)$, beside the condition $\alpha + \beta > 1$, we need the following smoothing effect: there exists a constant $c = c(\alpha, \beta) > 0$ such that

$$\int_{\mathbb{R}^d} |y|^\beta |D_y^k p_\alpha(t, y)| dy \leq \frac{c}{t^{[k-\beta]/\alpha}}, \quad t \in (0, 1], \quad k = 1, 2. \quad (\mathcal{P}_\beta)$$

where $D_y^1 p_\alpha(t, y) = D_y p_\alpha(t, y)$ and $D_y^2 p_\alpha(t, y)$ denote the first and second derivatives in the y -variable. \square

Remark 1. It is known that in the case of the fractional Laplacian $L_\alpha = \Delta^{\alpha/2}$ assumption (\mathcal{P}_β) is always verified for any $\beta \in (0, 1)$ and $\alpha \in (0, 1)$. On the other hand, in the more general class of non-degenerate symmetric stable operators, (\mathcal{P}_β) holds only if $0 < \beta < \alpha$ (see Proposition 1). This condition together with $\alpha + \beta > 1$ imposes $\alpha > 1/2$. We also manage to establish (\mathcal{P}_β) , $\beta \in (0, 1)$ for the non-symmetric relativistic stable operator.

Remark 2. We mention that **(NDb)** is specifically needed to handle the remainder perturbative term in the r.h.s. of (1.8) and can be viewed as a *sufficient* condition to cope with the supercritical case.

1.1.2 Non-degenerate symmetric stable operators

We now introduce a class of operators L_α which verify **(NDa)** and **(NDb)**. These operators L_α will be the generators of non-degenerate symmetric stable processes, i.e., L_α can be represented by (1.9) where $\nu = \nu_\alpha$ is a symmetric stable Lévy measure of order $\alpha \in (0, 1)$. If we now write in polar coordinates $y = \rho s$, $(\rho, s) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$, the previous measure ν_α decomposes as

$$\nu_\alpha(dy) = \frac{d\rho \tilde{\mu}(ds)}{\rho^{1+\alpha}}, \quad (1.13)$$

where $\tilde{\mu}$ is a symmetric measure on the \mathbb{S}^{d-1} which is a spherical part of ν_α . Again, if $\tilde{\mu}$ is precisely the Lebesgue measure on the sphere, then $L_\alpha = \Delta^{\frac{\alpha}{2}}$. It is easy to verify that $\int_{\mathbb{R}^d} (1 \wedge |x|) \nu_\alpha(dx) < \infty$.

The Lévy symbol associated with L_α is given by the Lévy-Khintchine formula

$$\Psi(\lambda) = \int_{\mathbb{R}^d} (e^{i\langle \lambda, y \rangle} - 1) \nu_\alpha(dy), \quad \lambda \in \mathbb{R}^d, \quad (1.14)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on \mathbb{R}^d (see, for instance Jacob [Jac05] or Sato [Sat99]). In the current symmetric setting, Theorem 14.10 in [Sat99] then yields:

$$\Psi(\lambda) = - \int_{\mathbb{S}^{d-1}} |\langle \lambda, s \rangle|^\alpha \mu(ds), \quad (1.15)$$

where $\mu = C_{\alpha,d} \tilde{\mu}$ for a positive constant $C_{\alpha,d}$. The spherical measure μ is called the *spectral measure* associated with ν_α . We suppose that μ is non-degenerate, i.e., there exists $\eta \geq 1$ s.t. for all $\lambda \in \mathbb{R}^d$,

$$\eta^{-1} |\lambda|^\alpha \leq \int_{\mathbb{S}^{d-1}} |\langle \lambda, s \rangle|^\alpha \mu(ds) \leq \eta |\lambda|^\alpha, \quad \alpha \in (0, 1). \quad (1.16)$$

We carefully point out that condition (1.16) is fulfilled by many types of spherical measures μ , from measure equivalent to the Lebesgue measure on \mathbb{S}^{d-1} (a *stable-like* case) to very singular ones, like sum of Dirac masses along the canonical directions, which would correspond to the pure cylindrical case (or equivalently to the sum of scalar fractional Laplacians):

$$\sum_{k=1}^d (\partial_{x_k x_k}^2)^{\alpha/2}. \quad (1.17)$$

For symmetric stable operators under (1.16), it is well known (see e.g. [Kol00]) that the associated convolution Markov semigroup (P_t^α) (see (1.10)) has a C^∞ -smooth density $p_\alpha(t, \cdot)$. Through Fourier inversion, we get for all $t > 0, y \in \mathbb{R}^d$:

$$p_\alpha(t, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-\langle y, \lambda \rangle) \exp\left(-t \int_{\mathbb{S}^{d-1}} |\langle \lambda, s \rangle|^\alpha \mu(ds)\right) d\lambda. \quad (1.18)$$

From (1.18) and (1.16) we derive directly **(NDa)**. Indeed we have the following scaling property: $p_\alpha(t, y) = t^{-d/\alpha} p_\alpha(1, t^{-1/\alpha} y)$, $t > 0, y \in \mathbb{R}^d$.

Moreover, we have the following global upper bound for the derivatives of the heat-kernel: there exists $C := C(\eta)$ s.t. for all $k \in \{0, 1, 2\}$ and for all $t > 0, y \in \mathbb{R}^d$,

$$|D_x^k p_\alpha(t, y)| \leq \frac{C}{t^{\frac{d+k}{\alpha}}}, \quad t > 0. \quad (1.19)$$

which in turns yields with the notations of (1.10): $\forall t > 0, x \in \mathbb{R}^d, |D_x^k P_t^\alpha h(x)| \leq C t^{-(d+k)/\alpha} \|h\|_\infty, t > 0$.

The validity of **(NDb)** for general symmetric non-degenerate stable operators follows by the next result (note that in this case we have the property for any $t > 0$ and not only for $t \in (0, 1)$).

Proposition 1. *Let $\alpha \in (0, 1)$. Assume (1.16) holds, then for any $0 \leq \gamma < \alpha$, there exists $C := C(\eta, \gamma)$ s.t. for all $\ell \in \{1, 2\}$,*

$$\int_{\mathbb{R}^d} |y|^\gamma |D_y^\ell p_\alpha(t, y)| dy \leq \frac{C}{t^{[\ell-\gamma]/\alpha}}, \quad t > 0. \quad (1.20)$$

This proposition can be proved following the arguments of Lemma 4.2 in [HMP19]. A complete proof is provided in Section 4 for the sake of completeness. Importantly, it gives for $\beta = \gamma$ with $\alpha + \beta > 1$ the constraint $\alpha > 1/2$ in Schauder estimates for any non-degenerate symmetric stable operator.

For a class of more regular non-degenerate symmetric stable operators including the fractional Laplacian we have the following better result.

Proposition 2. *Let $\alpha \in (0, 1)$. Assume (1.16) holds and that the spectral measure μ has a smooth density equivalent to the Lebesgue on \mathbb{S}^{d-1} . Then for any $\gamma \in [0, 1]$, there exists $C := C(\eta, \gamma, \mu)$ s.t. for all $\ell \in \{1, 2\}$,*

$$\int_{\mathbb{R}^d} |y|^\gamma |D_y^\ell p_\alpha(t, y)| dy \leq \frac{C}{t^{[\ell-\gamma]/\alpha}}, \quad t > 0. \quad (1.21)$$

This result can be derived using the estimates of Kolokoltsov [Kol00]. It extends to a wider class of spectral measure what was already known for the fractional Laplacian itself. Namely, since the derivatives of the associated heat-kernel exhibit a decay improvement at infinity, see e.g. Bogdan and Jakubowicz [BJ07], then (1.21) holds for any $\gamma \in [0, 1]$. We prove this in Section 4.

Thus for $\beta = \gamma$ with $\alpha + \beta > 1$ we can prove Schauder estimates for operators L_α as in Proposition 2 for any $\alpha \in (0, 1)$.

Remark 3. We will be also able to treat a non-degenerate symmetric stable operator perturbed by a bounded term after proving Schauder estimates for the non-degenerate symmetric stable operator; this is why we can also consider truncated stable operators. Namely, for a given truncation threshold $K > 0$, we can consider as well, with the notations of (1.13), a Lévy measure of the form

$$\nu_{\alpha, K}(dy) = \frac{d\rho\tilde{\mu}(ds)}{\rho^{1+\alpha}} \mathbb{1}_{\rho \in (0, K)}. \quad (1.22)$$

Remark 4. We believe that formula (1.21) with $\gamma \in [0, 1]$ may hold for other class of stable operators. Indeed estimates on the derivatives of densities of stable-like operators which could be useful to establish (1.21) are already given in [Pen17].

On the other hand, one can check that (1.20) does not hold for $\gamma = \alpha$ in the case of a cylindrical fractional Laplacian $\sum_{k=1}^2 (\partial_{x_k}^2)^{\alpha/2}$. Indeed in such case the density $p_\alpha(t, x) = q_\alpha(t, x_1)q_\alpha(t, x_2)$ ($q_\alpha(t, r)$ is the density of a one-dimensional fractional Laplacian) and so, for $t > 0$,

$$\int_{\mathbb{R}^2} |y|^\alpha |D p_\alpha(t, y)| dy \quad \text{is not finite.}$$

1.1.3 Relativistic Stable Operators

Let us consider here L_α corresponding to the relativistic stable operator with symbol

$$\Psi(\lambda) = -(|\lambda|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} + m, \quad (1.23)$$

for some $m > 0$, $\alpha \in (0, 1)$, $\lambda \in \mathbb{R}^d$. It appears to be an important object in the study of relativistic Schrödinger operators (see [Ryz02] and also the references therein).

The operator L_α can be represented by (1.9) with $\nu = \nu_{\alpha, m}$ which has density $C_{\alpha, d} |x|^{-d-\alpha} e^{-m^{1/\alpha} |x|} \cdot \phi(m^{1/\alpha} |x|)$, $x \neq 0$, with $0 \leq \phi(s) \leq c_{\alpha, d, m} (s^{\frac{d-1+\alpha}{2}} + 1)$, $s \geq 0$ (see Lemma 2 in [Ryz02]). It is clear that $\int_{\mathbb{R}^d} (1 \wedge |x|) \nu_\alpha(dx) < \infty$.

Let us fix $\alpha \in (0, 1)$. The heat-kernel $p_{\alpha, m}$ of such operator is given in formula (7) of [Ryz02] at page 4 (with the correspondence $2\beta = \alpha$):

$$p_{\alpha, m}(t, x) = e^{mt} \int_0^\infty g(u, x) e^{-m^{\frac{2}{\alpha}} u} \theta_\alpha(t, u) du, \quad (1.24)$$

where $g(u, x) = (4\pi u)^{-d/2} e^{-|x|^2/4u}$ is the Gaussian kernel. Moreover $\theta_\alpha(t, u)$, $u > 0$, is the density function of the strictly $\alpha/2$ -stable subordinator at time t (see (4) in the indicated reference). It is well known that $p_m(t, \cdot) \in C^\infty(\mathbb{R}^d)$, $t > 0$. In the limit case $m = 0$, one gets the density of $\Delta^{\alpha/2}$:

$$p_\alpha(t, x) = p_{\alpha,0}(t, x) = \int_0^\infty g(u, x) \theta_\alpha(t, u) du, \quad x \neq 0, \quad (1.25)$$

which is also considered in the proof of Lemma 5 in [BJ07]; note that $\theta_\alpha(t, u) \leq c_1 t u^{-1-\frac{\alpha}{2}}$. We obtain easily that, for $t \in (0, 1]$, $x \neq 0$,

$$p_{\alpha,m}(t, x) \leq e^m p_{\alpha,0}(t, x) \quad (1.26)$$

and so **(NDa)** holds. It will be shown in Section 4.3 that (\mathcal{P}_β) holds even in this non-symmetric case.

1.1.4 Hölder spaces and smoothness assumptions

In the following, for a fixed terminal time T and a Borel function $\psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^\ell$, $\ell \in \{1, \dots, d\}$ which is γ -Hölder continuous in the space variable, $\gamma \in (0, 1)$, uniformly in $t \in [0, T]$ we denote by

$$[\psi]_{\gamma,T} := \inf\{K > 0 : \forall(t, x, x') \in [0, T] \times (\mathbb{R}^d)^2, |\psi(t, x) - \psi(t, x')| \leq K|x - x'|^\gamma\}. \quad (1.27)$$

We write that $\psi \in L^\infty([0, T], C^\gamma(\mathbb{R}^d, \mathbb{R}^\ell))$ as soon as $[\psi]_{\gamma,T} < \infty$.

If additionally the function ψ is bounded, we write that $\psi \in L^\infty([0, T], C_b^\gamma(\mathbb{R}^d, \mathbb{R}^\ell))$, where the subscript precisely emphasizes the boundedness, and introduce the corresponding *norm*:

$$\|\psi\|_{L^\infty(C_b^\gamma)} := \|\psi\|_\infty + [\psi]_{L^\infty(C^\gamma)} = \|\psi\|_\infty + [\psi]_{\gamma,T}, \quad \|\psi\|_\infty := \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |\psi(t, x)|. \quad (1.28)$$

Again, $C_b^\gamma(\mathbb{R}^d, \mathbb{R}^\ell)$ is the *usual* Hölder space of index γ .

We also naturally define, accordingly the spaces $L^\infty([0, T], C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^\ell))$ and $C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^\ell)$ respectively endowed with the norms:

$$\begin{aligned} \|\psi\|_{L^\infty([0,T], C_b^{1+\gamma})} &:= \|\psi\|_\infty + \|D\psi\|_\infty + [D\psi]_{\gamma,T}, \quad \psi \in L^\infty([0, T], C_b^{1+\gamma}); \\ \|\varphi\|_{C_b^{1+\gamma}} &:= \|\varphi\|_\infty + \|D\varphi\|_\infty + [D\varphi]_\gamma \\ &:= \sup_{x \in \mathbb{R}^d} |\varphi(x)| + \sup_{x \in \mathbb{R}^d} |D\varphi(x)| + \sup_{(x,x') \in (\mathbb{R}^d)^2, x \neq x'} \frac{|D\varphi(x) - D\varphi(x')|}{|x - x'|^\gamma}, \end{aligned} \quad (1.29)$$

$\varphi \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^\ell)$. With these notations at hand, for a fixed stability index $\alpha \in (0, 1)$, and given final horizon $T > 0$ our assumptions concerning the smoothness of the coefficients in (1.1) are the following:

- (S)** There exists $\beta \in (0, 1)$ s.t. $\alpha + \beta > 1$ and the source $f \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$, $g \in C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R})$.
- (D)** The drift/transport term F verifies (1.3) with β as in **(S)** for some $K_0 > 0$.

We will say that assumption **(A)** is in force as soon as the above conditions **(S)**, **(D)**, **(NDa)** and **(NDb)** hold. In particular under **(A)** we have that property (\mathcal{P}_β) holds.

1.2 Main Results

The solutions of (1.1) will be sought in function spaces which are the natural extension in the current stable framework of those considered in the diffusive setting by Krylov and Priola [KP10]. Namely, we introduce $\mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ the set of functions $\psi(t, x)$ defined on $[0, T] \times \mathbb{R}^d$ such that:

- (i) The function ψ is continuous on $[0, T] \times \mathbb{R}^d$.
- (ii) For any $t \in [0, T]$ the function $\psi(t, \cdot) \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ and the norm $\|\psi(t, \cdot)\|_{C_b^{\alpha+\beta}}$ is bounded w.r.t $t \in [0, T]$, i.e., $\psi \in L^\infty([0, T], C_b^{\alpha+\beta}(\mathbb{R}^d))$.
- (iii) There exists a function $\varphi_\psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ s.t. for any smooth and compactly supported function $\eta \in C_0^\infty([0, T] \times \mathbb{R}^d)$, the product $(\varphi_\psi \eta)(t, x)$ is bounded and $\beta + \alpha - 1$ -Hölder continuous in space uniformly in $t \in [0, T]$ and for any $x \in \mathbb{R}^d$, $0 \leq t < s \leq T$, it holds that:

$$\psi(s, x) - \psi(t, x) = \int_t^s \varphi_\psi(v, x) dv.$$

For $\psi \in \mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$, we write $\partial_t \psi = \varphi_\psi$ which is actually the generalized derivative w.r.t. the time variable of the function ψ .

Accordingly, having a solution to (1.1) in $\mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ is equivalent to say that for $0 \leq t < s \leq T$, $x \in \mathbb{R}^d$,

$$u(t, x) = u(s, x) + \int_t^s dr f(r, x) - \int_t^s dr \left(L_\alpha + F(r, x) \cdot D \right) u(r, x). \quad (1.30)$$

Theorem 3 (Schauder Estimates). *Let $\alpha \in (0, 1)$ be fixed. Under **(A)** there exists a constant $C := C((\mathbf{A}), T, K_0)$ s.t. for any solution $u \in \mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ of (1.1), it holds that:*

$$\|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \leq C(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0, T], C_b^\beta)}). \quad (1.31)$$

In particular, the previous control provides uniqueness in the considered function space. Associated with an existence result developed in Section 3, we eventually derive the following theorem.

Theorem 4 (Existence and Uniqueness). *Let $\alpha \in (0, 1)$ be fixed. Under the assumptions of the previous theorem there exists a unique solution $u \in \mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ to (1.1) which also satisfies the estimate (1.31).*

We finish the section with some comments on the case $\alpha \in [1, 2)$.

Remark 5. When $\alpha \in [1, 2)$, a quite general class of generators of Lévy processes such that elliptic Schauder estimates hold is the one considered in Section 6 of [Pri18]. To introduce such class we define the Lévy generator

$$L\phi(x) = \int_{\mathbb{R}^d} (\phi(x+y) - \phi(x) - 1_{\{|y| \leq 1\}} D\phi(x) \cdot y) \nu(dy), \quad x \in \mathbb{R}^d, \quad \phi \in C_0^\infty(\mathbb{R}^d).$$

We assume that the Blumenthal-Gettoor exponent $\alpha_0 = \alpha_0(\nu)$, $\alpha_0 = \inf \{ \sigma > 0 : \int_{\{|x| \leq 1\}} |y|^\sigma \nu(dy) < \infty \}$ belongs to $(0, 2)$. Moreover, we require that the associated convolution semigroup (P_t) verifies: $P_t(C_b(\mathbb{R}^d)) \subset C_b^1(\mathbb{R}^d)$, $t > 0$, and, further, there exists $c_{\alpha_0} = c_{\alpha_0}(\nu) > 0$ such that

$$\sup_{x \in \mathbb{R}^d} |DP_t f(x)| \leq c_{\alpha_0} t^{-\frac{1}{\alpha_0}} \cdot \sup_{x \in \mathbb{R}^d} |f(x)|, \quad t \in (0, 1], \quad f \in C_b(\mathbb{R}^d). \quad (1.32)$$

Non-degenerate symmetric α -stable operators, relativistic α -stable operators, temperate α -stable operators verify the previous two assumptions with $\alpha = \alpha_0$.

According to Theorem 6.7 in [Pri18] we have, for $\alpha_0 \geq 1$, $\beta \in (0, 1)$ and $\beta + \alpha_0 > 1$,

$$\lambda \|w_\lambda\|_\infty + [Dw_\lambda]_{C_b^{\alpha_0+\beta-1}(\mathbb{R}^d)} \leq C_0 \|\lambda w - \mathcal{L}w - b \cdot Dw\|_{C_b^\beta(\mathbb{R}^d)}, \quad \lambda \geq 1, \quad (1.33)$$

assuming $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$. Such elliptic estimates could be extended to the parabolic setting without difficulties.

