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► **To cite this version:**

Marc Prévost, Tanguy Rivoal. Remainder Padé approximants for hypergeometric series. *Integral Transforms and Special Functions*, 2019, 30 (10), pp.833-843. 10.1080/10652469.2019.1624542 . hal-02017857

**HAL Id: hal-02017857**

**<https://hal.science/hal-02017857v1>**

Submitted on 13 Feb 2019

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# REMAINDER PADÉ APPROXIMANTS FOR HYPERGEOMETRIC SERIES

MARC PRÉVOST AND TANGUY RIVOAL

ABSTRACT. Remainder Padé Approximation (RPA) consists in adding to the  $n$ -th partial sum of a series a suitable Padé approximant of the asymptotic expansion in the variable  $1/n$  of the remainder term. In a previous paper, we proved the non-trivial property that the RPA of the exponential function is identical to the Padé approximant to the function  $e^z$ . In this paper, we extend this property to the hypergeometric series  ${}_1F_1(1; a; z)$  and  ${}_2F_0(b, 1; z)$ .

## 1. INTRODUCTION

**1.1. Remainder Padé Approximants.** In [5], the first author introduced a new kind of rational approximation, the Remainder Padé Approximants (RPA). They are defined as follows. Let  $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$ ; for simplicity of the exposition, in the discussion below  $f$  is assumed to be convergent in a disk  $D$  of positive radius <sup>(1)</sup>. For any integer  $n \geq 0$ , we assume the existence of certain functions  $b_n(z) \in \mathbb{C}(z)$  such that the normalized remainder series  $\Phi(z, n) := \frac{1}{b_n(z)} \sum_{k=n}^{\infty} a_k z^k$  admits, for any fixed  $z \in D \setminus \{\text{zeros of all the } b_n\}$ , an asymptotic expansion as  $n \rightarrow +\infty$  of the form

$$\Phi(z, n) \sim \widehat{\Phi}_z\left(\frac{1}{n}\right) := \sum_{k=0}^{\infty} \frac{\phi_k(z)}{n^k} \in \mathbb{C}(z) \left[ \left[ \frac{1}{n} \right] \right].$$

Let  $[p/q]_{\widehat{\Phi}_z(t)}$  denote the ordinary Padé approximant of degrees  $p$  and  $q$  of the formal power series  $\widehat{\Phi}_z(t)$  at  $t = 0$ ; it belongs to  $\mathbb{C}(z, t)$ . We thus have

$$f(z) = \sum_{k=0}^{n-1} a_k z^k + b_n(z) \Phi_n(z) \approx \sum_{k=0}^n a_k z^k + b_n(z) ([p/q]_{\widehat{\Phi}_z(t)})_{t=1/u(n)}. \quad (1.1)$$

At this stage, the  $\approx$  sign does not have any particular meaning, and the sequence of complex numbers  $u(n)$  is  $n + o(n)$  but not necessarily equal to  $n$ . We say that the rational fraction

$$\frac{A(z)}{B(z)} := \sum_{k=0}^{n-1} a_k z^k + b_n(z) ([p/q]_{\widehat{\Phi}_z(t)})_{t=1/u(n)}$$

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2010 *Mathematics Subject Classification.* Primary 41A21; Secondary 41A28.

*Key words and phrases.* Remainder Padé approximants, hypergeometric function, multiple Charlier polynomials.

<sup>1</sup>It is possible to extend the notion of RPA when  $f$  diverges everywhere but is summable in some sense; we present an exemple of this situation in Theorem 2.

is a Remainder Padé Approximant of  $f(z)$ , where  $A(z), B(z) \in \mathbb{C}[z]$  are of lowest degrees. In certain cases, there exists a sequence  $u(n)$  such that the corresponding RPA of  $f(z)$  coincides with an ordinary Padé approximant of  $f(z)$  at  $z = 0$ , say  $[N/D]_{f(z)}$ . It is not yet understood for which class of functions  $f$  this remarkable fact happens. When it does happen, the  $\approx$  sign in (1.1) can then be understood as follows: the order at  $z = 0$  of the remainder  $B(z)f(z) - A(z)$  is larger than or equal to  $N + D + 1$ , the order of Padé approximation.

