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► **To cite this version:**

Abdoul Karim Sane. Curves on surfaces and surgeries. European Journal of Combinatorics, 2021, 93, pp.103281. 10.1016/j.ejc.2020.103281 . hal-02018250v3

HAL Id: hal-02018250

<https://hal.science/hal-02018250v3>

Submitted on 22 Dec 2020 (v3), last revised 13 Jul 2021 (v4)

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CURVES ON SURFACES AND SURGERIES

Abdoul Karim SANE

Abstract

In this paper, we introduce a new surgery operation on the set of collections of curves, on a closed oriented surface, whose complement is a topological disk. We prove that any two such collections can be connected by a sequence of surgeries.

1 Introduction

In this article, Σ_g denotes a closed oriented surface of genus g . A collection $\Gamma := \{\gamma_1, \dots, \gamma_n\}$ of closed curves on Σ_g is *filling* if its complement in Σ_g is a union of topological disks. If $\Sigma_g - \Gamma$ is a single topological disk, we say that Γ is a *unicellular collection*. A unicellular collection Γ on Σ_g is equivalent to a regular 4-valent graph G on Σ_g with one disk on its complement, and when the degree of the vertices of G is unspecified, we speak of a *unicellular map*. An oriented edge of Γ has a right and left side according to the orientation of Σ_g .

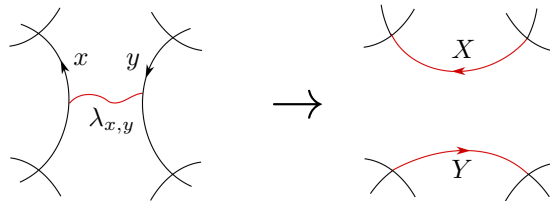


FIGURE 1 – Local modification of Γ along $\lambda_{x,y}$; X and Y are the new oriented edges obtained after the modification.

Now, let x and y be two oriented edges of a unicellular collection Γ . There is a unique simple arc $\lambda_{x,y}$ (up to isotopy with endpoints gliding in x

and y), whose interior is disjoint from Γ , leaving x from its right side and entering y by its right side. The local modification on Γ along $\lambda_{x,y}$, depicted in Figure 1, leads to a new collection $\Gamma' := \sigma_{x,y}(\Gamma)$. When Γ' is unicellular, we say that Γ' is obtained from Γ by a surgery between x and y . We give a necessary and sufficient condition (Lemma 3.1) for Γ' to be unicellular. As we will see, surgery is reminiscent to how a deck of cards is shuffled.

To the surgery operation, we associate the *combinatorial surgery graph* K_g whose vertices are unicellular collections on Σ_g (up to orientation-preserving homeomorphisms) and whose edges are pairs of unicellular collections which differ by a surgery. The graph K_g is finite and defined a metric space by taking every edge to be length one. In general, there are counting formula for *rooted* (distinguished oriented edge) unicellular maps on Σ_g (see [5], [8], [9], [10], [11], [12]). A. Goupil and G. Schaeffer ([4], Theorem 2.1) provided a counting formula for rooted unicellular maps on Σ_g with the same *degree partition*: the ordered list of the degree of the vertices. Restricting to unicellular collection, that formula greatly simplified to $\frac{(4g-2)!}{2^{2g-1}g!}$.

For $g = 1$, there is only one unicellular collection up to orientation-preserving homeomorphisms. So, K_1 consists of one vertex. Using the Goupil-Schaeffer formula for $g = 2$, one checks that there are exactly six¹ unicellular collections on Σ_2 . By inductively trying all possible surgeries, one obtains those six unicellular collections.

One notices that K_2 is connected (see Figure 2). Our main result is:

Theorem 1. *For every integer $g \geq 1$ the graph K_g is connected. Moreover, for $g \geq 2$, $2g - 1 \leq D_g \leq 3g^2 + 9g - 12$; where D_g is the diameter of K_g .*

Our proof is not straightforward as one may hope. We define a *connected sum* operation on unicellular collections that turns two unicellular collections on Σ_{g_1} and Σ_{g_2} into a new unicellular collection on $\Sigma_{g_1+g_2}$. We then consider the *surgery-sum graph* \widehat{K}_g as the union $\sqcup_{i \leq g} K_i$ where we also add an edge between Γ_1 and Γ_2 if Γ_2 can be realized as a connected sum of Γ_1 with the unique unicellular collection on the torus.

Theorem 2. *For every g , the graph \widehat{K}_g is connected.*

Our proof of Theorem 2 uses an interesting lemma: *the trisection lemma*, stated by G. Chapuy in [2]-Lemma 3 (the author was interested in combinatorial identities for unicellular maps). The proof of Theorem 1 is by induction on g and uses the fact that \widehat{K}_g is connected.

¹. The Goupil-Schaeffer formula gives 45 marked collections, but every unicellular collection corresponds to 3, 6, or 12 marked collections depending on the number of symmetries of the collection.

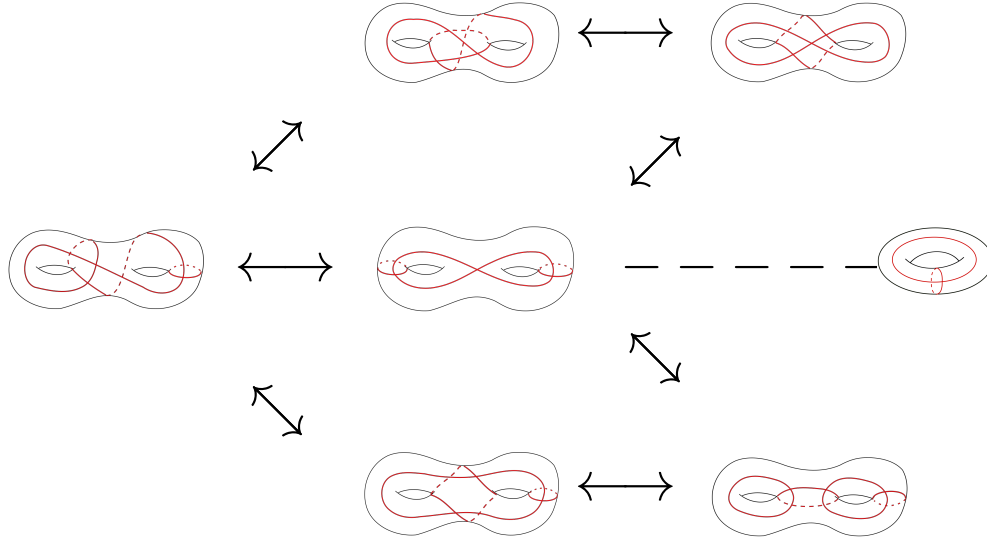


FIGURE 2 – The graphs K_2 and K_1 . The dashed edge indicates the vertex obtained by the connected sum of two copies of the unique collection in K_1 .

Let $\widehat{K}_\infty := \sqcup_{i \geq 1} \widehat{K}_i$ be the *infinite surgery-sum graph* obtained by taking the union of all surgery-sum graphs. We also have:

Theorem 3. *The infinite surgery-sum graph \widehat{K}_∞ is not Gromov hyperbolic.*

As mentioned above, unicellular collections (unicellular maps in general) are well-known for their counting formulas and different methods have been used to get these formulas. In a more topological context, T. Aougab and S. Huang [1] studied *minimally intersecting filling pairs*: those unicellular collections made of exactly two simple closed curves. They show that their number also grows exponentially with the genus.

Our approach is in some sense transverse to the previous works and in [7], we studied unicellular collections to partially answer a question about the shape of dual unit balls of intersection norms². Combinatorial surgery graphs widen the family of graphs associated to surfaces (curves complexe, arcs complexes, pant graphs...) and our results give a refreshing to the theory.

Outline of the paper: Section 2 recalls some facts on unicellular collections. In Section 3, we define the surgery operation and the connected sum.

². Initially, the motivation of this paper was to count unicellular collections, since I was unaware about the previous works on unicellular maps. This ignorance was in some sense benefic.

Section 4 prepares the proofs of Theorem 1, Theorem 2 and Theorem 3 stated in Section 5. Section 6 is about some open questions on the subject.

2 Descriptions of unicellular collections

Let Γ be a filling collection in general position on Σ_g ; that is the complement of Γ in Σ_g is a union of topological disks. One can consider Γ as a graph embedded in Σ_g . Let V be the number of vertices of Γ , E the number of edges and F the number of faces. The Euler characteristic of Σ_g is given by $\chi(\Sigma_g) = 2 - 2g = V - E + F$. As Γ defines a regular 4-valent graph we have $E = 2V$, which implies $V = 2g - 2 + F$. If Γ is unicellular, we then have

$$V = 2g - 1, \quad E = 4g - 2, \quad F = 1.$$

From this one can see that there is only one unicellular collection on the torus up to orientation-preserving homeomorphism; we denote it by $\Gamma_{\mathbb{T}}$ (see Figure 3).

Gluing pattern of unicellular collections: Let Γ be a unicellular collection on Σ_g , then $\Sigma_g - \Gamma$ is a polygon with $8g - 4$ edges, that we denote by P_{Γ} . The polygon P_{Γ} comes with a pairwise identification of its edges. Choosing an edge on P_{Γ} as an origin, one can label the edges of P_{Γ} from the origin in a clockwise manner; thus obtaining a word W_{Γ} on $8g - 4$ letters. If two edges are identified, we label them with the same letter with a bar ($\bar{}$) above the letter assigned to the second edge. The word W_{Γ} is a *gluing pattern* of (Σ_g, Γ) and two gluing patterns of (Σ_g, Γ) differ by a cyclic permutation and a relabeling. A letter of a gluing pattern associated to a unicellular collection corresponds to a side of an edge of Γ ; thus we can see a letter x as an *oriented edge* and \bar{x} as the same edge with opposite orientation.

