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REMAINDER PADÉ APPROXIMANTS FOR HYPERGEOMETRIC SERIES

MARC PRÉVOST AND TANGUY RIVOAL

ABSTRACT. Remainder Padé Approximation (RPA) consists in adding to the n -th partial sum of a series a suitable Padé approximant of the asymptotic expansion in the variable $1/n$ of the remainder term. In a previous paper, we proved the non-trivial property that the RPA of the exponential function is identical to the Padé approximant to the function e^z . In this paper, we extend this property to the hypergeometric series ${}_1F_1(1; a; z)$ and ${}_2F_0(b, 1; z)$.

1. INTRODUCTION

1.1. Remainder Padé Approximants. In [5], the first author introduced a new kind of rational approximation, the Remainder Padé Approximants (RPA). They are defined as follows. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$; for simplicity of the exposition, in the discussion below f is assumed to be convergent in a disk D of positive radius ⁽¹⁾. For any integer $n \geq 0$, we assume the existence of certain functions $b_n(z) \in \mathbb{C}(z)$ such that the normalized remainder series $\Phi(z, n) := \frac{1}{b_n(z)} \sum_{k=n}^{\infty} a_k z^k$ admits, for any fixed $z \in D \setminus \{\text{zeros of all the } b_n\}$, an asymptotic expansion as $n \rightarrow +\infty$ of the form

$$\Phi(z, n) \sim \widehat{\Phi}_z\left(\frac{1}{n}\right) := \sum_{k=0}^{\infty} \frac{\phi_k(z)}{n^k} \in \mathbb{C}(z) \left[\left[\frac{1}{n} \right] \right].$$

Let $[p/q]_{\widehat{\Phi}_z(t)}$ denote the ordinary Padé approximant of degrees p and q of the formal power series $\widehat{\Phi}_z(t)$ at $t = 0$; it belongs to $\mathbb{C}(z, t)$. We thus have

$$f(z) = \sum_{k=0}^{n-1} a_k z^k + b_n(z) \Phi_n(z) \approx \sum_{k=0}^n a_k z^k + b_n(z) ([p/q]_{\widehat{\Phi}_z(t)})_{t=1/u(n)}. \quad (1.1)$$

At this stage, the \approx sign does not have any particular meaning, and the sequence of complex numbers $u(n)$ is $n + o(n)$ but not necessarily equal to n . We say that the rational fraction

$$\frac{A(z)}{B(z)} := \sum_{k=0}^{n-1} a_k z^k + b_n(z) ([p/q]_{\widehat{\Phi}_z(t)})_{t=1/u(n)}$$

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¹It is possible to extend the notion of RPA when f diverges everywhere but is summable in some sense; we present an exemple of this situation in Theorem 2.

is a Remainder Padé Approximant of $f(z)$, where $A(z), B(z) \in \mathbb{C}[z]$ are of lowest degrees. In certain cases, there exists a sequence $u(n)$ such that the corresponding RPA of $f(z)$ coincides with an ordinary Padé approximant of $f(z)$ at $z = 0$, say $[N/D]_{f(z)}$. It is not yet understood for which class of functions f this remarkable fact happens. When it does happen, the \approx sign in (1.1) can then be understood as follows: the order at $z = 0$ of the remainder $B(z)f(z) - A(z)$ is larger than or equal to $N + D + 1$, the order of Padé approximation.

More generally, it also happens that, instead of Padé approximants, we recover some previously known functional approximations of $f(z)$, or even numerical approximations of certain of its values. For instance, the first author constructed certain sequences of RPA for $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2$ and $\zeta(3) = \sum_{k=1}^{\infty} 1/k^3$ (both can be viewed as values of the polylogarithms $\sum_{k=1}^{\infty} z^k/k^s$ at $z = 1$) and proved that they are exactly the sequences of rational numbers used by Apéry [2] to prove the irrationality of $\zeta(2)$ and $\zeta(3)$. In [7], the second author used a modification of these RPA to produce a sequence of fast converging rational approximations for Catalan's constant $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$. In [6], using certain discrete multiple orthogonal polynomials introduced in [3], which generalize the classical Charlier orthogonal polynomials, we proved that certain RPA of e^z provide the Padé approximants of the function e^z .