2 Proof of the main results through a perturbative approach

2.1 Frozen semi-group and associated smoothing effects

The key idea in our approach consists in considering a suitable *proxy* IPDE, for which we have *good* controls along which to expand a solution $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ to (1.1). Under **(A)**, which involves potentially unbounded drifts, we will use for the proxy IPDE a non zero first order term which involves a flow associated with the drift coefficient F (which in the current setting exists from the Peano theorem). This flow is, for given *freezing* parameters $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, defined as:

$$\theta_{s, \tau}(\xi) = \xi + \int_\tau^s F(v, \theta_{v, \tau}(\xi)) dv, \quad s \geq \tau, \quad (2.1)$$

$\theta_{s, \tau}(\xi) = \xi$ for $s < \tau$. For f and g as in Theorem 3 we then introduce our proxy IPDE:

$$\begin{aligned} \partial_t \tilde{u}(t, x) + L_\alpha \tilde{u}(t, x) + F(t, \theta_{t, \tau}(\xi)) \cdot D_x \tilde{u}(t, x) &= -f(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\ \tilde{u}(T, x) &= g(x), \quad \text{on } \mathbb{R}^d. \end{aligned} \quad (2.2)$$

Under **(A)**, it is clear that the time-dependent operator $L_\alpha + F(t, \theta_{t, \tau}(\xi)) \cdot D_x$ generates a family of transition probability (or two parameter transition semi-group) $(\tilde{P}_{s, t, \alpha}^{(\tau, \xi)})_{0 \leq t \leq s \leq T}$. For fixed $0 \leq t < s \leq T$, the associated heat-kernel writes:

$$\tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) = p_\alpha \left(s - t, y - m_{s, t}^{(\tau, \xi)}(x) \right), \quad m_{s, t}^{(\tau, \xi)}(x) := x + \int_t^s F(v, \theta_{v, \tau}(\xi)) dv, \quad (2.3)$$

(we remark that, for fixed $(t, x) \in [0, T] \times \mathbb{R}^d$ and $s \in [t, T]$, the corresponding process with frozen drift is $\tilde{X}_s^{\tau, \xi} = x + \int_t^s F(v, \theta_{v, \tau}(\xi)) dv + Z_{s-t}$) with generator L_α . Observe that from the above definition of the *shift* $m_{s,t}^{(\tau, \xi)}(x)$ we have the important property

$$m_{s,t}^{(\tau, \xi)}(x)|_{(\tau, \xi)=(t, x)} = \theta_{s,t}(x). \quad (2.4)$$

Let us now state the smoothing effect of the semi-group associated with (2.2).

Lemma 5 (Smoothing effects of the derivatives of the frozen semi-group). *Assume (NDa) and (NDb) (so property (\mathcal{P}_β) holds). There exists $C \geq 1$ s.t. for any $\varphi \in C^\beta(\mathbb{R}^d, \mathbb{R})$, any freezing couple (τ, ξ) , $\ell \in \{1, 2\}$ and all $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$:*

$$|D_x^\ell \tilde{P}_{s,t,\alpha}^{(\tau, \xi)} \varphi(x)| \leq \frac{C[\varphi]_\beta}{(s-t)^{\frac{\ell}{\alpha} - \frac{\beta}{\alpha}}}. \quad (2.5)$$

Proof. We recall that, with the notation of (2.3), $m_{s,t}^{(\tau, \xi)}(x) := x + \int_t^s F(v, \theta_{v, \tau}(\xi)) dv$. Again, if $(\tau, \xi) = (t, x)$ then $m_{s,t}^{(\tau, \xi)}(x) = \theta_{s,t}(x)$ but this will not be the only case to be considered. Let us prove (2.5). We use a cancellation argument (recall that $\int_{\mathbb{R}^d} dy \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) = 1$) and property (\mathcal{P}_β) to write:

$$\begin{aligned} |D_x \tilde{P}_{s,t,\alpha}^{(\tau, \xi)} \varphi(x)| &= \left| \int_{\mathbb{R}^d} dy D \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) [\varphi(y) - \varphi(m_{s,t}^{(\tau, \xi)}(x))] \right| \\ &= \left| \int_{\mathbb{R}^d} dy D_x p_\alpha(s-t, y - m_{s,t}^{(\tau, \xi)}(x)) [\varphi(y) - \varphi(m_{s,t}^{(\tau, \xi)}(x))] \right| \\ &\leq [\varphi]_\beta \int_{\mathbb{R}^d} dy |D_x p_\alpha(s-t, y - m_{s,t}^{(\tau, \xi)}(x))| |y - m_{s,t}^{(\tau, \xi)}(x)|^\beta \\ &\leq [\varphi]_\beta \int_{\mathbb{R}^d} d\tilde{y} |D_x p_\alpha(s-t, \tilde{y})| |\tilde{y}|^\beta \stackrel{(\mathcal{P}_\beta)}{\leq} \frac{C[\varphi]_\beta}{(s-t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}}. \end{aligned}$$

This proves the result for $\ell = 1$. The case $\ell = 2$ is dealt similarly. \square

We point out that, if we restrict to bounded functions $\varphi \in C_b^\beta(\mathbb{R}^d, \mathbb{R})$, the above result can also be derived directly from standard interpolation arguments provided (1.16) holds and without the integrability constraints of (\mathcal{P}_β) (see e.g. the proof of Theorem 3.3 in [Pri12]).

We now define our candidate to be the solution of the proxy IPDE. This candidate appears to be (if it exists and is smooth enough) the representation of the solution of (2.2) obtained through the Duhamel principle. Hence, we call it the Duhamel representation associated with (2.2). As it will appear, such a representation is robust enough to satisfy Schauder estimates under the assumptions of Theorem 3. It then indeed provides a solution to (2.2).

Definition 1 (Duhamel representation associated with (2.2)). *For any freezing couple $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, we call Duhamel representation associated with (2.2), and we denote it by $\tilde{u}^{(\tau, \xi)}$ the map*

$$\tilde{u}^{(\tau, \xi)} : [0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \tilde{P}_{T,t,\alpha}^{(\tau, \xi)} g(x) + \int_t^T ds \left(\tilde{P}_{s,t,\alpha}^{(\tau, \xi)} f(s, \cdot) \right)(x), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (2.6)$$

Proposition 6 (Schauder estimate for the frozen semi-group). *Under (A), for any freezing couple $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, the following Schauder estimate holds for the map $\tilde{u}^{(\tau, \xi)}$ defined by (2.6). There exists a constant $C := C((\mathbf{A}))$ s.t.*

$$\|\tilde{u}^{(\tau, \xi)}\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \leq C(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0, T], C_b^\beta)}). \quad (2.7)$$

Proof. We focus on the proof of (2.7) and proceed in three steps: control of the supremum norm of the function, of its gradient, of the Hölder modulus of the gradient.

(i) *Supremum norm of the solution.*

The control for the supremum norm readily follows from (2.6). Indeed, for a fixed frozen couple $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$ and any $(t, x) \in [0, T] \times \mathbb{R}^d$:

$$|\tilde{u}^{(\tau, \xi)}(t, x)| = \left| \tilde{P}_{T,t,\alpha}^{(\tau, \xi)} g(x) + \int_t^T ds \left(\tilde{P}_{s,t,\alpha}^{(\tau, \xi)} f(s, \cdot) \right)(x) \right| \leq \|g\|_\infty + T\|f\|_\infty. \quad (2.8)$$

(ii) *Supremum norm of the gradient.*

Let us now give a global bound for the gradient. Differentiating (2.6) yields:

$$\begin{aligned} |D_x \tilde{u}^{(\tau, \xi)}(t, x)| &= \left| D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x) + \int_t^T ds D_x \left(\tilde{P}_{s, t, \alpha}^{(\tau, \xi)} f(s, \cdot) \right) (x) \right| \\ &\leq \left| D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x) \right| + \left| \int_t^T ds D_x \left(\tilde{P}_{s, t, \alpha}^{(\tau, \xi)} f(s, \cdot) \right) (x) \right|. \end{aligned} \quad (2.9)$$

From (2.3) write for $g \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ recalling that $\alpha + \beta > 1$:

$$\begin{aligned} D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x) &= D_x \int_{\mathbb{R}^d} dy \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x, y) g(y) = D_x \int_{\mathbb{R}^d} dy p_\alpha(T - t, y - m_{T, t}^{(\tau, \xi)}(x)) g(y) \\ &= D_x \int_{\mathbb{R}^d} dy p_\alpha \left(T - t, y - \left(x + \int_t^T F(v, \theta_{v, \tau}(\xi)) dv \right) \right) g(y) \\ &= D_x \int_{\mathbb{R}^d} dz p_\alpha(T - t, z) g \left(z + \left(x + \int_t^T F(v, \theta_{v, \tau}(\xi)) dv \right) \right), \\ |D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x)| &\leq \|Dg\|_\infty. \end{aligned} \quad (2.10)$$

Let us now turn to control of the source in (2.9). From Lemma 5 we derive

$$|D_x (\tilde{P}_{s, t, \alpha}^{(\tau, \xi)} f(s, \cdot))(x)| \leq \frac{C[f]_{\beta, T}}{(s - t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}}.$$

Since $\alpha + \beta > 1$ we thus get that $1/\alpha - \beta/\alpha < 1$ so that the above singularity is integrable in time. We therefore derive:

$$\left| \int_t^T ds D_x \left(\tilde{P}_{s, t, \alpha}^{(\tau, \xi)} f(s, \cdot) \right) (x) \right| \leq C[f]_{\beta, T} \int_t^T \frac{ds}{(s - t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}} \leq C[f]_{\beta, T} (T - t)^{\frac{\alpha + \beta - 1}{\alpha}}. \quad (2.11)$$

Plugging (2.11) and (2.10) into (2.9) gives the following bound for the gradient:

$$|D_x \tilde{u}^{(\tau, \xi)}(t, x)| \leq \|Dg\|_\infty + C(T - t)^{\frac{\alpha + \beta - 1}{\alpha}} [f]_{\beta, T}. \quad (2.12)$$

(iii) *Hölder modulus of the solution.*

It now remains to control the $\alpha + \beta - 1$ Hölder modulus of the gradient of the solution. To this end we introduce for a given time $t \in [0, T]$ and given spatial points $x, x' \in \mathbb{R}^d$, the notion of *diagonal* and *off-diagonal* regime.

For the frozen semi-group, we say that the off-diagonal regime (resp. diagonal) holds when $T - t \leq c_0 |x - x'|^\alpha$ (resp. $T - t \geq c_0 |x - x'|^\alpha$). We use here a constant c_0 , which will be for our further perturbative analysis meant to be small for circular type arguments to work. Anyhow, for the frozen semi-group this constant could be arbitrary, for instance 1.

We first investigate the Hölder continuity in space of $x \mapsto D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x)$.

(iii) – (a) *Off-diagonal regime.* Let $T - t \leq c_0 |x - x'|^\alpha$. Observe from (2.3) that $D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x, y) = -D_y \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x, y)$. We therefore write:

$$\begin{aligned} &D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x) - D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)} g(x') \\ &= \left[\int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x, y) Dg(y) dy - \int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x', y) Dg(y) dy \right] \\ &= \left[\int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x, y) [Dg(y) - Dg(m_{t, T}^{(\tau, \xi)}(x))] dy \right. \\ &\quad \left. + [Dg(m_{t, T}^{(\tau, \xi)}(x)) - Dg(m_{t, T}^{(\tau, \xi)}(x'))] - \int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau, \xi)}(t, T, x', y) [Dg(y) - Dg(m_{t, T}^{(\tau, \xi)}(x'))] dy \right], \end{aligned}$$

recalling that $\tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, \cdot)$, $\tilde{p}_\alpha^{(\tau,\xi')}(t, T, x', \cdot)$ are probability densities for the last equality. Since Dg is $\alpha+\beta-1$ -Hölder continuous and $\alpha + \beta - 1 < \alpha$, we therefore get:

$$\begin{aligned} & |D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \\ & \leq [Dg]_{\alpha+\beta-1} \left(\int_{\mathbb{R}^d} dy p_\alpha(T-t, y - m_{T,t}^{(\tau,\xi)}(x)) |y - m_{T,t}^{(\tau,\xi)}(x)|^{\alpha+\beta-1} \right. \\ & \quad \left. + |m_{t,T}^{(\tau,\xi)}(x) - m_{t,T}^{(\tau,\xi)}(x')|^{\alpha+\beta-1} + \int_{\mathbb{R}^d} dy p_\alpha(T-t, y - m_{T,t}^{(\tau,\xi)}(x')) |y - m_{T,t}^{(\tau,\xi)}(x')|^{\alpha+\beta-1} \right) \\ & \leq [Dg]_{\alpha+\beta-1} \left(C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + |x-x'|^{\alpha+\beta-1} \right), \end{aligned}$$

using property **(NDa)** and recalling that the mapping $x \mapsto m_{T,t}^{(\tau,\xi)}(x)$ is affine for the last inequality. On the considered off-diagonal regime, the previous bound eventually yields:

$$|D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \leq [Dg]_{\alpha+\beta-1} (C c_0^{\frac{\alpha+\beta-1}{\alpha}} + 1) |x-x'|^{\alpha+\beta-1}, \quad (2.13)$$

which is the expected bound.

(iii) – (b) *Diagonal regime.* If $T-t > c_0|x-x'|^\alpha$, we directly write for all $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$:

$$\begin{aligned} & |D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \\ & \leq \left| \int_{\mathbb{R}^d} [\tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, y) - \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x', y)] Dg(y) dy \right| \\ & \leq \left| \int_0^1 d\mu \int_{\mathbb{R}^d} [D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x' + \mu(x-x'), y) \cdot (x-x')] Dg(y) dy \right| \\ & = \left| \int_0^1 d\mu \int_{\mathbb{R}^d} [D_x p_\alpha(T-t, y - m_{T,t}^\xi(x' + \mu(x-x'))) \cdot (x-x')] Dg(y) dy \right|. \end{aligned}$$

This contribution is again dealt through usual cancellation techniques recalling that since $\int_{\mathbb{R}^d} \tilde{p}_\alpha^\xi(t, T, x' + \mu(x-x'), y) dy = 1$ then for $\ell \in \{1, 2\}$, $D_x^\ell \int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x' + \mu(x-x'), y) dy = 0$. We get from the above estimate that denoting as well as in (2.3) by $m_{T,t}^{(\tau,\xi)}(x' + \mu(x-x')) := x' + \mu(x-x') + \int_t^T F(v, \theta_{v,\tau}(\xi)) dv$, we obtain:

$$\begin{aligned} & |D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \quad (2.14) \\ & \leq \left| \int_0^1 d\mu \int_{\mathbb{R}^d} [D_x p_\alpha(T-t, y - m_{T,t}^{(\tau,\xi)}(x') - \mu(x-x')) \cdot (x-x')] [Dg(y) - Dg(m_{T,t}^{(\tau,\xi)}(x') + \mu(x-x'))] dy \right| \\ & \leq [Dg]_{\alpha+\beta-1} \int_0^1 d\mu \int_{\mathbb{R}^d} |D_x p_\alpha(T-t, y - m_{T,t}^{(\tau,\xi)}(x') - \mu(x-x')) \cdot (x-x')| |y - m_{T,t}^{(\tau,\xi)}(x') - \mu(x-x')|^{\alpha+\beta-1} dy, \end{aligned}$$

since $g \in C_b^{\alpha+\beta}(\mathbb{R}^d)$. From Lemma 5, we obtain

$$\begin{aligned} & |D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \\ & \leq C [Dg]_{C_b^{\alpha+\beta-1}} (T-t)^{-\frac{1}{\alpha} + \frac{\alpha+\beta-1}{\alpha}} |x-x'| \leq C [Dg]_{\alpha+\beta-1} |x-x'|^{\alpha+\beta-1}, \quad (2.15) \end{aligned}$$

where the above constant C also depends on c_0 : since $c_0|x-x'|^\alpha < (T-t)$ and $\alpha + \beta < 2$, we indeed have $(T-t)^{-\frac{1}{\alpha} + \frac{\alpha+\beta-1}{\alpha}} |x-x'| \leq (c_0|x-x'|^\alpha)^{-\frac{1}{\alpha} + \frac{\alpha+\beta-1}{\alpha}} |x-x'| \leq C|x-x'|^{\alpha+\beta-1}$. Then, from equations (2.15) and (2.13) we get that, for all $x, x' \in \mathbb{R}^d$:

$$|D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)} g(x')| \leq C [Dg]_{\alpha+\beta-1} |x-x'|^{\alpha+\beta-1}, \quad (2.16)$$

which gives the expected control.

Let us now turn to the mapping $x \mapsto \int_t^T ds D_x \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s, \cdot) \right)(x)$. For separate points, i.e., $|x-x'| > 0$, while integrating in $s \in [t, T]$ we have that, accordingly with the previous terminology if $s \in [t, (t+c_0|x-x'|^\alpha) \wedge T]$ then the *off-diagonal* regime holds for the analysis of

$$D_x \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s, \cdot) \right)(x) - D_x \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s, \cdot) \right)(x') \quad (2.17)$$

and similarly, if $s \in [(t + c_0|x - x'|^\alpha) \wedge T, T]$ then the *diagonal* regime holds.

We now introduce the transition time t_0 defined as follows:

$$t_0 = (t + c_0|x - x'|^\alpha) \wedge T, \quad (2.18)$$

i.e., t_0 precisely corresponds to the critical time at which a change of regime occurs. Observe that, if $t_0 = T$ the contribution (2.17) is in the *off-diagonal* regime along the whole time interval.

Introduce the following family of Green kernels.

$$\forall 0 \leq v < r \leq T, \quad \tilde{G}_{r,v,\alpha}^{(\tau,\xi)} f(t,x) := \int_v^r ds \int_{\mathbb{R}^d} dy \tilde{p}_\alpha^{(\tau,\xi)}(t,s,x,y) f(s,y). \quad (2.19)$$

The off-diagonal contribution associated with the difference (2.17) now writes:

$$\begin{aligned} |D_x \tilde{G}_{t_0,t,\alpha}^{(\tau,\xi)} f(t,x) - D_x \tilde{G}_{t_0,t,\alpha}^{(\tau,\xi)} f(t,x')| &\leq \left| \int_t^{t_0} ds D_x \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s,\cdot) \right) (x) \right| + \left| \int_t^{t_0} ds D_x \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s,\cdot) \right) (x') \right| \\ &\leq C[f]_{T,\beta} \int_t^{t_0} ds (s-t)^{-\frac{1}{\alpha} + \frac{\beta}{\alpha}} \leq C[f]_{T,\beta} |x - x'|^{\alpha+\beta-1}, \end{aligned} \quad (2.20)$$

using equation (2.5) of Lemma 5 for the last but one inequality.

