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# ON LEIBNIZ COHOMOLOGY

JÖRG FELDVOSS AND FRIEDRICH WAGEMANN

ABSTRACT. In this paper we prove the Leibniz analogues of several vanishing theorems for the Chevalley-Eilenberg cohomology of Lie algebras. In particular, we obtain the second Whitehead lemma for Leibniz algebras. Our main tools are three spectral sequences. Two are Leibniz analogues of the Hochschild-Serre spectral sequence, one of which is an extension of the dual of a spectral sequence of Pirashvili for Leibniz homology from symmetric bimodules to arbitrary bimodules, and the other one is due to Beaudouin. A third spectral sequence (also due to Pirashvili in homology) relates the Leibniz cohomology of a Lie algebra to its Chevalley-Eilenberg cohomology.

## INTRODUCTION

In [1], the authors study the cohomology of semi-simple Leibniz algebras, i.e., the cohomology of finite-dimensional Leibniz algebras  $\mathfrak{L}$  with an ideal of squares  $\text{Leib}(\mathfrak{L})$  such that the corresponding canonical Lie algebra  $\mathfrak{g} := \mathfrak{L}/\text{Leib}(\mathfrak{L})$  is semi-simple, and conjecture that  $\text{HL}^2(\mathfrak{L}, \mathfrak{L}_{\text{ad}}) = 0$ . In [18], the authors determine the deviation of the second Leibniz cohomology of a complex Lie algebra with adjoint or trivial coefficients from the corresponding Chevalley-Eilenberg cohomology. With these motivations, we transpose in the present article systematically Pirashvili's results to cohomology, extend them and prove that we have  $\text{HL}^2(\mathfrak{L}, \mathfrak{L}_{\text{ad}}) = 0$  for a semi-simple Leibniz algebra  $\mathfrak{L}$  over a field of characteristic zero (see Corollary 4.6).

The main theme of the present paper is the correct analogue of the Hochschild-Serre spectral sequence for Leibniz algebras and its applications. Recall that the Hochschild-Serre spectral sequence [21, Sections 2 and 3] for a Lie algebra extension

$$0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{g} \rightarrow \mathfrak{q} \rightarrow 0$$

arises from a filtration of the standard cochain complex of  $\mathfrak{g}$  by cochains which vanish in case the first  $p$  arguments contain an element of  $\mathfrak{k}$ . When trying to generalize the Hochschild-Serre spectral sequence from Lie algebras to Leibniz algebras, one notices that on the one hand there are two ways to generalize the filtration from skew-symmetric cochains to all multilinear cochains, namely, by filtering from the left or from the right, and on the other hand that the resulting spectral sequences (see Theorems 3.2 and 3.5, the latter being an extension of the cohomological version of a homology spectral sequence of Pirashvili [33, Theorem C] from symmetric bimodules to arbitrary bimodules, while the former is due to Beaudouin [6,

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Théorème 3.2.16]) are less powerful than in the case of Lie algebra cohomology. Our belief is that the correct replacement of the Hochschild-Serre spectral sequence in the case of Leibniz algebras is not *one* spectral sequence, but the *two* spectral sequences of Theorems 3.2 and 3.5.

The main applications of the Hochschild-Serre spectral sequence for Lie algebras contain a complete description of the Chevalley-Eilenberg cohomology of semi-simple Lie algebras (see [21]) as well as vanishing theorems for the Chevalley-Eilenberg cohomology of nilpotent Lie algebras (see [13, [Théorème 1]]) and solvable Lie algebras with values in certain irreducible modules (see [3, Theorems 2 and 3]). As applications of the spectral sequences in Theorems 3.2 and 3.5, we generalize these results to Leibniz algebras. The method of proof for these theorems relies on the corresponding result for Lie algebras and works roughly as follows. The Leibniz algebra  $\mathfrak{L}$  and the  $\mathfrak{L}$ -bimodule  $M$  must be chosen such that the canonical Lie algebra  $\mathfrak{g} := \mathfrak{L}/\text{Leib}(\mathfrak{L})$  associated to  $\mathfrak{L}$  and the corresponding left  $\mathfrak{g}$ -module  $M$  satisfy the hypotheses of the relevant vanishing theorem for Lie algebras. Therefore we have vanishing Chevalley-Eilenberg cohomology  $H^\bullet(\mathfrak{g}, M)$ . An entirely Lie-theoretical spectral sequence (see Theorem 2.5) and the long exact sequence comparing Chevalley-Eilenberg and Leibniz cohomology of a Lie algebra for symmetric Leibniz bimodules (see Proposition 2.2) show that these cohomologies coincide. Thus the vanishing of  $H^\bullet(\mathfrak{g}, M)$  implies that of  $\text{HL}^\bullet(\mathfrak{g}, M)$ . The spectral sequence of Theorem 3.2 has  $\text{HL}^\bullet(\mathfrak{g}, M)$  in its  $E_2$ -term and converges to the relative cohomology which therefore vanishes. As a consequence of the long exact sequence linking the cohomologies of  $\mathfrak{L}$  and  $\mathfrak{g}$  (see Proposition 3.1),  $\text{HL}^\bullet(\mathfrak{L}, M)$  vanishes for symmetric  $\mathfrak{L}$ -bimodules  $M$ . Then a reduction of the Leibniz cohomology with coefficients in anti-symmetric bimodules to the Leibniz cohomology with coefficients in certain symmetric bimodules (see Lemma 1.5) in conjunction with the previous argument yields the vanishing of  $\text{HL}^\bullet(\mathfrak{L}, M)$  for anti-symmetric  $\mathfrak{L}$ -bimodules  $M$ . Finally, the general case can be obtained from the symmetric and the anti-symmetric case by applying the long exact cohomology sequence to the short exact sequence  $0 \rightarrow M_0 \rightarrow M \rightarrow M_{\text{sym}} \rightarrow 0$ , where  $M_0$  denotes an anti-symmetric sub-bimodule of the Leibniz bimodule  $M$  such that  $M_{\text{sym}} := M/M_0$  is symmetric. This method of proof is dual to the one used for Leibniz homology by Pirashvili (see the proof of Propositions 2.1 and 4.3 in [33]; note that Pirashvili considers only symmetric bimodules).

The main outcome are the following theorems. The Whitehead vanishing theorem for Leibniz cohomology (see Theorem 4.1) implies the following complete description of the Leibniz cohomology of a semi-simple Leibniz algebra with coefficients in a finite-dimensional bimodule (see Theorem 4.2):

**Theorem 1.** *Let  $\mathfrak{L}$  be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, and let  $M$  be a finite-dimensional  $\mathfrak{L}$ -bimodule. Then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 2$ , and there is a five-term exact sequence*

$$0 \rightarrow M_0 \rightarrow \text{HL}^0(\mathfrak{L}, M) \rightarrow M_{\text{sym}}^{\text{Lie}} \rightarrow \text{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\text{ad}, \ell}, M_0) \rightarrow \text{HL}^1(\mathfrak{L}, M) \rightarrow 0.$$

*Moreover, if  $M$  is symmetric, then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 1$ .*

Note that the case  $n = 2$  of Theorem 1 is the second Whitehead lemma for Leibniz algebras. We deduce in particular the rigidity of semi-simple Leibniz algebras in characteristic zero (see Corollary 4.6), show in examples that Theorem 1 fails in non-zero characteristic or for infinite-dimensional modules (see Examples A and B,

respectively), and prove that a semi-simple non-Lie Leibniz algebra in characteristic zero always possesses a non-inner derivation (see Corollary 4.5).

Applying the spectral sequence of Theorem 3.2 to the Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \text{Rad}(\mathfrak{L})$  of a Leibniz algebra  $\mathfrak{L}$  into a hemi-semidirect product of a semi-simple Lie algebra  $\mathfrak{s}$  and its radical  $\text{Rad}(\mathfrak{L})$  (see the main result of [4]), we obtain a reduction formula for the Leibniz cohomology of  $\mathfrak{L}$ , which can be seen as an analogue of the formula describing the Chevalley-Eilenberg cohomology of a finite-dimensional Lie algebra in [21, Theorem 13]. Our formula is less general as it takes only into account Leibniz cohomology with values in symmetric  $\mathfrak{s}$ -bimodules, or in other words, the radical  $\text{Rad}(\mathfrak{L})$  has to act trivially on the symmetric  $\mathfrak{L}$ -bimodule  $M$ . Under this assumption, we have the following result:

**Theorem 2.** *Let  $\mathfrak{L}$  be a finite-dimensional left Leibniz algebra with Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \text{Rad}(\mathfrak{L})$ , and let  $M$  be a symmetric  $\mathfrak{L}$ -bimodule such that the radical  $\text{Rad}(\mathfrak{L})$  acts trivially on  $M$ . Then for every non-negative integer  $n$  there are isomorphisms*

$$\text{HL}^n(\mathfrak{L}, M) \cong \begin{cases} M^{\mathfrak{s}} & \text{if } n = 0 \\ \text{HL}^{n-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\text{Rad}(\mathfrak{L}), M^{\mathfrak{s}})_s) & \text{if } n \geq 1 \end{cases}$$

of vector spaces.

Applying now Dixmier's theorem [13, Théorème 1] instead of Whitehead's theorem, we derive by the same method of proof a vanishing theorem for the Leibniz cohomology of nilpotent Leibniz algebras (see Theorem 5.1).

**Theorem 3.** *Let  $\mathfrak{L}$  be a left Leibniz algebra such that  $\mathfrak{L}_{\text{Lie}} := \mathfrak{L}/\text{Leib}(\mathfrak{L})$  is a finite-dimensional nilpotent Lie algebra. If  $M$  is a finite-dimensional  $\mathfrak{L}$ -bimodule such that  $M^{\mathfrak{L}} = 0$ , then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

The Leibniz cohomology of an abelian Lie algebra with trivial coefficients is known. In Example C we compute the Leibniz cohomology of the trivial bimodule for the smallest non-abelian nilpotent Lie algebra, namely, the three-dimensional Heisenberg algebra (cf. [33, Example 1.4. iv]) for the corresponding Leibniz homology), and in Example D we determine this cohomology for the smallest nilpotent non-Lie Leibniz algebra.

Using Barnes' vanishing theorem [3, Theorem 3], we obtain a vanishing theorem for the cohomology of supersolvable Leibniz algebras (see Theorem 5.3) whose natural setting is a field  $\mathbb{F}$  of characteristic  $p > 0$ .

**Theorem 4.** *Let  $\mathfrak{L}$  be a finite-dimensional supersolvable left Leibniz algebra over a field  $\mathbb{F}$ , and let  $M$  be a finite-dimensional irreducible  $\mathfrak{L}$ -bimodule such that  $\dim_{\mathbb{F}} M \neq 1$ . Then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every positive integer  $n$ . Moreover, if  $M$  is symmetric, then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

As a first indication what to expect for the Leibniz cohomology of supersolvable Leibniz algebras with coefficients in one-dimensional bimodules, in Example E we compute this Leibniz cohomology for the non-abelian two-dimensional Lie algebra and in Example F for the non-nilpotent two-dimensional non-Lie left Leibniz algebra.