All these examples provide various meanings to the \approx sign in (1.1) and we use the vocable *RPA phenomenon* to cover such non-trivial properties.

We recall the notion of type II (diagonal) Padé approximants. For a given family $(F_j(X))_{j=1, \dots, r}$ of formal series in $\mathbb{C}[[X]]$ and any integers p and q such that $p \geq (r-1)q \geq 0$, there exist (by linear algebra) some polynomials $P_1(x), \dots, P_r(x)$ and $Q(X)$ in $\mathbb{C}[X]$, not all 0, such that $\deg(P_j) \leq p$, $\deg(Q) \leq r \cdot q$ and

$$Q(X)F_j(X) - P_j(X) = \mathcal{O}(X^{p+q+1}), \quad 1 \leq j \leq r.$$

The fractions P_j/Q are unique and are by definition the type II Padé approximants of $(F_j(X))_{j=1, \dots, r}$ of parameters (p, q) . When $r = 1$, we recover the ordinary Padé approximants $[p/q]$ of $F_1(X)$.

1.2. The results. The goal of this paper is to show that the RPA phenomenon occurs (in various generality) for the hypergeometric series

$$E_a(z) := {}_1F_1(1; a; z) = \sum_{k=0}^{\infty} \frac{z^k}{(a)_k} \quad \text{and} \quad {}_2F_0(b, 1; z) = \sum_{k=0}^{\infty} (b)_k z^k.$$

We assume that $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The former series generalizes the exponential function ${}_1F_1(1; 1; z) = \exp(z)$, while the latter generalizes Euler's series ${}_2F_0(1, 1; z) = \sum_{k=0}^{\infty} k! z^k$. The variable z is a formal one in Theorems 1 and 2 below. However, it is important to have in mind that, though the series $E_a(z)$ is an entire function of z , the ${}_2F_0(b, 1; z)$ diverges for every z . To overcome this divergence, we shall first work the function

$$\mathcal{E}_b(z) := \frac{1}{\Gamma(b)} \int_0^{\infty} \frac{u^{b-1}}{1-uz} e^{-u} du$$

defined for $\Re(b) > 0$ and analytic for $z \in \mathbb{C} \setminus [0, +\infty)$. It admits an asymptotic expansion $\widehat{\mathcal{E}}_b(z)$ as $z \rightarrow 0$ in any angular sector centered at 0 that does not contain $[0, +\infty)$. It turns out that $\widehat{\mathcal{E}}_b(z) = {}_2F_0(b, 1; z)$ in such sectors.

To state our results, we need to introduce a certain number of notations. The parameters $\alpha_j, j = 1, \dots, r$, are pairwise distinct non-zero complex numbers, and we set $\underline{\alpha} := (\alpha_j)_{j=1, \dots, r}$. For any $z \in \mathbb{C}$ and any $t \in \mathbb{C}$ such that $1/t \notin \mathbb{N}$, we define

$$\Phi_z(t) := \sum_{k=0}^{\infty} \frac{z^k}{(1 - 1/t)_k}.$$

For any $j = 1, \dots, r$ and any $n \geq 0$, we have the trivial identity

$$\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k} = \sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \Phi_{\alpha_j z} \left(\frac{-1}{a - 1 + n} \right). \quad (1.2)$$

In [6], it has been proved that, for any fixed $z \in \mathbb{C}$, $\Phi_z(t)$ admits an asymptotic expansion $\widehat{\Phi}_z(t) := \sum_{k=0}^{\infty} \varphi_k(-z)t^k$ as $t \rightarrow 0, t < 0$, where the coefficients $\varphi_k(z)$ are *Touchard exponential polynomials of degree k* . The latter are defined by the exponential generating function $e^{z(e^X - 1)} = \sum_{k=0}^{\infty} \varphi_k(z) \frac{X^k}{k!}$ (see [10]).