More generally, it also happens that, instead of Padé approximants, we recover some previously known functional approximations of  $f(z)$ , or even numerical approximations of certain of its values. For instance, the first author constructed certain sequences of RPA for  $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2$  and  $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$  (both can be viewed as values of the polylogarithms  $\sum_{k=1}^{\infty} z^k/k^s$  at  $z = 1$ ) and proved that they are exactly the sequences of rational numbers used by Apéry [2] to prove the irrationality of  $\zeta(2)$  and  $\zeta(3)$ . In [7], the second author used a modification of these RPA to produce a sequence of fast converging rational approximations for Catalan's constant  $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$ . In [6], using certain discrete multiple orthogonal polynomials introduced in [3], which generalize the classical Charlier orthogonal polynomials, we proved that certain RPA of  $e^z$  provide the Padé approximants of the function  $e^z$ .

All these examples provide various meanings to the  $\approx$  sign in (1.1) and we use the vocable *RPA phenomenon* to cover such non-trivial properties.

We recall the notion of type II (diagonal) Padé approximants. For a given family  $(F_j(X))_{j=1, \dots, r}$  of formal series in  $\mathbb{C}[[X]]$  and any integers  $p$  and  $q$  such that  $p \geq (r-1)q \geq 0$ , there exist (by linear algebra) some polynomials  $P_1(x), \dots, P_r(x)$  and  $Q(X)$  in  $\mathbb{C}[X]$ , not all 0, such that  $\deg(P_j) \leq p$ ,  $\deg(Q) \leq r \cdot q$  and

$$Q(X)F_j(X) - P_j(X) = \mathcal{O}(X^{p+q+1}), \quad 1 \leq j \leq r.$$

The fractions  $P_j/Q$  are unique and are by definition the type II Padé approximants of  $(F_j(X))_{j=1, \dots, r}$  of parameters  $(p, q)$ . When  $r = 1$ , we recover the ordinary Padé approximants  $[p/q]$  of  $F_1(X)$ .

**1.2. The results.** The goal of this paper is to show that the RPA phenomenon occurs (in various generality) for the hypergeometric series

$$E_a(z) := {}_1F_1(1; a; z) = \sum_{k=0}^{\infty} \frac{z^k}{(a)_k} \quad \text{and} \quad {}_2F_0(b, 1; z) = \sum_{k=0}^{\infty} (b)_k z^k.$$

We assume that  $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ . The former series generalizes the exponential function  ${}_1F_1(1; 1; z) = \exp(z)$ , while the latter generalizes Euler's series  ${}_2F_0(1, 1; z) = \sum_{k=0}^{\infty} k! z^k$ . The variable  $z$  is a formal one in Theorems 1 and 2 below. However, it is important to have in mind that, though the series  $E_a(z)$  is an entire function of  $z$ , the  ${}_2F_0(b, 1; z)$  diverges for every  $z$ . To overcome this divergence, we shall first work the function

$$\mathcal{E}_b(z) := \frac{1}{\Gamma(b)} \int_0^{\infty} \frac{u^{b-1}}{1-uz} e^{-u} du$$

defined for  $\Re(b) > 0$  and analytic for  $z \in \mathbb{C} \setminus [0, +\infty)$ . It admits an asymptotic expansion  $\widehat{\mathcal{E}}_b(z)$  as  $z \rightarrow 0$  in any angular sector centered at 0 that does not contain  $[0, +\infty)$ . It turns out that  $\widehat{\mathcal{E}}_b(z) = {}_2F_0(b, 1; z)$  in such sectors.