Example 2.1. The word $W = ab\bar{a}\bar{b}$ is a gluing pattern for the unicellular collection $\Gamma_{\mathbb{T}}$ on the torus.

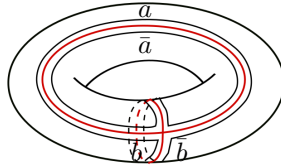


FIGURE 3 – The unicellular collection $\Gamma_{\mathbb{T}}$ on the torus.

Proposition 2.1. *Two unicellular collections Γ_1 and Γ_2 on Σ_g are the same up to orientation-preserving homeomorphisms if and only if they have the same gluing patterns up to cyclic permutation and relabeling.*

Proof. Let us assume that Γ_1 and Γ_2 have the same gluing patterns. Since $\Sigma_g - \Gamma_1$ and $\Sigma_g - \Gamma_2$ are disks, they are homeomorphic. The fact that Γ_1 and Γ_2 have the same gluing patterns implies that one can choose an homeomorphism

$$\tilde{\phi} : P_{\Gamma_1} \longrightarrow P_{\Gamma_2}$$

such that $\tilde{\phi}$ maps a couple of identified sides on P_{Γ_1} to a couple of identified sides on P_{Γ_2} . Therefore, $\tilde{\phi}$ factors to ϕ on the quotient (identification of sides) and $\phi(\Gamma_1) = \Gamma_2$.

Conversely, if $\phi(\Gamma_1) = \Gamma_2$ then Γ_1 and Γ_2 have the same gluing pattern. \square

Unicellular collections as permutations: Maps on surfaces can be described by a triple of permutations which satisfy some conditions (see [6] Section 1.3.3, for the combinatorial definition of maps). We restrict that definition to the case of unicellular collection.

To a unicellular collection we associate H : the set of oriented edges, an involution α of H which maps an oriented edge to the same edge with opposite orientation and a permutation μ whose cycles are oriented edges emanating from vertices when we turn counter-clockwise around them.

Definition 2.1. The elements (H, α, μ, γ) are the *permutations associated* to the unicellular collection Γ and $\gamma := \alpha\mu$ describes the face of Γ .

If W_Γ is a gluing pattern of Γ , we can take H to be the set of letters of W_Γ . The permutation γ is then the shift to the right and it corresponds to the unique face. The cycles of α and μ correspond to the edges and vertices of Γ , respectively. Moreover, if we fix an origin $x \in H$ we get a natural order from γ :

$$x < \gamma(x) < \dots < \gamma^{8g-3}(x).$$

Changing the origin, the order above changes cyclically.

The cycles of μ are in one to one correspondence with the vertices of Γ . If x and y are two oriented edges with $x < y$, they define two intervals in a gluing pattern for Γ : $[x, y] := \{a, x \leq a \leq y\}$ and $[y, x] := \{a, a \leq x\} \cup \{a, y \leq a\}$.

Exemple 2.2. The unicellular collection on the torus is given by the following permutations:

$$\alpha = (a\bar{a})(b\bar{b}); \quad \mu = (a\bar{b}\bar{a}b); \quad \gamma = (ab\bar{a}\bar{b}).$$

3 Surgery and connected sum on unicellular collections

In this section, we define two topological operations on the set of unicellular collections: the surgery and the connected sum.

Surgery on a unicellular collection: Let Γ be a unicellular collection on Σ_g , x and y be two oriented edges of Γ (x and y correspond to two sides of P_Γ). Since Γ is unicellular, there is a unique homotopy class of simple arcs whose interiors are disjoint from Γ and with endpoints in x and y ; let us denote it by $\lambda_{x,y}$. We obtain a new collection denoted by $\sigma_{x,y}(\Gamma)$ by "cutting-open" Γ along $\lambda_{x,y}$ (see Figure 1). The collection $\sigma_{x,y}(\Gamma)$ is not necessarily unicellular.

Definition 3.1. Let Γ be a unicellular collection, x and y be two oriented edges of Γ . We say that $\{x, y\}$ and $\{\bar{x}, \bar{y}\}$ are *intertwined* if \bar{x} and \bar{y} are not both in $[x, y]$ and not both in $[y, x]$ (see Figure 4). It means that Γ admits a gluing pattern of the form $w_1\mathbf{x}w_2\bar{\mathbf{x}}w_3\mathbf{y}w_4\bar{\mathbf{y}}$.

Otherwise, we say that $\{x, y\}$ and $\{\bar{x}, \bar{y}\}$ are *not intertwined*. By abuse, we will just say that x and y are intertwined or not intertwined.

Now, the following lemma gives a necessary and sufficient condition for the above operation to preserve the unicellular character.

Lemma 3.1 (Card shuffling). *Let Γ be a unicellular collection, x and y be two oriented edges of Γ . Then $\sigma_{x,y}(\Gamma)$ is unicellular if and only if x and y are intertwined. In this case, we call the operation a surgery on Γ between x and y .*

Moreover, if $w_1\mathbf{x}w_2\bar{\mathbf{x}}w_3\mathbf{y}w_4\bar{\mathbf{y}}$ is a gluing pattern for Γ then,

$$w_3\mathbf{X}w_2\bar{\mathbf{X}}w_1\mathbf{Y}w_4\bar{\mathbf{Y}}$$

is a gluing pattern for $\sigma_{x,y}(\Gamma)$.

Proof. Since the operation along $\lambda_{x,y}$ leads to a new collection $\Gamma' := \sigma_{x,y}(\Gamma)$, all we have to do is to prove that Γ' is unicellular. We use a cut and past argument similar to several proofs of the classification of surfaces (see Figure 4).

Assume first that x and y are intertwined. When we "cut-open" along $\lambda_{x,y}$, the edges $\{x, \bar{x}\}$ and $\{y, \bar{y}\}$ get replaced by new edges $\{X, \bar{X}\}$ and $\{Y, \bar{Y}\}$. When we cut along the two new edges (in the polygonal description) and

glue along the old ones (see Figure 4), we obtain a polygon; that is Γ' is unicellular with gluing pattern

$$w_3 X w_2 \bar{X} w_1 Y w_4 \bar{Y}.$$

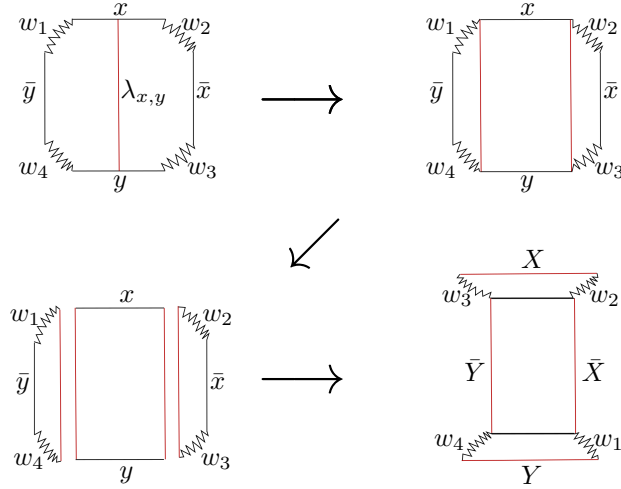


FIGURE 4 – Cut and paste on the polygon P_Γ .

On the other hand, if x and y are not intertwined, one constructs an essential curve disjoint from Γ' , so Γ' is not unicellular (see Figure 5).

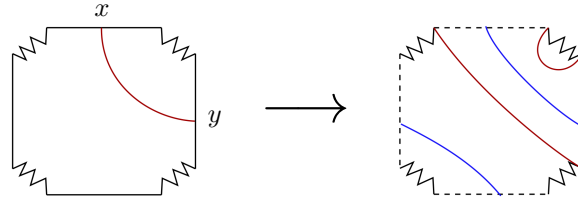


FIGURE 5 – The two arcs in blue define an essential closed curve on Σ_g disjoint from Γ' since it intersects Γ algebraically twice.

□

Remark 3.1. The word $W_{\sigma_{x,y}(\Gamma)}$ in Lemma 3.1 is obtained by permuting w_1 and w_3 . It is also equivalent to permute w_2 and w_4 .

Remark 3.2. If x and y are two intertwined oriented edges, then \bar{x} and \bar{y} are also intertwined. Moreover, one has $\sigma_{x,y}(\Gamma) = \sigma_{\bar{x},\bar{y}}(\Gamma)$. In fact, by Lemma 3.1, if $W_\Gamma = w_1 x w_2 \bar{x} w_3 y w_4 \bar{y}$ is a gluing pattern for Γ , $W_{\sigma_{\bar{x},\bar{y}}(\Gamma)} = w_1 \bar{X} w_4 \bar{X} w_3 Y w_2 \bar{Y}$ is a gluing pattern for $\sigma_{\bar{x},\bar{y}}(\Gamma)$, and it is equivalent to $W_{\sigma_{x,y}(\Gamma)} = w_3 X w_2 \bar{X} w_1 Y w_4 \bar{Y}$ up to cyclic permutation and relabeling.