Our observation is that Leibniz cohomology of Lie algebras is much more regular than Chevalley-Eilenberg cohomology. This is one reason Leibniz cohomology is

maybe more interesting. Future work in this direction is meant to illustrate our intuition.

In this paper we will follow the notation used in [17]. All tensor products are over the relevant ground field and will be denoted by  $\otimes$ . For a subset  $X$  of a vector space  $V$  over a field  $\mathbb{F}$  we let  $\langle X \rangle_{\mathbb{F}}$  be the subspace of  $V$  spanned by  $X$ . We will denote the space of linear transformations from an  $\mathbb{F}$ -vector space  $V$  to an  $\mathbb{F}$ -vector space  $W$  by  $\text{Hom}_{\mathbb{F}}(V, W)$ . In particular,  $V^* := \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$  will be the space of linear forms on a vector space  $V$  over a field  $\mathbb{F}$ . Moreover,  $S^2(V)$  will denote the symmetric square of a vector space  $V$ . Finally, the identity function on a set  $X$  will be denoted by  $\text{id}_X$ , and the set  $\{0, 1, 2, \dots\}$  of non-negative integers will be denoted by  $\mathbb{N}_0$ .

## 1. PRELIMINARIES

In this section we recall some definitions and collect several results that will be useful in the remainder of the paper.

A *left Leibniz algebra* is an algebra  $\mathfrak{L}$  such that every left multiplication operator  $L_x : \mathfrak{L} \rightarrow \mathfrak{L}, y \mapsto xy$  is a derivation. This is equivalent to the identity

$$(1.1) \quad x(yz) = (xy)z + y(xz)$$

for all  $x, y, z \in \mathfrak{L}$ , which in turn is equivalent to the identity

$$(1.2) \quad (xy)z = x(yz) - y(xz)$$

for all  $x, y, z \in \mathfrak{L}$ . We will call both identities the *left Leibniz identity*. There is a similar definition of a *right Leibniz algebra* but in this paper we will only consider left Leibniz algebras.

Every left Leibniz algebra has an important ideal, its Leibniz kernel, that measures how much the Leibniz algebra deviates from being a Lie algebra. Namely, let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ . Then

$$\text{Leib}(\mathfrak{L}) := \langle x^2 \mid x \in \mathfrak{L} \rangle_{\mathbb{F}}$$

is called the *Leibniz kernel* of  $\mathfrak{L}$ . The Leibniz kernel  $\text{Leib}(\mathfrak{L})$  is an abelian ideal of  $\mathfrak{L}$ , and  $\text{Leib}(\mathfrak{L}) \neq \mathfrak{L}$  when  $\mathfrak{L} \neq 0$  (see [17, Proposition 2.20]). Moreover,  $\mathfrak{L}$  is a Lie algebra if, and only if,  $\text{Leib}(\mathfrak{L}) = 0$ .

By definition of the Leibniz kernel,  $\mathfrak{L}_{\text{Lie}} := \mathfrak{L}/\text{Leib}(\mathfrak{L})$  is a Lie algebra which we call the *canonical Lie algebra* associated to  $\mathfrak{L}$ . In fact, the Leibniz kernel is the smallest ideal such that the corresponding factor algebra is a Lie algebra (see [17, Proposition 2.22]).

Next, we will briefly discuss left modules and bimodules of left Leibniz algebras. Let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ . A *left  $\mathfrak{L}$ -module* is a vector space  $M$  over  $\mathbb{F}$  with an  $\mathbb{F}$ -bilinear left  $\mathfrak{L}$ -action  $\mathfrak{L} \times M \rightarrow M, (x, m) \mapsto x \cdot m$  such that

$$(1.3) \quad (xy) \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m)$$

is satisfied for every  $m \in M$  and all  $x, y \in \mathfrak{L}$ .

By virtue of [17, Lemma 3.3], every left  $\mathfrak{L}$ -module is an  $\mathfrak{L}_{\text{Lie}}$ -module, and vice versa. This is the reason that in [6, Définition 1.1.14] left Leibniz modules are called Lie modules. Consequently, many properties of left Leibniz modules follow from the corresponding properties of modules for the canonical Lie algebra.

The correct concept of a module for a left Leibniz algebra  $\mathfrak{L}$  is the notion of a Leibniz bimodule. An  $\mathfrak{L}$ -bimodule is a left  $\mathfrak{L}$ -module  $M$  with an  $\mathbb{F}$ -bilinear right  $\mathfrak{L}$ -action  $M \times \mathfrak{L} \rightarrow M$ ,  $(m, x) \mapsto m \cdot x$  such that

$$(1.4) \quad (x \cdot m) \cdot y = x \cdot (m \cdot y) - m \cdot (xy)$$

and

$$(1.5) \quad (m \cdot x) \cdot y = m \cdot (xy) - x \cdot (m \cdot y)$$

are satisfied for every  $m \in M$  and all  $x, y \in \mathfrak{L}$  (see [17, Section 3] for the motivation behind this definition of a bimodule for a left Leibniz algebra).

The usual definitions of the notions of *sub-(bi)module*, *irreducibility*, *complete reducibility*, *composition series*, *homomorphism*, *isomorphism*, etc., hold for left Leibniz modules and Leibniz bimodules.

Let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ , and let  $M$  be an  $\mathfrak{L}$ -bimodule. Then  $M$  is said to be *symmetric* if  $m \cdot x = -x \cdot m$  for every  $x \in \mathfrak{L}$  and every  $m \in M$ , and  $M$  is said to be *anti-symmetric* if  $m \cdot x = 0$  for every  $x \in \mathfrak{L}$  and every  $m \in M$ . We call

$$M_0 := \langle x \cdot m + m \cdot x \mid x \in \mathfrak{L}, m \in M \rangle_{\mathbb{F}}$$

the *anti-symmetric kernel* of  $M$ . It is known that  $M_0$  is an anti-symmetric  $\mathfrak{L}$ -sub-bimodule of  $M$  (see [17, Proposition 3.12]) such that  $M/M_0$  is symmetric (see [17, Proposition 3.13]).

Moreover, for any  $\mathfrak{L}$ -bimodule  $M$  we will need its *space of right  $\mathfrak{L}$ -invariants*

$$M^{\mathfrak{L}} := \{m \in M \mid \forall x \in \mathfrak{L} : m \cdot x = 0\}$$

and the *annihilator*

$$\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M) := \{x \in \mathfrak{L} \mid \forall m \in M : x \cdot m = 0 = m \cdot x\}$$

Our first result will be useful in Sections 4 and 5.

**Lemma 1.1.** *Let  $\mathfrak{L}$  be a left Leibniz algebra, and let  $M$  be an  $\mathfrak{L}$ -bimodule such that  $M^{\mathfrak{L}} = 0$ . Then  $M$  is symmetric. In particular,  $\text{Leib}(\mathfrak{L}) \subseteq \text{Ann}_{\mathfrak{L}}^{\text{bi}}(M)$ .*

*Proof.* Since  $M_0$  is anti-symmetric, it follows from the hypothesis that

$$M_0 = M_0^{\mathfrak{L}} \subseteq M^{\mathfrak{L}} = 0.$$

Hence we obtain from the definition of  $M_0$  that  $M$  is symmetric. The second part is then an immediate consequence of [17, Lemma 3.10].  $\square$

It is clear from the definition of  $M^{\mathfrak{L}}$  that an  $\mathfrak{L}$ -bimodule  $M$  is anti-symmetric if, and only if,  $M^{\mathfrak{L}} = M$ . We will use Lemma 1.1 to show that the symmetry of non-trivial irreducible Leibniz bimodules can also be characterized by the behavior of their spaces of right invariants. As a preparation for this, we need to know that the latter space is a sub-bimodule.

**Lemma 1.2.** *Let  $\mathfrak{L}$  be a left Leibniz algebra, and let  $M$  be an  $\mathfrak{L}$ -bimodule. Then  $M^{\mathfrak{L}}$  is a sub-bimodule of  $M$ .*

*Proof.* It follows from (1.4) that  $M^{\mathfrak{L}}$  is invariant under the left action on  $M$ , and it follows from (1.5) that  $M^{\mathfrak{L}}$  is invariant under the right action on  $M$ .  $\square$

Now we can characterize the symmetry of a non-trivial irreducible Leibniz bimodule by the vanishing of its space of right invariants. In particular, for non-trivial irreducible Leibniz bimodules we obtain the converse of Lemma 1.1. (Recall that an irreducible bimodule  $M$  is a bimodule that has exactly two sub-bimodules, namely, 0 and  $M$ . In particular, an irreducible bimodule is by definition a non-zero vector space.)

**Corollary 1.3.** *Let  $\mathfrak{L}$  be a left Leibniz algebra, and let  $M$  be an irreducible  $\mathfrak{L}$ -bimodule. Then  $M$  is symmetric with non-trivial  $\mathfrak{L}$ -action if, and only if,  $M^{\mathfrak{L}} = 0$ .*

*Proof.* Since  $M$  is irreducible, we obtain from Lemma 1.2 that  $M^{\mathfrak{L}} = 0$  or  $M^{\mathfrak{L}} = M$ . Suppose first that  $M$  is symmetric with non-trivial  $\mathfrak{L}$ -action. Then we have that  $M^{\mathfrak{L}} = 0$ . On the other hand, the converse follows immediately from Lemma 1.1.  $\square$

We propose to define the faithfulness of a Leibniz bimodule as follows. We call an  $\mathfrak{L}$ -bimodule  $M$  *faithful* if  $\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M) \subseteq \text{Leib}(\mathfrak{L})$ . Note that this is a more general notion than the one used by Barnes in [5], and both definitions generalize this concept from Lie algebras to Leibniz algebras.

For a faithful  $\mathfrak{L}$ -bimodule  $M$ , the chain of inclusions

$$\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M) \subseteq \text{Leib}(\mathfrak{L}) \subseteq \text{Ann}_{\mathfrak{L}}(M)$$

follows from [17, Lemma 3.3], where

$$\text{Ann}_{\mathfrak{L}}(M) := \{x \in \mathfrak{L} \mid \forall m \in M : x \cdot m = 0\}$$

is the *annihilator* of the left  $\mathfrak{L}$ -module  $M$ . Note that for a completely reducible faithful Leibniz bimodule these inclusions are equalities.