To state our results, we need to introduce a certain number of notations. The parameters  $\alpha_j, j = 1, \dots, r$ , are pairwise distinct non-zero complex numbers, and we set  $\underline{\alpha} := (\alpha_j)_{j=1, \dots, r}$ . For any  $z \in \mathbb{C}$  and any  $t \in \mathbb{C}$  such that  $1/t \notin \mathbb{N}$ , we define

$$\Phi_z(t) := \sum_{k=0}^{\infty} \frac{z^k}{(1 - 1/t)_k}.$$

For any  $j = 1, \dots, r$  and any  $n \geq 0$ , we have the trivial identity

$$\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k} = \sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \Phi_{\alpha_j z} \left( \frac{-1}{a - 1 + n} \right). \quad (1.2)$$

In [6], it has been proved that, for any fixed  $z \in \mathbb{C}$ ,  $\Phi_z(t)$  admits an asymptotic expansion  $\widehat{\Phi}_z(t) := \sum_{k=0}^{\infty} \varphi_k(-z)t^k$  as  $t \rightarrow 0, t < 0$ , where the coefficients  $\varphi_k(z)$  are *Touchard exponential polynomials of degree  $k$* . The latter are defined by the exponential generating function  $e^{z(e^X - 1)} = \sum_{k=0}^{\infty} \varphi_k(z) \frac{X^k}{k!}$  (see [10]).

We provided in [6] explicit expressions for the type II Padé approximants at  $t = 0$  of parameters  $(rp - 1, p)$  for the formal power series  $\widehat{\Phi}_{\alpha_j z}(t)$  in  $\mathbb{C}[z][[t]]$  for  $j = 1, \dots, r$ : we denote them by  $P_{j, \underline{\alpha}, p}(t, z)/Q_{\underline{\alpha}, p}(t, z)$ , and the expressions of these polynomials in  $t$  are given in §2. It turns out that they are also polynomials in  $z$ .

We now “replace”  $\Phi_{\alpha_j z} \left( \frac{-1}{a-1+n} \right)$  in (1.2) by the corresponding simultaneous Padé approximant of parameters  $(rp - 1, p)$  of  $\widehat{\Phi}_{\alpha_j z}(t)$  evaluated at  $t = -1/(a - 1 + n)$  (this substitution is possible because  $Q_{\underline{\alpha}, p}(-1/(a - 1 + n), z)$  does not vanish identically as a function of  $z$ ). This alteration of (1.2) provides the formal Laurent series

$$R_{j, n, \underline{\alpha}, p}(z) := \sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k} - \sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} - \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j, \underline{\alpha}, p} \left( \frac{-1}{a-1+n}, z \right)}{Q_{\underline{\alpha}, p} \left( \frac{-1}{a-1+n}, z \right)}, \quad j = 1, \dots, r,$$

and we want to know in which sense they are  $\approx 0$ . By construction, for every  $j$ , the rational fraction

$$\sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j, \underline{\alpha}, p} \left( \frac{-1}{a-1+n}, z \right)}{Q_{\underline{\alpha}, p} \left( \frac{-1}{a-1+n}, z \right)} \quad (1.3)$$

is a RPA of  $E_a(\alpha_j z)$ . The degrees of its numerator and denominator are *a priori* bounded by  $n + rp - 1$  and  $rp$  respectively. A better bound holds for the numerator ((i) below) and it is crucial to prove (ii). (Throughout the paper,  $\mathcal{O}(z^c)$  denotes a Laurent series in  $z$  with order equal to  $c$  at  $z = 0$ .)

**Theorem 1.** *Let us fix the integers  $r \geq 1, n \geq 1, p \geq 0$ , such that  $n \geq p, a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \alpha_1, \dots, \alpha_r \in \mathbb{C}$  pairwise distinct.*

(i) The degrees of the numerator and denominator of the RPA in (1.3) are bounded by  $n + (r - 1)p - 1$  and  $rp$  respectively, and moreover  $R_{j,n,\alpha,p}(z) = \mathcal{O}(z^{n+rp})$ .

(ii) The collection of the RPA in (1.3),  $j = 1, \dots, r$ , is the type II Padé approximant of  $(E_a(\alpha_j z))_{j=1,\dots,r}$  of parameters  $(n + (r - 1)p - 1, p)$ .

(iii) For  $r = 1$  and  $\alpha_1 = 1$ , (ii) gives the following equality of rational fractions:

$$\sum_{k=0}^{n-1} \frac{z^k}{(a)_k} + \frac{z^n}{(a)_n} ([p - 1/p]_{\widehat{\Phi}_z(t)})_{t=-1/(a-1+n)} = [n - 1/p]_{E_a(z)},$$

For a generic value of  $a$ , our proof needs the assumption that  $n \geq p$ . However, if  $a = 1$ , it can be removed in the statement of Theorem 1, and we simply recover the results from [6]. The explanation of this fact is Identity (2.5) in §2.