For the *diagonal regime*, which appears if $t_0 < T$ and corresponds in that case to the difference

$$D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x) - D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x'),$$

we have to be more subtle and perform a Taylor expansion of $D_x(\tilde{P}_{s,t,\alpha}^\xi f(s,\cdot))$ similarly to what we did in (2.14). Namely:

$$\begin{aligned} |D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x) - D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x')| &\leq \int_{t_0}^T ds \left| \int_0^1 d\mu D_x^2 \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s,\cdot) \right) (x' + \mu(x-x')) \cdot (x-x') \right| \\ &\leq |x-x'| \int_{t_0}^T ds \int_0^1 d\mu \left| D_x^2 \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} f(s,\cdot) \right) (x' + \mu(x-x')) \right| \\ &\leq C[f]_{\beta,T} |x-x'| \int_{t_0}^T ds (s-t)^{-\frac{2}{\alpha} + \frac{\beta}{\alpha}}, \end{aligned}$$

using again Lemma 5, equation (2.5) for the last inequality. This finally yields, recalling that $t_0 = (t + c_0|x - x'|^\alpha) \wedge T$:

$$\begin{aligned} |D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x) - D_x \tilde{G}_{T,t_0,\alpha}^{(\tau,\xi)} f(t,x')| &\leq C[f]_{\beta,T} |x-x'| (|x-x'|^\alpha)^{1-\frac{2}{\alpha} + \frac{\beta}{\alpha}} \\ &\leq C[f]_{\beta,T} |x-x'|^{\alpha+\beta-1}. \end{aligned} \quad (2.21)$$

Gathering (2.20) and (2.21) gives the stated estimate for the Hölder modulus of the gradient of the Green kernel. We eventually derive:

$$|D_x \tilde{u}^{(\tau,\xi)}(t,x) - D_x \tilde{u}^{(\tau,\xi)}(t,x')| \leq C([Dg]_{\alpha+\beta-1} + [f]_{\beta,T}) |x-x'|^{\alpha+\beta-1}. \quad (2.22)$$

The estimate (2.7) of the proposition follows from (2.8), (2.12) and (2.22). □

Proposition 7. *Let \tilde{u} be the map defined in (2.6). Then,*

- (i) *if property (\mathcal{P}_β) holds, for any freezing couple $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$ the function $\tilde{u}^{(\tau,\xi)}$ defined in (2.6) belongs to $\mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ and solves (2.2);*
- (ii) *“conversely”, if for any freezing couple $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$, $\tilde{v}^{(\tau,\xi)}$ is a solution to (2.2) in $\mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ with bounded support, i.e., there exists a compact set $K \subset \mathbb{R}^d$ such that*

$$\text{Supp}(\tilde{v}^{(\tau,\xi)}(t, \cdot)) \subset K, \quad t \in [0, T]. \quad (2.23)$$

then $\tilde{v}^{(\tau,\xi)} = \tilde{u}^{(\tau,\xi)}$ defined by (2.6).

Proof. Assertion (i) is rather direct in view of Proposition 6: first, it follows from this result that for all (τ, ξ) the map $\tilde{u}^{(\tau, \xi)}$ belongs to $L^\infty([0, T], C_b^{\alpha+\beta})$. Secondly, we have from the proof of the aforementioned proposition that for all x in \mathbb{R}^d , the mapping $s \in (t, T] \mapsto (D_x + L_\alpha)[\tilde{P}_{s,t,\alpha}^{(\tau, \xi)}(f(s, \cdot))](x)$ is controlled by an integrable quantity on $(t, T]$. From the very definition of $\tilde{p}_\alpha^{(\tau, \xi)}$ (fundamental solution of (2.2)) one can hence invert the (time) differentiation and integral operator in second term in the r.h.s. of (2.6). Similar arguments apply for the first term $\tilde{P}_{T,t,\alpha}^{(\tau, \xi)}g(\cdot)$ in (2.6). This proves, on the one hand, that for all (τ, ξ) the map $\tilde{u}^{(\tau, \xi)}$ indeed belongs to $\mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$. On the other hand, using again the fact that the kernel $\tilde{p}_\alpha^{(\tau, \xi)}$ in $P_{\cdot, \cdot, \alpha}^{(\tau, \xi)}$ is a fundamental solution of (2.2) we obtain (inverting again differentiation and integration operator) that $\tilde{u}^{(\tau, \xi)}$ solves (2.2). The previous arguments are detailed in the diffusive setting in Lemma 3.3 in [KP10]. This concludes the proof of the first point.

Concerning (ii), we first fix τ and ξ and set $b(t) = F(t, \theta_{t,\tau}(\xi))$, $t \in [0, T]$. We define

$$h(t, x) = \tilde{u}(t, x - \int_t^T b(v)dv).$$

Note that, a.e. in t , $\partial_t h(t, x) + L_\alpha h(t, x) = -f(t, x - \int_t^T b(v)dv)$, $h(T, x) = g(x)$, $x \in \mathbb{R}^d$. Set $l(t, x) = f(t, x - \int_t^T b(v)dv)$.

We can apply the (partial) Fourier transform in the x -variable to $\partial_t h(t, x) + L_\alpha h(t, x)$ and obtain, a.e. in t ,

$$\partial_t v(t, \lambda) + \mathcal{F}(L_\alpha h(t, \cdot))(\lambda),$$

where $v(t, \lambda) = \mathcal{F}h(t, \cdot)(\lambda)$. Note that, for each $t \in [0, T]$, $L_\alpha h(t, \cdot) = \int_{|y| \leq 1} [h(t, \cdot + y) - h(t, \cdot)] \nu_\alpha(dy) + \int_{|y| > 1} [h(t, \cdot + y) - h(t, \cdot)] \nu_\alpha(dy)$ belongs to $L^p(\mathbb{R}^d)$, for any $p \geq 1$.

Now we use the symbol $\Psi(\lambda) = \Psi_\alpha(\lambda)$ of L_α given in (1.14). We find (cf. Section 3.3.2 in [App09])

$$v(s, \lambda) - v(t, \lambda) + \Psi(\lambda) \int_t^s v(r, \lambda) dr = - \int_t^s \hat{l}(r, \lambda), \quad \lambda \in \mathbb{R}^d, \quad t \leq s \leq T, \quad (2.24)$$

where $\mathcal{F}l(t, \cdot)(\lambda) = \hat{l}(t, \lambda)$ with the condition $v(T, \lambda) = \mathcal{F}(g)(\lambda) = \hat{g}(\lambda)$. The solution is given by

$$v(t, \lambda) = e^{(T-t)\Psi(\lambda)} \hat{g}(\lambda) + \int_t^T e^{(r-t)\Psi(\lambda)} \hat{l}(r, \lambda) dr.$$

Using the stable convolution semigroup $P_t = P_t^\alpha$ associated with L_α and the anti-Fourier transform we get

$$h(t, x) = P_{T-t}g(x) + \int_t^T P_{r-t}l(r, \cdot)(x)dr.$$

It follows that $\tilde{u}(t, x - \int_t^T b(v)dv) = P_{T-t}g(x) + \int_t^T P_{r-t}f(r, \cdot)(x - \int_r^T b(v)dv)dr$. Since $\int_r^T b(v)dv = \int_t^T b(v)dv - \int_t^r b(v)dv$, we arrive at

$$\tilde{u}(t, y) = P_{T-t}g(y + \int_t^T b(v)dv) + \int_t^T P_{r-t}f(r, \cdot)(y + \int_t^r b(v)dv)dr.$$

which gives (2.6). □

2.2 Duhamel type formulas

The central point is that we will use the auxiliary *proxy* IPDE (2.2) in order to derive appropriate quantitative controls on a solution $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ of (1.1). The parameters (τ, ξ) are set free and will be chosen in function of the control we aim to establish.

Importantly, we can exploit equation (2.2) in the following proposition which gives a Duhamel type representation of the solution of (1.1) involving precisely the *proxy* IPDE (2.2).

Proposition 8 (A first Duhamel type representation). *Let (A) hold. For a smooth non-negative spatial test function ρ which is equal to 1 on the ball $B(0, 1/2)$ and vanishes outside $B(0, 1)$, we introduce using the proxy parameters (τ, ξ) the following cut-off function*

$$\eta_{\tau, \xi}(s, y) = \rho(y - \theta_{s, \tau}(\xi)), \quad y \in \mathbb{R}^d, \quad s \in [0, T],$$

which precisely localizes around the frozen flow.

For $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ solving (1.1) the function $v_{\tau, \xi} := u\eta_{\tau, \xi}$ solves the equation

$$\begin{aligned} \partial_t v_{\tau, \xi}(t, x) + L_\alpha v_{\tau, \xi}(t, x) + F(t, \theta_{t, \tau}(\xi)) \cdot D_x v_{\tau, \xi}(t, x) &= -\left[(\eta_{\tau, \xi} f)(t, x) + \mathcal{R}_{\tau, \xi}(t, x) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\ v_{\tau, \xi}(T, x) &= g(x)\eta_{\tau, \xi}(T, x), \quad \text{on } \mathbb{R}^d, \end{aligned} \quad (2.25)$$

where, $\nu = \nu_\alpha$,

$$\begin{aligned} \mathcal{R}_{\tau, \xi}(t, x) &= \left[([F(t, x) - F(t, \theta_{t, \tau}(\xi))] \cdot D_x u(t, x)) \eta_{\tau, \xi}(t, x) \right] \\ &\quad - \left[u(t, x) L_\alpha \eta_{\tau, \xi}(t, x) + \int_{\mathbb{R}^d} (u(t, x+y) - u(t, x)) (\eta_{\tau, \xi}(t, x+y) - \eta_{\tau, \xi}(t, x)) \nu(dy) \right] \\ &=: R_{\tau, \xi}(t, x) + \mathcal{S}_{\tau, \xi}(t, x), \end{aligned} \quad (2.26)$$

and for $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ solving (1.1), $\mathcal{S}_{\tau, \xi} \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$.

Importantly, the following representations also hold:

$$\begin{aligned} v_{\tau, \xi}(t, x) &= \tilde{u}^{(\tau, \xi)}(t, x) + \int_t^T ds \tilde{P}_{s, t, \alpha}^{(\tau, \xi)} \left(R_{\tau, \xi}(s, \cdot) \right) (x), \\ D_x v_{\tau, \xi}(t, x) &= D_x \tilde{u}^{(\tau, \xi)}(t, x) + D_x \int_t^T ds \tilde{P}_{s, t, \alpha}^{(\tau, \xi)} \left(R_{\tau, \xi}(s, \cdot) \right) (x) \end{aligned} \quad (2.27)$$

where the function $\tilde{u}^{(\tau, \xi)}$ solves

$$\begin{aligned} \partial_t \tilde{u}^{(\tau, \xi)}(t, x) + L_\alpha \tilde{u}^{(\tau, \xi)}(t, x) + F(t, \theta_{t, \tau}(\xi)) \cdot D_x \tilde{u}^{(\tau, \xi)}(t, x) &= -\left(\eta_{\tau, \xi} f + \mathcal{S}_{\tau, \xi} \right) (t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\ \tilde{u}^{(\tau, \xi)}(T, x) &= g(x)\eta_{\tau, \xi}(T, x), \quad \text{on } \mathbb{R}^d, \end{aligned} \quad (2.28)$$

Eventually,

$$\left(D_x v_{\tau, \xi}(t, x) \right) \Big|_{(\tau, \xi)=(t, x)} = D_x u(t, x) = \left(D_x \tilde{u}^{(\tau, \xi)}(t, x) \right) \Big|_{(\tau, \xi)=(t, x)} + \int_t^T ds \left[D_x \tilde{P}_{s, t, \alpha}^{(\tau, \xi)} \left(R_{\tau, \xi}(s, \cdot) \right) (x) \right] \Big|_{(\tau, \xi)=(t, x)}, \quad (2.29)$$

Remark 6. The above representation formulas (2.27) are crucial in the sense that they allow to write any solution $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ of (1.1) localized with a cut-off along the flow in terms of the solution $\tilde{u}^{(\tau, \xi)}$ to equation (2.2) with modified source and terminal condition, namely $-(\eta_{\tau, \xi} f) + \mathcal{S}_{\tau, \xi}$ and $g\eta_{\tau, \xi}(T, \cdot)$ respectively, and the remainder term $\int_t^T ds \tilde{P}_{s, t, \alpha}^{(\tau, \xi)} \left(R_{\tau, \xi}(s, \cdot) \right)$. Roughly speaking, the regularity of $\tilde{u}^{(\tau, \xi)}$ follows from Proposition 6 whereas the control of the remainder will precisely be the difficult remaining part of the work for which we will also need to specify, in the representations, the appropriate values of the freezing parameters $(\tau, \xi) \in [0, T] \times \mathbb{R}^d$.

Proof. Let a solution $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ of (1.1) be given and let us prove that $v_{\tau, \xi} = u\eta_{\tau, \xi}$ satisfies (2.25). Observe that:

$$\begin{aligned} \partial_t v_{\tau, \xi}(t, x) &= \partial_t u(t, x) \eta_{\tau, \xi}(t, x) + u(t, x) \partial_t \eta_{\tau, \xi}(t, x) \\ &= \partial_t u(t, x) \eta_{\tau, \xi}(t, x) - u(t, x) D\rho(x - \theta_{t, \tau}(\xi)) \cdot F(t, \theta_{t, \tau}(\xi)), \\ F(t, \theta_{t, \tau}(\xi)) \cdot Dv_{\tau, \xi}(t, x) &= \left(F(t, \theta_{t, \tau}(\xi)) \cdot Du(t, x) \right) \eta_{\tau, \xi}(t, x) + u(t, x) F(t, \theta_{t, \tau}(\xi)) \cdot D\rho(x - \theta_{t, \tau}(\xi)), \\ L_\alpha(v_{\tau, \xi}(t, x)) &= (L_\alpha u(t, x)) \eta_{\tau, \xi}(t, x) + (L_\alpha \eta_{\tau, \xi}(t, x)) u(t, x) \\ &\quad + \int_{\mathbb{R}^d} (u(t, x+y) - u(t, x)) (\eta_{\tau, \xi}(t, x+y) - \eta_{\tau, \xi}(t, x)) \nu(dy) \\ &= (L_\alpha u(t, x)) \eta_{\tau, \xi}(t, x) - \mathcal{S}_{\tau, \xi}(t, x). \end{aligned}$$

Summing the above terms yields:

$$\begin{aligned} &\partial_t v_{\tau, \xi}(t, x) + F(t, \theta_{t, \tau}(\xi)) \cdot Dv_{\tau, \xi}(t, x) + L_\alpha(v_{\tau, \xi}(t, x)) \\ &= (\partial_t u + F(t, \theta_{t, \tau}(\xi)) \cdot Du(t, x) + L_\alpha u(t, x)) \eta_{\tau, \xi}(t, x) - \mathcal{S}_{\tau, \xi}(t, x) \\ &= (\partial_t u + F(t, x) \cdot Du(t, x) + L_\alpha u(t, x)) \eta_{\tau, \xi}(t, x) + \left((F(t, \theta_{t, \tau}(\xi)) - F(t, x)) \cdot Du(t, x) \right) \eta_{\tau, \xi}(t, x) - \mathcal{S}_{\tau, \xi}(t, x) \\ &= -\left(f(t, x) \eta_{\tau, \xi}(t, x) + R_{\tau, \xi}(t, x) + \mathcal{S}_{\tau, \xi}(t, x) \right) = -\left(f(t, x) \eta_{\tau, \xi} + \mathcal{R}_{\tau, \xi} \right) (t, x), \end{aligned}$$

which precisely gives equation (2.25). In the above right-hand side it is clear that $f\eta_{\tau,\xi} \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$. Also, the mapping

$$R_{\tau,\xi} : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto \left((F(t, \theta_{t,\tau}(\xi)) - F(t, x)) \cdot Du(t, x) \right) \eta_{\tau,\xi}(t, x)$$

belongs to $L^\infty([0, T], C_b^{\alpha+\beta-1}(\mathbb{R}^d, \mathbb{R}))$.

The lower regularity is precisely due to the fact that $Du \in L^\infty([0, T], C_b^{\alpha+\beta-1}(\mathbb{R}^d, \mathbb{R}))$. Anyhow, this is not a problem here since we are, for the moment, simply interested in finding the representation formulas in (2.27) which will in turn allow to investigate quantitative bounds related to the smoothness of u (gradient bounds and Hölder moduli). Let us now show similarly to the proof of Theorem 3.4 in [Pri12] that $\mathcal{S}_{\tau,\xi} \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$. In the quoted reference, the previous smoothness property is obtained for $\alpha = 1$.

Introduce the non-local operator $\mathcal{T}_{\tau,\xi}$ defined for $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$ as:

$$\mathcal{T}_{\tau,\xi} f(x) := \int_{\mathbb{R}^d} (f(x+y) - f(x)) (\eta_{\tau,\xi}(x+y) - \eta_{\tau,\xi}(x)) \nu(dy).$$

It is direct to check that $\mathcal{T}_{\tau,\xi}$ is continuous from $C_b^1(\mathbb{R}^d, \mathbb{R})$ to $C_b(\mathbb{R}^d, \mathbb{R})$. Observe now that for a function $f \in C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R})$:

$$\begin{aligned} |D\mathcal{T}_{\tau,\xi} f(x)| &\leq \left| \int_{\mathbb{R}^d} (Df(x+y) - Df(x)) (\eta_{\tau,\xi}(x+y) - \eta_{\tau,\xi}(x)) \nu(dy) \right| \\ &\quad + \left| \int_{\mathbb{R}^d} (f(x+y) - f(x)) (D\eta_{\tau,\xi}(x+y) - D\eta_{\tau,\xi}(x)) \nu(dy) \right| \\ &\leq \left(\|Df\|_{C_b^{\alpha+\beta-1}} \|\eta_{\tau,\xi}\|_{C_b^1} \int_{|y|\leq 1} |y|^{\alpha+\beta} \nu(dy) + 4\|Df\|_\infty \int_{|y|>1} \nu(dy) \right) \\ &\quad + \|Df\|_\infty \|\eta_{\tau,\xi}\|_{C_b^2} \int_{|y|\leq 1} |y|^2 \nu(dy) + 4\|f\|_\infty \|D\eta_{\tau,\xi}\|_\infty \int_{|y|>1} \nu(dy) \\ &\leq C \|\eta_{\tau,\xi}\|_{C_b^2} \|f\|_{C_b^{\alpha+\beta}}. \end{aligned}$$

Hence, \mathcal{T} is also continuous from $C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R})$ into $C_b^1(\mathbb{R}^d, \mathbb{R})$. We also recall the following interpolation equality between Hölder spaces:

$$\left(C_b^1(\mathbb{R}^d, \mathbb{R}), C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R}) \right)_{\beta, \infty} = C_b^{(1-\beta)+\beta(\alpha+\beta)}(\mathbb{R}^d, \mathbb{R}),$$

see e.g. Chapter 1 in Lunardi [Lun09]. Therefore the operator \mathcal{T} is also continuous from $C_b^{(1-\beta)+\beta(\alpha+\beta)}(\mathbb{R}^d, \mathbb{R})$ into $C_b^\beta(\mathbb{R}^d, \mathbb{R})$ (see Theorem 1.1.6 in [Lun09]). Recall that, since $\alpha + \beta > 1$, we indeed have $(1 - \beta) + \beta(\alpha + \beta) < \alpha + \beta$. To derive the stated smoothness of $\mathcal{S}_{\tau,\xi}$ we want to apply $\mathcal{T}_{\tau,\xi}$ to $u(t, \cdot)$ where $u \in L^\infty([0, T], C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R}))$. From the above computations and since $C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R}) \subset C_b^{(1-\beta)+\beta(\alpha+\beta)}(\mathbb{R}^d, \mathbb{R})$ we readily derive that $\mathcal{T}_{\tau,\xi} u \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$. The other term in $\mathcal{S}_{\tau,\xi}$, namely $L_\alpha \eta_{\tau,\xi}(t, x) u(t, x)$ is handled without difficulties. This concludes the proof of the statement that $\mathcal{S}_{\tau,\xi} \in L^\infty([0, T], C_b^\beta(\mathbb{R}^d, \mathbb{R}))$.

It now follows from Proposition 7 that

$$\begin{aligned} v_{\tau,\xi}(t, x) &= \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) + \int_t^T ds \tilde{P}_{s,t,\alpha}^{(\tau,\xi)}(\eta_{\tau,\xi} f + \mathcal{S}_{\tau,\xi}(s, \cdot))(x) + \int_t^T ds \tilde{P}_{s,t,\alpha}^{(\tau,\xi)}(R_{\tau,\xi}(s, \cdot))(x) \\ &= \tilde{u}^{(\tau,\xi)}(t, x) + \int_t^T ds \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R_{\tau,\xi}(s, \cdot) \right)(x), \end{aligned}$$

where

$$\tilde{u}^{(\tau,\xi)}(t, x) = \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) + \int_t^T ds \tilde{P}_{s,t,\alpha}^{(\tau,\xi)}(\eta_{\tau,\xi} f + \mathcal{S}_{\tau,\xi}(s, \cdot))(x).$$

Also, from Definition 2.6, Proposition 6 and 7, $\tilde{u}^{(\tau,\xi)}$ is the unique solution in $\mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ of (2.28). Therefore the first Duhamel representation formula in (2.27) holds.