**Lemma 1.4.** *Let  $\mathfrak{L}$  be a left Leibniz algebra. If  $M$  is completely reducible faithful  $\mathfrak{L}$ -bimodule, then  $\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M) = \text{Leib}(\mathfrak{L}) = \text{Ann}_{\mathfrak{L}}(M)$ .*

*Proof.* Since  $M$  is completely reducible, we have that  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$  is an irreducible  $\mathfrak{L}$ -bimodule for every  $i \in I$ . According to [17, Theorem 3.14], irreducible  $\mathfrak{L}$ -bimodules are symmetric or anti-symmetric, and therefore we obtain from the proof of [17, Lemma 3.10] that  $\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M_i) = \text{Leib}(\mathfrak{L}) = \text{Ann}_{\mathfrak{L}}(M_i)$  for every  $i \in I$ . Hence we conclude that

$$\text{Ann}_{\mathfrak{L}}^{\text{bi}}(M) = \bigcap_{i \in I} \text{Ann}_{\mathfrak{L}}^{\text{bi}}(M_i) = \text{Leib}(\mathfrak{L}) = \bigcap_{i \in I} \text{Ann}_{\mathfrak{L}}(M_i) = \text{Ann}_{\mathfrak{L}}(M).$$

This completes the proof.  $\square$

Recall that every left  $\mathfrak{L}$ -module  $M$  of a left Leibniz algebra  $\mathfrak{L}$  determines a unique symmetric  $\mathfrak{L}$ -bimodule structure on  $M$  by defining  $m \cdot x := -x \cdot m$  for every element  $m \in M$  and every element  $x \in \mathfrak{L}$  (see [17, Proposition 3.15 (b)]). We will denote this symmetric  $\mathfrak{L}$ -bimodule by  $M_s$ . Similarly, every left  $\mathfrak{L}$ -module  $M$  with trivial right action is an anti-symmetric  $\mathfrak{L}$ -bimodule (see [17, Proposition 3.15 (a)]). We will denote this module by  $M_a$ .

Similar to the boundary map in [29] for the homology of a right Leibniz algebra with coefficients in a right module one can also introduce a coboundary map  $\tilde{d}^\bullet$  for the cohomology of a left Leibniz algebra with coefficients in a left module as follows.

Let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ , and let  $M$  be a left  $\mathfrak{L}$ -module. For any non-negative integer  $n$  set  $\text{CL}^n(\mathfrak{L}, M) := \text{Hom}_{\mathbb{F}}(\mathfrak{L}^{\otimes n}, M)$  and consider the linear transformation  $\tilde{d}^n : \text{CL}^n(\mathfrak{L}, M) \rightarrow \text{CL}^{n+1}(\mathfrak{L}, M)$  defined by

$$\begin{aligned} (\tilde{d}^n f)(x_1, \dots, x_{n+1}) &:= \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1}) \end{aligned}$$

for any  $f \in \text{CL}^n(\mathfrak{L}, M)$  and all elements  $x_1, \dots, x_{n+1} \in \mathfrak{L}$ . (Here and in the remainder of the paper we identify the tensor power  $\mathfrak{L}^{\otimes n}$  with the corresponding Cartesian power.)

Now let  $M$  be an  $\mathfrak{L}$ -bimodule and for any non-negative integer  $n$  consider the linear transformation  $d^n : \text{CL}^n(\mathfrak{L}, M) \rightarrow \text{CL}^{n+1}(\mathfrak{L}, M)$  defined by

$$\begin{aligned} (d^n f)(x_1, \dots, x_{n+1}) &:= \sum_{i=1}^n (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ (-1)^{n+1} f(x_1, \dots, x_n) \cdot x_{n+1} \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1}) \end{aligned}$$

for any  $f \in \text{CL}^n(\mathfrak{L}, M)$  and all elements  $x_1, \dots, x_{n+1} \in \mathfrak{L}$ .

It is proved in [12, Lemma 1.3.1] that  $\text{CL}^\bullet(\mathfrak{L}, M) := (\text{CL}^n(\mathfrak{L}, M), d^n)_{n \in \mathbb{N}_0}$  is a cochain complex, i.e.,  $d^{n+1} \circ d^n = 0$  for every non-negative integer  $n$ . Of course, the original idea of defining Leibniz cohomology as the cohomology of such a cochain complex for right Leibniz algebras is due to Loday [30, Section 1.8]. Hence one can define the *cohomology of  $\mathfrak{L}$  with coefficients in an  $\mathfrak{L}$ -bimodule  $M$*  by

$$\text{HL}^n(\mathfrak{L}, M) := \text{H}^n(\text{CL}^\bullet(\mathfrak{L}, M)) := \text{Ker}(d^n) / \text{Im}(d^{n-1})$$

for every non-negative integer  $n$ . (Note that  $d^{-1} := 0$ .)

If  $M$  is a symmetric  $\mathfrak{L}$ -bimodule, then we have the identity  $\tilde{d}^n = d^n$  for any non-negative integer  $n$ . Namely,

$$\begin{aligned} (\tilde{d}^n f)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^n (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ (-1)^{n+2} x_{n+1} \cdot f(x_1, \dots, x_n) \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_{n+1}) \\ &= (d^n f)(x_1, \dots, x_{n+1}) \end{aligned}$$

for any  $f \in \text{CL}^n(\mathfrak{L}, M)$  and all elements  $x_1, \dots, x_{n+1} \in \mathfrak{L}$ . In particular, as mentioned earlier, any left  $\mathfrak{L}$ -module  $M$  can be turned into a symmetric  $\mathfrak{L}$ -module  $M_s$ , and the fact that  $d^\bullet$  is a coboundary map for  $\text{CL}^\bullet(\mathfrak{L}, M_s)$  shows that  $\widetilde{\text{CL}}^\bullet(\mathfrak{L}, M) := (\text{CL}^n(\mathfrak{L}, M), \tilde{d}^n)_{n \in \mathbb{N}_0}$  is a cochain complex, i.e.,  $\tilde{d}^{n+1} \circ \tilde{d}^n = 0$  for every non-negative integer  $n$ . Hence one can define the *cohomology of  $\mathfrak{L}$  with coefficients in a left  $\mathfrak{L}$ -module  $M$*  by

$$\widetilde{\text{HL}}^n(\mathfrak{L}, M) := \text{H}^n(\widetilde{\text{CL}}^\bullet(\mathfrak{L}, M)) := \text{Ker}(\tilde{d}^n) / \text{Im}(\tilde{d}^{n-1})$$

for every non-negative integer  $n$ . (Note that as before  $\tilde{d}^{-1} := 0$ .)

Now we are ready to state the next result (see [33, Lemma 2.2] for the analogous result in Leibniz homology) whose second part generalizes [17, Corollary 4.4 (b)] to arbitrary degrees and which will be crucial in Sections 4 and 5. (Note that the first part has already been obtained in [12, Section 1.3.4]). For the convenience of the reader we include a detailed proof.

**Lemma 1.5.** *Let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ , and let  $M$  be a left  $\mathfrak{L}$ -module. Then the following statements hold:*

(a) *If  $M$  is considered as a symmetric  $\mathfrak{L}$ -bimodule  $M_s$ , then*

$$\mathrm{HL}^n(\mathfrak{L}, M_s) = \widetilde{\mathrm{HL}}^n(\mathfrak{L}, M)$$

*for every integer  $n \geq 0$ .*

(b) *If  $M$  is considered as an anti-symmetric  $\mathfrak{L}$ -bimodule  $M_a$ , then*

$$\mathrm{HL}^0(\mathfrak{L}, M_a) = M$$

*and*

$$\mathrm{HL}^n(\mathfrak{L}, M_a) \cong \widetilde{\mathrm{HL}}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)) = \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s)$$

*for every integer  $n \geq 1$ , where  $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$  is a left  $\mathfrak{L}$ -module via*

$$(x \cdot f)(y) := x \cdot f(y) - f(xy)$$

*for every  $f \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$  and any elements  $x, y \in \mathfrak{L}$ .*

*Proof.* By virtue of the computation before Lemma 1.5, we only need to prove part (b). Note that the first part of (b) is just [17, Corollary 4.2 (b)].

First, we show that  $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$  is a left  $\mathfrak{L}$ -module via the given action. Let  $f \in \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)$  and  $x, y, z \in \mathfrak{L}$  be arbitrary. Then we obtain from the defining identity of a left Leibniz module (1.3) and the left Leibniz identity (1.2) that

$$\begin{aligned} ((xy) \cdot f)(z) &= (xy) \cdot f(z) - f((xy)z) \\ &= x \cdot (y \cdot f(z)) - y \cdot (x \cdot f(z)) - f(x(yz)) + f(y(xz)), \end{aligned}$$

and

$$\begin{aligned} (x \cdot (y \cdot f))(z) &= x \cdot (y \cdot f)(z) - (y \cdot f)(xz) \\ &= x \cdot (y \cdot f(z)) - x \cdot f(yz) - y \cdot f(xz) + f(y(xz)), \end{aligned}$$

as well as

$$\begin{aligned} (y \cdot (x \cdot f))(z) &= y \cdot (x \cdot f)(z) - (x \cdot f)(yz) \\ &= y \cdot (x \cdot f(z)) - y \cdot f(xz) - x \cdot f(yz) + f(x(yz)). \end{aligned}$$

Hence  $((xy) \cdot f)(z) = (x \cdot (y \cdot f))(z) - (y \cdot (x \cdot f))(z)$  for every  $z \in \mathfrak{L}$ , or equivalently,  $(xy) \cdot f = x \cdot (y \cdot f) - y \cdot (x \cdot f)$ .

Now we will prove the second part of (b). Let  $n$  be any positive integer. Consider the linear transformations  $\varphi^n : \mathrm{CL}^n(\mathfrak{L}, M) \rightarrow \mathrm{CL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M))$  defined by  $\varphi^n(f)(x_1, \dots, x_{n-1})(x) := f(x_1, \dots, x_{n-1}, x)$  for any elements  $x_1, \dots, x_{n-1}, x \in \mathfrak{L}$  and  $\psi^n : \mathrm{CL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)) \rightarrow \mathrm{CL}^n(\mathfrak{L}, M)$  defined by  $\psi^n(g)(x_1, \dots, x_{n-1}, x_n) := g(x_1, \dots, x_{n-1})(x_n)$  for any elements  $x_1, \dots, x_{n-1}, x_n \in \mathfrak{L}$ . Then  $\varphi^n$  and  $\psi^n$  are inverses of each other.