We provided in [6] explicit expressions for the type II Padé approximants at $t = 0$ of parameters $(rp - 1, p)$ for the formal power series $\widehat{\Phi}_{\alpha_j z}(t)$ in $\mathbb{C}[z][[t]]$ for $j = 1, \dots, r$: we denote them by $P_{j, \underline{\alpha}, p}(t, z)/Q_{\underline{\alpha}, p}(t, z)$, and the expressions of these polynomials in t are given in §2. It turns out that they are also polynomials in z .

We now “replace” $\Phi_{\alpha_j z} \left(\frac{-1}{a-1+n} \right)$ in (1.2) by the corresponding simultaneous Padé approximant of parameters $(rp - 1, p)$ of $\widehat{\Phi}_{\alpha_j z}(t)$ evaluated at $t = -1/(a - 1 + n)$ (this substitution is possible because $Q_{\underline{\alpha}, p}(-1/(a - 1 + n), z)$ does not vanish identically as a function of z). This alteration of (1.2) provides the formal Laurent series

$$R_{j, n, \underline{\alpha}, p}(z) := \sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k} - \sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} - \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j, \underline{\alpha}, p} \left(\frac{-1}{a-1+n}, z \right)}{Q_{\underline{\alpha}, p} \left(\frac{-1}{a-1+n}, z \right)}, \quad j = 1, \dots, r,$$

and we want to know in which sense they are ≈ 0 . By construction, for every j , the rational fraction

$$\sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j, \underline{\alpha}, p} \left(\frac{-1}{a-1+n}, z \right)}{Q_{\underline{\alpha}, p} \left(\frac{-1}{a-1+n}, z \right)} \quad (1.3)$$

is a RPA of $E_a(\alpha_j z)$. The degrees of its numerator and denominator are *a priori* bounded by $n + rp - 1$ and rp respectively. A better bound holds for the numerator ((i) below) and it is crucial to prove (ii). (Throughout the paper, $\mathcal{O}(z^c)$ denotes a Laurent series in z with order equal to c at $z = 0$.)

Theorem 1. *Let us fix the integers $r \geq 1, n \geq 1, p \geq 0$, such that $n \geq p, a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, \alpha_1, \dots, \alpha_r \in \mathbb{C}$ pairwise distinct.*

(i) The degrees of the numerator and denominator of the RPA in (1.3) are bounded by $n + (r - 1)p - 1$ and rp respectively, and moreover $R_{j,n,\alpha,p}(z) = \mathcal{O}(z^{n+rp})$.

(ii) The collection of the RPA in (1.3), $j = 1, \dots, r$, is the type II Padé approximant of $(E_a(\alpha_j z))_{j=1,\dots,r}$ of parameters $(n + (r - 1)p - 1, p)$.

(iii) For $r = 1$ and $\alpha_1 = 1$, (ii) gives the following equality of rational fractions:

$$\sum_{k=0}^{n-1} \frac{z^k}{(a)_k} + \frac{z^n}{(a)_n} ([p - 1/p]_{\widehat{\Phi}_z(t)})_{t=-1/(a-1+n)} = [n - 1/p]_{E_a(z)},$$

For a generic value of a , our proof needs the assumption that $n \geq p$. However, if $a = 1$, it can be removed in the statement of Theorem 1, and we simply recover the results from [6]. The explanation of this fact is Identity (2.5) in §2.

Let us now describe our results for the function $\mathcal{E}_b(z)$. For any $n \geq 0$, b such that $\Re(b) > 0$ and $z \in \mathbb{C} \setminus [0, +\infty)$, we have the identity (proved in §3)

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n \Psi_z \left(\frac{1}{b - 1 + n} \right) \quad (1.4)$$

where

$$\Psi_z(t) = \frac{1}{\Gamma(1 + 1/t)} \int_0^\infty \frac{u^{1/t}}{1 - uz} e^{-u} du$$

is defined for any $z \in \mathbb{C} \setminus (0, +\infty)$ and any t such that $\Re(1/t) > -1$ (in particular, for any $t > 0$). We shall prove in §3 that when $z < 0$, $\Psi_z(t)$ admits an asymptotic expansion as $t \rightarrow 0$, $t > 0$, given by (2)

$$\widehat{\Psi}_z(t) := -\frac{1}{z} \sum_{k=0}^{\infty} \varphi_k(1/z) t^{k+1} = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t). \quad (1.5)$$