Let us now describe our results for the function  $\mathcal{E}_b(z)$ . For any  $n \geq 0$ ,  $b$  such that  $\Re(b) > 0$  and  $z \in \mathbb{C} \setminus [0, +\infty)$ , we have the identity (proved in §3)

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n \Psi_z \left( \frac{1}{b-1+n} \right) \quad (1.4)$$

where

$$\Psi_z(t) = \frac{1}{\Gamma(1 + 1/t)} \int_0^\infty \frac{u^{1/t}}{1 - uz} e^{-u} du$$

is defined for any  $z \in \mathbb{C} \setminus (0, +\infty)$  and any  $t$  such that  $\Re(1/t) > -1$  (in particular, for any  $t > 0$ ). We shall prove in §3 that when  $z < 0$ ,  $\Psi_z(t)$  admits an asymptotic expansion as  $t \rightarrow 0$ ,  $t > 0$ , given by (2)

$$\widehat{\Psi}_z(t) := -\frac{1}{z} \sum_{k=0}^{\infty} \varphi_k(1/z) t^{k+1} = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t). \quad (1.5)$$

It then immediately follows that the Padé approximant  $[p/p]$  at  $t = 0$  of the formal power series  $\widehat{\Psi}_z(t) \in \mathbb{C}[1/z][[t]]$  is  $-\frac{t}{z} \frac{P_{j,1,p}(t, -1/z)}{Q_{1,p}(t, -1/z)}$  where the polynomials are those used to construct the Padé approximants  $[p - 1/p]$  of  $\widehat{\Phi}_z(t)$  when  $r = 1$ .

In (1.4), we now “replace”  $\mathcal{E}_b(z)$  by  $\widehat{\mathcal{E}}_b(z)$ , and  $\Psi_z \left( \frac{1}{b-1+n} \right)$  by the Padé approximant  $[p/p]$  of  $\widehat{\Psi}_z(t)$  evaluated at  $t = 1/(b-1+n)$  (this substitution is possible because  $Q_{1,p}(1/(b-1+n), 1/z)$  does not vanish identically as a function of  $z$ ). This alteration of (1.4) provides the formal Laurent series

$$S_{n,p}(z) := \widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k + \frac{(b)_n z^{n-1}}{(b-1+n)} \frac{P_{j,1,p}\left(\frac{1}{b-1+n}, -\frac{1}{z}\right)}{Q_{1,p}\left(\frac{1}{b-1+n}, -\frac{1}{z}\right)},$$

<sup>2</sup>This identity is reminiscent of André’s duality between solutions of  $E$ -operators at  $z = 0$  and  $z = \infty$ ; see [1]. The former involve function like  $E_a(z)$ , while the latter involve functions like  $\widehat{\mathcal{E}}_b(1/z)$ , with  $a, b \in \mathbb{Q}$ .

and we want to know in which sense it is  $\approx 0$ . Because of (1.4), the rational fraction

$$\sum_{k=0}^{n-1} (b)_k z^k - \frac{(b)_n z^{n-1}}{(b-1+n)} \frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})} \quad (1.6)$$

can be viewed as a RPA of  $\mathcal{E}_b(z)$  in an extended sense. The degrees of its numerator and denominator are *a priori* bounded by  $n+p-1$  and  $p$  respectively. Again, a better bound holds for the numerator ((i) below) and again it is crucial to prove (ii).

**Theorem 2.** *Let us fix  $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ , and the integers  $n \geq 1$ ,  $p \geq 0$  such that  $n \geq p$ .*

(i) *The degrees of the numerator and denominator of the RPA in (1.6) are bounded by  $n-1$  and  $p$  respectively, and moreover  $S_{n,p}(z) = \mathcal{O}(z^{n+p})$  exists and is finite.*

(ii) *The RPA in (1.6) coincides with the Padé approximant  $[n-1/p]$  of  $\widehat{\mathcal{E}}_b(z)$ . In other words,*

$$\sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\widehat{\Psi}_z(t)})_{t=1/(b-1+n)} = [n-1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

It seems that a RPA phenomenon similar to (ii) in Theorem 1 does not hold for any family  $(\widehat{\mathcal{E}}_b(\beta_j))_{j=1,\dots,r}$  with  $r \geq 2$ .