Remark 3.3. Given a unicellular collection, there are always intertwined pairs unless it is the unicellular collection of the torus. Indeed, if all pairs of Γ are not intertwined a gluing pattern for Γ is given by

$$W_\Gamma = x_1 x_2 \dots x_{4g-2} \bar{x}_1 \bar{x}_2 \dots \bar{x}_{4g-2}.$$

After identifying the sides of P_Γ , all the vertices of P_Γ get identified. Thus Γ has only one self-intersection point. It follows that $g = 1$ and that Γ is the only unicellular collection with one self-intersection point, namely $\Gamma_{\mathbb{T}}$.

Connected sum: Let Γ_1 and Γ_2 be two unicellular collections on two surfaces Σ_1 and Σ_2 , respectively. Let D_1 and D_2 be two open disks on Σ_1 and Σ_2 , disjoint from Γ_1 and Γ_2 , respectively. Let $\Sigma_{g_1} \# \Sigma_{g_2}$ be the connected sum along D_1 and D_2 . Then $(\Sigma_{g_1} \# \Sigma_{g_2}, \Gamma_1 \cup \Gamma_2)$ is a genus $g_1 + g_2$ surface endowed with a collection $\Gamma_1 \cup \Gamma_2$. Since Γ_1 and Γ_2 are unicellular, the complement of $\Gamma_1 \cup \Gamma_2$ in $\Sigma_{g_1} \# \Sigma_{g_2}$ is an annulus.

Now, let x and y be two oriented edges of Γ_1 and Γ_2 respectively, and $\lambda_{x,y}$ a simple arc on $\Sigma_{g_1} \# \Sigma_{g_2}$ from x to y whose interior is disjoint from $\Gamma_1 \cup \Gamma_2$. The arc $\lambda_{x,y}$ joins the two boundary components of $\Sigma_{g_1} \# \Sigma_{g_2} - \Gamma_1 \cup \Gamma_2$. Therefore the graph $\Gamma_1 \cup \Gamma_2 \cup \lambda_{x,y}$ fills $\Sigma_{g_1} \# \Sigma_{g_2}$ with one disk in its complement.

Thus, the collection $\Gamma' := (\Gamma_1 \cup \Gamma_2 \cup \lambda_{x,y}) / \lambda_{x,y}$ —the quotient here means the contraction of $\lambda_{x,y}$ into a point— (see Figure 6) is a unicellular collection. We say that Γ' is the *connected sum* of the marked collections (Γ_1, x) and (Γ_2, y) .

Definition 3.2. Let Γ be a unicellular collection, x and y be two oriented edges of Γ . We say that x and y are *symmetric* if the gluing pattern for Γ starting at x is the same as the one starting at y up to relabeling.

Exemple 3.1. On $\Gamma_{\mathbb{T}}$, any two oriented edges are symmetric.

The following lemma states how we obtain a gluing pattern for $(\Gamma_1, x) \# (\Gamma_2, y)$ from gluing patterns for Γ_1 and Γ_2 .

Lemma 3.2. *If $W_{\Gamma_1} = \mathbf{x} w_1 \bar{\mathbf{x}} w_2$ (respectively $W_{\Gamma_2} = \mathbf{y} w'_1 \bar{\mathbf{y}} w'_2$) is a gluing pattern for Γ_1 (respectively Γ_2), then*

$$\mathbf{x}_1 w_1 \bar{\mathbf{x}}_1 \mathbf{x}_2 w_2 \bar{\mathbf{x}}_2 \mathbf{y}_1 w'_1 \bar{\mathbf{y}}_1 \mathbf{y}_2 w'_2 \bar{\mathbf{y}}_2$$

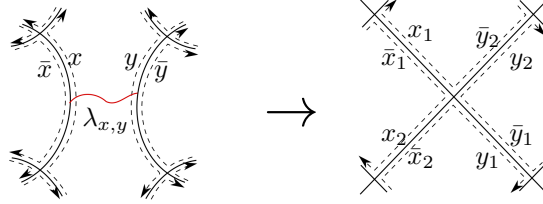


FIGURE 6 – Connected sum.

is a gluing pattern for $(\Gamma_1, x) \# (\Gamma_2, y)$.

Moreover, $(\Gamma_1, x) \# (\Gamma_2, y)$ and $(\Gamma_1, x) \# (\Gamma_2, y')$ are topologically equivalent if y and y' are symmetric.

Proof. The proof can be read on Figure 6. □

Lemma 3.2 implies that the connected sum of a unicellular collection Γ with $\Gamma_{\mathbb{T}}$ depends only on the oriented edge we choose on Γ , since all oriented edges of $\Gamma_{\mathbb{T}}$ are symmetric.

4 Some reductions on unicellular collections

Unicellular collections are easy to see when they have many simple curves. In this section, we show how one can reduce unicellular collections to those with many simple curves. These reductions will make possible to isolate toroidal parts and will provide an induction argument for the proof of our theorems.

Simplification of unicellular collections: Let Γ be a unicellular collection, (H, α, μ, γ) the permutations associated to Γ (Definition 2.1) and x an oriented edge of Γ . Then the oriented edges x and $C(x) := \gamma\alpha\gamma(x)$ belong to the same curve $\beta \in \Gamma$; x and $C(x)$ are consecutive along β (see Figure 7). Moreover, the sequence $(C^n(x))_n$ is periodic and it travels through all edges of β .

Definition 4.1. Let Γ be a unicellular collection and $\theta \in \Gamma$ a simple curve. The curve θ is *1-simple* if θ intersects exactly one time $\Gamma - \theta$.

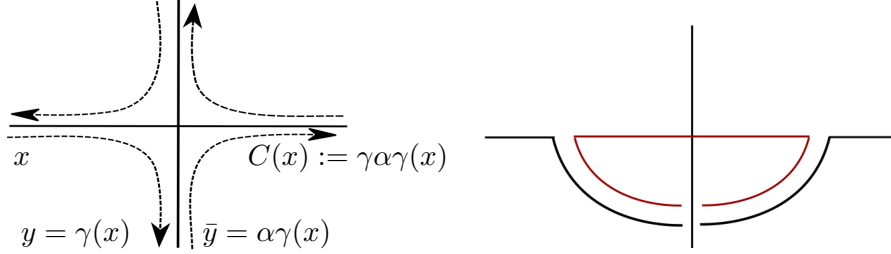


FIGURE 7 – A simplification ; x and $C(x)$ are consecutive.

Note that we have $C(x) = x$ if and only if x is a side of a 1-simple curve $\theta \in \Gamma$. We denote by S_Γ the number of 1-simple closed curves in Γ .

Definition 4.2. Assume that x is an edge not lying on a 1-simple component of Γ , and that x and $C(x)$ are intertwined. The operation transforming Γ into $\sigma_{x,C(x)}(\Gamma)$ is called a *simplification*.

The name is explained by:

Lemma 4.1. *If $\Gamma' := \sigma_{x,C(x)}(\Gamma)$ is a simplification, then $S_{\Gamma'} = S_\Gamma + 1$. In other words, a surgery on Γ between x and $C(x)$ creates an additional 1-simple curve in Γ' .*

Proof. Suppose that x and $C(x)$ are intertwined, a gluing pattern for Γ is given by:

$$W_\Gamma = (tw'_1)\mathbf{x}(yw'_2\bar{t})\bar{\mathbf{x}}(w'_3\bar{y})\mathbf{C}(\mathbf{x})w_4\overline{\mathbf{C}(\mathbf{x})}.$$

Therefore, by Lemma 3.1 a gluing pattern for $\Gamma' := \sigma_{x,C(x)}(\Gamma)$ is given by:

$$W_{\Gamma'} = w'_3\bar{y}\mathbf{X}yw'_2\bar{t}\bar{\mathbf{X}}tw'_1\mathbf{Z}w_4\bar{\mathbf{Z}}.$$

So, in Γ' we have $C(X) = \gamma\alpha\gamma(X) = \gamma\alpha(y) = \gamma(\bar{y}) = X$. It implies that X is the side of simple curve which intersects Γ' only once. \square

Now if θ_1 and θ_2 are two 1-simple curves of a unicellular collection, then θ_1 and θ_2 are disjoint; otherwise $\theta_1 \cup \theta_2$ would be disjoint from Γ , that is absurd since Γ is connected. So the number S_Γ of 1-simple curves on a unicellular collection Γ is bounded by the genus g of the underlying surface. Therefore a sequence of simplifications on a unicellular collection stabilizes at a collection on which no simplification can be applied anymore.

Definition 4.3. A collection Γ is *non-simplifiable* if one cannot do a simplification from it, i.e, x and $C(x)$ are always non intertwined.

Order around vertices of a non-simplifiable collection: In this paragraph, we will show that vertices of non-simplifiable unicellular collections are of certain types.

Let Γ be a unicellular collection and (H, α, μ, γ) the permutations associated to Γ ; H being the set of letters of a gluing pattern for Γ .

If we fix an origin $x_0 \in H$, we then get an order on H :

$$x_0 < \gamma(x_0) < \dots < \gamma^{8g-3}(x_0).$$

Therefore, if v is a vertex of Γ defined by a cycle $(txyz)$ of μ , we get a local order around v by comparing t, x, y and z . Since each letter corresponds to an oriented edge which leaves an angular sector of v (see Figure 8), the local order around v corresponds also to a local order on the four angular sectors around v when running around Γ with γ .