It then immediately follows that the Padé approximant $[p/p]$ at $t = 0$ of the formal power series $\widehat{\Psi}_z(t) \in \mathbb{C}[1/z][[t]]$ is $-\frac{t}{z} \frac{P_{j,1,p}(t, -1/z)}{Q_{1,p}(t, -1/z)}$ where the polynomials are those used to construct the Padé approximants $[p - 1/p]$ of $\widehat{\Phi}_z(t)$ when $r = 1$.

In (1.4), we now “replace” $\mathcal{E}_b(z)$ by $\widehat{\mathcal{E}}_b(z)$, and $\Psi_z \left(\frac{1}{b-1+n} \right)$ by the Padé approximant $[p/p]$ of $\widehat{\Psi}_z(t)$ evaluated at $t = 1/(b - 1 + n)$ (this substitution is possible because $Q_{1,p}(1/(b - 1 + n), 1/z)$ does not vanish identically as a function of z). This alteration of (1.4) provides the formal Laurent series

$$S_{n,p}(z) := \widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k + \frac{(b)_n z^{n-1}}{(b - 1 + n)} \frac{P_{j,1,p}\left(\frac{1}{b-1+n}, -\frac{1}{z}\right)}{Q_{1,p}\left(\frac{1}{b-1+n}, -\frac{1}{z}\right)},$$

²This identity is reminiscent of André’s duality between solutions of E -operators at $z = 0$ and $z = \infty$; see [1]. The former involve function like $E_a(z)$, while the latter involve functions like $\widehat{\mathcal{E}}_b(1/z)$, with $a, b \in \mathbb{Q}$.

and we want to know in which sense it is ≈ 0 . Because of (1.4), the rational fraction

$$\sum_{k=0}^{n-1} (b)_k z^k - \frac{(b)_n z^{n-1}}{(b-1+n)} \frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})} \quad (1.6)$$

can be viewed as a RPA of $\mathcal{E}_b(z)$ in an extended sense. The degrees of its numerator and denominator are *a priori* bounded by $n+p-1$ and p respectively. Again, a better bound holds for the numerator ((i) below) and again it is crucial to prove (ii).

Theorem 2. *Let us fix $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and the integers $n \geq 1$, $p \geq 0$ such that $n \geq p$.*

(i) *The degrees of the numerator and denominator of the RPA in (1.6) are bounded by $n-1$ and p respectively, and moreover $S_{n,p}(z) = \mathcal{O}(z^{n+p})$ exists and is finite.*

(ii) *The RPA in (1.6) coincides with the Padé approximant $[n-1/p]$ of $\widehat{\mathcal{E}}_b(z)$. In other words,*

$$\sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\widehat{\Psi}_z(t)})_{t=1/(b-1+n)} = [n-1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

It seems that a RPA phenomenon similar to (ii) in Theorem 1 does not hold for any family $(\widehat{\mathcal{E}}_b(\beta_j))_{j=1,\dots,r}$ with $r \geq 2$.

The proofs of Theorems 1 and 2 are given in §2 and §3 respectively.

2. PROOF OF THEOREM 1

Proof of (i). We recall the expressions of the type II approximants of the functions $\widehat{\Phi}_{\alpha_j z}(t)$ (obtained in [6]):

$$P_{j,\underline{\alpha},p}(t, z) = (-t)^{rp-1} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (-1/t)_{k_1+\dots+k_r} \sum_{i=1}^{k_1+\dots+k_r} \frac{(\alpha_j z)^{i-1}}{(-1/t)_i}, \quad (2.1)$$

and

$$Q_{\underline{\alpha},p}(t, z) = (-t)^{rp} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (-1/t)_{k_1+\dots+k_r}. \quad (2.2)$$

The P 's have degree $rp-1$ in both t and z , while the Q 's have degree rp in both t and z . Moreover, $Q_{\underline{\alpha},p}(0, 1) = 1$.