The proofs of Theorems 1 and 2 are given in §2 and §3 respectively.

## 2. PROOF OF THEOREM 1

**Proof of (i).** We recall the expressions of the type II approximants of the functions  $\widehat{\Phi}_{\alpha_j z}(t)$  (obtained in [6]):

$$P_{j,\underline{\alpha},p}(t, z) = (-t)^{rp-1} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (-1/t)_{k_1+\dots+k_r} \sum_{i=1}^{k_1+\dots+k_r} \frac{(\alpha_j z)^{i-1}}{(-1/t)_i}, \quad (2.1)$$

and

$$Q_{\underline{\alpha},p}(t, z) = (-t)^{rp} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (-1/t)_{k_1+\dots+k_r}. \quad (2.2)$$

The  $P$ 's have degree  $rp-1$  in both  $t$  and  $z$ , while the  $Q$ 's have degree  $rp$  in both  $t$  and  $z$ . Moreover,  $Q_{\underline{\alpha},p}(0, 1) = 1$ .

For the sake of simplicity, we set  $K_r = k_1 + k_2 + \dots + k_r$ . Substituting  $-1/(a-1+n)$  for  $t$  (which is possible because  $a \notin \mathbb{Z}_{\leq 0}$  and  $n \geq 1$ ), we get

$$Q_{\underline{\alpha},p}\left(\frac{-1}{a-1+n}, z\right) = (a+n-1)^{-rp} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r}$$

and

$$P_{j,\alpha,p}\left(\frac{-1}{a-1+n}, z\right) = \frac{(a-1+n)^{1-rp}(a)_{n-1}}{(\alpha_j z)^n} \\ \times \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} (S_{n+K_r-1}(\alpha_j z) - S_{n-1}(\alpha_j z))$$

where  $S_n(z) = \sum_{k=0}^n z^k / (a)_k$ . (Note that  $Q_{\alpha,p}(\frac{-1}{a-1+n}, z)$  is clearly not identically zero, as stated in the Introduction.)

Hence, after some simplifications, the RPAs of the functions  $E_{\alpha_j}(z)$ ,  $j = 1, \dots, r$ , are given by

$$\sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j,\alpha,p}(\frac{-1}{a-1+n}, z)}{Q_{\alpha,p}(\frac{-1}{a-1+n}, z)} \\ = \frac{\sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} S_{n+K_r-1}(\alpha_j z)}{\sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r}}. \quad (2.3)$$

At first sight, the numerator and denominator of this rational fraction have degrees bounded by  $n + rp - 1$  and  $rp$  in  $z$  respectively, but a better estimate holds for the numerator. Indeed, we shall now prove that its degree is bounded by  $n + (r-1)p - 1$ . Indeed, with  $K'_{j,r} = k_1 + \dots + k_{j-1} + k_{j+1} + \dots + k_r$  and  $\underline{k}'_{j,r} = (k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r)$ , the numerator of (2.3) can be expressed as

$$\sum_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r=0}^p \left( \prod_{i=1, i \neq j}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K'_{j,r}} A_{j, \underline{k}'_{j,r}}(z) \quad (2.4)$$

with

$$A_{j, \underline{k}'_{j,r}}(z) = \sum_{k_j=0}^p \binom{p}{k_j} (-\alpha_j z)^{p-k_j} (a-1+n + K'_{j,r})_{k_j} S_{n+K_r-1}(\alpha_j z).$$