Definition 4.4. Let v be a vertex defined by the oriented edges $(t, x := \mu(t), y := \mu^2(t), z := \mu^3(t))$ with $t = \min\{t, x, y, z\}$ relatively to an order of edges on Γ . Then,

- v is a vertex of *Type 1* if $t < x < y < z$;
- v is a vertex of *Type 2* if $t < z < y < x$.

Otherwise, the vertex v is a vertex of *Type 3*.

Up to rotation and change of origin, we have the three cases depicted on Figure 8.

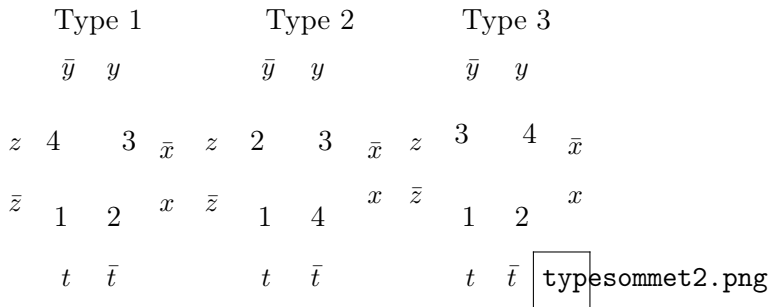


FIGURE 8 – Different types of vertices of a unicellular collection.

Lemma 4.2. ³ *A unicellular collection Γ is non-simplifiable if and only if all of its vertices are of Type 1 or Type 2.*

³ It is exactly after the proof of this lemma that I met G. Chapuy's works and all the other papers on unicellular maps

Proof. All we have to do is to write the possible gluing patterns of Γ by figuring out the order of the edges around a vertex and look when consecutive edges are intertwined or not.

Case 1: If v is a vertex of Type 1, then a gluing pattern for Γ is given by:

$$W_\Gamma = w_1 \bar{z} t w_2 \bar{t} x w_3 \bar{x} y w_4 \bar{y} z.$$

Therefore, one checks that \bar{x} and $C(\bar{x}) = z$ are not intertwined; so are \bar{t} and $C(\bar{t}) = y$. Hence, no simplification is possible around v .

Case 2 : If v is a vertex of Type 2, the gluing pattern for Γ is

$$W_\Gamma = w_1 \bar{z} t w_2 \bar{y} z w_3 \bar{x} y w_4 \bar{t} x.$$

Then, \bar{x} and $C(\bar{x}) = z$ are not intertwined; so are \bar{t} and $C(\bar{t}) = y$. Again, no simplification is possible around v in this case.

Case 3: If v is a vertex of Type 3, then

$$W_\Gamma = w_1 \bar{z} t w_2 \bar{t} x w_3 \bar{y} z w_4 \bar{x} y.$$

Here, \bar{t} and $C(\bar{t}) = y$ are intertwined and a simplification is possible.

So Γ is non-simplifiable if and only all its vertices are of Type 1 or Type 2. \square

Number of vertices of Type 1 and 2 in a non-simplifiable unicellular collection: In [2], G. Chapuy has defined a notion which catches the topology of a unicellular map: *trisection*. We recall one of his results about trisection.

Let G be a unicellular map and (H, α, μ, γ) the permutations associated to G . Let v be a degree d vertex of G defined by a cycle $(x_1 x_2 \dots x_d)$ of μ , with $x_1 = \min\{x_1, \dots, x_d\}$ relatively to an order on H .

If $x_i > x_{i+1}$ we say that we have a *down-step*. Since $x_1 = \min\{x_1, \dots, x_d\}$, one has $x_d > x_1$; the other down-steps around v are called *non trivial*.

Definition 4.5 (G. Chapuy). A *trisection* is a down-step which is not a trivial one.

Lemma 4.3 (The trisection lemma; G. Chapuy [2]). *Let G be a unicellular map on a genus g surface. Then G has exactly $2g$ trisections.*

Applying the trisection lemma to unicellular collections, we get:

Corollary 4.1. *A non-simplifiable unicellular collection on Σ_g has g vertices of Type 2 and $g - 1$ vertices of Type 1.*

Proof. A vertex of Type 2 (respectively a vertex of Type 1) has two trisections (respectively zero trisection) (see Figure 8). If N_i is the number of vertices of Type i ($i=1,2$), by the trisection lemma we have $2N_2 = 2g$; so $N_2 = g$.

Since $V_\Gamma = N_1 + N_2 = 2g - 1$, it follows that $N_1 = g - 1$. \square

Repertition of vertices on a non-simplifiable collection: Now, we show that using surgeries, we can re-order the vertices of a non-simplifiable collection.

Definition 4.6. Let G be a graph. Two vertices are *adjacent* if they share an edge.

If v_1 and v_2 are two vertices represented by the cycles $(abcd)$ and $(efgh)$, respectively, they are adjacent if and only if there exists $x \in \{a, b, c, d\}$ such that $\bar{x} \in \{e, f, g, h\}$.

We now show that some configurations of vertices "hide" simplifications; that is from those configurations we can create new 1-simple curves after a suitable surgery, without touching the old ones.

Lemma 4.4. *Let Γ be a non-simplifiable unicellular collection. If Γ contains two vertices of Type 2 which are adjacent, then there is a sequence of surgeries $\Gamma = \Gamma_0 \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ from Γ to Γ_n such that Γ_n is non-simplifiable and $S_\Gamma < S_{\Gamma_n}$.*

Proof. Let v_1 and v_2 be two adjacent vertices of Type 2 defined by the cycles $(b\bar{f}\bar{g}\bar{a})$ and $(c\bar{d}\bar{e}\bar{b})$, respectively (see Figure 9).

Let us fix an oriented edge as an origin, so that

$$a = \min\{a, \bar{c}, d, e, f, g\};$$

that is the first time we enter in the local configuration is by the oriented edge a . Then we have the following order:

$$a < b < c < g < \bar{a} < f < \bar{g} < e < \bar{b} < \bar{f} < d < \bar{e} < \bar{c} < \bar{d}.$$

Otherwise, it would contradict the fact that the two vertices are of Type 2. A gluing pattern for Γ is given by:

$$W_\Gamma = w_1abcw_2g\bar{a}w_3f\bar{g}w_4e\bar{b}\bar{f}w_5d\bar{e}w_6\bar{c}\bar{d}.$$

The oriented edges b and \bar{g} are intertwined, so we can define $\Gamma' := \sigma_{b,\bar{g}}(\Gamma)$. By Lemma 3.1, a gluing pattern for Γ' is:

$$W_{\Gamma'} = \bar{a}w_3f\mathbf{B}cw_2\mathbf{G}\bar{f}w_5d\bar{e}w_6\bar{c}\bar{d}w_1a\bar{\mathbf{G}}w_4e\bar{\mathbf{B}}.$$

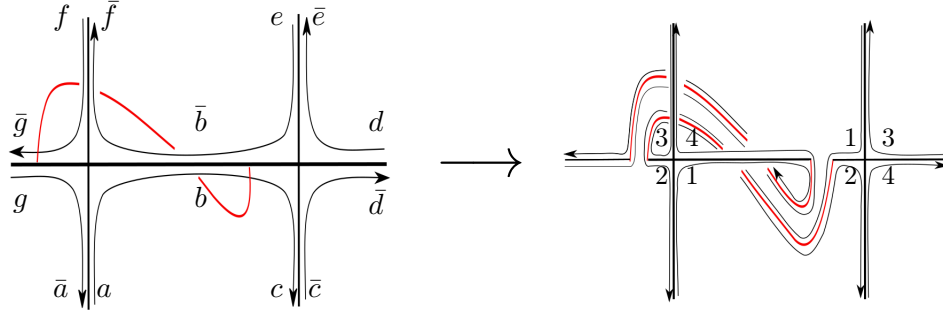


FIGURE 9 – Surgery which creates new simplifications. On the left figure, $a < g < f < e < d < \bar{c}$ is the order by which we pass through the eight sectors. On the figure on the right, we focus on the angular order around v_1 and v_2 . The order on the figure on the right comes from that of the figure on the left. At each time we leave the local configuration on the figure on the right, we come back on it in the same way like in the figure on the left.

The cycles $(\bar{G}\bar{f}B\bar{a})$ and $(\bar{B}c\bar{d}\bar{e})$ define the two vertices of Γ' in Figure 9 and the orders around these two vertices are:

$$\bar{G} < \bar{a} < B < \bar{f}; \quad \bar{B} < c < \bar{e} < \bar{d}.$$

Therefore, the vertex $(\bar{B}c\bar{d}\bar{e})$ is a vertex of Type 3 and it implies that Γ' is simplifiable. Indeed the operation on Figure 9 does not touch any 1-simple curve of Γ and each simplification increases strictly the number of 1-simple. Let Γ_n be a non-simplifiable collection obtained after finitely many simplifications on Γ' ; so $S_\Gamma < S_{\Gamma_n}$. \square

Let v_1 and v_2 be two vertices of Type 1 and Type 2 defined by the cycles $(\bar{c}d\bar{e}f)$ and $(g\bar{a}bc)$, respectively, such that v_1 and v_2 are adjacent. The local configuration in this case is depicted on Figure 10 and we assume that

$$a = \min\{a, \bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}.$$

Lemma 4.5. *If $\min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\} \neq \bar{g}$, then there is a sequence $\Gamma = \Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \dots \longrightarrow \Gamma_n$ from Γ to Γ_n such that Γ_n is non-simplifiable and $S_\Gamma < S_{\Gamma_n}$.*

Proof. Since v_2 is a vertex of Type 2, $\min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}$ is different from \bar{b} and \bar{f} .