For the sake of simplicity, we set $K_r = k_1 + k_2 + \dots + k_r$. Substituting $-1/(a-1+n)$ for t (which is possible because $a \notin \mathbb{Z}_{\leq 0}$ and $n \geq 1$), we get

$$Q_{\underline{\alpha},p}\left(\frac{-1}{a-1+n}, z\right) = (a+n-1)^{-rp} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r}$$

and

$$P_{j,\alpha,p}\left(\frac{-1}{a-1+n}, z\right) = \frac{(a-1+n)^{1-rp}(a)_{n-1}}{(\alpha_j z)^n} \\ \times \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} (S_{n+K_r-1}(\alpha_j z) - S_{n-1}(\alpha_j z))$$

where $S_n(z) = \sum_{k=0}^n z^k / (a)_k$. (Note that $Q_{\alpha,p}(\frac{-1}{a-1+n}, z)$ is clearly not identically zero, as stated in the Introduction.)

Hence, after some simplifications, the RPAs of the functions $E_{\alpha_j}(z)$, $j = 1, \dots, r$, are given by

$$\sum_{k=0}^{n-1} \frac{(\alpha_j z)^k}{(a)_k} + \frac{(\alpha_j z)^n}{(a)_n} \frac{P_{j,\alpha,p}(\frac{-1}{a-1+n}, z)}{Q_{\alpha,p}(\frac{-1}{a-1+n}, z)} \\ = \frac{\sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} S_{n+K_r-1}(\alpha_j z)}{\sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r}}. \quad (2.3)$$

At first sight, the numerator and denominator of this rational fraction have degrees bounded by $n + rp - 1$ and rp in z respectively, but a better estimate holds for the numerator. Indeed, we shall now prove that its degree is bounded by $n + (r-1)p - 1$. Indeed, with $K'_{j,r} = k_1 + \dots + k_{j-1} + k_{j+1} + \dots + k_r$ and $\underline{k}'_{j,r} = (k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r)$, the numerator of (2.3) can be expressed as

$$\sum_{k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_r=0}^p \left(\prod_{i=1, i \neq j}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K'_{j,r}} A_{j, \underline{k}'_{j,r}}(z) \quad (2.4)$$

with

$$A_{j, \underline{k}'_{j,r}}(z) = \sum_{k_j=0}^p \binom{p}{k_j} (-\alpha_j z)^{p-k_j} (a-1+n + K'_{j,r})_{k_j} S_{n+K_r-1}(\alpha_j z).$$

We claim that the degree in z of the polynomial $A_{j, \underline{k}'_{j,r}}(z)$ is at most $\max(p, n + K'_{j,r}) - 1$. To prove this observation, which is non-trivial, we first observe that, for any integers m, q such that $m \geq q \geq 0$,

$$(t^{a-1} S_m(xt)) \Big|_{t=1}^{(q)} = x^q S_{m-q}(x) + U_q(x) \quad (2.5)$$

where

$$U_q(x) := \sum_{j=0}^{q-1} (-1)^{q-j} (1-a)_{q-j} x^j.$$

(Above and below in this section, all the differentiations are with respect to t .) Observe that $U_q(x) \equiv 0$ if $a = 1$, but in general $U_q(x)$ has degree $q - 1$ in x . Since $(a - 1 + m)_k = (-1)^k (t^{-a+1-m})|_{t=1}^{(k)}$, it follows that

$$\begin{aligned} A_{j, \underline{k}_{j,r}}(z) &= (-1)^p \sum_{k=0}^p \binom{p}{k} (t^{-a+1-n-K'_{j,r}})|_{t=1}^{(k)} (t^{a-1} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p-k)} \\ &\quad + \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z) \\ &= (-1)^p (t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p)} \end{aligned} \quad (2.6)$$

$$+ \sum_{k=0}^p (-1)^{p-k} \binom{p}{k} (a-1+n+K'_{j,r})_k U_{p-k}(\alpha_j z). \quad (2.7)$$