We claim that the degree in  $z$  of the polynomial  $A_{j, \underline{k}'_{j,r}}(z)$  is at most  $\max(p, n + K'_{j,r}) - 1$ . To prove this observation, which is non-trivial, we first observe that, for any integers  $m, q$  such that  $m \geq q \geq 0$ ,

$$(t^{a-1} S_m(xt)) \Big|_{t=1}^{(q)} = x^q S_{m-q}(x) + U_q(x) \quad (2.5)$$

where

$$U_q(x) := \sum_{j=0}^{q-1} (-1)^{q-j} (1-a)_{q-j} x^j.$$

(Above and below in this section, all the differentiations are with respect to  $t$ .) Observe that  $U_q(x) \equiv 0$  if  $a = 1$ , but in general  $U_q(x)$  has degree  $q - 1$  in  $x$ . Since  $(a - 1 + m)_k = (-1)^k (t^{-a+1-m})|_{t=1}^{(k)}$ , it follows that

$$\begin{aligned} A_{j, \underline{k}_{j,r}}(z) &= (-1)^p \sum_{k=0}^p \binom{p}{k} (t^{-a+1-n-K'_{j,r}})|_{t=1}^{(k)} (t^{a-1} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p-k)} \\ &\quad + \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z) \\ &= (-1)^p (t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p)} \end{aligned} \quad (2.6)$$

$$+ \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z). \quad (2.7)$$

The polynomial in (2.7) obviously has degree at most  $p - 1$  in  $z$ . On the other hand

$$\begin{aligned} (t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p)} &= \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} (t^{j-n-K'_{j,r}})|_{t=1}^{(p)} \\ &= (-1)^p \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} (n+K'_{j,r}-j)_p. \end{aligned}$$

Since  $(n+K'_{j,r}-j)_p = 0$  for any  $j \in \{n+K'_{j,r}, \dots, n+K'_{j,r}+p-1\}$ , it follows that the polynomial in (2.6) has degree at most  $n+K'_{j,r}-1$ , so that the degree of  $A_{j, \underline{k}_{j,r}}(z)$  is at most  $\max(p, n+K'_{j,r}) - 1$  as claimed. Since  $n \geq p$  by assumption, we deduce that the degree of the numerator of the RPAs given by (2.4) is at most  $n+(r-1)p-1$ , as expected.

To prove that  $R_{j,n,\alpha,p}(z) = \mathcal{O}(z^{n+rp})$ , we have to multiply the denominator of (2.3) by the series  $\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}$  and prove that the product is the sum of a polynomial of degree  $n+(r-1)p-1$  and a function which is  $\mathcal{O}(z^{n+rp})$ . This amounts to proving that the coefficient  $c_m$  of  $z^m$  in the Taylor expansion of

$$\left( \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} \right) \cdot \sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}$$

is zero for  $m \in \{n+(r-1)p, \dots, n+rp-1\}$ . Let us assume that  $m \geq n+(r-1)p$ . Then

$$c_m = \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i)^{p-k_i} \right) (a-1+n)_{K_r} \frac{\alpha_j^k}{(a)_k} \quad (2.8)$$

where in (2.8) the integer  $k := m + K_r - rp$  is  $\geq 0$  (because  $m \geq n+(r-1)p \geq rp$ ).



We now also assume that  $m \leq n + rp - 1$ . In that case, we use the identity

$$\frac{(a-1+n)_{K_r}}{(a)_k} = \frac{1}{(a)_{n-1}} \left( t^{a+n+K_r-2} \right)_{|t=1}^{(n+rp-m-1)}$$

in (2.8) and get

$$\begin{aligned} c_m &= \frac{1}{(a)_{n-1}} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (-\alpha_i)^{p-k_i} \right) \alpha_j^{m-rp+K_r} \left( t^{a+n+K_r-2} \right)_{|t=1}^{(n+rp-m-1)} \\ &= \frac{1}{(a)_{n-1}} \left( t^{a-1+n-rp} \prod_{i=1}^r (\alpha_j - \alpha_i t)^p \right)_{|t=1}^{(n+rp-m-1)}. \end{aligned}$$

Since  $t = 1$  is a zero of order  $p$  of the product  $\prod_{i=1}^r (\alpha_j - \alpha_i t)^p$ , it follows that  $c_m = 0$  when  $n + rp - m - 1 \leq p - 1$ , i.e. when  $m \geq n + (r - 1)p$ . This complete the proof of (i).