Next, we will show that  $\tilde{d}^{n-1} \circ \varphi^n = \varphi^{n+1} \circ d^n$ . Compute

$$\begin{aligned}
(\tilde{d}^{n-1} \circ \varphi^n)(f)(x_1, \dots, x_n)(x) &= \tilde{d}^{n-1}(\varphi^n(f))(x_1, \dots, x_n)(x) \\
&= \sum_{i=1}^n (-1)^{i+1} (x_i \cdot \varphi^n(f))(x_1, \dots, \hat{x}_i, \dots, x_n)(x) \\
&\quad + \sum_{1 \leq i < j \leq n} (-1)^i \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n)(x) \\
&= \sum_{i=1}^n (-1)^{i+1} x_i \cdot \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_n)(x) \\
&\quad - \sum_{i=1}^n (-1)^{i+1} \varphi^n(f)(x_1, \dots, \hat{x}_i, \dots, x_n)(x_i x) \\
&\quad + \sum_{i \leq i < j \leq n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x) \\
&= \sum_{i=1}^n (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_n, x) \\
&\quad + \sum_{i=1}^n (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i x) \\
&\quad + \sum_{1 \leq i < j \leq n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x)
\end{aligned}$$

and

$$\begin{aligned}
(\varphi^{n+1} \circ d^n)(f)(x_1, \dots, x_n)(x) &= \varphi^{n+1}(d^n(f))(x_1, \dots, x_n)(x) \\
&= d^n(f)(x_1, \dots, x_n, x) \\
&= \sum_{i=1}^n (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_n, x) \\
&\quad + (-1)^{n+1} f(x_1, \dots, x_n) \cdot x \\
&\quad + \sum_{1 \leq i < j \leq n} (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_i x_j, \dots, x_n, x) \\
&\quad + \sum_{i=1}^n (-1)^i f(x_1, \dots, \hat{x}_i, \dots, x_n, x_i x)
\end{aligned}$$

for any elements  $x_1, \dots, x_n, x \in \mathfrak{L}$ . Since  $M$  is anti-symmetric, the second of the last four summands vanishes, and thus the two compositions are equal. From the identity  $\tilde{d}^{n-1} \circ \varphi^n = \varphi^{n+1} \circ d^n$  for every integer  $n \geq 1$  we obtain that

$$\varphi^n(\text{Ker}(d^n)) \subseteq \text{Ker}(\tilde{d}^{n-1})$$

and

$$\varphi^n(\text{Im}(d^{n-1})) \subseteq \text{Im}(\tilde{d}^{n-2})$$

for every integer  $n \geq 1$ . Hence  $\varphi^n$  induces an isomorphism of vector spaces between  $\text{HL}^n(\mathfrak{L}, M)$  and  $\widetilde{\text{HL}}^{n-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\mathfrak{L}, M))$  for every integer  $n \geq 1$ . In order to see the remainder of the assertion, apply part (a).  $\square$

In the special case of the trivial one-dimensional Leibniz bimodule we obtain from Lemma 1.5 the following result which will be needed in Section 4 (see [29, Exercise E.10.6.1] for the analogous result in Leibniz homology).

**Corollary 1.6.** *Let  $\mathfrak{L}$  be a left Leibniz algebra over a field  $\mathbb{F}$ . Then for every integer  $n \geq 1$  there are isomorphisms*

$$\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \widehat{\mathrm{HL}}^{n-1}(\mathfrak{L}, \mathfrak{L}^*) = \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s)$$

of vector spaces, where  $\mathfrak{L}^* := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, \mathbb{F})$  is a left  $\mathfrak{L}$ -module via  $(x \cdot f)(y) := -f(xy)$  for every linear form  $f \in \mathfrak{L}^*$  and any elements  $x, y \in \mathfrak{L}$ .

**Remark.** Note that [22, Theorem 3.5] is an immediate consequence of the case  $n = 2$  of Corollary 1.6 and [17, Corollary 4.4 (a)].

## 2. A RELATION BETWEEN CHEVALLEY-EILENBERG COHOMOLOGY AND LEIBNIZ COHOMOLOGY FOR LIE ALGEBRAS

Let  $\mathfrak{g}$  be a Lie algebra, and let  $M$  be a left  $\mathfrak{g}$ -module, also viewed as a symmetric  $\mathfrak{g}$ -bimodule  $M_s$ . In this section, we will develop some links between the Chevalley-Eilenberg cohomology  $\mathrm{H}^\bullet(\mathfrak{g}, M)$  and the Leibniz cohomology  $\mathrm{HL}^\bullet(\mathfrak{g}, M_s)$ . We follow the analogous treatment for homology given in [33] very closely.

The Chevalley-Eilenberg cohomology of a Lie algebra  $\mathfrak{g}$  with trivial coefficients is not isomorphic (up to a degree shift) to the Chevalley-Eilenberg cohomology of  $\mathfrak{g}$  with coadjoint coefficients as it is the case for Leibniz cohomology (see Corollary 1.6). Instead these cohomologies are only related by a long exact sequence (see Proposition 2.1). The cohomology measuring the deviation from such an isomorphism will appear in a spectral sequence (see Theorem 2.5) which can be used to relate the Leibniz cohomology of a Lie algebra to its Chevalley-Eilenberg cohomology (see Proposition 2.2).

The exterior product map  $m : \Lambda^n \mathfrak{g} \otimes \mathfrak{g} \rightarrow \Lambda^{n+1} \mathfrak{g}$  given on homogeneous tensors by  $x_1 \wedge \dots \wedge x_n \otimes x \mapsto x_1 \wedge \dots \wedge x_n \wedge x$  induces a monomorphism

$$m^\bullet : \overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})[-1] \hookrightarrow \mathrm{C}^\bullet(\mathfrak{g}, \mathfrak{g}^*),$$

where  $\overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})$  is the truncated cochain complex

$$\overline{\mathrm{C}}^0(\mathfrak{g}, \mathbb{F}) := 0 \text{ and } \overline{\mathrm{C}}^n(\mathfrak{g}, \mathbb{F}) := \mathrm{C}^n(\mathfrak{g}, \mathbb{F}) \text{ for every integer } n > 0.$$

The complex  $\mathrm{CR}^\bullet(\mathfrak{g})$  is defined by  $\mathrm{CR}^\bullet(\mathfrak{g}) := \mathrm{Coker}(m^\bullet)[-1]$ . Observe that classes in  $\mathrm{CR}^n(\mathfrak{g})$  are represented by cochains of degree  $n + 1$  with values in  $\mathfrak{g}^*$ , i.e., they have  $n + 2$  arguments. From the short exact sequence

$$0 \rightarrow \overline{\mathrm{C}}^\bullet(\mathfrak{g}, \mathbb{F})[-1] \rightarrow \mathrm{C}^\bullet(\mathfrak{g}, \mathfrak{g}^*) \rightarrow \mathrm{CR}^\bullet(\mathfrak{g})[1] \rightarrow 0$$

of cochain complexes we obtain a long exact cohomology sequence:

**Proposition 2.1.** *For every Lie algebra  $\mathfrak{g}$  over a field  $\mathbb{F}$  there is a long exact sequence*

$$\begin{aligned} 0 &\rightarrow \mathrm{H}^2(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{H}^1(\mathfrak{g}, \mathfrak{g}^*) \rightarrow \mathrm{HR}^0(\mathfrak{g}) \\ &\rightarrow \mathrm{H}^3(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{H}^2(\mathfrak{g}, \mathfrak{g}^*) \rightarrow \mathrm{HR}^1(\mathfrak{g}) \rightarrow \dots \end{aligned}$$

and an isomorphism  $\mathrm{H}^1(\mathfrak{g}, \mathbb{F}) \cong \mathrm{H}^0(\mathfrak{g}, \mathfrak{g}^*)$ .

**Remark.** If we assume that the characteristic of the ground field  $\mathbb{F}$  is not 2, then  $\mathrm{HR}^0(\mathfrak{g}) \cong [S^2(\mathfrak{g})^*]^{\mathfrak{g}}$  is the space of invariant symmetric bilinear forms on  $\mathfrak{g}$  (see [33, p. 403]). As a consequence, we obtain from Proposition 2.1 in the case that  $\mathrm{char}(\mathbb{F}) \neq 2$  the five-term exact sequence

$$0 \rightarrow \mathrm{H}^2(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{H}^1(\mathfrak{g}, \mathfrak{g}^*) \rightarrow [S^2(\mathfrak{g})^*]^{\mathfrak{g}} \rightarrow \mathrm{H}^3(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{H}^2(\mathfrak{g}, \mathfrak{g}^*),$$

which generalizes [16, Proposition 1.3(1) & (3)]. Note that the map  $[S^2(\mathfrak{g})^*]^{\mathfrak{g}} \rightarrow \mathrm{H}^3(\mathfrak{g}, \mathbb{F})$  is the classical Cartan-Koszul map defined by  $\omega \mapsto \bar{\omega} + \mathrm{B}^3(\mathfrak{g}, \mathbb{F})$ , where  $\bar{\omega}(x \wedge y \wedge z) := \omega(xy, z)$  for any elements  $x, y, z \in \mathfrak{g}$  (see [33, p. 403]).

For a Lie algebra  $\mathfrak{g}$  and a left  $\mathfrak{g}$ -module  $M$  viewed as a symmetric Leibniz  $\mathfrak{g}$ -bimodule  $M_s$ , we have a natural monomorphism

$$\mathrm{C}^\bullet(\mathfrak{g}, M) \hookrightarrow \mathrm{CL}^\bullet(\mathfrak{g}, M_s).$$

The cokernel of this morphism is by definition (up to a shift in degree) the complex  $\mathrm{C}_{\mathrm{rel}}^\bullet(\mathfrak{g}, M)$ :

$$\mathrm{C}_{\mathrm{rel}}^\bullet(\mathfrak{g}, M) := \mathrm{Coker}(\mathrm{C}^\bullet(\mathfrak{g}, M) \rightarrow \mathrm{CL}^\bullet(\mathfrak{g}, M_s))[-2].$$

We therefore have a long exact sequence in cohomology. (For degrees 0 and 1 see [17, Corollary 4.2(a)] and [17, Corollary 4.4(a)].)

**Proposition 2.2.** *Let  $\mathfrak{g}$  be a Lie algebra, and let  $M$  be a left  $\mathfrak{g}$ -module considered as a symmetric Leibniz  $\mathfrak{g}$ -bimodule  $M_s$ . Then there are a long exact sequence*

$$\begin{aligned} 0 &\rightarrow \mathrm{H}^2(\mathfrak{g}, M) \rightarrow \mathrm{HL}^2(\mathfrak{g}, M_s) \rightarrow \mathrm{H}_{\mathrm{rel}}^0(\mathfrak{g}, M) \\ &\rightarrow \mathrm{H}^3(\mathfrak{g}, M) \rightarrow \mathrm{HL}^3(\mathfrak{g}, M_s) \rightarrow \mathrm{H}_{\mathrm{rel}}^1(\mathfrak{g}, M) \rightarrow \cdots \end{aligned}$$

and isomorphisms

$$\mathrm{HL}^0(\mathfrak{g}, M_s) \cong \mathrm{H}^0(\mathfrak{g}, M), \quad \mathrm{HL}^1(\mathfrak{g}, M_s) \cong \mathrm{H}^1(\mathfrak{g}, M).$$

**Remark.** If we again assume that the characteristic of the ground field  $\mathbb{F}$  is not 2, it follows from Theorem 2.5 in conjunction with [17, Proposition 4.1] and the remark after Proposition 2.1 that  $\mathrm{H}_{\mathrm{rel}}^0(\mathfrak{g}, \mathbb{F}) \cong \mathrm{HR}^0(\mathfrak{g}) \cong [S^2(\mathfrak{g})^*]^{\mathfrak{g}}$  is the space of invariant symmetric bilinear forms on  $\mathfrak{g}$ . Hence when  $\mathrm{char}(\mathbb{F}) \neq 2$  we obtain the five-term exact sequence

$$0 \rightarrow \mathrm{H}^2(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{HL}^2(\mathfrak{g}, \mathbb{F}) \rightarrow [S^2(\mathfrak{g})^*]^{\mathfrak{g}} \rightarrow \mathrm{H}^3(\mathfrak{g}, \mathbb{F}) \rightarrow \mathrm{HL}^3(\mathfrak{g}, \mathbb{F})$$

as a special case of Proposition 2.2 (cf. [22, Proposition 3.2] for fields of characteristic zero). Note that it follows from Corollary 1.6 that the second terms of the five-term exact sequences in Proposition 2.1 and Proposition 2.2 are isomorphic, but Example E in Section 5 shows that the fifth terms are not always isomorphic.