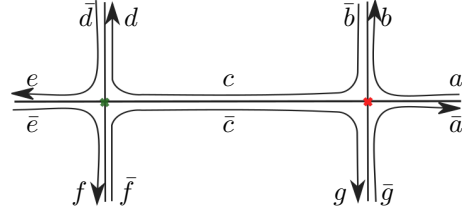


FIGURE 10 – Local configuration around $v_1 := (\bar{c}def)$ and $v_2 := (g\bar{a}bc)$

Case 1: If $\min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\} = \bar{d}$, the fact that the vertices are of Type 1 and 2 implies that the local order is either

$$a < b < \bar{d} < e < \bar{e} < f < \bar{g} < \bar{a} < \bar{f} < \bar{c} < g < \bar{b} < c < d,$$

or

$$a < b < \bar{d} < e < \bar{g} < \bar{a} < \bar{e} < f < \bar{f} < \bar{c} < g < \bar{b} < c < d.$$

Sub-case 1: If $a < b < \bar{d} < e < \bar{e} < f < \bar{g} < \bar{a} < \bar{f} < \bar{c} < g < \bar{b} < c < d$, then

$$W_\Gamma = w_1 \mathbf{a} b w_2 \bar{\mathbf{d}} e w_3 \bar{\mathbf{e}} f w_4 \bar{\mathbf{g}} \bar{\mathbf{a}} w_5 \bar{\mathbf{f}} \bar{\mathbf{c}} w_6 \bar{\mathbf{b}} c d$$

is a gluing pattern for Γ .

The oriented edges a and \bar{e} are intertwined and a gluing pattern for $\Gamma' := \sigma_{a, \bar{e}}(\Gamma)$ is given by

$$W_{\Gamma'} = w_3 \mathbf{A} b w_2 \bar{\mathbf{d}} \mathbf{E} w_5 \bar{\mathbf{f}} \bar{\mathbf{c}} g w_6 \bar{\mathbf{b}} c d w_1 \bar{\mathbf{E}} f w_4 \bar{\mathbf{g}} \bar{\mathbf{A}}.$$

The cycles $(bcg\bar{A})$ and $(E\bar{f}\bar{c}d)$ define the two vertices in Figure 11. Moreover, $b < g < c < \bar{A}$ and $E < \bar{c} < d < f$ that is they are vertices of Type 3.

Therefore, Γ' is simplifiable and there is sequence of simplification from Γ'

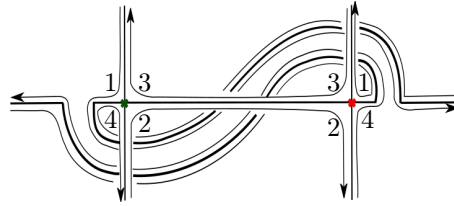


FIGURE 11 – Local order around v_1 and v_2 after the surgery $\sigma_{a, \bar{e}}(\Gamma)$

to Γ_n such that Γ_n is non-simplifiable and $S_{\Gamma_n} > S_{\Gamma'} = S_{\Gamma}$. The equality $S_{\Gamma'} = S_{\Gamma}$ holds since the surgery in this case does not touch a 1-simple curve.

Sub-case 2: If $a < b < \bar{d} < e < \bar{g} < \bar{a} < \bar{e} < f < \bar{f} < \bar{c} < g < \bar{b} < c < d$, then a gluing pattern for Γ is

$$W_{\Gamma} = w_1 \mathbf{a} b w_2 \bar{d} e w_3 \bar{g} \bar{a} w_4 \bar{e} \mathbf{f} w_5 \bar{f} \bar{c} g w_6 \bar{b} c d.$$

Here again, the oriented edges a and f are intertwined and a gluing pattern for $\Gamma' := \sigma_{a,f}(\Gamma)$ is given by:

$$W_{\Gamma'} = w_4 \bar{e} \mathbf{A} b w_2 \bar{d} e w_3 \bar{g} \bar{\mathbf{A}} \bar{c} g w_6 \bar{b} c d w_1 \mathbf{F} w_5 \bar{\mathbf{F}}.$$

The two vertices in Figure 12 are defined by the cycles $(bcg\bar{A})$ and $(A\bar{c}de)$. Moreover, $b < \bar{A} < g < c$ and $A < e < \bar{c} < d$. It follows that the vertex $(A\bar{c}de)$ is a vertex of Type 3 and therefore, Γ' is simplifiable.

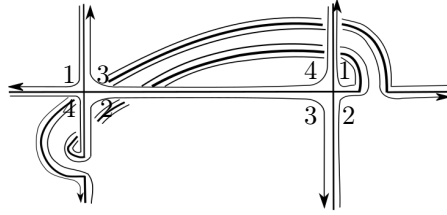


FIGURE 12 – Local order around v_1 and v_2 after the surgery $\sigma_{a,f}(\Gamma)$.

Case 2: if $\min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\} = \bar{e}$, then the local order is given by:

$$a < b < \bar{e} < f < \bar{g} < \bar{a} < \bar{f} < \bar{c} < \bar{g} < \bar{b} < c < d < \bar{d} < e,$$

and a gluing pattern for Γ is given by:

$$W_4 = w_1 \mathbf{a} b w_2 \bar{e} f w_3 \bar{g} \bar{a} w_4 \bar{f} \bar{c} g w_5 \bar{b} c d w_6 \bar{\mathbf{d}} e.$$

In this case, a and d are intertwined. A gluing pattern for $\Gamma' := \sigma_{a,d}(\Gamma)$ is given by:

$$W_{\Gamma'} = w_4 \bar{f} \bar{c} g w_5 \bar{b} c \mathbf{A} b w_2 \bar{e} f w_3 \bar{g} \bar{\mathbf{A}} e w_1 \mathbf{D} w_6 \bar{\mathbf{D}}.$$

The cycles $(\bar{c}Aef)$ and $(g\bar{A}bc)$ represent the two vertices in Figure 13 and $\bar{c} < A < f < e$. So, the vertex $(\bar{c}Aef)$ is a vertex of Type 3 and Γ' is simplifiable. \square

Definition 4.7. Let v_1 and v_2 be two adjacent vertices of Type 1 and 2, respectively. We say that we have a *good order* around v_1 and v_2 if $\bar{g} = \min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}$ (see Figure 10).

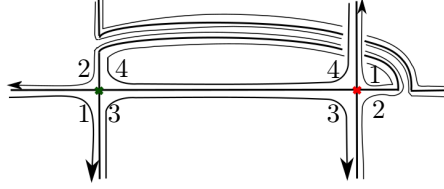


FIGURE 13 – Local order around v_1 and v_2 after the surgery $\sigma_{a,d}(\Gamma)$.

Definition 4.8. A unicellular collection Γ is *almost toral* if:

- Γ is non-simplifiable,
- no two vertices of Type 2 are adjacent,
- the local orders around two adjacent vertices of Type 1 and 2 are good.

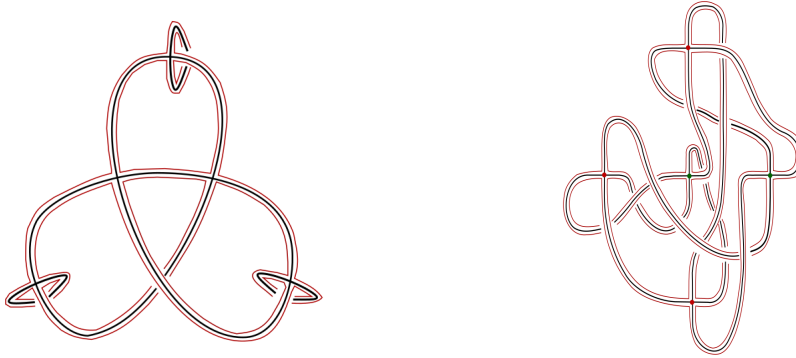


FIGURE 14 – An almost toral unicellular collection on the left with two vertices of Type 1 and three vertices of Type 2 on the three 1-simple curves. A non-simplifiable unicellular collection without 1-simple curve on the right. Here, unicellular collections are represented just by showing their tubular neighborhood (in red) on the surface. One can recover the local orders around each vertex by following the boundary of the tubular neighborhood.

Lemma 4.6. *Let Γ be a non-simplifiable unicellular collection. Then there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \dots \rightarrow \Gamma_n$ such that Γ_n is an almost toral unicellular collection.*

Proof. If Γ is not almost toral, by Lemma 4.4 and Lemma 4.5 there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ such that Γ_n is non-

simplifiable and $S_\Gamma < S_{\Gamma_n}$, i.e, we create new 1-simple curves after some suitable surgeries. Since the number of 1-simple curves is bounded by the genus, those operations stabilize to an almost toral unicellular collection. \square

Now, we are going to improve the configuration of the vertices of an almost toral unicellular collection.

Lemma 4.7. *Let Γ be an almost toral unicellular collection, v_1 and v_2 be two adjacent vertices of Type 1 and Type 2, respectively; with $a = \min\{a, \bar{b}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}$ (see Figure 10).*

If $x := \min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}\}$ is adjacent to a vertex of Type 2, then there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ such that Γ_n is almost toral and $S_\Gamma < S_{\Gamma_n}$.