**Proof of (ii).** By unicity of type II Padé approximants, the bounds for the degrees of the simultaneous RPAs and the order of  $R_{j,n,\alpha,p}$  at 0 ensure that (2.3) provides the expression of the type II Padé approximants of the family  $(E_a(\alpha_j z))_{j=1, \dots, r}$  of parameters  $(n + (r - 1)p - 1, p)$  when  $n \geq p$ .

### 3. PROOF OF THEOREM 2

Let  $b$  such that  $\Re(b) > 0$ . We first prove (1.4) of the Introduction, i.e. that for any  $n \geq 0$  and any  $z \in \mathbb{C} \setminus [0, +\infty)$ ,

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n \Psi_z \left( \frac{1}{b-1+n} \right).$$

This follows from

$$\begin{aligned} (b)_n z^n \Psi_z \left( \frac{1}{b-1+n} \right) &= \frac{(b)_n z^n}{\Gamma(b+n)} \int_0^\infty \frac{u^{b+n-1}}{1-uz} e^{-u} du \\ &= \frac{(b)_n}{\Gamma(b+n)} \int_0^\infty \frac{u^{b-1}}{1-uz} e^{-u} du + \frac{(b)_n}{\Gamma(b+n)} \int_0^\infty u^{b-1} \frac{(uz)^n - 1}{1-uz} e^{-u} du \\ &= \mathcal{E}_b(z) - \frac{(b)_n}{\Gamma(b+n)} \sum_{k=0}^{n-1} z^k \int_0^\infty u^{k+b-1} e^{-u} du \\ &= \mathcal{E}_b(z) - \sum_{k=0}^{n-1} \frac{(b)_n \Gamma(b+k)}{\Gamma(b+n)} z^k = \mathcal{E}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k. \end{aligned}$$

We now prove the following lemma.

**Lemma 1.** *Let us fix  $z \in (-\infty, 0)$ . The function  $\Psi_z(t)$  admits an asymptotic expansion as  $t \rightarrow 0, t > 0$ , given by*

$$\widehat{\Psi}_z(t) := -\frac{1}{z} \sum_{k=0}^{\infty} \varphi_k(1/z) t^{k+1} = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t),$$

where the coefficients  $\varphi_k(X)$  are Touchard exponential polynomials of degree  $k$  in  $X$ .

*Proof.* We first use Proposition 2 in [8] to get the following alternative expression for  $\Psi_z(t)$ : for any  $z \in (-\infty, 0)$  and  $t > 0$ ,

$$\Psi_z(t) = \int_0^\infty \frac{e^{-u}}{(1-zu)^{1+1/t}} du.$$

We now set  $v = \ln(1 - uz)$ , so that

$$\Psi_z(t) = -\frac{1}{z} \int_0^\infty e^{(e^v-1)/z} e^{-v/t} dv \quad (3.1)$$

for  $z \in (-\infty, 0)$  and  $t > 0$ . Since  $v \mapsto e^{(e^v-1)/z}$  is  $C^\infty$  at  $v = 0$ , non-zero at  $v = 0$ , and

$$\int_0^\infty |e^{(e^v-1)/z} e^{-v/t}| dv < \infty,$$

we can apply Watson's lemma ([4, Chapter 2, §2]) to (3.1): the function  $\Psi_z(t)$  has the asymptotic expansion

$$\Psi_z(t) \sim -\frac{1}{z} \sum_{k=0}^\infty \left( \frac{d^k}{dv^k} e^{(e^v-1)/z} \right)_{|v=0} t^{k+1} = -\frac{t}{z} \sum_{k=0}^\infty \varphi_k(1/z) t^k, \quad t \rightarrow 0, t > 0.$$