The polynomial in (2.7) obviously has degree at most $p - 1$ in z . On the other hand

$$\begin{aligned} (t^{-n-K'_{j,r}} S_{n+K'_{j,r}+p-1}(\alpha_j z t))|_{t=1}^{(p)} &= \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} (t^{j-n-K'_{j,r}})|_{t=1}^{(p)} \\ &= (-1)^p \sum_{j=0}^{n+K'_{j,r}+p-1} \frac{(\alpha_j z)^j}{(a)_j} (n+K'_{j,r}-j)_p. \end{aligned}$$

Since $(n+K'_{j,r}-j)_p = 0$ for any $j \in \{n+K'_{j,r}, \dots, n+K'_{j,r}+p-1\}$, it follows that the polynomial in (2.6) has degree at most $n+K'_{j,r}-1$, so that the degree of $A_{j, \underline{k}_{j,r}}(z)$ is at most $\max(p, n+K'_{j,r}) - 1$ as claimed. Since $n \geq p$ by assumption, we deduce that the degree of the numerator of the RPAs given by (2.4) is at most $n+(r-1)p-1$, as expected.

To prove that $R_{j,n,\alpha,p}(z) = \mathcal{O}(z^{n+rp})$, we have to multiply the denominator of (2.3) by the series $\sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}$ and prove that the product is the sum of a polynomial of degree $n+(r-1)p-1$ and a function which is $\mathcal{O}(z^{n+rp})$. This amounts to proving that the coefficient c_m of z^m in the Taylor expansion of

$$\left(\sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i z)^{p-k_i} \right) (a-1+n)_{K_r} \right) \cdot \sum_{k=0}^{\infty} \frac{(\alpha_j z)^k}{(a)_k}$$

is zero for $m \in \{n+(r-1)p, \dots, n+rp-1\}$. Let us assume that $m \geq n+(r-1)p$. Then

$$c_m = \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i)^{p-k_i} \right) (a-1+n)_{K_r} \frac{\alpha_j^k}{(a)_k} \quad (2.8)$$

where in (2.8) the integer $k := m + K_r - rp$ is ≥ 0 (because $m \geq n+(r-1)p \geq rp$).

We now also assume that $m \leq n + rp - 1$. In that case, we use the identity

$$\frac{(a-1+n)_{K_r}}{(a)_k} = \frac{1}{(a)_{n-1}} \left(t^{a+n+K_r-2} \right)_{|t=1}^{(n+rp-m-1)}$$

in (2.8) and get

$$\begin{aligned} c_m &= \frac{1}{(a)_{n-1}} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (-\alpha_i)^{p-k_i} \right) \alpha_j^{m-rp+K_r} \left(t^{a+n+K_r-2} \right)_{|t=1}^{(n+rp-m-1)} \\ &= \frac{1}{(a)_{n-1}} \left(t^{a-1+n-rp} \prod_{i=1}^r (\alpha_j - \alpha_i t)^p \right)_{|t=1}^{(n+rp-m-1)}. \end{aligned}$$

Since $t = 1$ is a zero of order p of the product $\prod_{i=1}^r (\alpha_j - \alpha_i t)^p$, it follows that $c_m = 0$ when $n + rp - m - 1 \leq p - 1$, i.e. when $m \geq n + (r - 1)p$. This complete the proof of (i).

Proof of (ii). By unicity of type II Padé approximants, the bounds for the degrees of the simultaneous RPAs and the order of $R_{j,n,\alpha,p}$ at 0 ensure that (2.3) provides the expression of the type II Padé approximants of the family $(E_a(\alpha_j z))_{j=1, \dots, r}$ of parameters $(n + (r - 1)p - 1, p)$ when $n \geq p$.