Proof. If $x := \min\{\bar{b}, \bar{d}, \bar{e}, \bar{f}\}$ is adjacent to a vertex $v := (x\bar{y}\bar{t}\bar{u})$, a gluing pattern of Γ is given by (see Figure 15):

$$W_\Gamma = w_1 a w_2 \bar{a} w_3 u x w_4 \bar{x} \bar{y}.$$

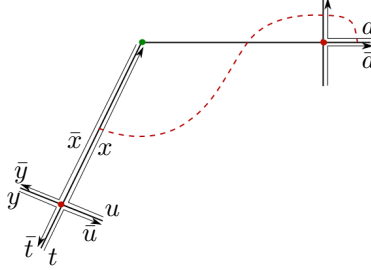


FIGURE 15 – Surgery between x and a ; where x , adjacent to a vertex of Type 2, is the oriented edge by which we come back to the local configuration around v_1 and v_2 for the third time after a and \bar{a} .

Since v is a vertex of Type 2, $x < \bar{u} < \bar{t} < \bar{y}$. It implies that $\bar{t} \in w_4$ and $\bar{u} \in w_4$.

The oriented edges a and x are intertwined and

$$W_{\Gamma'} = w_3 u A w_2 \bar{A} \bar{y} w_1 X w_4 \bar{X}$$

is a gluing pattern for $\Gamma' := \sigma_{a,x}(\Gamma)$. The vertex v in Γ' is defined by the cycle $(A\bar{y}\bar{t}\bar{u})$ and one checks that $A < \bar{y} < \bar{u} < \bar{t}$; that is v is a vertex of Type 3 and Γ' is simplifiable. Hence, there is a sequence of simplification which strictly increases the number of 1-simple curves. \square

Definition 4.9. Let Γ be a unicellular collection. We say that Γ is a *toral unicellular collection* if Γ is an almost toral unicellular collection and if every vertex of Type 1 is adjacent to at most two vertices of Type 2.

Figure 2 shows two toral unicellular maps on Σ_2 : the one with four simple curves, and the one with two 1-simple curves and one non-simple curve.

Lemma 4.8. *Let Γ be an almost toral unicellular collection. Then there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ such that Γ_n is a toral unicellular collection.*

Proof. Let v be a vertex of Γ of Type 1. If v is adjacent to 4 vertices of Type 2 or 3 vertices all of which are of Type 2, Lemma 4.7 implies that there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ such that $S_\Gamma < S_{\Gamma_n}$. Since the number of 1-simple curves is bounded by the genus g , there is a sequence of surgeries $\Gamma_0 = \Gamma \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_n$ such that Γ_n is almost toral and such that every vertex of Type 1 adjacent to three vertices of Type 2 is also adjacent to a fourth of Type 1. The local configuration around those vertices is depicted in Figure 16, with $\bar{e} = \min\{\bar{d}, \bar{e}, \bar{f}\}$. A gluing pattern for Γ_n is given by:

$$W_{\Gamma_n} = w_1 a w_2 \bar{a} w_3 d w_4 \bar{d}.$$

The oriented edges a and d are intertwined and the surgery $\sigma_{a,d}(\Gamma_n)$ decreases the number of adjacent vertices to v (see Figure 16). Following this process, we get a toral unicellular collection after finitely many surgeries.

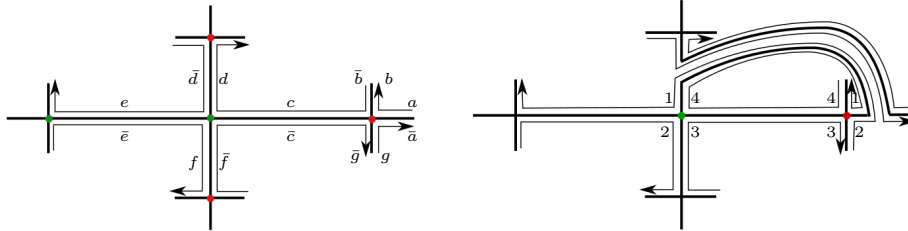


FIGURE 16 – Local configuration around a vertex v of Type 1 adjacent to three vertices of Type 2 and one vertex of Type 1. The surgery $\sigma_{a,d}(\Gamma)$ decreases the number of adjacent vertices to v .

□

Lemma 4.9. *Let Γ be a toral unicellular collection in Σ_{g+1} . Then $\Gamma = (\Gamma', x) \# \Gamma_{\mathbb{T}}$ (up to a surgery); where Γ' is a unicellular collection in Σ_g .*

Proof. We have to show that there is a vertex of Type 2 which is adjacent to exactly one vertex of Type 1.

Assume that every vertex of Type 2 is adjacent to at least two vertices of Type 1. Let N_1 and N_2 be the number of vertices of Type 1 and Type 2 respectively, and let $N_{1,2}$ be the number of pairs of vertices of Type 1 and Type 2 which are adjacent.

Since Γ is a toral unicellular collection, any vertex of Type 1 has at most two vertices of Type 2. It implies that,

$$N_{1,2} < 2N_1.$$

On the other part, we have assumed that every vertex of Type 2 is adjacent to at least two vertices of Type 1. Therefore,

$$2N_2 \leq N_{1,2}.$$

Combining the two inequalities above, we get $N_2 \leq N_1$ which contradicts the fact that we have $g - 1$ vertices of Type 1 and g vertices of Type 2 in a non-simplifiable unicellular collection.

So, there is a vertex v_0 of Type 2 which is adjacent to exactly one vertex v_1 of Type 1. As vertices of Type 2 are not adjacent, v_0 lies on a 1-simple curve. (*)

Next, we show that v_1 can be transformed into a self-intersection point (if it is not the case) by a surgery. Assume that v_1 is not a self-intersection point. Then a gluing pattern for Γ is given by:

$$W_\Gamma = w_1 x \mathbf{y} w_2 \bar{\mathbf{y}} z w_3 \bar{z} t w_4 \bar{t} \bar{x};$$

where v_1 is defined by the cycle $(yzt\bar{x})$ (Figure 17).

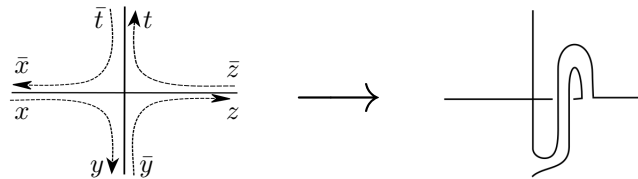


FIGURE 17 – Transforming a Type 1 vertex to a self-inter. point.

The oriented edges y and z are intertwined and a gluing pattern for $\Gamma' := \sigma_{y,z}(\Gamma)$ is given by:

$$W_{\Gamma'} = \mathbf{Y} w_2 \bar{\mathbf{Y}} t w_4 \bar{t} \bar{x} w_1 x \mathbf{Z} w_3 \bar{\mathbf{Z}}$$

The vertex v_1 in Γ' is defined by the cycle $(Yt\bar{x}Z)$ and $Y < t < \bar{x} < Z$; that is v_1 is still a vertex of Type 1. Moreover, v_1 get transformed to a self-intersection point. So, the surgery on Γ between y and z has transformed v_1 to a self-intersection point of Type 1. (**)

Finally, (*) and (**) implies that

$$\Gamma = (\Gamma', x) \# (\Gamma_{\mathbb{T}}, x_0);$$

with Γ' a unicellular collection on Σ_{g-1} . □

5 Proof of the main theorems

In this section, we prove Theorem 1, Theorem 2 and Theorem 3. We recall that the graph K_g is the graph whose vertices are homeomorphism classes of unicellular collections on Σ_g , and on which two vertices Γ_1 and Γ_2 are connected by an edge if there is a surgery which transforms Γ_1 into Γ_2 (if a surgery on Γ fix Γ , we do not put a loop). The graph $\widehat{K}_g := \sqcup_{i \leq g} K_i$ is the disjoint union of the graphs K_i on which we add an edge between two unicellular collections Γ_1 and Γ_2 on Σ_i and Σ_{i+1} respectively if Γ_2 is a connected sum of Γ_1 with the unicellular collection of the torus.

The following proposition is the main technical result: it directly implies Theorem 2. It also implies Theorem 1 with a bit of extra-work and its proof uses most lemmas of Section 4.

Proposition 5.1. *Let Γ be a unicellular collection on Σ_{g+1} . Then there is a finite sequence of surgeries $\Gamma := \Gamma_0 \longrightarrow \dots \longrightarrow \Gamma_n$ and a unicellular collection Γ' on Σ_g with a marked edge x such that $\Gamma_n = (\Gamma', x) \# \Gamma_{\mathbb{T}}$.*

Proof. Let Γ be a unicellular collection on Σ_g , there is a sequence of surgeries $\Gamma \longrightarrow \dots \longrightarrow \Gamma_1$ such that Γ_1 is non-simplifiable. By Lemma 4.6, there is a sequence of surgeries $\Gamma_1 \longrightarrow \dots \longrightarrow \Gamma_2$ such that Γ_2 is almost toral and by Lemma 4.7, there is a sequence of surgeries $\Gamma_2 \longrightarrow \dots \longrightarrow \Gamma_3$ such that Γ_3 is toral. Lemma 4.9 implies that $\Gamma_3 = (\Gamma', x) \# \Gamma_{\mathbb{T}}$ with Γ' a unicellular collection in Σ_{g-1} . □

We can now prove Theorem 2, which states that for every g the graph \widehat{K}_g is connected.