Observe that as for  $\mathrm{CR}^n(\mathfrak{g})$ , representatives of classes in  $\mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M)$  have  $n+2$  arguments.

On the quotient complex  $\mathrm{C}_{\mathrm{rel}}^\bullet(\mathfrak{g}, M)$  there is the following filtration

$$\mathcal{F}^p \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) = \{[c] \in \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) \mid c(x_1, \dots, x_{n+2}) = 0 \text{ if } \exists j \leq p+1 : x_{j-1} = x_j\}.$$

Note that the condition is independent of the representative  $c$  of the class  $[c]$ . This defines a finite decreasing filtration

$$\mathcal{F}^0 \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) = \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) \supset \mathcal{F}^1 \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) \supset \cdots \supset \mathcal{F}^{n+1} \mathrm{C}_{\mathrm{rel}}^n(\mathfrak{g}, M) = \{0\},$$

whose associated spectral sequence converges thus in the strong (i.e., finite) sense to  $\mathrm{H}_{\mathrm{rel}}^n(\mathfrak{g}, M)$  thanks to the following lemma:

**Lemma 2.3.** *This filtration is compatible with the Leibniz coboundary map  $d^\bullet$ .*

*Proof.* The Leibniz coboundary map, acting on a cochain  $c$ , is an alternating sum of operators  $d_{ij}(c)$ ,  $\delta_i(c)$  and  $\partial(c)$ , where  $d_{ij}(c)$  is the term involving the product of the  $i$ -th and the  $j$ -th element,  $\delta_i(c)$  is the term involving the left action of the  $i$ -th element, and  $\partial(c)$  is the term involving the right action of the  $(n+1)$ -th element. As the module is symmetric, the term involving the right action can be counted among the terms involving the left actions.

We have to show that  $d^\bullet(\mathcal{F}^p C_{\text{rel}}^n(\mathfrak{g}, M)) \subseteq \mathcal{F}^p C_{\text{rel}}^{n+1}(\mathfrak{g}, M)$ . We thus consider the different terms of  $d^\bullet(c)$  with two equal elements as arguments in the first  $p+1$  slots and have to show that all terms are zero. For  $d_{ij}(c)$  with  $i, j \leq p+1$ , the assertion is clear because either the two equal elements do not occur in the product, and then it is correct, or at least one of them occurs, and then the product generates out of the sum of  $d_{ij}$  and  $d_{ij+1}$  (or  $d_{ij-1}$ ) an element  $x_i x \otimes x + x \otimes x_i x$ , which is a sum of symmetric elements thanks to

$$x_i x \otimes x + x \otimes x_i x = (x_i x + x) \otimes (x_i x + x) - x_i x \otimes x_i x - x \otimes x.$$

Even more elementary, the assertion is clear for  $d_{ij}(c)$  with  $i, j \geq p+1$ . For  $d_{ij}(c)$  with  $i \leq p+1$  and  $j \geq p+2$ , the assertion is clear in case  $x_i$  is not one of the equal elements. In case it is, the two terms corresponding to the product action of the two equal elements cancel (as they are equal and have different sign).

For the action terms  $\delta_i(c)$  the reasoning is similar. In case  $i \leq p+1$ , either the two equal elements do not occur and the assertion is clear, or both occur and cancel each other because of the alternating sign. For  $\delta_i(c)$  with  $i \geq p+2$ , the assertion is clear in any case.  $\square$

The next step is then to compute the 0-th term of the corresponding spectral sequence, i.e., the associated graded vector space of the filtration:

$$E_0^{p,q} := \mathcal{F}^p C_{\text{rel}}^{p+q}(\mathfrak{g}, M) / \mathcal{F}^{p+1} C_{\text{rel}}^{p+q}(\mathfrak{g}, M).$$

Observe that the filtration can be expressed as

$$\mathcal{F}^p C_{\text{rel}}^{p+q}(\mathfrak{g}, M) = \{[c] \in C_{\text{rel}}^n(\mathfrak{g}, M) \mid c|_{(\text{Ker}(\otimes^{p+1} \mathfrak{g} \rightarrow \Lambda^{p+1} \mathfrak{g})) \otimes (\otimes^{q+1} \mathfrak{g})} = 0\}.$$

This is useful, because by elementary linear algebra, we have

$$F^\perp / G^\perp = \text{Hom}_{\mathbb{F}}(G/F, M),$$

where  $F^\perp := \{f : E \rightarrow M \mid f|_F = 0\}$  and  $G^\perp := \{f : E \rightarrow M \mid f|_G = 0\}$  for  $F \subseteq G \subseteq E$ .

In order to be able to find  $E_0^{p,q}$ , we therefore have to compute

$$(\text{Ker}(\otimes^{p+2} \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g}) \otimes (\otimes^q \mathfrak{g})) / (\text{Ker}(\otimes^{p+1} \mathfrak{g} \rightarrow \Lambda^{p+1} \mathfrak{g}) \otimes (\otimes^{q+1} \mathfrak{g})).$$

Using the isomorphism (see the proof of Theorem A in [33])

$$\text{Ker}(\otimes^{p+2} \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g}) / (\text{Ker}(\otimes^{p+1} \mathfrak{g} \rightarrow \Lambda^{p+1} \mathfrak{g}) \otimes \mathfrak{g}) \cong \text{Ker}(\Lambda^{p+1} \mathfrak{g} \otimes \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g}).$$

we obtain

$$\begin{aligned} E_0^{p,q} &= \{c : \text{Ker}(\Lambda^{p+1} \mathfrak{g} \otimes \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g}) \otimes \text{CL}_q(\mathfrak{g}) \rightarrow M\} / C^{p+q+2}(\mathfrak{g}, M) \\ &= \text{Ker}(\Lambda^{p+1} \mathfrak{g} \otimes \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g})^* \otimes \text{CL}^q(\mathfrak{g}, M) / C^{p+q+2}(\mathfrak{g}, M). \end{aligned}$$

It remains to observe that the first tensor factor is the kernel of the exterior multiplication map  $m$ , and thus

$$\text{Ker}(\Lambda^{p+1} \mathfrak{g} \otimes \mathfrak{g} \rightarrow \Lambda^{p+2} \mathfrak{g})^* = \text{Ker}(m)^* = \text{Coker}(m^*) = \text{CR}^p(\mathfrak{g}).$$

Therefore the term  $E_0^{p,q}$  takes the form

$$E_0^{p,q} = \text{CR}^p(\mathfrak{g}) \otimes \text{CL}^q(\mathfrak{g}, M).$$

Next, let us find the differential on  $E_0^{p,q}$ .

**Lemma 2.4.** *The differential on  $E_0^{\bullet,\bullet}$  induced from  $d^\bullet$  on  $C_{\text{rel}}^\bullet(\mathfrak{g}, M)$  identifies with  $\text{id}_{\text{CR}^p(\mathfrak{g})}^{\otimes(p+2)} \otimes d_{\text{CL}^q(\mathfrak{g}, M)}^q$ .*

*Proof.* By definition, the differential  $d_0$  of the spectral sequence is the differential which is induced from the Leibniz coboundary map  $d^\bullet$  on the associated graded quotients

$$d_0 : \mathcal{F}^p C_{\text{rel}}^{p+q}(\mathfrak{g}, M) / \mathcal{F}^{p+1} C_{\text{rel}}^{p+q}(\mathfrak{g}, M) \rightarrow \mathcal{F}^p C_{\text{rel}}^{p+q+1}(\mathfrak{g}, M) / \mathcal{F}^{p+1} C_{\text{rel}}^{p+q+1}(\mathfrak{g}, M).$$

In order to examine which terms  $d_{ij}(c)$ ,  $\delta_i(c)$  and  $\partial(c)$  are zero for a cochain  $c \in \mathcal{F}^p C_{\text{rel}}^{p+q}(\mathfrak{g}, M)$ , we have to insert two consecutive equal elements in the arguments of  $c$  within the first  $p+2$  arguments.

Now, by the same reasoning as in the proof of Lemma 2.3, the terms  $d_{ij}(c)$  are zero in case  $i, j \leq p+2$ , because in case the equal elements are not involved, the formula for  $d_{ij}(c)$  diminishes the number of arguments by one and as  $c$  is of degree  $p$  in the filtration, this gives then zero. In case the elements occur, they create once again a symmetric element of the form  $x_i x \otimes x + x \otimes x_i x$ . Also for  $d_{ij}(c)$  with  $i \leq p+2$  and  $j \geq p+3$ , the terms are zero when the equal elements are not involved and are zero in addition with  $d_{ij+1}(c)$  (or  $d_{ij-1}(c)$ ), in case of multiplying with one of the equal elements. The terms  $\delta_i(c)$  for  $i \leq p+1$  are zero as the corresponding formula diminishes the number of arguments by one in case the equal elements do not occur and annihilate each other in case they occur.

There remain thus the terms  $d_{ij}(c)$  with  $i, j \geq p+3$ ,  $\delta_i(c)$  with  $i \geq p+3$ , and  $\partial(c)$ , which form together the coboundary map of the cochain complex  $\text{CL}^\bullet(\mathfrak{g}, M)$ .  $\square$

In conclusion, we have as second term of the spectral sequence

$$E_2^{p,q} = \text{HR}^p(\mathfrak{g}) \otimes \text{HL}^q(\mathfrak{g}, M_s),$$

because the differential  $d_1$  is again induced by  $d^\bullet$ , but all terms involving terms of  $\text{HL}^\bullet(\mathfrak{g}, M)$  are zero, because a Leibniz algebra acts trivially on its cohomology. This follows from the Cartan relations for Leibniz cohomology (see [30, Proposition 3.1] for the case of right Leibniz algebras and [12, Proposition 1.3.2] for the case of left Leibniz algebras).

This discussion proves the following result:

**Theorem 2.5.** *Let  $\mathfrak{g}$  be a Lie algebra, and let  $M$  be a left  $\mathfrak{g}$ -module considered as a symmetric Leibniz  $\mathfrak{g}$ -bimodule  $M_s$ . Then there is a spectral sequence converging to  $\text{H}_{\text{rel}}^\bullet(\mathfrak{g}, M)$  with second term*

$$E_2^{p,q} = \text{HR}^p(\mathfrak{g}) \otimes \text{HL}^q(\mathfrak{g}, M_s).$$

**Remark.** As the spectral sequence is the spectral sequence of a filtered complex, the higher differentials in the spectral sequence are again induced by the total Leibniz cohomology differential  $d^\bullet$ . We will see in Example C below an instance of a concrete computation of  $d_2$ .