3. PROOF OF THEOREM 2

Let b such that $\Re(b) > 0$. We first prove (1.4) of the Introduction, i.e. that for any $n \geq 0$ and any $z \in \mathbb{C} \setminus [0, +\infty)$,

$$\mathcal{E}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n \Psi_z \left(\frac{1}{b-1+n} \right).$$

This follows from

$$\begin{aligned} (b)_n z^n \Psi_z \left(\frac{1}{b-1+n} \right) &= \frac{(b)_n z^n}{\Gamma(b+n)} \int_0^\infty \frac{u^{b+n-1}}{1-uz} e^{-u} du \\ &= \frac{(b)_n}{\Gamma(b+n)} \int_0^\infty \frac{u^{b-1}}{1-uz} e^{-u} du + \frac{(b)_n}{\Gamma(b+n)} \int_0^\infty u^{b-1} \frac{(uz)^n - 1}{1-uz} e^{-u} du \\ &= \mathcal{E}_b(z) - \frac{(b)_n}{\Gamma(b+n)} \sum_{k=0}^{n-1} z^k \int_0^\infty u^{k+b-1} e^{-u} du \\ &= \mathcal{E}_b(z) - \sum_{k=0}^{n-1} \frac{(b)_n \Gamma(b+k)}{\Gamma(b+n)} z^k = \mathcal{E}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k. \end{aligned}$$

We now prove the following lemma.

Lemma 1. *Let us fix $z \in (-\infty, 0)$. The function $\Psi_z(t)$ admits an asymptotic expansion as $t \rightarrow 0, t > 0$, given by*

$$\widehat{\Psi}_z(t) := -\frac{1}{z} \sum_{k=0}^{\infty} \varphi_k(1/z) t^{k+1} = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t),$$

where the coefficients $\varphi_k(X)$ are Touchard exponential polynomials of degree k in X .

Proof. We first use Proposition 2 in [8] to get the following alternative expression for $\Psi_z(t)$: for any $z \in (-\infty, 0)$ and $t > 0$,

$$\Psi_z(t) = \int_0^\infty \frac{e^{-u}}{(1-zu)^{1+1/t}} du.$$

We now set $v = \ln(1 - uz)$, so that

$$\Psi_z(t) = -\frac{1}{z} \int_0^\infty e^{(e^v-1)/z} e^{-v/t} dv \quad (3.1)$$

for $z \in (-\infty, 0)$ and $t > 0$. Since $v \mapsto e^{(e^v-1)/z}$ is C^∞ at $v = 0$, non-zero at $v = 0$, and

$$\int_0^\infty |e^{(e^v-1)/z} e^{-v/t}| dv < \infty,$$

we can apply Watson's lemma ([4, Chapter 2, §2]) to (3.1): the function $\Psi_z(t)$ has the asymptotic expansion

$$\Psi_z(t) \sim -\frac{1}{z} \sum_{k=0}^\infty \left(\frac{d^k}{dv^k} e^{(e^v-1)/z} \right)_{|v=0} t^{k+1} = -\frac{t}{z} \sum_{k=0}^\infty \varphi_k(1/z) t^k, \quad t \rightarrow 0, t > 0.$$

This complete the proof of the lemma. \square

From now on, we view $\widehat{\Psi}_z(t)$ as a formal series in $\mathbb{C}[1/z][[t]]$. As observed in the Introduction, the identity $\widehat{\Psi}_z(t) = -\frac{t}{z} \widehat{\Phi}_{-1/z}(t)$ enables us to immediately get the Padé approximant $[p/p]$ of $\widehat{\Psi}_z(t)$ at $t = 0$. More generally, let $\underline{\beta} := (\beta_j)_{j=1, \dots, r}$ denote a family of pairwise distinct complex numbers. Then, without further efforts, we even obtain the type II Padé approximants at $t = 0$ of parameters (rp, p) of the formal power series $(\widehat{\Psi}_{\beta_j z}(t))_{j=1, \dots, r}$ for $j = 1, \dots, r$. They are given by

$$-\frac{t}{\beta_j z} \frac{P_{j,1/\underline{\beta},p}(t, -1/z)}{Q_{1/\underline{\beta},p}(t, -1/z)}, \quad j = 1, \dots, r$$

where

$$\begin{aligned} & P_{j,1/\underline{\beta},p}(t, -1/z) \\ &= (-t)^{rp-1} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1/t)_{k_1+\dots+k_r} \sum_{i=1}^{k_1+\dots+k_r} \frac{(-\beta_j z)^{1-i}}{(-1/t)_i}, \\ & Q_{1/\underline{\beta},p}(t, -1/z) = (-t)^{rp} \sum_{k_1, \dots, k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1/t)_{k_1+\dots+k_r}. \end{aligned}$$