Proof of Theorem 2. Let $\Gamma \in K_g$. By Proposition 5.1, there is path in K_g from Γ to $(\Gamma', x) \# \Gamma_{\mathbb{T}}$ where $\Gamma' \in K_{g-1}$. Thus, there is path in \widehat{K}_g from Γ to Γ' . By induction on g , we deduce a path from Γ to $\Gamma_{\mathbb{T}}$. So, \widehat{K}_g is connected. □

Now, we turn to the proof of Theorem 1. Let us start with some preliminaries.

Lemma 5.1. *Let Γ be a unicellular collection on Σ_g and x an oriented edge of Γ . Then there is a surgery from $(\Gamma, x)\#\Gamma_{\mathbb{T}}$ to $(\Gamma, \bar{x})\#\Gamma_{\mathbb{T}}$.*

Lemma 5.1 states that up to surgery the connected sum of Γ with $\Gamma_{\mathbb{T}}$ depends only on the edge we choose on Γ but not on its orientation.

Proof. Let $W_{\Gamma} := w_1 x w_2 \bar{x}$ be a gluing pattern for Γ , $\Gamma_1 := (\Gamma, x)\#\Gamma_{\mathbb{T}}$ and $\Gamma_2 := (\Gamma, \bar{x})\#\Gamma_{\mathbb{T}}$. We recall that $W_{\Gamma_{\mathbb{T}}} = ab\bar{a}\bar{b}$ is a gluing pattern for $\Gamma_{\mathbb{T}}$. By Lemma 3.2,

$$W_{\Gamma_1} = x_1 w_1 \bar{x}_1 x_2 w_2 \bar{x}_2 a_1 \bar{b} \bar{a}_1 a_2 \bar{b} \bar{a}_2$$

and

$$W_{\Gamma_2} = x_1 w_2 \bar{x}_1 x_2 w_1 \bar{x}_2 a_1 \bar{b} \bar{a}_1 a_2 \bar{b} \bar{a}_2$$

are gluing patterns of Γ_1 and Γ_2 , respectively.

Thus, in W_{Γ_1} , x_1 and x_2 are intertwined, $\Gamma' := \sigma_{x_1, x_2}(\Gamma_1)$ is unicellular, with gluing pattern

$$W_{\Gamma'} = x_1 w_2 \bar{x}_1 x_2 w_1 \bar{x}_2 a_1 \bar{b} \bar{a}_1 a_2 \bar{b} \bar{a}_2.$$

We check that $W_{\Gamma'} = W_{\Gamma_1}$. So, $\sigma_{x_1, x_2}(\Gamma_1) = \Gamma_2$. □

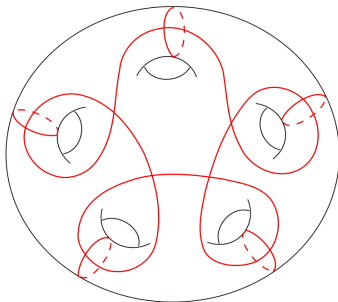


FIGURE 18 – The 5-necklace N_5 .

Definition 5.1. We call *g-necklace* the homeomorphism class of the unicellular collection on Σ_g , denoted by N_g , with g 1-simple curves and one spiraling curve η with $g - 1$ self intersection points (see Figure 18 for the 5-necklace).

Remark 5.1. Intersection points between 1-simple curves and γ in N_g are of Type 2; the others are Type 1 vertices. There are $g - 1$ vertices of Type 2 which are adjacent to exactly one vertex of Type 1 and one *special vertex* of Type 2 which is adjacent to two vertices of Type 1.

We can now prove Theorem 1, namely given Γ_1 and Γ_2 are two unicellular collections on a genus g surface Σ_g there is a finite sequence of surgeries from Γ_1 to Γ_2 .

Proof of Theorem 1: connectedness. We give a proof by induction on g . Assume K_g is connected. Let Γ be a unicellular collection on Σ_{g+1} . By Proposition 5.1, there exists a sequence of surgeries $\Gamma = \Gamma_0 \rightarrow \dots \rightarrow \Gamma_n$ where Γ_n is of the form $(\Gamma', x) \# \Gamma_{\mathbb{T}}$.

Since we have assumed that K_g is connected, then there is a sequence of surgeries $\Gamma' = \Gamma'_0 \rightarrow \dots \rightarrow \Gamma'_n = N_g$ from Γ' to N_g (the g -necklace). This sequence lifts to a sequence $\Gamma = (\Gamma', x) \# \Gamma_{\mathbb{T}} \rightarrow \dots \rightarrow (N_g, x_n) \# \Gamma_{\mathbb{T}}$ of surgeries on Σ_{g+1} . Indeed if x is an oriented edge of Γ , then x brokes into two oriented edges x_1 and x_2 . The surgery $\sigma_{x,y}(\Gamma')$ (respectively $\sigma_{z,y}(\Gamma')$) lift to $\sigma_{x_1,y}(\Gamma)$ (respectively $\sigma_{z,y}(\Gamma)$).

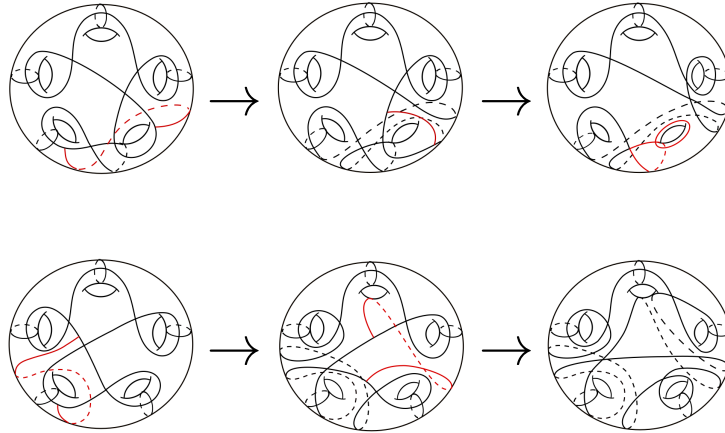


FIGURE 19 – Sequence of surgeries to the necklace. The sequence on the first arrow goes from the case where $\Gamma_{\mathbb{T}}$ is glued on a 1-simple curve to the case where $\Gamma_{\mathbb{T}}$ is glued between one vertex of Type 1 and one vertex of Type 2. The second arrow leads to the 5-necklace. The red arcs are the arcs on which we apply surgeries.

By Lemma 5.1, up to surgery the way we glue $\Gamma_{\mathbb{T}}$ on N_g depends only on the edges of N_g but not on their sides. It follows that there are three situations depending whether:

- x_n is the side of an edge connecting two vertices of Type 1, or one vertex of Type 1 and the special vertex of Type 2,
- x_n is the side of an edge connecting one vertex of Type 1 and one vertex of Type 2 which is not the special one.
- x_n lies on a 1-simple closed curve.

The first situation leads to the $(g + 1)$ -necklace N_{g+1} , and for the other two situations there is a path to the $(g + 1)$ -necklace. We give the paths for the genus 5 case in Figure 19; the other cases inductively follow the same sequence of surgeries.

Since K_1 (a single vertex) is connected, by induction K_g is also connected. \square

Now we turn to the question of the diameter of K_g that we denote by D_g . We prove

Proof of Theorem 1: Diameter. Let $d_g := \max\{d(\Gamma, N_g)\}$ be the maximal distance to the necklace. By Proposition 5.1, if Γ is a unicellular collection, there is sequence s_n of surgeries from Γ to Γ_n such that Γ_n is toral. In this sequence, we have three kind of steps:

- making an apparent simplification on a vertex of Type 3; let m be their numbers,
- making a *hidden simplification*, that is a simplification which follows a suitable surgery as in Lemma 4.4 (Figure 14 shows a collection on which making a hidden simplification is necessary); let n be their numbers,
- making a surgery which are not followed by simplification as in Figure 16; let k be their numbers.

It follows that the length $l(s_n)$ is equal to $m + 2n + k$; with $m + n \leq g$ and $k \leq g - 1$ (since the last step correspond to a surgery around vertices of Type 1).

The maximum is reached when every simplification follows a suitable surgery; that is $m = 0$ and $n = g$. So we have $l(s_n) \leq 3g - 1$.

Since $\Gamma_n = (\Gamma', x) \# \Gamma_{\mathbb{T}}$, it follows that Γ is at most at distance $3g - 1 + d_{g-1}$ of $(N_{g-1}, y) \# \Gamma_{\mathbb{T}}$. So,

$$d(\Gamma, N_g) \leq 3g + 3 + d_{g-1};$$

since $(N_{g-1}, y) \# \Gamma_{\mathbb{T}}$ is at most at distance 4 of N_g (see Figure 19). Hence

$$d_g \leq 3g + 3 + d_{g-1};$$

and by induction on g

$$D_g \leq 2d_g \leq 3g^2 + 9g - 12.$$

The lower bound comes from the following remark. Let Γ (respectively Γ') be a unicellular collection with k curves (respectively k' curves) such that Γ' is obtained after a surgery on Γ . Then, $|k' - k| \leq 1$; depending on whether the surgery is between two oriented edges on different curves or not. It follows from this remark that if Γ (respectively Γ') is a unicellular collection with k curve (respectively k'), $d(\Gamma, \Gamma') \geq |k - k'|$. Since for $g \geq 2$, there are unicellular collections with $2g$ simple curves (see Figure 2) and unicellular collections with one curve (see Figure 2), the inequality follows. \square

Non-hyperbolicity of \widehat{K}_∞ : Let (X, d) be a totally geodesic metric space, that is every pair of points in X are joint by a geodesic.