Our main application of the spectral sequence will be the next theorem:

**Theorem 2.6.** *Let  $\mathfrak{g}$  be a Lie algebra, let  $M$  be a left  $\mathfrak{g}$ -module considered as a symmetric Leibniz  $\mathfrak{g}$ -bimodule  $M_s$ , and let  $n$  be a non-negative integer. If  $H^k(\mathfrak{g}, M) = 0$  for every integer  $k$  with  $0 \leq k \leq n$ , then  $HL^k(\mathfrak{g}, M_s) = 0$  for every integer  $k$  with  $0 \leq k \leq n$  and  $HL^{n+1}(\mathfrak{g}, M_s) \cong H^{n+1}(\mathfrak{g}, M)$  as well as  $HL^{n+2}(\mathfrak{g}, M_s) \cong H^{n+2}(\mathfrak{g}, M)$ . In particular,  $H^\bullet(\mathfrak{g}, M) = 0$  implies that  $HL^\bullet(\mathfrak{g}, M_s) = 0$ .*

*Proof.* The proof follows the proof of Corollary 1.3 in [33] very closely.

According to Proposition 2.2, it suffices to prove that  $H^k(\mathfrak{g}, M) = 0$  for every integer  $k$  with  $0 \leq k \leq n$  implies that  $H_{\text{rel}}^n(\mathfrak{g}, M) = 0$  for every integer  $k$  with  $0 \leq k \leq n$ . We proceed by induction on  $n$ . In the case  $n = 0$ , the hypothesis yields that  $E_2^{0,0} = 0$  for the second term of the spectral sequence of Theorem 2.5, and therefore we obtain from the convergence of the spectral sequence that  $H_{\text{rel}}^0(\mathfrak{g}, M) = 0$  initializing the induction.

So suppose now that  $n \geq 1$  and  $H^k(\mathfrak{g}, M_s) = 0$  for every integer  $k$  with  $0 \leq k \leq n + 1$ . By induction hypothesis, we obtain that  $H_{\text{rel}}^n(\mathfrak{g}, M) = 0$  for every integer  $k$  with  $0 \leq k \leq n$ . Hence it follows from Proposition 2.2 that  $HL^k(\mathfrak{g}, M_s) = 0$  for every integer  $k$  with  $0 \leq k \leq n$  and  $HL^{n+1}(\mathfrak{g}, M_s) \cong H^{n+1}(\mathfrak{g}, M) = 0$ . Consequently, the second term  $E_2^{p,q}$  of the spectral sequence in Theorem 2.5 is zero for  $p + q \leq n + 1$ , and therefore  $H_{\text{rel}}^{n+1}(\mathfrak{g}, M) = 0$ .  $\square$

**Remark.** Note that the converse of Theorem 2.6 is also true, namely,  $H^k(\mathfrak{g}, M) = 0$  for every integer  $k$  with  $0 \leq k \leq n$  if, and only if,  $HL^k(\mathfrak{g}, M_s) = 0$  for every integer  $k$  with  $0 \leq k \leq n$ . In particular,  $H^\bullet(\mathfrak{g}, M) = 0$  if, and only if,  $HL^\bullet(\mathfrak{g}, M_s) = 0$ .

We conclude this section by proving an extension of a result by Fialowski, Magnin, and Mandal (see Corollary 2 in [18]), namely the fact that the vanishing of the center  $C(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$  implies  $HL^2(\mathfrak{g}, \mathfrak{g}_{\text{ad}}) = H^2(\mathfrak{g}, \mathfrak{g})$ , where  $\mathfrak{g}_{\text{ad}}$  denotes the adjoint Leibniz  $\mathfrak{g}$ -bimodule induced by the left and right multiplication operator. Observe that for Lie algebras, this bimodule is indeed symmetric. The spectral sequence of Theorem 2.5 with adjoint coefficients has as its  $E_2$ -term

$$E_2^{p,q} = HR^p(\mathfrak{g}) \otimes HL^q(\mathfrak{g}, \mathfrak{g}_{\text{ad}}).$$

We note that  $HL^0(\mathfrak{g}, \mathfrak{g}_{\text{ad}}) = C(\mathfrak{g})$  (see [17, Corollary 4.2(a)]). Therefore it is an immediate consequence of the case  $n = 0$  of Theorem 2.6 that the vanishing of the center implies that  $HL^2(\mathfrak{g}, \mathfrak{g}_{\text{ad}}) = H^2(\mathfrak{g}, \mathfrak{g})$ . By the same token for  $n = 1$ , we can extend this to complete Lie algebras, i.e., to those Lie algebras  $\mathfrak{g}$  for which  $H^0(\mathfrak{g}, \mathfrak{g}) = H^1(\mathfrak{g}, \mathfrak{g}) = 0$ :

**Corollary 2.7.** *Let  $\mathfrak{g}$  be a complete Lie algebra. Then*

$$HL^2(\mathfrak{g}, \mathfrak{g}_{\text{ad}}) \cong H^2(\mathfrak{g}, \mathfrak{g}) \text{ and } HL^3(\mathfrak{g}, \mathfrak{g}_{\text{ad}}) \cong H^3(\mathfrak{g}, \mathfrak{g}).$$

A class of examples of complete Lie algebras over an algebraically closed field  $\mathbb{F}$  of characteristic zero consists of those finite-dimensional Lie algebras  $\mathfrak{g}$  for which  $\mathfrak{g}$  has the same dimension as its Lie algebra of derivations and  $\dim_{\mathbb{F}} \mathfrak{g}/\mathfrak{g}^2 > 1$  (see [10, Proposition 3.1]). Another example is the two-sided Witt algebra over a field of characteristic zero. Indeed, this infinite-dimensional simple Lie algebra is complete (see [15, Theorem A.1.1]). Hence we obtain from [34, Theorem 3.1] and [15, Theorem 4.1] in conjunction with the case  $n = 3$  of Theorem 2.6 the following result:

**Corollary 2.8.** *Let  $\mathcal{W} := \text{Der}(\mathbb{F}[t, t^{-1}])$  be the two-sided Witt algebra over a field  $\mathbb{F}$  of characteristic zero. Then  $\text{HL}^2(\mathcal{W}, \mathcal{W}_{\text{ad}}) = 0$  and  $\text{HL}^3(\mathcal{W}, \mathcal{W}_{\text{ad}}) = 0$ . Moreover,*

$$\text{HL}^4(\mathcal{W}, \mathcal{W}_{\text{ad}}) \cong \text{H}^4(\mathcal{W}, \mathcal{W}) \quad \text{and} \quad \text{HL}^5(\mathcal{W}, \mathcal{W}_{\text{ad}}) \cong \text{H}^5(\mathcal{W}, \mathcal{W}).$$

**Remark.** Very recently, Camacho, Omirov, and Kurbanbaev also proved that the second adjoint Leibniz cohomology of  $\mathcal{W}$  vanishes (see [9, Theorem 4]) by explicitly showing that every adjoint Leibniz 2-cocycle (resp. Leibniz 2-coboundary) is an adjoint Chevalley-Eilenberg 2-cocycle (resp. Chevalley-Eilenberg 2-coboundary) for  $\mathcal{W}$ .

### 3. TWO SPECTRAL SEQUENCES FOR LEIBNIZ COHOMOLOGY

In this section we consider two Leibniz analogues of the Hochschild-Serre spectral sequence for the Chevalley-Eilenberg cohomology of Lie algebras that converge to the same relative cohomology. They will both play a role in Sections 4 and 5.

Let  $\pi : \mathfrak{L} \rightarrow \mathfrak{Q}$  be an epimorphism of left Leibniz algebras, and let  $M$  be a  $\mathfrak{Q}$ -bimodule. Then  $M$  is also an  $\mathfrak{L}$ -bimodule via  $\pi$ . Moreover, the epimorphisms  $\pi^{\otimes n} : \mathfrak{L}^{\otimes n} \rightarrow \mathfrak{Q}^{\otimes n}$  induce a monomorphism  $\text{CL}^\bullet(\mathfrak{Q}, M) \rightarrow \text{CL}^\bullet(\mathfrak{L}, M)$  of cochain complexes. Now set

$$\text{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M) := \text{Coker}(\text{CL}^\bullet(\mathfrak{Q}, M) \rightarrow \text{CL}^\bullet(\mathfrak{L}, M))[-1]$$

and

$$\text{HL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M) := \text{H}^\bullet(\text{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)).$$

Then by applying the long exact cohomology sequence to the short exact sequence

$$0 \rightarrow \text{CL}^\bullet(\mathfrak{Q}, M) \rightarrow \text{CL}^\bullet(\mathfrak{L}, M) \rightarrow \text{CL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)[1] \rightarrow 0$$

of cochain complexes one obtains the following result (see also [33, Proposition 4.1] for the corresponding result on Leibniz homology).

**Proposition 3.1.** *For every epimorphism  $\pi : \mathfrak{L} \rightarrow \mathfrak{Q}$  of left Leibniz algebras and every  $\mathfrak{Q}$ -bimodule  $M$  there exists a long exact sequence*

$$\begin{aligned} 0 &\rightarrow \text{HL}^1(\mathfrak{Q}, M) \rightarrow \text{HL}^1(\mathfrak{L}, M) \rightarrow \text{HL}^0(\mathfrak{L}|\mathfrak{Q}, M) \\ &\rightarrow \text{HL}^2(\mathfrak{Q}, M) \rightarrow \text{HL}^2(\mathfrak{L}, M) \rightarrow \text{HL}^1(\mathfrak{L}|\mathfrak{Q}, M) \rightarrow \dots \end{aligned}$$

We now come to the first of the two spectral sequences (see [6, Théorème 3.2.16]).

**Theorem 3.2.** *Let  $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{L} \xrightarrow{\pi} \mathfrak{Q} \rightarrow 0$  be a short exact sequence of left Leibniz algebras, and let  $M$  be an  $\mathfrak{L}$ -bimodule such that  $\mathfrak{J} \subseteq \text{Ann}_{\mathfrak{L}}^{\text{bi}}(M)$ . Then  $M$  is a  $\mathfrak{Q}$ -bimodule via  $x \cdot m := \pi^{-1}(x) \cdot m$  and  $m \cdot x := m \cdot \pi^{-1}(x)$  for every element  $x \in \mathfrak{Q}$  and every element  $m \in M$ . Moreover, there is a spectral sequence converging to  $\text{HL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)[1]$  with second term*

$$E_2^{p,q} = \begin{cases} \text{HL}^{q-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\mathfrak{J}, \text{HL}^p(\mathfrak{Q}, M))_s) & \text{if } p \geq 0, q \geq 1 \\ 0 & \text{if } p \geq 0, q = 0. \end{cases}$$

**Remark.** According to [17, Lemma 3.10], Theorem 3.2 applies to  $\mathfrak{J} := \text{Leib}(\mathfrak{L})$  and  $\mathfrak{Q} := \mathfrak{L}_{\text{Lie}}$ . Note that the  $\mathfrak{L}$ -bimodule  $\text{Hom}_{\mathbb{F}}(\mathfrak{J}, \text{HL}^p(\mathfrak{Q}, M))_s$  is viewed as a *symmetric* bimodule in this theorem.