Since both $D\tilde{v}_{\tau,\xi}(t, x)$ and $D\tilde{u}^{(\tau,\xi)}(t, x)$ exist (recall indeed that $\tilde{v}_{\tau,\xi} = u\eta_{\tau,\xi}$), we deduce that

$$D_x \int_t^T ds \left(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R_{\tau,\xi}(s, \cdot) \right)(x)$$

is also meaningful. This proves the second assertions of (2.27). Eventually, recall now that under (\mathcal{P}_β) , one has for $(\tau, \xi) = (t, x)$:

$$\begin{aligned}
& \left(|D_x(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R_{\tau,\xi}(s, \cdot))(x)| \right) \Big|_{(\tau,\xi)=(t,x)} \\
& \leq \int_{\mathbb{R}^d} dy \left(|D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, s, x, y)(F(s, y) - F(s, \theta_{s,\tau}(\xi))) Du(s, y) \eta_{\tau,\xi}(y)| \right) \Big|_{(\tau,\xi)=(t,x)} \\
& \leq K_0 \|Du\|_\infty \int_{\mathbb{R}^d} dy \left(|D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, s, x, y)| |y - \theta_{s,\tau}(\xi)|^\beta \right) \Big|_{(\tau,\xi)=(t,x)} \\
& \leq K_0 \|Du\|_\infty \int_{\mathbb{R}^d} dy \left(|D_x p_\alpha(s-t, y - m_{s,t}^{(\tau,\xi)}(x))| |y - \theta_{s,\tau}(\xi)|^\beta \right) \Big|_{(\tau,\xi)=(t,x)} \\
& \leq K_0 \|Du\|_\infty \int_{\mathbb{R}^d} dy |D_x p_\alpha(s-t, z)| \Big|_{z=y-\theta_{s,t}(x)} |y - \theta_{s,t}(x)|^\beta \\
& \stackrel{(\mathcal{P}_\beta)}{\leq} \frac{CK_0 \|Du\|_\infty}{(s-t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}}, \tag{2.30}
\end{aligned}$$

recalling from (2.4) that $m_{s,t}^{(\tau,\xi)}(x) \Big|_{(\tau,\xi)=(t,x)} = \theta_{s,t}(x)$ for the last but one inequality. Hence, we derive the following important control

$$\begin{aligned}
& \left(D_x \int_t^T ds (\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R_{\tau,\xi}(s, \cdot))(x) \right) \Big|_{(\tau,\xi)=(t,x)} \\
& = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \left(\int_t^T ds \int_{\mathbb{R}^d} dy \left[\tilde{p}_\alpha(s-t, y - (x + \varepsilon + \int_t^s F(v, \theta_{s,\tau}(\xi)) dv) \right. \right. \\
& \quad \left. \left. - \tilde{p}_\alpha(s-t, y - (x + \int_t^s F(v, \theta_{s,\tau}(\xi)) dv) \right) \right] \left\{ F(s, y) - F(s, \theta_{s,\tau}(\xi)) \right\} Du(s, y) \eta_{\tau,\xi}(s, y) \right) \Big|_{(\tau,\xi)=(t,x)} \\
& = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \left(\int_t^T ds \int_{\mathbb{R}^d} dy \left[\tilde{p}_\alpha(s-t, y - (x + \varepsilon + \int_t^s F(v, \theta_{s,t}(x)) dv) \right. \right. \\
& \quad \left. \left. - \tilde{p}_\alpha(s-t, y - (x + \int_t^s F(v, \theta_{s,t}(x)) dv) \right) \right] \left\{ F(s, y) - F(s, \theta_{s,t}(x)) \right\} Du(s, y) \eta_{t,x}(s, y) \right) \\
& = \left(\int_t^T ds \int_{\mathbb{R}^d} dy \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \left[\tilde{p}_\alpha(s-t, y - (x + \varepsilon + \int_t^s F(v, \theta_{s,t}(x)) dv) \right. \right. \\
& \quad \left. \left. - \tilde{p}_\alpha(s-t, y - (x + \int_t^s F(v, \theta_{s,t}(x)) dv) \right) \right] \left\{ F(s, y) - F(s, \theta_{s,t}(x)) \right\} Du(s, y) \eta_{t,x}(s, y) \right) \\
& = \int_t^T ds \left(D(\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R_{\tau,\xi}(s, \cdot))(x) \right) \Big|_{(\tau,\xi)=(t,x)}, \tag{2.31}
\end{aligned}$$

where (2.30) allowed to use the bounded convergence theorem in the last but one equality. Equation (2.31) in turn yields (2.29). Indeed, $D_x v_{\tau,\xi}(t, x) = D_x u(t, x) \eta_{\tau,\xi}(t, x) + u(t, x) D_x \eta_{\tau,x}(x)$ and for $(\tau, \xi) = (t, x)$ one has $(\eta_{\tau,\xi}(t, x))_{(\tau,\xi)=(t,x)} = 1$, $(D_x \eta_{\tau,\xi}(t, x))_{(\tau,\xi)=(t,x)} = 0$. \square

Remark 7 (*About the localization in the Duhamel formula*). We mention that the localization with the cut-off $\eta_{\tau,\xi}$ is precisely needed because we imposed in (1.3) a local Hölder continuity condition. Anyhow, even if we had assumed a global Hölder assumption, such a localization would still be needed to give a meaning to the first identity in (2.27) when $\alpha < 1/2$ (recall that (\mathcal{P}_β) involves the derivatives of the heat-kernel). For $\alpha > 1/2$ and $\beta < \alpha$ such a localization could have been avoided for a globally β -Hölder continuous drift F .

2.3 Derivation of the main *a priori* estimates

From the representations (2.27) and (2.29) in Proposition 8 we see that, since we also know from Proposition 6 that $\tilde{u}^{(\tau,\xi)}$ is itself *smooth*, the main term which remains to be investigated is the remainder $\int_t^T ds (\tilde{P}_{s,t,\alpha}^{(\tau,\xi)} R(s, \cdot))(x)$.

In the following, we first give bounds for the solution $u \in L^\infty([0, T], C_b^{\alpha+\beta}(\mathbb{R}^d, \mathbb{R}))$. Estimates for the supremum norm of the solution and its gradient are given in Lemma 9 and the control of the Hölder modulus of the gradient are stated in Lemma 10. Then, we eventually prove Theorem 3 in paragraph 2.3.2.

We emphasize that as the proof of Lemma 10 requires a thorough analysis (namely a refinement of Proposition 8, in order to consider different freezing points in function of the diagonal and off-diagonal regimes introduced in the proof of Proposition 6), it will be postponed to the next subsection 2.4.

2.3.1 Control of the supremum norms for the solution and its gradient and associated Hölder modulus

As an important corollary of Proposition 8, we get the next estimates for the supremum norm of u and its gradient.

Lemma 9 (Control of the supremum norm of the solution and the gradient). *Assume **(A)** (thus property (\mathcal{P}_β) holds). Let $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ be a solution of (1.1). There exists a constant $C := C(\mathbf{A})$ s.t. for all $(t, x) \in [0, T] \times \mathbb{R}^d$:*

$$|u(t, x)| \leq \|g\|_\infty + T\|f\|_\infty + K_0\|Du\|_\infty T, \quad (2.32)$$

$$\begin{aligned} |Du(t, x)| &\leq \|Dg\|_\infty + C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} ([f]_{\beta, T} + \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} + K_0\|Du\|_\infty) \\ &\leq \|Dg\|_\infty + C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} ([f]_{\beta, T} + \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} (1 + K_0)). \end{aligned} \quad (2.33)$$

Proof. Equation (2.32) readily follows from (2.27) taking $(\tau, \xi) = (t, x)$ and observing that $v_{(\tau, \xi)}(t, x)|_{(\tau, \xi)=(t, x)} = u(t, x)\eta_{t, x}(t, x) = u(t, x)$. To derive (2.33) we start from (2.29) to write:

$$|D_x u(t, x)| \leq |D_x \tilde{u}^{(\tau, \xi)}(t, x)|_{(\tau, \xi)=(t, x)} + \left| \int_t^T ds \left(D(\tilde{P}_{s, t, \alpha}^{(\tau, \xi)} R(s, \cdot))(x) \right) \Big|_{(\tau, \xi)=(t, x)} \right|.$$

From equation (2.28) and the proof of Proposition 6 (see equation (2.12)) and (2.30) we thus derive, recalling that $1/\alpha - \beta/\alpha < 1$:

$$\begin{aligned} |D_x u(t, x)| &\leq \|Dg\|_\infty + C \left((T-t)^{\frac{\alpha+\beta-1}{\alpha}} ([f\eta_{\tau, \xi} + \mathcal{S}_{\tau, \xi}]_{\beta, T}) \Big|_{(\tau, \xi)=(t, x)} + K_0\|Du\|_\infty \int_t^T \frac{ds}{(s-t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}} \right) \\ &\leq \|Dg\|_\infty + C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} \left(([f\eta_{\tau, \xi} + \mathcal{S}_{\tau, \xi}]_{\beta, T}) \Big|_{(\tau, \xi)=(t, x)} + K_0\|Du\|_\infty \right). \end{aligned} \quad (2.34)$$

It therefore remains to precise the quantity $([f\eta_{\tau, \xi} + \mathcal{S}_{\tau, \xi}]_{\beta, T}) \Big|_{(\tau, \xi)=(t, x)} \leq ([f\eta_{\tau, \xi}]_{\beta, T} + [\mathcal{S}_{\tau, \xi}]_{\beta, T}) \Big|_{(\tau, \xi)=(t, x)}$.

We have,

$$\begin{aligned} [f\eta_{\tau, \xi}]_{\beta, T} &\leq [f]_{\beta, T} + C\|f\|_\infty \leq C\|f\|_{L^\infty([0, T], C_b^\beta)}, \\ [\mathcal{S}_{\tau, \xi}]_{\beta, T} &\leq [(L_\alpha \eta_{\tau, \xi})u]_{\beta, T} + [\mathcal{I}_{\tau, \xi}]_{\beta, T} \leq C \left(\|u\|_\infty + \|Du\|_\infty + \|u\|_{L^\infty([0, T], C_b^{(1-\beta)+\beta(\alpha+\beta)})} \right) \\ &\leq C_{\beta, \alpha} ((1 + \varepsilon^{-1})(\|u\|_\infty + \|Du\|_\infty) + \varepsilon\|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}), \end{aligned} \quad (2.35)$$

for any $\varepsilon \in (0, 1)$ using for the last inequality that for all $t \in [0, T]$, the usual interpolation inequality

$$[u(t, \cdot)]_{(1-\beta)+\beta(\alpha+\beta)} \leq [u(t, \cdot)]_{\alpha+\beta}^{s/(\alpha+\beta)} [u(t, \cdot)]_1^{1-s/(\alpha+\beta)},$$

for $s = (1 - \beta) + \beta(\alpha + \beta) - 1$ (see e.g. [Kry96], p. 40, (3.3.7)) then yields from the Young inequality $[u(t, \cdot)]_{(1-\beta)+\beta(\alpha+\beta)} \leq C_{\beta, \alpha} (\varepsilon[u(t, \cdot)]_{\alpha+\beta} + \varepsilon^{-1}[u(t, \cdot)]_1)$. Plugging (2.35) with $\varepsilon = 1/2$ into (2.34) gives (2.33). This completes the proof. \square

Remark 8 (On the β -Hölder modulus of $\mathcal{S}_{\tau, \xi}$). We eventually mention that equation (2.35) will also be crucial, for a parameter ε sufficiently small, when investigating the Hölder modulus of the gradient, in order to make the circular argument working.

Concerning the Hölder modulus of the gradient of the solution we have the following control whose proof is presented in the next section.

Lemma 10 (Hölder modulus of the gradient). *Assume **(A)**. Let $u \in \mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ be a solution of (1.1). There exists two constant $C_1 := C(\mathbf{A}) > 0$ and $C_2 := C(\mathbf{A}) > 0$ such that*

$$\begin{aligned} [Du]_{\beta+\alpha-1, T} &\leq C_1 \left\{ \left((1 + c_0)\|g\|_{C_b^{\alpha+\beta}} + \left(c_0^{\frac{\alpha+\beta-1}{\alpha}} [f]_{\beta, T} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)} \right) \right. \right. \\ &\quad \left. \left. + K_0 \left(c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}} \right) \|Du\|_{L^\infty} \right\} \\ &\quad + \left(\frac{1}{4} + C_2 c_0^{\frac{\alpha+\beta-1}{\alpha}} (1 + K_0) \right) \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}. \end{aligned} \quad (2.36)$$

2.3.2 Final derivation of Theorem 3: Schauder estimate for the solution of (1.1)

Observe carefully from the above Lemmas that the norm of u appears in the r.h.s of the previous controls. However, those contributions are multiplied either by a constant c_0 , either by a function of T or a small constant. Provided these quantities can be chosen small enough, we can conclude the proof of our main estimates through a circular argument, i.e., the norms in the r.h.s. will be absorbed by those on the l.h.s. When doing so, we end up with Schauder estimates in small time only. To extend it to an arbitrary fixed horizon T , we eventually use the fact that Schauder estimates precisely provide a kind of stability result in the class $\mathcal{C}^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ so that the final bound follows by inductive application of the estimate in small time.

Note now that thanks to (2.32), (2.33) we have from (2.36) that

$$\begin{aligned} & \|u\|_{L^\infty} + \|Du\|_{L^\infty} + [Du]_{\beta+\alpha-1, T} \\ & \leq C_1 \left\{ \left((1 + c_0 + K_0(c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}})) + K_0 T \right) \|g\|_{C_b^{\alpha+\beta}} \right. \\ & \quad \left. + \left(c_0^{\frac{\alpha+\beta-1}{\alpha}} [f]_{\beta, T} + (1 + K_0(c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}})) + K_0 T \right) (T-t)^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)} \right\} \\ & \quad + \Psi(K_0, c_0, \alpha, \beta, (\mathbf{A})) \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \end{aligned}$$

up to a modification of C_1 and where

$$\Psi(K_0, c_0, \alpha, \beta, (\mathbf{A})) = \frac{1}{4} + C_2 c_0^{\frac{\alpha+\beta-1}{\alpha}} (1 + K_0) + C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} (1 + K_0) \left(1 + K_0 T + C_1 K_0 (c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}}) \right).$$

We can hence choose c_0 small enough so that $C_2 c_0^{\frac{\alpha+\beta-1}{\alpha}} (1 + K_0) \leq 1/4$ and then T small enough so that $C(T-t)^{\frac{\alpha+\beta-1}{\alpha}} (1 + K_0) \left(1 + K_0 T + C_1 K_0 (c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}}) \right) \leq 1/4$. This eventually yields that there exists $\tilde{C} := \tilde{C}((\mathbf{A}), c_0)$ s.t.

$$\begin{aligned} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} & \leq \tilde{C} \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0, T], C_b^\beta)} \right) + \frac{3}{4} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \\ & \leq 4\tilde{C} \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0, T], C_b^\beta)} \right). \end{aligned} \quad (2.37)$$

Equation (2.37) provides the control of Theorem 3 for T small enough, i.e., for $T \leq T_0((\mathbf{A}), (\mathcal{P}_\beta))$. The result is extended to an arbitrary time T considering N subintervals of $[0, T]$ s.t. $T/N \leq T_0$ and applying inductively (2.37) going backwards in time on the time intervals $[(i-1)T/N, iT/N]$, $i \in \llbracket 1, N \rrbracket$ considering as *final* condition on the current time interval the function $g_i(x) := u(iT/n, x)$ which precisely belongs to $C_b^{\alpha+\beta}$ from the previous application of the Schauder estimate (2.37) if $i < N$ or by (1.1) if $i = N$. This proves Theorem 3 is complete provided Lemma 10 holds. \square

2.4 Proof of Lemma 10

Let $t \in [0, T]$ be fixed. For this part of the analysis, we distinguish two cases, either the given points $(x, x') \in \mathbb{R}^d$ are for a fixed $t \in [0, T)$ in a globally *off-diagonal* regime, i.e., $c_0|x - x'|^\alpha \geq (T-t)$ for a constant c_0 to be specified but meant to be small. This means that the spatial distance is larger than the characteristic time-scale up to a prescribed constant which will be useful to equilibrate the computations. In this case, we will *mainly* use the controls of Lemma 9.

In the *diagonal* case, $c_0|x - x'|^\alpha \leq (T-t)$, the spatial points are closer than the typical time-scale magnitude but in the time integration for the source and the perturbative term (see e.g. (2.45) below), when $(s-t) \leq c_0|x - x'|^\alpha$ there is again a *local off-diagonal regime*. The key point is that to handle these terms properly it will be useful to be able to change freezing point, i.e., it seems reasonable that, when the spatial points are in a *local diagonal regime*, i.e., $(s-t) \geq c_0|x - x'|^\alpha$, the auxiliary frozen densities are considered for the same freezing parameter and conversely that in the *locally off-diagonal regime* the densities are frozen along their own spatial argument (similarly to equation (2.29) in Lemma 9). We are thus faced with a change of freezing point in the Duhamel formulation. This approach was already used in [CHM18] to obtain Schauder estimates for degenerate local Kolmogorov equations and can be used in the current setting.

2.4.1 Off-Diagonal Regime

Let $x, x' \in \mathbb{R}^d$ be s.t. $c_0|x - x'|^\alpha \geq (T-t)$. In that case, we claim that:

$$|Du(t, x) - Du(t, x')| \leq C|x - x'|^{\alpha+\beta-1} \left([Dg]_{\beta+\alpha-1} (1+c_0) + c_0^{\frac{\alpha+\beta-1}{\alpha}} ([f]_{\beta, T} + \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} (1+K_0)) \right). \quad (2.38)$$

Indeed, we readily get from (2.29), (2.33) and the proof of Lemma 9 that:

$$\begin{aligned}
|Du(t, x) - Du(t, x')| &\leq C \left(\left| D\tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) \Big|_{(\tau,\xi)=(t,x)} - D\tilde{P}_{T,t,\alpha}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \Big|_{(\tau,\xi')=(t,x')} \right| \right. \\
&\quad \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, s, x, y) (\eta_{\tau,\xi} f - \mathcal{S}_{\tau,\xi}(s, y)) \right) \Big|_{(\tau,\xi)=(t,x)} \right. \\
&\quad \left. + \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, s, x', y) (\eta_{\tau,\xi'} f - \mathcal{S}_{\tau,\xi'}(s, y)) \right) \Big|_{(\tau,\xi')=(t,x')} \right. \\
&\quad \left. + \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, s, x, y) (F(s, y) - F(s, \theta_{s,\tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau,\xi}(s, y) \right|_{(\tau,\xi)=(t,x)} \right. \\
&\quad \left. + \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau,\xi')} (t, s, x', y) (F(s, y) - F(s, \theta_{s,\tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau,\xi'}(s, y) \right|_{(\tau,\xi')=(t,x')} \right) \\
&\leq C \left(\left| D\tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) \Big|_{(\tau,\xi)=(t,x)} - D\tilde{P}_{T,t,\alpha}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \Big|_{(\tau,\xi')=(t,x')} \right| \right. \\
&\quad \left. + (T-t)^{\frac{\alpha+\beta-1}{\alpha}} ([f]_{\beta,T} + \|u\|_{L^\infty([0,T], C_b^{\alpha+\beta})} (1+K_0)) \right) \\
&\quad C \left(\left| D\tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) \Big|_{(\tau,\xi)=(t,x)} - D\tilde{P}_{T,t,\alpha}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \Big|_{(\tau,\xi')=(t,x')} \right| \right. \\
&\quad \left. + c_0^{\frac{\alpha+\beta-1}{\alpha}} |x-x'|^{\alpha+\beta-1} ([f]_{\beta,T} + \|u\|_{L^\infty([0,T], C_b^{\alpha+\beta})} (1+K_0)) \right). \quad (2.39)
\end{aligned}$$

The last term in the r.h.s. gives a *good* control in the sense that provided c_0 is small, the above bound is compatible with the previously indicated circular argument to absorb the norms of u in the r.h.s. Hence to conclude in that case it remains to handle the difference of the frozen semi-groups observed at different freezing points in space.