We now substitute $1/(b-1+n)$ for t in these expressions, which is possible because $\Re(b) > 0$ and $n \geq 1$. Since $Q_{1/\underline{\beta},p}(1/(b-1+n), -1/z)$ is clearly not identically zero, we

get simultaneous RPAs of the family $(\mathcal{E}_b(\beta_j z))_{j=1,\dots,r}$ given by the formulas:

$$\begin{aligned} \sum_{k=0}^{n-1} (b)_k (\beta_j z)^k - (b)_n \frac{(\beta_j z)^{n-1}}{(b-1+n)} \frac{P_{j,1/\beta,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1/\beta,p}(\frac{1}{b-1+n}, -\frac{1}{z})} \\ = \frac{z^{rp} \sum_{k_1,\dots,k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1)^{K_r} T_{n-K_r-1}(\beta_j z) / (b)_{n-K_r}}{z^{rp} \sum_{k_1,\dots,k_r=0}^p \left(\prod_{i=1}^r \binom{p}{k_i} (\beta_i z)^{k_i-p} \right) (-1)^{K_r} / (b)_{n-K_r}}. \end{aligned} \quad (3.2)$$

where $T_n(z) := \sum_{k=0}^n (b)_k z^k$.

We are now in position to complete the proof of Theorem 2. We take $r = 1$ and $\beta_1 = 1$ in (3.2).

Proof of (i). It is immediate that the degrees of the numerator and denominator of the rational function in (3.2) are bounded by $n - 1$ and p respectively. We now want to prove that, formally,

$$\widehat{\mathcal{E}}_b(z) - \sum_{k=0}^{n-1} (b)_k z^k + \frac{(b)_n z^{n-1}}{(b-1+n)} \frac{P_{j,1,p}(\frac{1}{b-1+n}, -\frac{1}{z})}{Q_{1,p}(\frac{1}{b-1+n}, -\frac{1}{z})} = \mathcal{O}(z^{n+p}). \quad (3.3)$$

The denominator of the right-hand side of (3.2) is

$$Q(z) := \sum_{k=0}^p \frac{(-1)^k \binom{p}{k}}{(b)_{n-k}} z^k \quad (3.4)$$

so that to prove (3.3) it is enough to prove that the formal series expansion of

$$\left(\sum_{k=0}^p \frac{(-1)^k \binom{p}{k}}{(b)_{n-k}} z^k \right) \widehat{\mathcal{E}}_b(z)$$

is the sum of a polynomial $P(z)$ of degree $\leq n - 1$ and a formal power series of order $\geq n + p$ at $z = 0$. This in fact means that we want to prove that $P(z)/Q(z)$ is the $[n - 1/p]$ Padé approximant of $\widehat{\mathcal{E}}_b(z)$. This is indeed the case because we recognize $Q(z)$ in (3.4) as being of the form given in [9] and unicity of Padé approximant enables us to conclude.

Proof of (ii). This is simply a summary of what was proved in (i). We have

$$\widehat{\mathcal{E}}_b(z) = \sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} + \mathcal{O}(z^{n+p}) \quad (3.5)$$

and the bounds for the degrees show that

$$\sum_{k=0}^{n-1} (b)_k z^k + (b)_n z^n ([p/p]_{\Psi_z(t)})_{|t=1/(b-1+n)} = [n - 1/p]_{\widehat{\mathcal{E}}_b(z)}.$$

The above results are *a priori* proved under the assumption that $\Re(b) > 0$. They hold under the weaker assumption that $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ because all these identities between rational fractions extend to this case.

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