This complete the proof of the lemma.  $\square$

From now on, we view  $\widehat{\Psi}_z(t)$  as a formal series in  $\mathbb{C}[1/z][[t]]$ . As observed in the Introduction, the identity  $\widehat{\Psi}_z(t) = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t)$  enables us to immediately get the Padé approximant  $[p/p]$  of  $\widehat{\Psi}_z(t)$  at  $t = 0$ . More generally, let  $\underline{\beta} := (\beta_j)_{j=1, \dots, r}$  denote a family of pairwise distinct complex numbers. Then, without further efforts, we even obtain the type II Padé approximants at  $t = 0$  of parameters  $(rp, p)$  of the formal power series  $(\widehat{\Psi}_{\beta_j z}(t))_{j=1, \dots, r}$  for  $j = 1, \dots, r$ . They are given by

$$-\frac{t}{\beta_j z} \frac{P_{j,1/\underline{\beta},p}(t, -1/z)}{Q_{1/\underline{\beta},p}(t, -1/z)}, \quad j = 1, \dots, r$$

where

$$\begin{aligned} & P_{j,1/\underline{\beta},p}(t, -1/z) \\ &= (-t)^{rp-1} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1/t)_{k_1+\dots+k_r} \sum_{i=1}^{k_1+\dots+k_r} \frac{(-\beta_j z)^{1-i}}{(-1/t)_i}, \\ & Q_{1/\underline{\beta},p}(t, -1/z) = (-t)^{rp} \sum_{k_1, \dots, k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1/t)_{k_1+\dots+k_r}. \end{aligned}$$

We now substitute  $1/(b-1+n)$  for  $t$  in these expressions, which is possible because  $\Re(b) > 0$  and  $n \geq 1$ . Since  $Q_{1/\underline{\beta},p}(1/(b-1+n), -1/z)$  is clearly not identically zero, we

get simultaneous RPAs of the family  $(\mathcal{E}_b(\beta_j z))_{j=1,\dots,r}$  given by the formulas:

$$\begin{aligned} \sum_{k=0}^{n-1} (b)_k (\beta_j z)^k - (b)_n \frac{(\beta_j z)^{n-1}}{(b-1+n)} \frac{P_{j,1/\beta,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1/\beta,p}(\frac{1}{b-1+n}, -\frac{1}{z})} \\ = \frac{z^{rp} \sum_{k_1,\dots,k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1)^{K_r} T_{n-K_r-1}(\beta_j z) / (b)_{n-K_r}}{z^{rp} \sum_{k_1,\dots,k_r=0}^p \left( \prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1)^{K_r} / (b)_{n-K_r}}. \end{aligned} \quad (3.2)$$

where  $T_n(z) := \sum_{k=0}^n (b)_k z^k$ .

We are now in position to complete the proof of Theorem 2. We take  $r = 1$  and  $\beta_1 = 1$  in (3.2).

**Proof of (i).** It is immediate that the degrees of the numerator and denominator of the rational function in (3.2) are bounded by  $n - 1$  and  $p$  respectively. We now want to prove that, formally,

$$\widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k + \frac{(b)_n z^{n-1}}{(b-1+n)} \frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})} = \mathcal{O}(z^{n+p}). \quad (3.3)$$

The denominator of the right-hand side of (3.2) is

$$Q(z) := \sum_{k=0}^p \frac{(-1)^k \binom{p}{k}}{(b)_{n-k}} z^k \quad (3.4)$$

so that to prove (3.3) it is enough to prove that the formal series expansion of

$$\left( \sum_{k=0}^p \frac{(-1)^k \binom{p}{k}}{(b)_{n-k}} z^k \right) \widehat{\mathcal{E}}_b(z)$$

is the sum of a polynomial  $P(z)$  of degree  $\leq n - 1$  and a formal power series of order  $\geq n + p$  at  $z = 0$ . This in fact means that we want to prove that  $P(z)/Q(z)$  is the  $[n - 1/p]$  Padé approximant of  $\widehat{\mathcal{E}}_b(z)$ . This is indeed the case because we recognize  $Q(z)$  in (3.4) as being of the form given in [9] and unicity of Padé approximant enables us to conclude.

**Proof of (ii).** This is simply a summary of what was proved in (i). We have

$$\widehat{\mathcal{E}}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} + \mathcal{O}(z^{n+p}) \quad (3.5)$$

and the bounds for the degrees show that

$$\sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} = [n - 1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

The above results are *a priori* proved under the assumption that  $\Re(b) > 0$ . They hold under the weaker assumption that  $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$  because all these identities between rational fractions extend to this case.

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