Recall from (2.3) that $D_x \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, y) = -D_y \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, y)$ and write:

$$\begin{aligned}
&\left(D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \right) \\
&= \left[\int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, y) D(g\eta_{\tau,\xi})(y) dy - \int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau,\xi')} (t, T, x', y) D(g\eta_{\tau,\xi'})(y) dy \right] \\
&= \left[\int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, y) [D(g\eta_{\tau,\xi})(y) - D(g\eta_{\tau,\xi})(\theta_{T,t}(\xi))] dy \right. \\
&\quad \left. + [D(g\eta_{\tau,\xi})(\theta_{T,t}(\xi)) - D(g\eta_{\tau,\xi'})(T, \theta_{T,t}(\xi'))] - \int_{\mathbb{R}^d} \tilde{p}_\alpha^{(\tau,\xi')} (t, T, x', y) [D(g\eta_{\tau,\xi'})(y) - D(g\eta_{\tau,\xi'})(\theta_{T,t}(\xi'))] dy \right],
\end{aligned}$$

recalling that $\tilde{p}_\alpha^{(\tau,\xi)}(t, T, x, \cdot)$, $\tilde{p}_\alpha^{(\tau,\xi')} (t, T, x', \cdot)$ are probability densities for the last equality. Taking $\tau = t, \xi = x, \xi' = x'$ we now write from (2.3) (recall that $\alpha + \beta - 1 < \alpha$) and observing from the definition of the cut-off functions $\eta_{\tau,\xi}, \eta_{\tau,\xi'}$ that $\eta_{\tau,\xi}(\theta_{T,\tau}(\xi)) \Big|_{(\tau,\xi)=(t,x)} = \eta_{\tau,\xi'}(\theta_{T,\tau}(\xi')) \Big|_{(\tau,\xi')=(t,x')} = 1$, $D\eta_{\tau,\xi}(\theta_{T,\tau}(\xi)) \Big|_{(\tau,\xi)=(t,x)} = D\eta_{\tau,\xi'}(\theta_{T,\tau}(\xi')) \Big|_{(\tau,\xi')=(t,x')} = 0$:

$$\begin{aligned}
&\left| D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) - D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \Big|_{(\tau,\xi,\xi')=(t,x,x')} \right. \\
&\leq C \left[\int_{\mathbb{R}^d} p_\alpha(T-t, y - m_{T,t}^\xi(x)) ([Dg]_{\beta+\alpha-1} + 1) |y - \theta_{T,t}(\xi)|^{\beta+\alpha-1} dy + [Dg]_{\beta+\alpha-1} |\theta_{T,t}(\xi) - \theta_{T,t}(\xi')|^{\alpha+\beta-1} \right. \\
&\quad \left. + \int_{\mathbb{R}^d} p_\alpha(T-t, y - m_{T,t}^{\xi'}(x')) ([Dg]_{\beta+\alpha-1} + 1) |y - \theta_{T,t}(\xi')|^{\beta+\alpha-1} dy \right] \Big|_{(\xi,\xi')=(x,x')}, \\
&\leq C \left[\int_{\mathbb{R}^d} p_\alpha(T-t, y - \theta_{T,t}(x)) ([Dg]_{\beta+\alpha-1} + 1) |y - \theta_{T,t}(x)|^{\beta+\alpha-1} dy + [Dg]_{\beta+\alpha-1} |\theta_{T,t}(x) - \theta_{T,t}(x')|^{\alpha+\beta-1} \right. \\
&\quad \left. + \int_{\mathbb{R}^d} p_\alpha(T-t, y - \theta_{T,t}(x')) ([Dg]_{\beta+\alpha-1} + 1) |y - \theta_{T,t}(x')|^{\beta+\alpha-1} dy \right].
\end{aligned}$$

With the corresponding scaling (cf. **(NDa)**) we derive:

$$\begin{aligned} & \left| D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) - D_x \tilde{P}_{T,t}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \right|_{(\tau,\xi,\xi')=(t,x,x')} \\ & \leq C([Dg]_{\beta+\alpha-1} + 1) \left[(T-t)^{\frac{\beta+\alpha-1}{\alpha}} + |\theta_{T,t}(x) - \theta_{T,t}(x')|^{\alpha+\beta-1} \right]. \end{aligned} \quad (2.40)$$

From the spatial regularity of F we have the following key result whose proof is postponed to the Appendix.

Lemma 11 (Controls on the flows). *Let $\alpha + \beta > 1$ and F satisfying (1.3). Then there exists a constant $C \geq 1$ s.t. for all $0 \leq t \leq s \leq T \leq 1$, $(x, x') \in (\mathbb{R}^d)^2$:*

$$|\theta_{s,t}(x) - \theta_{s,t}(x')| \leq C(|x - x'| + (s-t)^{\frac{1}{1-\beta}}) \leq C(|x - x'| + (s-t)^{\frac{1}{\alpha}}). \quad (2.41)$$

From (2.41) with $s = T$ and (2.40), we therefore derive that:

$$\begin{aligned} & \left| D_x \tilde{P}_{T,t,\alpha}^{(\tau,\xi)}(g\eta_{\tau,\xi}(T, \cdot))(x) - D_x \tilde{P}_{T,t}^{(\tau,\xi')} (g\eta_{\tau,\xi'}(T, \cdot))(x') \right|_{(\tau,\xi,\xi')=(t,x,x')} \\ & \leq C([Dg]_{\beta+\alpha-1} + 1) \left[(T-t)^{\frac{\beta+\alpha-1}{\alpha}} + |x - x'|^{\alpha+\beta-1} \right] \\ & \leq C([Dg]_{\beta+\alpha-1} + 1)(1 + c_0)|x - x'|^{\alpha+\beta-1}. \end{aligned} \quad (2.42)$$

Plugging (2.42) into (2.39) eventually yields for the off-diagonal regime that (2.38) holds.

2.4.2 Diagonal-Regime

It is here assumed that for given points $(t, x, x') \in [0, T] \times (\mathbb{R}^d)^2$, $c_0|x - x'|^\alpha \leq (T - t)$. **All the statements of the paragraph tacitly assume this condition holds.** We first need here to consider a Duhamel representation formula for which we change freezing point along the time integration variable. With the previous notations of Proposition 8 the following expansion of u and its gradient holds.

Proposition 12 (Duhamel formula with change of freezing points). *Let $u \in \mathcal{C}_b^{\alpha+\beta}([0, T] \times \mathbb{R}^d)$ be a solution of (1.1). For fixed $(t, x') \in [0, T] \times \mathbb{R}^d$ and any freezing parameters $(\tau, \xi', \tilde{\xi}') \in [0, T] \times (\mathbb{R}^d)^2$, for all $\tau_0 \in [t, T]$:*

$$\begin{aligned} v_{\tau,\xi'}(t, x') &= \tilde{P}_{T,t,\alpha}^{(\tau,\tilde{\xi}')} (g\eta_{\tau,\tilde{\xi}'}(T, \cdot))(x') + \tilde{G}_{\tau_0,t,\alpha}^{(\tau,\xi')} (f\eta_{\tau,\xi'} - \mathcal{S}_{\tau,\xi'})(t, x') + \tilde{G}_{T,\tau_0,\alpha}^{(\tau,\tilde{\xi}')} (f\eta_{\tau,\tilde{\xi}'} - \mathcal{S}_{\tau,\tilde{\xi}'})(t, x') \\ &+ \tilde{P}_{\tau_0,t,\alpha}^{(\tau,\xi')} v_{\tau,\xi'}(\tau_0, x') - \tilde{P}_{\tau_0,t,\alpha}^{(\tau,\tilde{\xi}')} v_{\tau,\tilde{\xi}'}(\tau_0, x') \\ &+ \int_t^T ds \int_{\mathbb{R}^d} dy \left(\mathbb{I}_{s \leq \tau_0} \tilde{p}_\alpha^{(\tau,\xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s,\tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau,\xi'}(s, y) \right. \\ &\left. + \mathbb{I}_{s > \tau_0} \tilde{p}_\alpha^{(\tau,\tilde{\xi}')} (t, s, x', y) \left((F(s, y) - F(s, \theta_{s,\tau}(\tilde{\xi}'))) \cdot Du(s, y) \right) \eta_{\tau,\tilde{\xi}'}(s, y) \right), \end{aligned} \quad (2.43)$$

where $v_{\tau,\xi'}(t, x') = (u\eta_{\tau,\xi'})(t, x')$ and we recall from (2.19) that:

$$\forall 0 \leq v < r \leq T, \quad \tilde{G}_{r,v,\alpha}^{(\tau,\xi')} f(t, x) := \int_v^r ds \int_{\mathbb{R}^d} dy \tilde{p}_\alpha^{(\tau,\xi')}(t, s, x', y) f(s, y).$$

Let now $x, x' \in \mathbb{R}^d$ be s.t. $c_0|x - x'|^\alpha \leq T - t$. Then we can differentiate the previous expression for suitable freezing parameters. Namely:

$$\begin{aligned} & (D_x v_{\tau,\xi'}(t, x'))|_{(\tau,\xi')=(t,x)} \\ &= (D_x \tilde{P}_{T,t,\alpha}^{(\tau,\tilde{\xi}')} (g\eta_{\tau,\tilde{\xi}'}(T, \cdot))(x'))|_{(\tau,\tilde{\xi}')=(t,x)} + (D \tilde{G}_{\tau_0,t,\alpha}^{(\tau,\xi')} (f\eta_{\tau,\xi'} - \mathcal{S}_{\tau,\xi'})(t, x'))|_{(\tau_0,\tau,\xi')=(t_0,t,x')} \\ &+ (D_x \tilde{G}_{T,\tau_0,\alpha}^{(\tau,\tilde{\xi}')} (f\eta_{\tau,\tilde{\xi}'} - \mathcal{S}_{\tau,\tilde{\xi}'})(t, x'))|_{(\tau_0,\tau,\tilde{\xi}')=(t_0,t,x)} \\ &+ (D_x \tilde{P}_{\tau_0,t,\alpha}^{(\tau,\xi')} v_{\tau,\xi'}(\tau_0, x'))|_{(\tau_0,\tau,\xi')=(t_0,t,x')} - (D \tilde{P}_{\tau_0,t,\alpha}^{(\tau,\tilde{\xi}')} v_{\tau,\tilde{\xi}'}(\tau_0, x'))|_{(\tau_0,\tau,\tilde{\xi}')=(t_0,t,x)} \\ &+ \int_t^T ds \int_{\mathbb{R}^d} dy \left(\mathbb{I}_{s \leq \tau_0} D \tilde{p}_\alpha^{(\tau,\xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s,\tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau,\xi'}(s, y) \right. \\ &\left. + \mathbb{I}_{s > \tau_0} D \tilde{p}_\alpha^{(\tau,\tilde{\xi}')} (t, s, x', y) \left((F(s, y) - F(s, \theta_{s,\tau}(\tilde{\xi}'))) \cdot Du(s, y) \right) \eta_{\tau,\tilde{\xi}'}(s, y) \right) \Big|_{(\tau_0,\tau,\xi',\tilde{\xi}')=(t_0,t,x',x)}, \end{aligned}$$

where $t_0 = t + c_0|x - x'|^\alpha$ as in (2.18).

The previous proposition thus emphasizes that changing the freezing point according to the current (local) diagonal or off-diagonal regime can actually been done up to an additional discontinuity term.

Proof. Restarting from (2.27) we can indeed rewrite for given $(t, x') \in [0, T] \times \mathbb{R}^d$ and any $r \in (t, T], (\tau, \xi') \in [0, T] \times \mathbb{R}^d$:

$$\begin{aligned} v_{\tau, \xi'}(t, x') &= \tilde{P}_{r, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(r, x) + \tilde{G}_{r, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') \\ &\quad + \int_t^r ds \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) (F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \eta_{\tau, \xi'}(s, y). \end{aligned} \quad (2.44)$$

According to Proposition 5, we obtain that for a.e. $r \in (t, T]$ for any $\xi' \in \mathbb{R}^{nd}$:

$$\begin{aligned} 0 &= \partial_r [\tilde{P}_{r, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(r, x')] + \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \xi')}(t, r, x', y) (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(r, y) \\ &\quad + \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \xi')}(t, r, x', y) \left((F(r, y) - F(r, \theta_{r, \tau}(\xi))) \cdot Du(r, y) \right) \eta_{\tau, \xi'}(r, y). \end{aligned} \quad (2.45)$$

Integrating (2.45) with respect to r between t and $t_0 \in (t, T]$ for a first given ξ' and between t_0 and T with a possibly different $\tilde{\xi}'$ yields:

$$\begin{aligned} 0 &= \tilde{P}_{t_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(t_0, x') - v_{\tau, \xi'}(t, x') + \int_t^{t_0} ds \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(s, y) \\ &\quad + \int_t^{t_0} ds \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \\ &\quad + \tilde{P}_{T, t, \alpha}^{(\tau, \tilde{\xi}')} v_{\tau, \tilde{\xi}'}(T, x') - \tilde{P}_{t_0, t, \alpha}^{(\tau, \tilde{\xi}')} v_{\tau, \tilde{\xi}'}(t_0, x') + \int_{t_0}^T ds \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, s, x', y) (f\eta_{\tau, \tilde{\xi}'} - \mathcal{S}_{\tau, \tilde{\xi}'})(s, y) \\ &\quad + \int_{t_0}^T ds \int_{\mathbb{R}^d} dy \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\tilde{\xi}))) \cdot Du(s, y) \right) \eta_{\tau, \tilde{\xi}'}(s, y). \end{aligned}$$

Since $v_{\tau, \tilde{\xi}'}(T, x') = (g\eta_{\tau, \tilde{\xi}'}(T, \cdot))(x')$ (terminal condition), using the notations of (2.44), the above equation rewrites:

$$\begin{aligned} v_{\tau, \xi'}(t, x') &= \tilde{P}_{T, t, \alpha}^{(\tau, \tilde{\xi}')} (g\eta_{\tau, \tilde{\xi}'}(T, \cdot))(x') + \tilde{G}_{t_0, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') + \tilde{G}_{T, t_0, \alpha}^{(\tau, \tilde{\xi}')} (f\eta_{\tau, \tilde{\xi}'} - \mathcal{S}_{\tau, \tilde{\xi}'})(t, x') \\ &\quad + \tilde{P}_{t_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(t_0, x') - \tilde{P}_{t_0, t, \alpha}^{(\tau, \tilde{\xi}')} v_{\tau, \tilde{\xi}'}(t_0, x') \\ &\quad + \int_t^T ds \int_{\mathbb{R}^d} dy \left(\mathbb{1}_{s \leq t_0} \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \right. \\ &\quad \left. + \mathbb{1}_{s > t_0} \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\tilde{\xi}))) \cdot Du(s, y) \right) \eta_{\tau, \tilde{\xi}'}(s, y) \right). \end{aligned} \quad (2.46)$$

This gives (2.46). Expression (2.43) can then, for the indicated freezing parameters, be differentiated in space reproducing the arguments used to derive (2.29) and noting that when $s > \tau_0 > t$ the bounded convergence theorem readily applies for the last contribution. This gives (2.44). \square

We can from (2.43) express the full expression of the difference to be investigated. Namely, for all $(t, x, x') \in$

$[0, T] \times (\mathbb{R}^d)^2$, recalling that for $(\tau, \xi, \xi') = (t, x, x')$, $\eta_{\tau, \xi}(t, x) = \eta_{\tau, \xi'}(t, x') = 1$ we get:

$$\begin{aligned}
& |D_x u(t, x') - D_x u(t, x)| \\
&= \left| D_x(u\eta_{\tau, \xi'})(t, x') \Big|_{(\tau, \xi')=(t, \xi')} - D_x(u\eta_{\tau, \xi})(t, x) \Big|_{(\tau, \xi)=(t, x)} \right| \\
&\leq \left| \left[D_x \tilde{P}_{T, t, \alpha}^{(\tau, \tilde{\xi}')}(\eta_{\tau, \tilde{\xi}'}(T, \cdot)g)(x') - D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)}(\eta_{\tau, \xi}(T, \cdot)g)(x) \right] \Big|_{(\tau, \tilde{\xi}', \xi)=(t, x, x)} \right. \\
&\quad + \left| D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(\tau_0, x') - D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi)} v_{\tau, \xi}(\tau_0, x') \right|_{(\tau_0, \tau, \xi', \xi)=(t_0, t, x', x)} \\
&\quad + |D_x \tilde{G}_{\tau_0, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') + D_x \tilde{G}_{T, \tau_0, \alpha}^{(\tau, \xi)} (f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) \\
&\quad - D_x \tilde{G}_{T, t, \alpha}^{(\tau, \xi)} (f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) \Big|_{(\tau_0, \tau, \xi', \xi)=(t_0, t, x', x, x)} \\
&\quad + \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(\mathbb{1}_{s \leq \tau_0} D_x \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \right. \right. \\
&\quad \left. \left. + \mathbb{1}_{s > \tau_0} D_x \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau, \tilde{\xi}'}(s, y) \right. \right. \\
&\quad \left. \left. - D_x \tilde{p}_{\alpha}^{(\tau, \xi)}(t, s, x, y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi}(s, y) \right) \right|_{(\tau_0, \tau, \xi', \xi)=(t_0, t, x', x, x)}.
\end{aligned}$$

Since we have initially assumed T small and we are currently in the diagonal-regime, $|x - x'| \leq [(T - t)/c_0]^{\frac{1}{\alpha}}$, we may assume $|x - x'| \leq 1$ provided c_0 is small enough. In such case, we obtain the following estimate.

Lemma 13. *There exists $C \geq 1$ s.t. for all $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, s.t. $c_0|x - x'|^\alpha \leq T - t$, one has for $g \in C^{\beta+\alpha}(\mathbb{R}^d, \mathbb{R})$:*

$$\begin{aligned}
& |D_x u(t, x') - D_x u(t, x)| \\
&\leq |x - x'|^{\alpha+\beta-1} \left\{ C \left(([Dg]_{C_b^{\beta+\alpha-1}} + 1) + \|g\|_{C_b^{\alpha+\beta}} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)} \right) \right. \\
&\quad \left. + c_0^{\frac{\alpha+\beta-1}{\alpha}} ([Du]_{\beta+\alpha-1, T} + \|u\|_\infty)(1 + K_0) + K_0(c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}}) \|Du\|_{L^\infty} \right\} + \frac{1}{4} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}
\end{aligned}$$

The proof of Lemma 10 then follows from above Lemma and control (2.38). The remainder of this part is dedicated to the proof of the above estimate.

Controls of the frozen semi-group. Note that the freezing points $\tilde{\xi}' = \tilde{\xi} = x$. Hence, the first contribution in the r.h.s. of (2.47) can be handled similarly to the proof of Proposition 6 (see equations (2.13)-(2.16)). Namely, we get the following lemma.