Definition 5.2. A geodesic triangle is a triple (η_1, η_2, η_3) of geodesics $\eta_i : [0, 1] \rightarrow X$ such that:

$$\eta_1(1) = \eta_2(0); \quad \eta_2(1) = \eta_3(0); \quad \eta_3(1) = \eta_1(0).$$

Let $\delta \in \mathbb{R}_+$ and $T := (\eta_1, \eta_2, \eta_3)$ geodesic triangle of X . We say that T is δ -thin if the δ -neighborhood of the union of two geodesics of T contain the third.

A metric space (X, d) is *Gromov hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle T is δ -thin.

For more details on Gromov hyperbolic spaces, see [3].

We show that \widehat{K}_∞ is not Gromov hyperbolic by giving triangles on \widehat{K}_∞ which are not δ -thin for a given δ . We denote by d the distance on \widehat{K}_∞ . We recall that for a unicellular collection Γ , S_Γ denotes the number of 1-simple curves of Γ .

Lemma 5.2. *Let Γ_1 and Γ_2 two unicellular collections. Then,*

$$d(\Gamma_1, \Gamma_2) \geq \frac{1}{2} |S_{\Gamma_1} - S_{\Gamma_2}|.$$

Proof. If $\Gamma' = (\Gamma, x) \# \Gamma_{\mathbb{T}}$, then $|S_{\Gamma} - S_{\Gamma'}|$ is equal to 0 or 1 depending on whether x is a side of a 1-simple curve or not.

On the other side, if $\Gamma' = \sigma_{x,y}(\Gamma)$ then $|S_{\Gamma} - S_{\Gamma'}| \leq 2$, that is a surgery creates at most two 1-simple curves or eliminates at most two 1-simple curves.

Since a path in \widehat{K}_{∞} is a sequence of surgery and connected sum, then we need at least $\frac{1}{2}|S_{\Gamma} - S_{\Gamma'}|$ steps from Γ_1 to Γ_2 . \square

Let $A := \Gamma_{\mathbb{T}}$ and X_{2g} be the unicellular collection obtained by gluing g -copies of $\Gamma_{\mathbb{T}}$ on the necklace N_g , each copy being glue on a 1-simple curve of C_g (see Figure 20). The collection X_{2g} has g 1-simple curves.

Let Y_{2g} be the unicellular collection on Σ_{2g} obtained by gluing $g - 1$ copies of $\Gamma_{\mathbb{T}}$ to the necklace N_{g+1} as in figure 20.

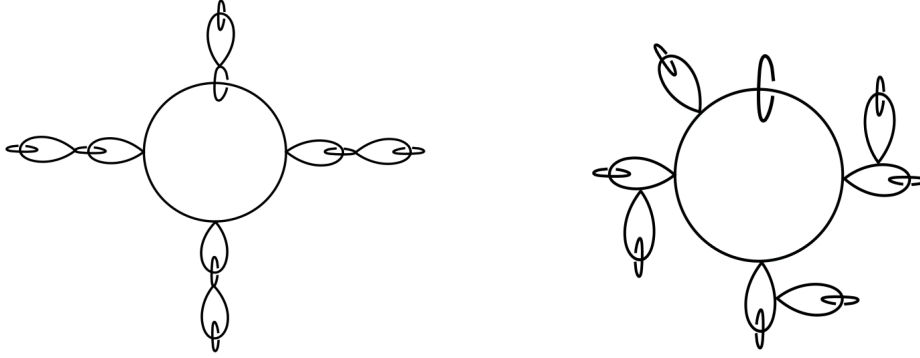


FIGURE 20 – The unicellular collections X_8 (on the left) and Y_8 (on the right).

Lemma 5.3. *For every $g \geq 1$, $d(X_{2g}, \Gamma_{\mathbb{T}}) = d(Y_{2g}, \Gamma_{\mathbb{T}}) = 2g$. Moreover,*

$$\frac{g}{2} \leq d(X_{2g}, Y_{2g}) \leq 2g.$$

Proof. The collections X_{2g} and Y_{2g} are in the $2g$ -th level of \widehat{K}_{∞} , and are obtained by gluing $2g$ copies of $\Gamma_{\mathbb{T}}$. Therefore, $d(X_{2g}, \Gamma_{\mathbb{T}}) = d(Y_{2g}, \Gamma_{\mathbb{T}}) = 2g$.

On X_{2g} , we cut the copies of $\Gamma_{\mathbb{T}}$ gluing on 1-simple curves and glue them again in an appropriate manner to obtain Y_{2g} . Doing so, we obtained a path on \widehat{K}_{∞} , from X_{2g} to Y_{2g} of length $2g$. Therefore,

$$d(X_{2g}, Y_{2g}) \leq 2g.$$

On the other side, we have $|S_{X_{2g}} - S_{Y_{2g}}| = 2g$, so $d(X_{2g}, Y_{2g}) \geq \frac{g}{2}$. \square

Let T_{2g} be a triangle with endpoints A , X_{2g} and Y_{2g} . The points X_{2g} and Y_{2g} are in the same level $K_{4,2g}$, but we do not know whether a geodesic from X_{2g} to Y_{2g} stays in $K_{4,2g}$ or not. The level $K_{4,2g}$ is maybe not geodesic. Nonetheless, Lemma 5.3 tells us that the geodesic $(X_{2g}Y_{2g})$ does not go down the level $K_{4,g}$, that is

$$d(\Gamma_{\mathbb{T}}, (X_{2g}Y_{2g})) \geq g.$$

In fact, since X_{2g} and Y_{2g} are in the same level, if the geodesic $(X_{2g}Y_{2g})$ goes down in level k times, it must go up in level k times and it implies that

$$2k \leq d(X_{2g}, Y_{2g}) \leq 2g \implies k \leq g.$$

This fact on T_{2g} is crucial and it allows us to show that the sequence of triangles $(T_{2g})_{g \in \mathbb{N}}$ is not δ -thin for any $\delta \geq 0$.

Proof of Theorem 3. Let $\mathcal{D}_{2k} := (X_{2k}, \rightarrow)$ (respectively $\mathcal{D}'_{2k} := (Y_{2k}, \rightarrow)$) be the half-geodesic passing through all the points X_{2m} (respectively Y_{2m}) for $m \geq k$.

Since $d(X_{2g}, Y_{2g}) \geq \frac{g}{2}$, then $d(\mathcal{D}_k, \mathcal{D}'_k) \rightarrow +\infty$. So, the δ -neighborhoods $V_\delta(\mathcal{D}_k)$ and $V_\delta(\mathcal{D}'_k)$ are disjoint for k sufficiently large and

$$d(V_\delta(\mathcal{D}_k), V_\delta(\mathcal{D}'_k)) \rightarrow +\infty.$$

It follows that for k_0 big enough,

$$d(\Gamma_{\mathbb{T}}, (X_{4k_0}Y_{4k_0})) > 2k_0, \quad d(V_\delta(\mathcal{D}_k), V_\delta(\mathcal{D}'_k)) \geq 1.$$

The geodesic (X_{4k_0}, Y_{4k_0}) is not contained in $V_\delta(\mathcal{D}_k) \cup V_\delta(\mathcal{D}'_k)$. So, $(X_{4k_0}Y_{4k_0})$ is not contained in $V_\delta(\Gamma_{\mathbb{T}}X_{4k_0}) \cup V_\delta(\Gamma_{\mathbb{T}}Y_{4k_0})$. Hence, \widehat{K}_∞ is not Gromov hyperbolic. \square

6 Questions

In this section, we state some open questions that could be interesting for further investigations.

The characterization of a surgery on a unicellular collection (Lemma 3.1) still holds for the general case, namely for unicellular maps. A unicellular map comes with its *degree partition* which is the ordered list $d := (d_1, \dots, d_n)$ of the degree of its vertices. The degree partition is left invariant by surgeries and for every degree partition d , we associate the combinatorial surgery graph $K_{d,g}$ on unicellular maps with degree partition d of Σ_g .

Question 1. *For which degree partition d , the graph $K_{d,g}$ is connected ?*

If one consider a closed surface $\Sigma_{g,n}$ with n punctures, a unicellular map is just a collection which separates $\Sigma_{g,n}$ into a topological disk with n punctures. If one considers unicellular maps up to orientation-preserving homeomorphisms that globally fix the punctures, the surgery still being well-defined and our results still holds. In the case where unicellular maps are considered up to orientation-preserving homeomorphisms that fix pointwise the punctures, the arc $\lambda_{x,y}$ between x and y is no longer unique and this case need to be fully consider.

Among unicellular collections, there is a big class of those made by only simple curves. We know that their number grows exponentially with the genus [1].

Question 2. *Is the combinatorial surgery graph on unicellular collections made by simple curves connected ? Is the combinatorial surgery graph on minimally intersecting filling pairs connected ?*

In the first case, surgeries are allowed only between intertwined oriented edges belonging to the same curve or to two disjoint curves. For minimally intersecting filling pairs, surgery are allowed only between intertwined oriented edges on the same curve with opposite orientation.

Given a graph G , the Cheeger constant of G denoted by $h(G)$ measures how strong is the connectedness of the graph. A sequence (G_n) of graphs is called an *expander* if the sequence of Cheeger constants $(h(G_n))$ is bounded from bellow.

Question 3. *Is the diameter D_g linear on g ? Is the family (K_g) an expander ?*

Acknowledgments: I am thankful to my supervisors P. Dehornoy and J.-C Sikorav for their support during this work. I am also grateful to Gregory Miermont for his indication to the works of G. Chapuy.

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