*Proof.* This theorem is taken from Beaudouin's thesis. For the convenience of the reader we include a proof. We consider the following filtration on the complex  $\mathrm{CL}^\bullet(\mathfrak{L}, M)$ :

$$\mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M) := \{c \in \mathrm{CL}^n(\mathfrak{L}, M) \mid c(x_1, \dots, x_n) = 0 \text{ if } \exists i \geq n - p + 1 : x_i \in \mathfrak{J}\}.$$

This defines a finite decreasing filtration

$$\begin{aligned} \mathcal{F}^0 \mathrm{CL}^n(\mathfrak{L}, M) &= \mathrm{CL}^n(\mathfrak{L}, M) \supset \mathcal{F}^1 \mathrm{CL}^n(\mathfrak{L}, M) \supset \dots \\ &\dots \supset \mathcal{F}^n \mathrm{CL}^n(\mathfrak{L}, M) = \mathrm{CL}^n(\mathfrak{Q}, M), \end{aligned}$$

whose associated spectral sequence converges in the strong (i.e., finite) sense to the cohomology  $\mathrm{HL}^n(\mathfrak{L}|\mathfrak{Q}, M)$  of the quotient complex  $\mathrm{CL}^n(\mathfrak{L}, M) / \mathrm{CL}^n(\mathfrak{Q}, M)$  thanks to the following lemma:

**Lemma 3.3.** *The filtration is compatible with the Leibniz coboundary map  $d^\bullet$ .*

*Proof.* We have to prove that  $d^\bullet(\mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M)) \subseteq \mathcal{F}^p \mathrm{CL}^{n+1}(\mathfrak{L}, M)$ . For this, we consider the different terms  $d_{ij}(c)$ ,  $\delta_i(c)$ , and  $\partial(c)$ , which constitute the differential  $d^\bullet(c)$ , where we have inserted an element of  $\mathfrak{J}$  within the last  $p$  arguments. The vanishing is clear for the terms  $d_{ij}(c)$  with  $i, j \geq n - p + 1$ , because even if the element of  $\mathfrak{J}$  is concerned in the product, the product will again be in the ideal  $\mathfrak{J}$ . The vanishing is also clear for the terms  $d_{ij}(c)$  with  $i, j \leq n - p$ . The terms  $d(c)_{ij}(c)$  with  $i \leq n - p$  and  $j \geq n - p + 1$  are also zero, because  $\mathfrak{J}$  is an ideal.

The action terms follow a similar pattern. The terms  $\delta_i(c)$  with  $i \geq n - p + 1$  are zero, because either the element of  $\mathfrak{J}$  occurs in the arguments, or it acts on  $M$ , which is zero by assumption. The same holds for the term  $\partial(c)$ . The terms  $\delta_i(c)$  with  $i \leq n - p$  are again zero, because the element of  $\mathfrak{J}$  occurs in the arguments.  $\square$

We get for the 0-th term of the spectral sequence

$$E_0^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^q \otimes \mathfrak{Q}^p, M) / \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^{q-1} \otimes \mathfrak{Q}^{p+1}, M) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}^{q-1} \otimes \mathfrak{J} \otimes \mathfrak{Q}^p, M)$$

if  $q \geq 1$ , and  $= 0$  in case  $q = 0$ . This isomorphism is induced by the inclusion  $\mathfrak{J} \hookrightarrow \mathfrak{L}$ .

In order to describe the first and second term of the spectral sequence, note that  $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{J} \otimes \mathfrak{Q}^p, M)$  is a symmetric  $\mathfrak{L}$ -bimodule with respect to the usual left action

$$(x \cdot f)(y, z_1, \dots, z_p) = x \cdot f(y, z_1, \dots, z_p) - f(xy, z_1, \dots, z_p) - \sum_{i=1}^p f(y, z_1, \dots, xz_i, \dots, z_p).$$

**Lemma 3.4.** *The differential  $d_0$  on  $E_0^{p,q}$  reads  $d^{q-1}|_{\mathrm{CL}^{q-1}(\mathfrak{L}, \mathrm{Hom}(\mathfrak{J} \otimes \mathfrak{Q}^p, M))}$  for  $q \geq 1$  and is zero for  $q = 0$ .*

*Proof.* The differential

$$d_0 : \mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M) / \mathcal{F}^{p+1} \mathrm{CL}^n(\mathfrak{L}, M) \rightarrow \mathcal{F}^p \mathrm{CL}^{n+1}(\mathfrak{L}, M) / \mathcal{F}^{p+1} \mathrm{CL}^{n+1}(\mathfrak{L}, M)$$

is the differential induced by  $d^\bullet$ . Thus we have to examine which terms  $d_{ij}(c)$ ,  $\delta_i(c)$  and  $\partial(c)$  composing the differential  $d_0(c)$  are non-zero in case we put an element of  $\mathfrak{J}$  within the last  $p + 1$  entries.

It is clear that  $d_{ij}(c) = 0$  for  $i, j \geq n - p$ , because this is true if the element of  $\mathfrak{J}$  is not involved in the product as the number of elements is diminished by one, and it is also true if the element of  $\mathfrak{J}$  occurs in the product, because  $\mathfrak{J}$  is an ideal. Considering the terms  $d_{ij}(c)$  for  $i \leq n - p - 1$  and  $j \geq n - p$  together with  $\delta_i(c)$

for  $i \leq n - p - 1$ , we obtain the terms composing the above displayed action of  $\mathfrak{L}$  on  $\text{Hom}_{\mathbb{F}}(\mathfrak{J} \otimes \mathfrak{Q}^p, M)$ . Note that we are forced to view the latter as a symmetric bimodule, because, while in the present expression all action terms are from the left, that last action term has to be switched in order to make appear the Leibniz coboundary operator. Furthermore, the terms  $\delta_i(c)$  for  $i \geq n - p$  are zero, because if the element of  $\mathfrak{J}$  acts on  $M$ , it acts as zero, and otherwise the number of elements is reduced by one. The same holds for  $\partial(c)$ . We therefore are left with the terms composing the differential on  $\text{CL}^{q-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\mathfrak{J} \otimes \mathfrak{Q}^p, M))$ .  $\square$

The lemma readily implies that the first term of the spectral sequence reads for any integer  $q \geq 1$ :

$$E_1^{p,q} = \text{HL}^{q-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\mathfrak{J}, \text{CL}^p(\mathfrak{Q}, M))),$$

and we obtain for the second term of the spectral sequence for  $q \geq 1$ :

$$E_2^{p,q} = \text{HL}^{q-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\mathfrak{J}, \text{HL}^p(\mathfrak{Q}, M))).$$

This follows by examining the remaining potentially non-zero terms  $d_{ij}(c)$  for  $i, j \geq n - p$ ,  $\delta_i(c)$  for  $i \geq n - p$ , and  $\partial(c)$  of the differential  $d^\bullet$ , which did not contribute to the  $E_1$ -term. As  $\mathfrak{J}$  is an ideal acting trivially on  $M$ , the corresponding term in  $d^\bullet$  is zero, and we are left with  $(\text{id}_{\mathfrak{L}^*})^{\otimes q-1} \otimes \text{id}_{\mathfrak{J}^*} \otimes d^p|_{\text{CL}^p(\mathfrak{Q}, M)}$ .  $\square$

Finally, we derive a spectral sequence for certain short exact sequences of left Leibniz algebras that is dual to and an extension of one of Pirashvili's spectral sequence for Leibniz homology (see [33, Theorem C]). While Pirashvili considers only symmetric bimodules, we extend the dual of his spectral sequence to all bimodules. The filtration considered here is complementary to the filtration considered in Theorem 3.2.

**Theorem 3.5.** *Let  $0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{L} \xrightarrow{\pi} \mathfrak{Q} \rightarrow 0$  be a short exact sequence of left Leibniz algebras such that  $\mathfrak{J} \subseteq C_\ell(\mathfrak{L})$ . Then  $\mathfrak{J}$  is a  $\mathfrak{Q}$ -bimodule via  $x \cdot y := \pi^{-1}(x)y$  and  $y \cdot x := y\pi^{-1}(x)$  for every element  $x \in \mathfrak{Q}$  and every element  $y \in \mathfrak{J}$ , and there is a similar bimodule structure for its dual  $\mathfrak{J}^*$ . Moreover, there is a spectral sequence converging to  $\text{HL}^\bullet(\mathfrak{L}|\mathfrak{Q}, M)$  with second term*

$$E_2^{p,q} = \begin{cases} \text{HL}^p(\mathfrak{Q}, \text{Hom}_{\mathbb{F}}(\mathfrak{J}, \text{HL}^0(\mathfrak{L}, M))_s) & \text{if } p \geq 0, q = 0 \\ \text{HL}^p(\mathfrak{Q}, (\mathfrak{J}^*)_s) \otimes \text{HL}^q(\mathfrak{L}, M) & \text{if } p \geq 0, q \geq 1 \end{cases}$$

for every  $\mathfrak{Q}$ -bimodule  $M$ .

**Corollary 3.6.** *If in the above situation the  $\mathfrak{Q}$ -bimodule  $M$  is symmetric, then for any integers  $p, q \geq 0$  the  $E_2$ -term of the spectral sequence simply reads*

$$E_2^{p,q} = \text{HL}^p(\mathfrak{Q}, (\mathfrak{J}^*)_s) \otimes \text{HL}^q(\mathfrak{L}, M).$$

**Remarks.**

- (a) According to [17, Proposition 2.13], Theorem 3.5 applies to  $\mathfrak{J} := \text{Leib}(\mathfrak{L})$  and  $\mathfrak{Q} := \mathfrak{L}_{\text{Lie}}$  (see [33, Remark 4.2] for the analogous statement for Leibniz homology). Note that in the cohomology space  $\text{HL}^p(\mathfrak{Q}, (\mathfrak{J}^*)_s)$ , the left  $\mathfrak{Q}$ -module  $\mathfrak{J}^*$  is here viewed as a *symmetric* bimodule (while it is naturally an antisymmetric  $\mathfrak{Q}$ -bimodule).
- (b) The higher differentials in the spectral sequence are again induced by the the total Leibniz cohomology differential  $d^\bullet$ . Observe that the spectral

sequence of the corollary is isomorphic to the spectral sequence of the bi-complex  $\mathrm{CL}^\bullet(\mathfrak{Q}, \mathfrak{J}^*) \otimes \mathrm{CL}^\bullet(\mathfrak{L}, M)$ . Therefore the description of the higher differentials can be adapted from [23] (see in particular Remark 3.2 therein). For example, it is clear that in case one of the two differentials in the bi-complex is zero, all higher differentials vanish. We will see an instance of this case in Example D below.