Lemma 14. *There exists $C \geq 1$ s.t. for all $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, s.t. $c_0|x - x'|^\alpha \leq T - t$, one has for $g \in C^{\beta+\alpha}(\mathbb{R}^d, \mathbb{R})$:*

$$\begin{aligned}
& \left| D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)}(g\eta_{\tau, \xi}(T, \cdot))(x) - D_x \tilde{P}_{T, t, \alpha}^{(\tau, \tilde{\xi}')} (g\eta_{\tau, \tilde{\xi}'}(T, \cdot))(x') \right|_{(\tau, \tilde{\xi}', \xi)=(t, x, x)} \\
&= \left| D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)}(g\eta_{\tau, \xi}(T, \cdot))(x) - D_x \tilde{P}_{T, t, \alpha}^{(\tau, \xi)}(g\eta_{\tau, \xi}(T, \cdot))(x') \right|_{(\tau, \xi)=(t, x)} \\
&\leq C([D(g\eta_{\tau, \xi})(T, \cdot)]_{C_b^{\beta+\alpha-1}}) \Big|_{(\tau, \xi)=(t, x)} |x - x'|^{\alpha+\beta-1} \leq C([Dg]_{C_b^{\beta+\alpha-1}} + 1) |x - x'|^{\alpha+\beta-1}.
\end{aligned}$$

Smoothing effects associated with the Green kernel. Let us recall that in the proof of the Schauder estimates for the *frozen* operator (Proposition 6), to control in the *global* diagonal-regime the Hölder norms of the Green kernel, we split into two parts the time integrals according to the position of the time integration variable w.r.t. the change of regime time t_0 (see (2.18)) *a posteriori* chosen to be $t_0 := t + c_0|x - x'|^\alpha$. This is again the splitting according to the (now local) off-diagonal and diagonal regime.

Lemma 15. Under **(A)** and for T small enough, there exists a constant $C := C((\mathbf{A}), T)$ s.t. for fixed points $(t, x, x') \in [0, T] \times (\mathbb{R}^d)^2$, s.t. $c_0|x - x'|^\alpha \leq T - t$ and for all $f \in L^\infty([0, T], C^\beta(\mathbb{R}^d, \mathbb{R}))$:

$$\begin{aligned} & \left(|D_x \tilde{G}_{\tau_0, t, \alpha}^{(\tau, \xi)}(f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) - D_x \tilde{G}_{\tau_0, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') \right) \Big|_{(\tau_0, \tau, \xi, \xi') = (t_0, t, x, x')} \\ & + |D_x \tilde{G}_{T, \tau_0, \alpha}^{(\tau, \xi)}(f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) - D_x \tilde{G}_{T, \tau_0, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') \Big|_{(\tau_0, \tau, \xi, \xi') = (t_0, t, x, x')} \\ & \leq \left(C(\|g\|_{C_b^{\alpha+\beta}} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)}) + \frac{1}{4} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \right) |x - x'|^{\alpha+\beta-1}. \end{aligned} \quad (2.47)$$

Proof of Lemma 15. Let us first control the Hölder moduli of the arguments of the Green kernel. Observe that:

$$[f\eta_{\tau, \xi}]_{\beta, T} \leq [f]_{\beta, T} + C\|f\|_\infty \leq C\|f\|_{L^\infty([0, T], C_b^\beta)}. \quad (2.48)$$

On the other hand, from (2.35) and Lemma 9 we derive that for any $\varepsilon \in (0, 1)$:

$$\begin{aligned} [\mathcal{S}_{\tau, \xi}]_{\beta, T} & \leq C_{\beta, \alpha}((1 + \varepsilon^{-1})(\|u\|_\infty + \|Du\|_\infty) + \varepsilon\|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}), \\ & \leq C_{\beta, \alpha}((1 + \varepsilon^{-1})(\|g\|_\infty + T\|f\|_\infty + \|Du\|_\infty(1 + K_0)) + \varepsilon\|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}) \\ & \leq C_{\beta, \alpha}(1 + K_0) \left((1 + \varepsilon^{-1})(\|g\|_{\alpha+\beta} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)}) \right. \\ & \quad \left. + ((1 + \varepsilon^{-1})T^{\frac{\alpha+\beta-1}{\alpha}}(1 + K_0) + \varepsilon)\|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \right). \end{aligned}$$

Now, for ε small enough and T small enough w.r.t. ε it is clear from the above equation that

$$[\mathcal{S}_{\tau, \xi}]_{\beta, T} \leq C_{\beta, \alpha}(1 + K_0) \left((1 + \varepsilon^{-1})(\|g\|_{\alpha+\beta} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)}) \right) + \frac{c_0}{8} \frac{2 - (\alpha + \beta)}{\alpha} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})}. \quad (2.49)$$

Let us then consider for the difference of the Green kernels the *off-diagonal regime*. We readily get from Lemma 5 (with $\tau = t$) and the above equations (2.48) and (2.49) that

$$\begin{aligned} & \left| D_x \tilde{G}_{\tau_0, t, \alpha}^{(\tau, \xi)}(f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) - D_x \tilde{G}_{\tau_0, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') \right| \Big|_{(\tau_0, \tau, \xi, \xi') = (t_0, t, x, x')} \\ & \leq \left| \int_t^{t_0} ds D_x \tilde{P}_{s, t, \alpha}^{(\tau, \xi)}(f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(s, x) \right| \Big|_{(\tau, \xi) = (t, x)} + \left| \int_t^{t_0} ds D_x \tilde{P}_{s, t, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(s, x') \right| \Big|_{\xi' = x'} \\ & \leq C([f\eta_{\tau, \xi}]_{\beta, T} + [\mathcal{S}_{\tau, \xi}]_{\beta, T} + [f\eta_{\tau, \xi'}]_{\beta, T} + [\mathcal{S}_{\tau, \xi'}]_{\beta, T}) \Big|_{(\tau, \xi, \xi') = (t, x, x')} \int_t^{t_0} ds (s - t)^{-\frac{1}{\alpha} + \frac{\beta}{\alpha}} \\ & \leq \left(C(\|g\|_{C_b^{\alpha+\beta}} + T^{\frac{\alpha+\beta-1}{\alpha}} \|f\|_{L^\infty([0, T], C_b^\beta)}) + \frac{1}{8} \|u\|_{L^\infty([0, T], C_b^{\alpha+\beta})} \right) |x - x'|^{\alpha+\beta-1}, \end{aligned} \quad (2.50)$$

recalling that $c_0 \leq 1$ for the last inequality.

For the *diagonal regime*, we proceed as in the proof of Lemma 6 (see equation (2.21)) using again the previous controls (2.48) and (2.49) to bound $[f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi}]_{\beta, T}$. We obtain a control similar to (2.50) for:

$$\left| D_x \tilde{G}_{T, \tau_0, \alpha}^{(\tau, \xi)}(f\eta_{\tau, \xi} - \mathcal{S}_{\tau, \xi})(t, x) - D_x \tilde{G}_{T, \tau_0, \alpha}^{(\tau, \xi')} (f\eta_{\tau, \xi'} - \mathcal{S}_{\tau, \xi'})(t, x') \right| \Big|_{(\tau_0, \tau, \xi, \xi') = (t_0, t, x, x')}$$

which then gives the statement. \square

Smoothing effects associated with the discontinuity term. It now remains to control the contribution arising from the change of freezing point in equation (2.47). The main result of this section is the next lemma.

Lemma 16 (Control of the discontinuity terms). *There exists $C := C((\mathbf{A}))$ s.t. for all $(t, x, x') \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, s.t. $c_0|x - x'|^\alpha \leq T - t$, for $t_0 = (t + c_0|x - x'|^\alpha) \leq T$ as in (2.18),*

$$\begin{aligned} & \left| D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(\tau_0, x') - D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi)} v_{\tau, \xi}(\tau_0, x') \right| \Big|_{(\tau_0, \xi', \xi') = (t_0, x', x)} \\ & \leq C c_0^{\frac{\alpha+\beta-1}{\alpha}} ([Du]_{\beta+\alpha-1, T} + \|u\|_\infty)(1 + K_0) |x - x'|^{\alpha+\beta-1}. \end{aligned}$$

We can write:

$$\begin{aligned}
& \left(D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(\tau_0, x') - D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \tilde{\xi}')} v_{\tau, \tilde{\xi}'}(t_0, x') \right)_{(\tau_0, \tau, \xi', \tilde{\xi}') = (t_0, t, x', x)} \\
&= \left[\int_{\mathbb{R}^d} \tilde{p}_{\alpha}^{(\tau, \xi')}(t, \tau_0, x', y) Dv_{\tau, \xi'}(\tau_0, y) dy - \int_{\mathbb{R}^d} \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, \tau_0, x', y) Dv_{\tau, \tilde{\xi}'}(\tau_0, y) dy \right]_{(\tau_0, \tau, \xi', \tilde{\xi}') = (t_0, t, x', x)} \\
&= \left[\int_{\mathbb{R}^d} \tilde{p}_{\alpha}^{(\tau, \xi')}(t, \tau_0, x', y) [Dv_{\tau, \xi'}(t_0, y) - Du(\tau_0, m_{\tau_0, t}^{(\tau, \xi')}(x'))] dy \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, \tau_0, x', y) [Dv_{\tau, \tilde{\xi}'}(\tau_0, y) - Dv_{\tau, \tilde{\xi}'}(\tau_0, m_{\tau_0, t}^{(\tau, \tilde{\xi}')} (x'))] dy \right]_{(\tau_0, \tau, \xi', \tilde{\xi}') = (t_0, t, x', x)} \\
&\quad + |Dv_{\tau, \tilde{\xi}'}(\tau_0, m_{\tau_0, t}^{(\tau, \tilde{\xi}')} (x')) - Dv_{\tau, \xi'}(\tau_0, m_{\tau_0, t}^{(\tau, \xi')} (x'))|_{(\tau_0, \tau, \xi', \tilde{\xi}') = (t_0, t, x', x)}.
\end{aligned}$$

Exploiting now the regularity of Du and the integrability property **(NDa)** we then derive:

$$\begin{aligned}
& |D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \xi')} v_{\tau, \xi'}(\tau_0, x') - D_x \tilde{P}_{\tau_0, t, \alpha}^{(\tau, \tilde{\xi}')} v_{\tau, \tilde{\xi}'}(\tau_0, x')|_{(\tau_0, \tau, \xi', \tilde{\xi}') = (t_0, t, x', x)} \\
&\leq C([Du]_{\alpha+\beta-1, T} + \|u\|_{\infty}) \left[(t_0 - t)^{\frac{\alpha+\beta-1}{\alpha}} + |m_{t_0, t}^{(t, x)}(x') - m_{t_0, t}^{(t, x')}(x')|^{\alpha+\beta-1} \right] \\
&\leq C([Du]_{\alpha+\beta-1, T} + \|u\|_{\infty}) \left[c_0^{\frac{\alpha+\beta-1}{\alpha}} |x - x'|^{\alpha+\beta-1} + |m_{t_0, t}^{(t, x)}(x') - m_{t_0, t}^{(t, x')}(x')|^{\alpha+\beta-1} \right], \quad (2.51)
\end{aligned}$$

recalling from (2.18) that $t_0 = t + c_0|x - x'|^{\alpha}$ for the last inequality. It therefore remains to control $m_{t_0, t}^{(t, x)}(x') - m_{t_0, t}^{(t, x')}(x')$. Write:

$$\begin{aligned}
|m_{t_0, t}^{(t, x)}(x') - m_{t_0, t}^{(t, x')}(x')| &= |[x' + \int_t^{t_0} F(v, \theta_{v, t}(x)) dv] - [x' + \int_t^{t_0} F(v, \theta_{v, t}(x')) dv]| \\
&\leq K_0 \int_t^{t_0} dv |\theta_{v, t}(x) - \theta_{v, t}(x')|^{\beta} \leq CK_0 (t_0 - t) (|x - x'| + (t - t_0)^{\frac{1}{\alpha}})^{\beta} \\
&\leq CK_0 c_0 |x - x'|^{\alpha+\beta} \leq CK_0 c_0 |x - x'|, \quad (2.52)
\end{aligned}$$

using Lemma 11 for the second inequality. We have also assumed w.l.o.g. that $|x - x'| \leq 1$ and exploited as well that $\alpha + \beta > 1$. Plugging (2.52) into (2.51) yields the statement of the lemma.

Smoothing effects associated with the perturbative term. This section is dedicated to the investigation of the spatial Hölder continuity of the perturbative term in (2.47). We prove the following Lemma which is the most difficult part of the proof of Theorem 3.

Lemma 17. *Under **(A)**, there exists a constant $C := C((\mathbf{A}), T)$ s.t. for fixed $(t, x, x') \in [0, T] \times (\mathbb{R}^d)^2$ s.t. $c_0|x - x'|^{\alpha} \leq T - t$, we have that:*

$$\begin{aligned}
& \left| \int_t^T ds \int_{\mathbb{R}^d} dy \left(\mathbb{I}_{s \leq t_0} D_x \tilde{p}_{\alpha}^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \right. \right. \\
&\quad \left. \left. + \mathbb{I}_{s > t_0} D_x \tilde{p}_{\alpha}^{(\tau, \tilde{\xi}')} (t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\tilde{\xi}'))) \cdot Du(s, y) \right) \eta_{\tau, \tilde{\xi}'}(s, y) \right) \right. \\
&\quad \left. - D_x \tilde{p}_{\alpha}^{(\tau, \xi)}(t, s, x, y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi}(s, y) \right|_{(\tau_0, \tau, \xi, \tilde{\xi}') = (t_0, t, x, x', x)} \\
&\leq CK_0 (c_0^{1+\frac{\beta-2}{\alpha}} + c_0^{\frac{\alpha+\beta-1}{\alpha}}) \|Du\|_{L^{\infty}} |x - x'|^{\alpha+\beta-1}. \quad (2.53)
\end{aligned}$$

As already used for the semi-group and the Green kernel, we split the investigations into two parts: the first one is done when the system is in the *local off-diagonal* regime w.r.t. the current integration time s (i.e. for $s \leq t_0$) and the other one when the system is in the *local diagonal* regime (i.e., for $s > t_0$). We also recall that the critical time giving the change of regime is (chosen after potential differentiation) $t_0 = t + c_0|x - x'|^{\alpha} \leq T$ (in the current global diagonal regime).

• Control of (2.53): *Local Off-Diagonal case*. Arguing as in Lemma 15 we write for the *local off-diagonal* regime:

$$\begin{aligned}
& \left| D_x \Delta_{\text{off-diag}}^{\xi, \xi'}(t, x, x') \right|_{(\xi, \xi')=(x, x')} \\
:= & \left| \int_t^{t_0} ds \int_{\mathbb{R}^d} \left(D_x \tilde{p}_\alpha^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \right. \right. \\
& \left. \left. - D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) \left((F(s, y) - F(s, \theta_{s, \tau}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi}(s, y) \right) \right|_{(\tau, \xi, \xi')=(t, x, x')} \\
\leq & CK_0 \|Du\|_\infty \left[\left| \int_t^{t_0} ds \int_{\mathbb{R}^d} dy |D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y)| |y - \theta_{s, t}(\xi)|^\beta \right|_{(\tau, \xi)=(t, x)} \right. \\
& \left. + \left| \int_t^{t_0} ds \int_{\mathbb{R}^d} dy |D_x \tilde{p}_\alpha^{(\tau, \xi')}(t, s, x', y)| |y - \theta_{s, t}(\xi')|^\beta \right|_{(\tau, \xi')=(t, x')} \right]. \tag{2.54}
\end{aligned}$$

We readily get since $t_0 = t + c_0|x - x'|^\alpha \leq T$ using the integrability property (\mathcal{P}_β) :

$$\begin{aligned}
\left| D_x \Delta_{\text{off-diag}}^{\xi, \xi'}(t, x, x') \right|_{(\xi, \xi')=(x, x')} & \leq CK_0 \int_t^{t_0} \frac{ds}{(s-t)^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}} \|Du\|_{L^\infty} \\
& \leq CK_0 \|Du\|_{L^\infty} c_0^{\frac{\alpha+\beta-1}{\alpha}} |x - x'|^{\alpha+\beta-1}. \tag{2.55}
\end{aligned}$$

• Control of (2.53): *Local Diagonal case*. Let us now turn to the control of

$$\begin{aligned}
& \left| D_x \Delta_{\text{diag}}^{\xi, \xi'}(t, T, x, x') \right|_{(\xi, \xi')=(x, x)} \tag{2.56} \\
:= & \left| \int_{t_0}^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau, \xi')}(t, s, x', y) \left((F(s, y) - F(s, \theta_{s, t}(\xi'))) \cdot Du(s, y) \right) \eta_{\tau, \xi'}(s, y) \right. \right. \\
& \left. \left. - D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) \left((F(s, y) - F(s, \theta_{s, t}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi}(s, y) \right) \right|_{(\tau, \xi, \xi')=(t, x, x)} \\
:= & \left| \int_{t_0}^T ds \int_{\mathbb{R}^d} dy \left(D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x', y) - D_x \tilde{p}_\alpha^{(\tau, \xi)}(t, s, x, y) \right) \right. \\
& \left. \times \left((F(s, y) - F(s, \theta_{s, t}(\xi))) \cdot Du(s, y) \right) \eta_{\tau, \xi}(s, y) \right|_{(\tau, \xi)=(t, x)} \\
\leq & \|Du\|_\infty |x - x'| \int_{t_0}^T ds \int_0^1 d\mu \int_{\mathbb{R}^d} dy |D^2 p_\alpha(s-t, y - \theta_{s, t}(x) + \mu(x-x'))| |F(s, y) - F(s, \theta_{s, t}(x))| \eta_{t, x}(s, y),
\end{aligned}$$

recalling (2.3) for the last inequality. We finally get:

$$\begin{aligned}
& \left| D_x \Delta_{\text{diag}}^{\xi, \xi'}(t, T, x, x') \right|_{(\xi, \xi')=(x, x)} \\
\leq & \|Du\|_\infty K_0 |x - x'| \int_{t_0}^T ds \int_0^1 d\mu \int_{\mathbb{R}^d} dy |D^2 p_\alpha(s-t, y - \theta_{s, t}(x) + \mu(x-x'))| |y - \theta_{s, t}(x)|^\beta \mathbb{1}_{|y - \theta_{s, t}(x)| \leq 2} \\
\leq & \|Du\|_\infty K_0 |x - x'| \int_{t_0}^T ds \int_0^1 d\mu \int_{\mathbb{R}^d} dy |D^2 p_\alpha(s-t, y + \mu(x-x'))| |y|^\beta \\
\leq & \|Du\|_\infty K_0 |x - x'| \int_{t_0}^T ds \int_0^1 d\mu \left[\int_{\mathbb{R}^d} dy |D^2 p_\alpha(s-t, y + \mu(x-x'))| |y + \mu(x-x')|^\beta \right. \\
& \left. + \int_{\mathbb{R}^d} dy |D^2 p_\alpha(s-t, y + \mu(x-x'))| |\mu(x-x')|^\beta \right]
\end{aligned}$$

Now since $s \in [t_0, T]$ and $t_0 = t + c_0|x - x'|^\alpha$ we have $s - t \geq c_0|x - x'|^\alpha$ and so

$$|x - x'| \leq (c_0)^{-1/\alpha} (s - t)^{1/\alpha} \leq K (s - t)^{\frac{1}{\alpha}} \tag{2.57}$$

if the threshold K is chosen large enough. Applying property (\mathcal{P}_β) twice with $\gamma = \beta$ and $\gamma = 0$ we eventually

derive:

$$\begin{aligned}
|D_x \Delta_{\mathbf{diag}}^{\xi, \tilde{\xi}'}(t, x, x')|_{(\xi, \tilde{\xi}')=(x, x')} &\leq CK_0 \|Du\|_{\infty} |x - x'| \int_{t_0}^T \frac{ds}{(s-t)^{\frac{2}{\alpha} - \frac{\beta}{\alpha}}} \\
&\leq CK_0 \|Du\|_{L^\infty} (t_0 - t)^{1 + \frac{\beta}{\alpha} - \frac{2}{\alpha}} |x - x'| \\
&\leq CK_0 \|Du\|_{L^\infty} |x - x'|^{\alpha + \beta - 1},
\end{aligned} \tag{2.58}$$

where $C = C(c_0, (\mathbf{A})) > 0$. Equations (2.55) and (2.58) give the statement of the lemma.

3 Existence result.

We point out here that the classical continuity method, which is direct from the a priori estimate, and which was successfully used in [Pri12] to establish existence in the elliptic setting, does not work for $\alpha \in (0, 1)$. The key point is that when one tries to write:

$$\partial_t u(t, x) + L_\alpha u + \delta_0 F(t, x) \cdot Du(t, x) = -f(t, x) + (\delta_0 - \delta) F(t, x) \cdot Dv(t, x),$$

where $v \in C_b^{\alpha + \beta}(\mathbb{R}^d, \mathbb{R})$ then, the product $F(t, x) \cdot Dv(t, x)$ has under **(A)** a Hölder-regularity of order $\beta + \alpha - 1 < \beta$, since $\alpha \in (0, 1)$. Therefore, we cannot in this framework readily apply our a priori estimate in a fixed point perspective.

We will proceed through a vanishing viscosity approach, as it was also for instance considered by Silvestre in [Sil12] or by Zhang and Zhao in [ZZ18]. Namely, we consider for a given parameter $\varepsilon > 0$ the IPDE:

$$\begin{aligned}
\partial_t u(t, x) + L_\alpha u(t, x) + F_\varepsilon(t, x) \cdot D_x u(t, x) + \varepsilon \Delta^{\frac{1}{2}} u(t, x) &= -f_\varepsilon(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\
u(T, x) &= g_\varepsilon(x), \quad \text{on } \mathbb{R}^d,
\end{aligned} \tag{3.1}$$

where $f_\varepsilon, g_\varepsilon$ are mollified version of the initial sources and terminal condition f and g in time-space and space respectively which satisfy uniformly w.r.t. the mollification procedure assumption **(A)**. Also, F_ε stands for a mollified truncation of F so that for any fixed $\varepsilon > 0$, F_ε is smooth, bounded and uniformly β -Hölder continuous in space and satisfies assumption **(A)** uniformly w.r.t. the mollification procedure as well.

The procedure is the following we aim at showing that, for any fixed $\varepsilon > 0$, there is a unique solution $u := u^\varepsilon$ to equation (3.1) which belongs to the function space $\mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d)$ (where for the regularity of the generalized time derivative in point iii) of the corresponding definition at p. 6, the parameter $\alpha + \beta - 1$ has to be replaced by β because, from the $\varepsilon \Delta^{\frac{1}{2}}$ regularization term, we go back to the sub-critical case). The next step then consists in rewriting (3.1) as:

$$\begin{aligned}
\partial_t u(t, x) + L_\alpha u(t, x) + F_\varepsilon(t, x) \cdot D_x u(t, x) &= -f_\varepsilon(t, x) - \varepsilon \Delta^{\frac{1}{2}} u(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\
u(T, x) &= g_\varepsilon(x), \quad \text{on } \mathbb{R}^d,
\end{aligned} \tag{3.2}$$

and to establish that $\varepsilon \|\Delta^{\frac{1}{2}} u\|_{L^\infty([0, T], C_b^\beta)}$ is controlled uniformly in ε allowing thus to expand u along the frozen semi-groups $(\tilde{P}_{s,t}^{(\tau, \xi)})_{0 \leq s, t \leq T}$ as in Section 2 to establish that the solution satisfies the Schauder estimates uniformly in ε . The existence of a solution in $\mathcal{C}_b^{\alpha + \beta}([0, T] \times \mathbb{R}^d)$ then follows from a standard compactness argument letting ε go to 0. We also point out that, although inefficient in our case, the continuity method described above will be used several times for the analysis of the equation (3.1).

3.1 A priori controls for the regularized equation

3.1.1 Estimates for a generic source in $L^\infty([0, T], C_b^\beta)$

We focus in this section on an equation of the form

$$\begin{aligned}
\partial_t v(t, x) + F_\varepsilon(t, x) \cdot D_x v(t, x) + \varepsilon \Delta^{\frac{1}{2}} v(t, x) &= -\bar{f}_\varepsilon(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\
v(T, x) &= g_\varepsilon(x), \quad \text{on } \mathbb{R}^d,
\end{aligned} \tag{3.3}$$

where $\bar{f}_\varepsilon, F_\varepsilon$ is in $C_b^\beta([0, T] \times \mathbb{R}^d)$ (i.e., $f_\varepsilon, F_\varepsilon$ are β -Hölder continuous in both time and space) and $g_\varepsilon \in C^{1+\beta}(\mathbb{R}^d)$.

In this framework, it follows from [MP14] that there exists a unique solution $v := v^\varepsilon$ to (3.3) in $\mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d)$ which satisfies:

$$\|v\|_{L^\infty([0, T], C_b^{1+\beta})} \leq C(\Theta_1(\varepsilon)\|g_\varepsilon\|_{C_b^{1+\beta}} + \Theta_2(\varepsilon)\|\bar{f}_\varepsilon\|_{L^\infty([0, T], C_b^\beta)}). \quad (3.4)$$

With respect to the previously described procedure, we actually need to precisely quantify how the $(\Theta_i(\varepsilon))_{i \in \{1, 2\}}$ behave when ε goes to 0. This behavior actually depends on the smoothing effects associated with $P_{s, 1}^\varepsilon$ which denotes the semi-group associated with $\varepsilon\Delta^{\frac{1}{2}}$. We can therefore appeal to the results of Section 2 considering appropriate scaling arguments. Namely, let us consider $w := w^\varepsilon$ solving:

$$\begin{aligned} \partial_t w(t, x) + \varepsilon\Delta^{\frac{1}{2}}w(t, x) &= -\bar{f}_\varepsilon(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\ w(T, x) &= g_\varepsilon(x), \quad \text{on } \mathbb{R}^d. \end{aligned}$$

Setting $w(t, x) =: \bar{w}(t, x/\varepsilon)$ then

$$\begin{aligned} \partial_t \bar{w}(t, y) + \Delta^{\frac{1}{2}}\bar{w}(t, y) &= -\bar{f}_\varepsilon(t, \varepsilon y), \quad \text{on } [0, T] \times \mathbb{R}^d, \\ \bar{w}(T, y) &= g_\varepsilon(\varepsilon y), \quad \text{on } \mathbb{R}^d, \end{aligned}$$

From the proof of point *iii*) of Proposition 6 (see also Lemma 5), we get:

$$\varepsilon^{1+\beta}[Dw(t, \cdot)]_\beta = [D\bar{w}(t, \cdot)]_\beta \leq C(\varepsilon^{1+\beta}[Dg]_\beta + \varepsilon^\beta[f]_{\beta, T}).$$

We thus derive that there exists $C := C(\mathbf{A})$ independent of ε s.t. :

$$\|w\|_{L^\infty([0, T], C_b^{1+\beta})} \leq C\left(\|g_\varepsilon\|_{C_b^{1+\beta}} + \varepsilon^{-1}\|\bar{f}_\varepsilon\|_{L^\infty([0, T], C_b^\beta)}\right). \quad (3.5)$$

We can then follow the arguments of [KP10] to establish from the continuity method that a similar bound will also hold for the unique solution v in $\mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d)$ of the drifted equation (3.3). Namely,

$$\|v\|_{L^\infty([0, T], C_b^{1+\beta})} \leq C\left(\|g_\varepsilon\|_{C_b^{1+\beta}} + \varepsilon^{-1}\|\bar{f}_\varepsilon\|_{L^\infty([0, T], C_b^\beta)}\right), \quad (3.6)$$

where the above C also depends on $[F]_{\beta, T}$. Equation (3.6) specifies that the functions $(\Theta_i(\varepsilon))_{i \in \{1, 2\}}$ in (3.4) actually write $\Theta_1(\varepsilon) = 1, \Theta_2(\varepsilon) = \varepsilon^{-1}$.

3.1.2 Solvability of equation (3.1)

In order to prove existence we would like to benefit from the results of the previous section. From equation (3.1) we introduce for a parameter $\lambda \in [0, 1]$:

$$\begin{aligned} \partial_t u(t, x) + F_\varepsilon(t, x) \cdot D_x u(t, x) + \varepsilon\Delta^{\frac{1}{2}}u(t, x) &= -f_\varepsilon(t, x) - \lambda L_\alpha u(t, x), \quad \text{on } [0, T] \times \mathbb{R}^d, \\ u(T, x) &= g_\varepsilon(x), \quad \text{on } \mathbb{R}^d, \end{aligned} \quad (3.7)$$

which can be viewed as a particular case of (3.3) with $\bar{f}_\varepsilon = f_\varepsilon + \lambda L_\alpha u$.

We now recall a useful inequality. For $\theta \in (0, 1]$ consider an operator L_θ with symbol of the form (1.14) satisfying **(ND)** and $\gamma \in (0, 1)$ s.t $\theta + \gamma > 1$. There exists $C_{\theta, \gamma}$ s.t. for a function $\varphi \in C_b^{\gamma+\theta}$, it holds that:

$$\|L_\theta \varphi\|_{C_b^\gamma} \leq C_{\theta, \gamma} \|\varphi\|_{C_b^{\gamma+\theta}}. \quad (3.8)$$

Recalling that Hölder spaces can be viewed as Besov spaces, this inequality is as a direct consequence of norm equivalences on Besov spaces (see e.g. Triebel [Tri83]). We also provide a direct proof of (3.8) in Appendix A.2 for a self-contained presentation.

From (3.8), we derive that for $v \in \mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d), L_\alpha v \in L^\infty([0, T], C_b^{1+\beta-\alpha})$. Hence, the continuity method will also give from the previous estimates that for any fixed $\varepsilon > 0$ there exists a unique solution $u^\varepsilon = u \in \mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d)$ to (3.7) for $\lambda \in [0, 1]$ and therefore to (3.1) corresponding to $\lambda = 1$. Also, for the unique solution of (3.1) in $\mathcal{C}_b^{1+\beta}([0, T] \times \mathbb{R}^d)$ it holds that:

$$\|u\|_{L^\infty([0, T], C_b^{1+\beta})} \leq C(\|g_\varepsilon\|_{C_b^{1+\beta}} + \varepsilon^{-1}\|f_\varepsilon\|_{L^\infty([0, T], C_b^\beta)}), C := C(\mathbf{A}), \quad (3.9)$$

and for all $0 \leq t < s \leq T, x \in \mathbb{R}^d$,

$$u(t, x) = u(s, x) + \int_t^s dr f_\varepsilon(r, x) - \int_t^s dr \left(\varepsilon\Delta^{\frac{1}{2}} + L_\alpha + F_\varepsilon(r, x) \cdot D \right) u(r, x). \quad (3.10)$$

3.2 Viscosity viewed as a source and compactness arguments

We now rewrite equation (3.1) viewing the viscous perturbation as a source. Namely, as in (3.2). We now observe from (3.9) and (3.8) that:

$$\begin{aligned} \varepsilon \|\Delta^{\frac{1}{2}} u\|_{L^\infty([0,T],C_b^\beta)} &\leq C_{1,\beta} \varepsilon \|u\|_{L^\infty([0,T],C_b^{1+\beta})} \leq C_{1,\beta} C(\varepsilon \|g_\varepsilon\|_{C_b^{1+\beta}} + \|f_\varepsilon\|_{L^\infty([0,T],C_b^\beta)}) \\ &\leq C_{1,\beta} C(\varepsilon \|g_\varepsilon\|_{C_b^{1+\beta}} + \|f\|_{L^\infty([0,T],C_b^\beta)}). \end{aligned}$$

Hence, reproducing the previous expansion of Section 2 we derive that $u := u^\varepsilon$ solving (3.1) satisfy the estimate:

$$\begin{aligned} \|u^\varepsilon\|_{L^\infty([0,T],C_b^{\alpha+\beta})} &\leq C \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0,T],C_b^\beta)} + \varepsilon \|g_\varepsilon\|_{C_b^{1+\beta}} \right) \\ &\leq C \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0,T],C_b^\beta)} + h(\varepsilon) \|g\|_{C_b^{\alpha+\beta}} \right) \\ &\leq C \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0,T],C_b^\beta)} \right), \end{aligned} \quad (3.11)$$

where $h(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$, considering a suitable regularization of g , i.e., s.t. $\varepsilon \|g_\varepsilon\|_{C_b^{1+\beta}} \leq h(\varepsilon) \|g\|_{C_b^{\alpha+\beta}}$, for the second inequality.

Note that proceeding as in Corollary 4.2 of [KP10], we also have local γ -Hölder continuity in $[0, T] \times K$, for some small $\gamma > 0$, for u^ε and Du^ε for any compact set $K \subset \mathbb{R}^d$ (with a control of the Hölder norm independent of ε).

From the Ascoli-Arzelà theorem, we deduce that there exists a subsequence $\varepsilon_n \xrightarrow{n} 0$ and a continuous function u on $[0, T] \times \mathbb{R}^d$ having bounded and continuous derivatives with respect to x such that

$$(u^{\varepsilon_n}, Du^{\varepsilon_n}) \rightarrow (u, Du)$$

as $n \rightarrow \infty$, uniformly on bounded subsets of $[0, T] \times \mathbb{R}^d$. We also have that $u \in L^\infty([0, T], C_b^{\alpha+\beta})$ satisfies the last inequality of (3.11). Rewrite now (3.10) along the considered subsequence:

$$u^{\varepsilon_n}(t, x) = u^{\varepsilon_n}(s, x) + \int_t^s dr f_{\varepsilon_n}(r, x) - \int_t^s dr \left(\varepsilon_n \Delta^{\frac{1}{2}} + L_\alpha + F_{\varepsilon_n}(r, x) \cdot D \right) u^{\varepsilon_n}(r, x). \quad (3.12)$$

It is readily seen that, in order to pass to the limit in the previous equation, the only *delicate* term to analyze is $\varepsilon_n \Delta^{\frac{1}{2}} u^{\varepsilon_n}(t, x)$. Observe that:

$$\begin{aligned} |\Delta^{\frac{1}{2}} u^{\varepsilon_n}(t, x)| &\leq \left| \int_{|z| \leq 1} \left(u^{\varepsilon_n}(t, x+z) - u^{\varepsilon_n}(t, x) - Du^{\varepsilon_n}(t, x) \cdot z \right) \frac{dz}{|z|^2} \right| + \int_{|z| \geq 1} \frac{2 \|u^{\varepsilon_n}\|_\infty}{|z|^2} dz \\ &\leq C \left([Du^{\varepsilon_n}]_{\alpha+\beta-1, T} + \|u^{\varepsilon_n}\|_\infty \right) \leq C \left(\|g\|_{C_b^{\alpha+\beta}} + \|f\|_{L^\infty([0,T],C_b^\beta)} \right). \end{aligned}$$

We can then pass to the limit in (3.12) and so u satisfies:

$$u(t, x) = u(s, x) + \int_t^s dr f(r, x) - \int_t^s dr \left(L_\alpha + F(r, x) \cdot D \right) u(r, x). \quad (3.13)$$

The Schauder estimates of Theorem 3 then gives uniqueness. This also proves Theorem 4.

4 Proof of the Property (\mathcal{P}_β) in the indicated case

4.1 Proof of Proposition 2 on stable like operators close to $\Delta^{\alpha/2}$, $\alpha \in (0, 1)$

In other words, the Lévy measure ν in (1.9) rewrites:

$$\nu(dy) = \nu_\alpha(dy) = f\left(\frac{y}{|y|}\right) \frac{dy}{|y|^{d+\alpha}}.$$

We have to prove (1.21) for all $\alpha \in (0, 1)$, and $\gamma \in [0, 1]$; this will rely on global estimates on the derivatives of $p_\alpha(t, \cdot)$, $t > 0$, which can be deduced from the work of Kolokoltsov [Kol00]. First by [Kol00, formula (2.38) in Proposition 2.6] we know that there exists $c = c(\alpha, \eta)$ (where η denotes the non degeneracy constant associated with the spectral measure in (1.15)) such that

$$|D_y p_\alpha(t, y)| \leq c \left(\frac{1}{t^{1/\alpha}} \wedge \frac{1}{|y|} \right) p_\alpha(t, y), \quad y \in \mathbb{R}^d, t > 0. \quad (4.14)$$

We need the following result for the second derivatives.

Lemma 18. For any positive $K > 0$, there exists $c = c(\alpha, \eta, K) > 0$ such that, for $|y| \leq Kt^{1/\alpha}$ we have

$$|D_y^2 p_\alpha(t, y)| \leq \frac{c}{t^{2/\alpha}} p_\alpha(t, y), \quad t > 0, \quad (4.15)$$

and for $|y| > Kt^{1/\alpha}$ we have

$$|D_y^2 p_\alpha(t, y)| \leq \frac{c}{t} \cdot \frac{1}{|y|^{2-\alpha}} p_\alpha(t, y), \quad t > 0. \quad (4.16)$$

Proof. As in formula (2.23) of [Kol00] we consider with $b = 2$, $\sigma = t > 0$,

$$\phi_{2,\theta}(x, \alpha, t\mu) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[\int_{\mathbb{S}^{d-1}} \langle p, s \rangle^2 \theta(ds) \right] \exp \left(-t \int_{\mathbb{S}^{d-1}} |\langle p, s \rangle|^\alpha \mu(ds) \right) e^{-i\langle p, x \rangle} dp.$$

where θ is a finite positive measure on \mathbb{S}^{d-1} such that $\theta(\mathbb{S}^{d-1}) \leq M$ for some $M > 0$.

Then by [Kol00, Proposition 2.5] (see in particular estimates (2.29) and (2.30)) for any positive $K > 0$, there exists $c = c(\alpha, \eta, K, M) > 0$ such that, for $|x| \leq Kt^{1/\alpha}$ we have

$$|\phi_{2,\nu}(x, \alpha, t\mu)| \leq \frac{c}{t^{2/\alpha}} p_\alpha(t, x), \quad x \in \mathbb{R}^d, t > 0, \quad (4.17)$$

and for $|x| > Kt^{1/\alpha}$ we have

$$|\phi_{2,\nu}(x, \alpha, t\mu)| \leq \frac{c}{t} \cdot \frac{1}{|x|^{2-\alpha}} p_\alpha(t, x), \quad x \in \mathbb{R}^d, t > 0 \quad (4.18)$$

(recalling that in [Kol00] $p_\alpha(t, x)$ is denoted by $S(x, \alpha, t\mu)$). Let us fix $M > 0$. Let $h \in \mathbb{R}^d$, $h \neq 0$, with $|h| \leq \sqrt{M}$. Choosing the measure $\theta = |h|^2 \delta_{\frac{h}{|h|}}$ we find that

$$\phi_{2,\theta}(x, \alpha, t\mu) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \langle p, h \rangle^2 \exp \left(-t \int_{\mathbb{S}^{d-1}} |\langle p, s \rangle|^\alpha \mu(ds) \right) e^{-i\langle p, x \rangle} dp = -D_{hh}^2 p_\alpha(t, x),$$

i.e., we are considering the second derivative of $p_\alpha(t, \cdot)$ in the direction h . Assertions (4.15) and (4.16) now readily follow from (4.17), (4.18). \square

Proof of Proposition 2. Let $l = 1$. Using (4.14) we find

$$\begin{aligned} \int_{|y| \leq t^{1/\alpha}} |y|^\gamma |D_y p_\alpha(t, y)| dy &\leq \int_{|y| \leq t^{1/\alpha}} t^{\gamma/\alpha} |D_y p_\alpha(t, y)| dy \\ &\leq c \int_{|y| \leq t^{1/\alpha}} t^{\gamma/\alpha} t^{-1/\alpha} p_\alpha(t, y) dy \leq ct^{[\gamma-1]/\alpha}. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{|y| > t^{1/\alpha}} |y|^\gamma |D_y p_\alpha(t, y)| dy &\leq c \int_{|y| > t^{1/\alpha}} |y|^{\gamma-1} p_\alpha(t, y) dy \\ &\leq c \int_{|y| > t^{1/\alpha}} t^{[\gamma-1]/\alpha} p_\alpha(t, y) dy \leq ct^{[\gamma-1]/\alpha}. \end{aligned}$$

Let $k = 2$. Using (4.15) and (4.16) we find

$$\int_{|y| \leq t^{1/\alpha}} |y|^\gamma |D_y^2 p_\alpha(t, y)| dy \leq c \int_{|y| \leq t^{1/\alpha}} t^{\gamma/\alpha} t^{-2/\alpha} p_\alpha(t, y) dy \leq ct^{[\gamma-2]/\alpha}.$$

Since $\alpha + \gamma < 2$, we find

$$\begin{aligned} \int_{|y| > t^{1/\alpha}} |y|^\gamma |D_y^2 p_\alpha(t, y)| dy &\leq c \int_{|y| > t^{1/\alpha}} \frac{1}{t} |y|^{\gamma-2+\alpha} p_\alpha(t, y) dy \\ &\leq c \int_{|y| > t^{1/\alpha}} \frac{1}{t} t^{[\gamma-2+\alpha]/\alpha} p_\alpha(t, y) dy \leq ct^{[\gamma-2]/\alpha}. \end{aligned}$$

and the assertion follow. \square

We can mention that, in the specific case of the rotationally invariant heat kernel, corresponding to the fractional Laplacian $\Delta^{\alpha/2}$, the previous computations could have been shortened exploiting explicitly the concentration gain for the derivatives of the heat kernel. Namely, it is known from Lemma 5 and Remark 6 in Bogdan and Jakubowicz [BJ07] that in that case:

$$|D_x p_\alpha(t, z)| \leq \frac{C}{t^{\frac{1}{\alpha}}} \frac{1}{t^{\frac{d}{\alpha}}} \frac{1}{(1 + \frac{|z|}{t^{\frac{1}{\alpha}}})^{d+\alpha+1}} =: \frac{C}{t^{\frac{1}{\alpha}}} \bar{p}_\alpha(t, z). \quad (4.19)$$

In other words, the derivative induces a concentration gain. The same arguments as in [BJ07], see also the decompositions in [Wat07], also yield the corresponding result for $|D_x^2 p_\alpha(t, z)| \leq C t^{-2/\alpha} \bar{p}_\alpha(t, z)$.

4.2 Proof of Proposition 1 on symmetric stable operators with $\alpha \in (1/2, 1)$

Here we are considering symmetric stable operators such that (1.16) holds. We want to prove (1.21) for all $\alpha \in (0, 1)$, and $\gamma \in [0, \alpha)$. The result follows easily from the following key lemma.

Lemma 19 (Bounds and Sensitivities of the Stable Heat Kernel). *There exists $C := C(\mathbf{A})$ s.t. for all $\ell \in \{1, 2\}$, $t > 0$, and $y \in \mathbb{R}^d$:*

$$|D_y^\ell p_\alpha(t, y)| \leq \frac{C}{t^{\ell/\alpha}} q(t, y), \quad (4.20)$$

where $(q(t, \cdot))_{t>0}$ is a family of probability densities on \mathbb{R}^d such that $q(t, y) = t^{-d/\alpha} q(1, t^{-1/\alpha} y)$, $t > 0$, $\in \mathbb{R}^d$ and for all $\gamma \in [0, \alpha)$, there exists a constant $c := c(\alpha, \eta, \gamma)$ s.t.

$$\int_{\mathbb{R}^N} q(t, y) |y|^\gamma dy \leq C_\gamma t^{\frac{\gamma}{\alpha}}, \quad t > 0, \quad (4.21)$$

Remark 9. From now on, for the family of stable densities $(q(t, \cdot))_{t>0}$, we also use the notation $q(\cdot) := q(1, \cdot)$, i.e., without any specified argument $q(\cdot)$ stands for the density q at time 1.

Proof. We denote by $(S_t)_{t \geq 0}$ a stable process defined on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{\geq 0}, \mathbb{P})$ whose Lévy exponent is given by (1.14). For $t > 0$, the heat kernel $p_\alpha(t, \cdot)$ associated with L_α given in (1.18) is then precisely the density of S_t .

Let us recall that, for a given fixed $t > 0$, we can use an Itô-Lévy decomposition at the associated characteristic stable time scale (i.e., the truncation is performed at the threshold $t^{\frac{1}{\alpha}}$) to write $S_t := M_t + N_t$ where M_t and N_t are independent random variables. More precisely,

$$N_s = \int_0^s \int_{|x| > t^{\frac{1}{\alpha}}} x P(du, dx), \quad M_s = S_s - N_s, \quad s \geq 0, \quad (4.22)$$

where P is the Poisson random measure associated with the process S ; for the considered fixed $t > 0$, M_t and N_t correspond to the *small jumps part* and *large jumps part* respectively. A similar decomposition has been already used in [Wat07], [Szt10] and [HM16], [HMP19] (see in particular Lemma 4.3 therein). It is useful to note that the cutting threshold in (4.22) precisely yields for the considered $t > 0$ that:

$$N_t \stackrel{(\text{law})}{=} t^{\frac{1}{\alpha}} N_1 \quad \text{and} \quad M_t \stackrel{(\text{law})}{=} t^{\frac{1}{\alpha}} M_1. \quad (4.23)$$

To check the assertion about N we start with

$$\mathbb{E}[e^{i\langle p, N_t \rangle}] = \exp\left(t \int_{\mathbb{S}^{d-1}} \int_{t^{\frac{1}{\alpha}}}^\infty \left(\cos(\langle p, r\xi \rangle) - 1\right) \frac{dr}{r^{1+\alpha}} \tilde{\mu}_S(d\xi)\right), \quad p \in \mathbb{R}^d$$

(see (1.14) and [Sat99]). Changing variable $\frac{r}{t^{\frac{1}{\alpha}}} = s$ we get that $\mathbb{E}[e^{i\langle p, N_t \rangle}] = \mathbb{E}[e^{i\langle p, t^{\frac{1}{\alpha}} N_1 \rangle}]$ for any $p \in \mathbb{R}^d$ and this shows the assertion (similarly we get the statement for M). The density of S_t then writes

$$p_S(t, x) = \int_{\mathbb{R}^d} p_M(t, x - \xi) P_{N_t}(d\xi), \quad (4.24)$$

where $p_M(t, \cdot)$ corresponds to the density of M_t and P_{N_t} stands for the law of N_t . From Lemma A.2 in [HMP19] (see as well Lemma B.1 in [HM16]), $p_M(t, \cdot)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ and satisfies that for all $m \geq 1$ and all $\ell \in \{0, 1, 2\}$, there exist constants \bar{C}_m, C_m s.t. for all $t > 0$, $x \in \mathbb{R}^d$:

$$|D_x^\ell p_M(t, x)| \leq \frac{\bar{C}_m}{t^{\frac{\ell}{\alpha}}} p_{\bar{M}}(t, x), \quad \text{where} \quad p_{\bar{M}}(t, x) := \frac{C_m}{t^{\frac{d}{\alpha}}} \left(1 + \frac{|x|}{t^{\frac{1}{\alpha}}}\right)^{-m} \quad (4.25)$$

where C_m is chosen in order that $p_{\bar{M}}(t, \cdot)$ be a probability density.

We carefully point out that, to establish the indicated results, since we are led to consider potentially singular spherical measures, we only focus on integrability properties similarly to [HMP19] and not on pointwise density estimates as for instance in [HM16]. The main idea thus consists in exploiting (4.22), (4.24) and (4.25). The derivatives on which we want to obtain quantitative bounds will be expressed through derivatives of $p_M(t, \cdot)$, which also give the corresponding time singularities. However, as for general stable processes, the integrability restrictions come from the large jumps (here N_t) and only depend on its index α . A crucial point then consists in observing that the convolution $\int_{\mathbb{R}^d} p_{\bar{M}}(t, x - \xi) P_{N_t}(d\xi)$ actually corresponds to the density of the random variable

$$\bar{S}_t := \bar{M}_t + N_t, \quad t > 0 \quad (4.26)$$

(where \bar{M}_t has density $p_{\bar{M}}(t, \cdot)$ and is independent of N_t ; to have such decomposition one can define each \bar{S}_t on a product probability space). Then, the integrability properties of $\bar{M}_t + N_t$, and more generally of all random variables appearing below, come from those of \bar{M}_t and N_t .

One can easily check that $p_{\bar{M}}(t, x) = t^{-\frac{d}{\alpha}} p_{\bar{M}}(1, t^{-\frac{1}{\alpha}} x)$, $t > 0$, $x \in \mathbb{R}^d$. Hence

$$\bar{M}_t \stackrel{(\text{law})}{=} t^{\frac{1}{\alpha}} \bar{M}_1, \quad N_t \stackrel{(\text{law})}{=} t^{\frac{1}{\alpha}} N_1.$$

By independence of \bar{M}_t and N_t , using the Fourier transform, one can easily prove that

$$\bar{S}_t \stackrel{(\text{law})}{=} t^{\frac{1}{\alpha}} \bar{S}_1. \quad (4.27)$$

Moreover, $\mathbb{E}[|\bar{S}_t|^\gamma] = \mathbb{E}[|\bar{M}_t + N_t|^\gamma] \leq C_\gamma t^{\frac{\gamma}{\alpha}} (\mathbb{E}[|\bar{M}_1|^\gamma] + \mathbb{E}[|N_1|^\gamma]) \leq C_\delta t^{\frac{\gamma}{\alpha}}$, $\gamma \in (0, \alpha)$. This shows that the density of \bar{S}_t verifies (4.21). The controls on the derivatives are derived similarly using 4.25 for $\ell \in \{1, 2\}$ and the same previous argument. \square

4.3 Proof of Property (\mathcal{P}_β) for the relativistic stable operator

In order to prove the required integrability properties we first have to differentiate the density.

Starting from (1.24) and similarly to the proof of Lemma 5 in [BJ07] we can differentiate under the integral sign and obtain, for $x \neq 0$,

$$\begin{aligned} |D_x p_{\alpha, m}(t, x)| &= \left| e^{mt} \int_0^\infty D_x g(u, x) e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du \right| = \left| e^{mt} \frac{-x}{2} \int_0^\infty \frac{g(u, x)}{u} e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du \right| \\ &\leq e^{mt} \frac{|x|}{2} \int_0^\infty \frac{g(u, x)}{u} \theta_\alpha(t, u) du = e^{mt} |D_x p_{\alpha, 0}(t, x)|. \end{aligned} \quad (4.28)$$

The spatial concentration gain induced by the differentiation of $p_{\alpha, 0}(t, x)$ then provides the required integrability property for the first derivative. Similarly,

$$\begin{aligned} D_x^2 p_{\alpha, m}(t, x) &= e^{mt} \int_0^\infty D_x^2 g(u, x) e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du \\ &= e^{mt} \frac{-1}{2} \int_0^\infty \frac{g(u, x)}{u} e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du + e^{mt} \frac{|x|^2}{4} \int_0^\infty \frac{g(u, x)}{u^2} e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du. \end{aligned}$$

Since

$$\left| e^{mt} \frac{-1}{2} \int_0^\infty \frac{g(u, x)}{u} e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du \right| \leq \frac{e^m |D_x p_{\alpha, 0}(t, x)|}{|x|}, \quad x \neq 0,$$

and

$$e^{mt} \frac{|x|^2}{4} \int_0^\infty \frac{g(u, x)}{u^2} e^{-m \frac{2}{\alpha} u} \theta_\alpha(t, u) du \leq e^m \frac{|x|^2}{4} \int_0^\infty \frac{g(u, x)}{u^2} \theta_\alpha(t, u) du \leq e^m |D_x^2 p_{\alpha, 0}(t, x)| + \frac{e^m |D_x p_{\alpha, 0}(t, x)|}{|x|},$$

to prove **(NDb)** for $k = 2$, we concentrate on $\frac{|D_x p_{\alpha, 0}(t, x)|}{|x|}$. The estimate

$$\frac{1}{|z|} |D_x p_{\alpha, 0}(t, z)| \leq \frac{C}{t^{\frac{1}{\alpha}}} \frac{1}{t^{\frac{d}{\alpha}}} \frac{1}{(1 + \frac{|z|}{t^{\frac{1}{\alpha}}})^{d+\alpha+1}} \frac{1}{\frac{|z|}{t^{\frac{1}{\alpha}}}},$$

easily yields (\mathcal{P}_β) for $k = 2$.

A Proof of some technical results

A.1 Proof of the flow lemma 11

We first assume for the proof that the control (1.3) holds globally, i.e., the function F is globally β -Hölder continuous with constant K_0 . The point is then to write for $0 \leq t \leq s \leq T$, $(x, x') \in (\mathbb{R}^d)^2$:

$$\begin{aligned} \theta_{s,t}(x) - \theta_{s,t}(x') &= x - x' + \int_t^s [F(u, \theta_{u,t}(x)) - F(u, \theta_{u,t}(x'))] du \\ &= x - x' + \int_t^s [F(u, \theta_{u,t}(x)) - F_\delta(u, \theta_{u,t}(x))] du \\ &\quad + \int_t^s [F_\delta(u, \theta_{u,t}(x)) - F_\delta(u, \theta_{u,t}(x'))] du + \int_t^s [F(u, \theta_{u,t}(x')) - F_\delta(u, \theta_{u,t}(x'))] du, \end{aligned}$$

where for all $y \in \mathbb{R}^d$, and $\delta > 0$, $F_\delta(y) := \int_{\mathbb{R}^d} F(y-z)\phi_\delta(z)dz$ where $\phi_\delta(z) = \frac{1}{\delta^d}\phi(\frac{z}{\delta})$, where $\phi : \mathbb{R}^d \in \mathbb{R}^+$ is a standard mollifier, i.e., a smooth non negative function s.t. $\int_{\mathbb{R}^d} \phi(z)dz = 1$. It is then clear from the fact that $F \in L^\infty(C^\beta)$ that:

$$\begin{aligned} \forall (u, z) \in [0, T] \times \mathbb{R}^d, \quad |F_\delta(u, z) - F(u, z)| &\leq K_0\delta^\beta, \\ |DF_\delta|_\infty &\leq K_0\delta^{\beta-1}. \end{aligned} \tag{A.29}$$

We therefore get:

$$\begin{aligned} |\theta_{s,t}(x) - \theta_{s,t}(x')| &\leq |x - x'| + 2K_0\delta^\beta(s-t) + K_0\delta^{-1+\beta} \int_t^s |\theta_{u,t}(x) - \theta_{u,t}(x')| \\ &\leq \left(|x - x'| + 2K_0\delta^\beta(s-t)\right) \exp(K_0\delta^{-1+\beta}(s-t)), \end{aligned}$$

using the Gronwall lemma for the last inequality. Choose now $\delta^{-1+\beta}(s-t) = 1 \iff (s-t) = \delta^{1-\beta}$ to equilibrate the previous contributions. We eventually derive:

$$\begin{aligned} |\theta_{s,t}(x) - \theta_{s,t}(x')| &\leq \left(|x - x'| + 2K_0(s-t)^{1+\frac{\beta}{1-\beta}}\right) \exp(K_0), \\ &\leq C(|x - x'| + (s-t)^{\frac{1}{1-\beta}}) \leq C(|x - x'| + (s-t)^{\frac{1}{\alpha}}), \end{aligned}$$

recalling for the last inequality that $(s-t) \leq T \leq 1$ and $\frac{1}{\alpha} < \frac{1}{1-\beta}$ since $\alpha + \beta > 1$. This gives the result when F is globally β -Hölder continuous.

Recalling now that we appeal to this result when $|x - x'| \leq ((T-t)/c_0)^{1/\alpha}$ where the r.h.s. is small, provided that T is, it is plain to localize the above computations and to observe that the previous global results are actually also local when the final time horizon T is small enough and the initial points are close w.r.t. the time scale. This concludes the proof of the Lemma.

A.2 Proof of equation (3.8)

Let $\varphi \in C_b^{\gamma+\theta}(\mathbb{R}^d)$ where we recall $\gamma + \theta > 1$ with $\theta \in (0, 1]$, $\gamma \in (0, 1)$. We aim at proving:

$$\|L_\theta \varphi\|_{C_b^\gamma} \leq C_{\theta, \gamma} \|\varphi\|_{C_b^{\gamma+\theta}}.$$

Write first, for all $x \in \mathbb{R}^d$:

$$\begin{aligned} |L_\theta \varphi(x)| &\leq \left| \int_{|z| \leq 1} \left(\varphi(x+z) - \varphi(x) - D\varphi(x) \cdot z \mathbb{I}_{\theta=1} \right) \nu_\theta(dz) \right| + \int_{|z| \geq 1} \|\varphi\|_\infty \nu_\theta(dz) \\ &\leq \left(\int_0^1 d\lambda \int_{\mathbb{S}^{d-1}} \tilde{\mu}(d\xi) \int_{\rho \in (0,1]} \frac{d\rho}{\rho^{1+\theta}} [D\varphi(x + \lambda\xi\rho) - D\varphi(x)\mathbb{I}_{\theta=1}] \cdot \xi\rho + C_\theta \|\varphi\|_\infty \right) \\ &\leq C_{\theta, \gamma} (\|D\varphi\|_\infty \mathbb{I}_{\theta \in (0,1)} + [D\varphi]_\gamma \mathbb{I}_{\theta=1} + \|\varphi\|_\infty) \leq C_{\theta, \gamma} \|\varphi\|_{C_b^{\gamma+\theta}}. \end{aligned}$$

This gives the control for the supremum norm. Let us now turn to the Hölder modulus. Fix, $x, x' \in \mathbb{R}^d$, $x \neq x'$. We first consider the case $\theta \in (0, 1)$ for simplicity. Write:

$$\begin{aligned}
& |L_\theta \varphi(x) - L_\theta \varphi(x')| \\
& \leq \left| \int_0^1 d\lambda \int_{|z| \leq |x-x'|} \left(D\varphi(x + \lambda z) - D\varphi(x' + \lambda z) \right) \cdot z \nu_\theta(dz) \right| \\
& \quad + \left| \int_0^1 d\lambda \int_{|z| \geq |x-x'|} \left(D\varphi(x' + z + \lambda(x-x')) - D\varphi(x' + \lambda(x-x')) \right) \cdot (x-x') \nu_\theta(dz) \right| \\
& \leq C \left(\int_{\rho \in (0, |x-x'|]} \frac{d\rho}{\rho^{1+\theta}} [D\varphi]_{\theta+\gamma-1} |x-x'|^{\theta+\gamma-1} \rho + \int_{\rho \geq |x-x'|} \frac{d\rho}{\rho^{1+\theta}} [D\varphi]_{\alpha+\gamma-1} \rho^{\theta+\gamma-1} |x-x'| \right) \\
& \leq C_{\theta, \gamma} [D\varphi]_{\theta+\gamma-1} |x-x'|^\gamma.
\end{aligned}$$

The only modifications needed for $\theta = 1$ concern the *small jumps*. Indeed, we can introduce the compensator only up to the threshold $|x-x'|$. We are simply led to analyze:

$$\begin{aligned}
& \left| \int_0^1 d\lambda \int_{|z| \leq |x-x'|} \left([D\varphi(x + \lambda z) - D\varphi(x)] - [D\varphi(x' + \lambda z) - D\varphi(x')] \right) \cdot z \nu_1(dz) \right| \\
& \leq C \int_{\rho \in (0, |x-x'|]} \frac{d\rho}{\rho^2} [D\varphi]_\gamma \rho^{(1+\gamma-1)+1} \leq C_{1, \gamma} [D\varphi]_\gamma |x-x'|^\gamma.
\end{aligned}$$

The other contribution can be handled as above. We have therefore established for all $\theta \in (0, 1]$, $\gamma \in (0, 1)$ s.t. $\theta + \gamma > 1$:

$$|L_\theta \varphi(x) - L_\theta \varphi(x')| \leq C_{\theta, \gamma} [D\varphi]_{\theta+\gamma-1} |x-x'|^\gamma.$$

This completes the proof of inequality (3.8).

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