*Proof.* We consider the following filtration on the complex  $\mathrm{CL}^\bullet(\mathfrak{L}, M)[-1]$ .

$$\mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M)[-1] := \{c \in \mathrm{CL}^{n+1}(\mathfrak{L}, M) \mid c(x_1, \dots, x_{n+1}) = 0 \text{ if } \exists i \leq p : x_i \in \mathfrak{J}\}.$$

This defines a finite decreasing filtration

$$\begin{aligned} \mathcal{F}^0 \mathrm{CL}^n(\mathfrak{L}, M)[-1] &= \mathrm{CL}^n(\mathfrak{L}, M)[-1] \supset \mathcal{F}^1 \mathrm{CL}^n(\mathfrak{L}, M)[-1] \supset \dots \\ &\dots \supset \mathcal{F}^{n+1} \mathrm{CL}^n(\mathfrak{L}, M)[-1] = \mathrm{CL}^n(\mathfrak{Q}, M)[-1], \end{aligned}$$

whose associated spectral sequence converges in the strong (i.e., finite) sense to the cohomology  $\mathrm{HL}^n(\mathfrak{L}|\mathfrak{Q}, M)$  of the quotient complex  $\mathrm{CL}^n(\mathfrak{L}, M)[-1] / \mathrm{CL}^n(\mathfrak{Q}, M)[-1]$  thanks to the following result:

**Lemma 3.7.** *The filtration is compatible with the Leibniz coboundary map  $d^\bullet$ .*

*Proof.* We have to prove that  $d^\bullet(\mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M)[-1]) \subseteq \mathcal{F}^p \mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1]$ . For this, we consider the different terms  $d_{ij}(c)$ ,  $\delta_i(c)$ , and  $\partial(c)$ , which constitute the differential  $d_0(c)$ , where we have inserted an element of  $\mathfrak{J}$  within the first  $p$  arguments. The vanishing is clear for the terms  $d_{ij}(c)$  with  $i, j \leq p$ , because even if the element of  $\mathfrak{J}$  occurs in the product, the product will again be in the ideal  $\mathfrak{J}$ . The vanishing is also clear for the terms  $d_{ij}(c)$  with  $i, j \geq p+1$ . Concerning the terms  $d_{ij}(c)$  with  $i \leq p$  and  $j \geq p+1$ , we use the condition  $\mathfrak{J} \subseteq C_\ell(\mathfrak{L})$  to conclude that these are zero.

The action terms follow a similar pattern. The terms  $\delta_i(c)$  with  $i \leq p$  are zero, because either the element of  $\mathfrak{J}$  occurs in the arguments, or it acts on  $M$ , which is zero by assumption. The terms  $\delta_i(c)$  with  $i \geq p+1$  are zero for elementary reasons, as is the term  $\partial(c)$ .  $\square$

We get for the 0-th term of the spectral sequence

$$E_0^{p,q} = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{L}^{q+1}, M) / \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^{p+1} \otimes \mathfrak{L}^q, M) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{J} \otimes \mathfrak{L}^q, M),$$

where the isomorphism is induced by the inclusion  $\mathfrak{J} \hookrightarrow \mathfrak{L}$ .

**Lemma 3.8.** *The differential  $d_0$  on  $E_0^{p,q}$  reads  $\mathrm{id}_{\mathrm{CL}^{p+1}(\mathfrak{Q}, \mathfrak{J}^*)}^{\otimes p+1} \otimes d^q|_{\mathrm{CL}^q(\mathfrak{L}, M)}$ .*

*Proof.* The differential

$$\begin{aligned} d_0 : \mathcal{F}^p \mathrm{CL}^n(\mathfrak{L}, M)[-1] / \mathcal{F}^{p+1} \mathrm{CL}^n(\mathfrak{L}, M)[-1] &\rightarrow \\ &\rightarrow \mathcal{F}^p \mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1] / \mathcal{F}^{p+1} \mathrm{CL}^{n+1}(\mathfrak{L}, M)[-1] \end{aligned}$$

is the differential induced by  $d^\bullet$ . Thus we have to examine which terms  $d_{ij}(c)$ ,  $\delta_i(c)$ , and  $\partial(c)$  composing the differential  $d_0(c)$  are non-zero in case we put an element of  $\mathfrak{J}$  within the first  $p+1$  entries.

It is clear that  $d_{ij}(c) = 0$  for  $i, j \leq p+1$ , because this is true if the element of  $\mathfrak{J}$  is not involved in the product as the number of elements is diminished by one, and it is also true if the element of  $\mathfrak{J}$  is in the product as  $\mathfrak{J}$  is an ideal. We then have  $d_{ij}(c) = 0$  for  $i \leq p+1$  and  $j \geq p+2$  because in case the element of  $\mathfrak{J}$  acts in the

product, it acts as zero by  $\mathfrak{J} \subseteq C_\ell(\mathfrak{L})$ . Furthermore, the terms  $\delta_i(c)$  for  $i \leq p+1$  are zero, because if the element of  $\mathfrak{J}$  acts on  $M$ , it acts as zero.

Note that in all action terms on  $\mathfrak{J}^*$  the action is from the left, thus, in order to make appear the Leibniz coboundary operator with values in  $\mathfrak{J}^*$ , we have to switch around the last action term. This is the reason why we view  $\mathfrak{J}^*$  here as a symmetric  $\mathfrak{Q}$ -bimodule. We therefore are left with the terms composing the differential  $d_0|_{\text{CL}^q(\mathfrak{L}, M)}$ .  $\square$

The first term of the spectral sequence is thus

$$E_1^{p,q} = \text{Hom}_{\mathbb{F}}(\mathfrak{Q}^p \otimes \mathfrak{J}, \mathbb{F}) \otimes \text{HL}^q(\mathfrak{L}, M),$$

and the second term reads for  $q > 0$

$$E_2^{p,q} = \text{HL}^p(\mathfrak{Q}, (\mathfrak{J}^*)_s) \otimes \text{HL}^q(\mathfrak{L}, M),$$

because a Leibniz algebra acts trivially on its cohomology. This follows again from the Cartan relations for Leibniz cohomology (see [30, Proposition 3.1] for the case of right Leibniz algebras and [12, Proposition 1.3.2] for the case of left Leibniz algebras), but one needs to be careful since the Cartan relations do only hold for  $q \geq 1$ . Therefore, for an arbitrary bimodule  $M$ ,  $\mathfrak{Q}$  will act non-trivially on  $\text{HL}^0(\mathfrak{L}, M)$ . In case the bimodule  $M$  is symmetric however, the action is indeed trivial on  $\text{HL}^0(\mathfrak{L}, M)$ . This ends the proof of Theorem 3.5.  $\square$

#### 4. COHOMOLOGY OF SEMI-SIMPLE LEIBNIZ ALGEBRAS

The first result in this section is the Leibniz analogue of Whitehead's vanishing theorem for the Chevalley-Eilenberg cohomology of finite-dimensional semi-simple Lie algebras over a field of characteristic zero (see [11, Theorem 24.1] or [21, Theorem 10]).

**Theorem 4.1.** *Let  $\mathfrak{L}$  be a left Leibniz algebra over a field of characteristic zero such that the canonical Lie algebra  $\mathfrak{L}_{\text{Lie}}$  associated to  $\mathfrak{L}$  is finite-dimensional and semi-simple. If  $M$  is a finite-dimensional  $\mathfrak{L}$ -bimodule such that  $M^{\mathfrak{L}} = 0$ , then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

*Proof.* As  $M^{\mathfrak{L}} = 0$ , we obtain from Lemma 1.1 that  $M$  is a symmetric  $\mathfrak{L}$ -bimodule and  $\text{Leib}(\mathfrak{L}) \subseteq \text{Ann}_{\mathfrak{L}}^{\text{bi}}(M)$ . In particular, we have that  $M$  is a symmetric  $\mathfrak{L}_{\text{Lie}}$ -bimodule. Since  $M$  is a symmetric  $\mathfrak{L}$ -bimodule, we conclude from [17, Proposition 4.1 and Corollary 4.2 (a)] that  $M^{\mathfrak{L}_{\text{Lie}}} = 0$ . Hence Whitehead's vanishing theorem for the Chevalley-Eilenberg cohomology of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero implies that  $\text{H}^n(\mathfrak{L}_{\text{Lie}}, M) = 0$  for every non-negative integer  $n$ . Since  $M$  is a symmetric  $\mathfrak{L}_{\text{Lie}}$ -bimodule, it follows from Theorem 2.6 that  $\text{HL}^n(\mathfrak{L}_{\text{Lie}}, M) = 0$  for every non-negative integer  $n$ . As a consequence, for the spectral sequence in Theorem 3.2 we have that  $E_2^{p,q} = 0$  for all non-negative integers  $p$  and  $q$ . Hence the relative cohomology  $\text{HL}^n(\mathfrak{L}|\mathfrak{L}_{\text{Lie}}, M)$  vanishes for every non-negative integer  $n$ , and therefore we obtain from [17, Proposition 4.1] and Proposition 3.1 that  $\text{HL}^n(\mathfrak{L}, M) \cong \text{HL}^n(\mathfrak{L}_{\text{Lie}}, M) = 0$  for every non-negative integer  $n$ .  $\square$

Next, we derive the following generalization of Ntolo's vanishing theorem for the Leibniz cohomology of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero (see [32, Théorème 2.6]). We essentially proceed along the lines of Pirashvili's argument for the analogous result on Leibniz homology (see the proof

of [33, Proposition 2.1]), but in the first step of our proof we also use Theorem 4.1 twice. Recall that a left Leibniz algebra  $\mathfrak{L}$  is called *semi-simple* if  $\text{Leib}(\mathfrak{L})$  contains every solvable ideal of  $\mathfrak{L}$  (see [17, Section 7]).

**Theorem 4.2.** *Let  $\mathfrak{L}$  be a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, and let  $M$  be a finite-dimensional  $\mathfrak{L}$ -bimodule. Then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 2$ , and there is a five-term exact sequence*

$$0 \rightarrow M_0 \rightarrow \text{HL}^0(\mathfrak{L}, M) \rightarrow M_{\text{sym}}^{\mathfrak{L}\text{Lie}} \rightarrow \text{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\text{ad}, \ell}, M_0) \rightarrow \text{HL}^1(\mathfrak{L}, M) \rightarrow 0.$$

Moreover, if  $M$  is symmetric, then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 1$ .

*Proof.* The proof is divided into three steps. First, we will prove the assertion for symmetric  $\mathfrak{L}$ -bimodules. So suppose that  $M$  is symmetric. Since  $M$  is finite-dimensional, it has a composition series. It is clear that sub-bimodules and homomorphic images of a symmetric bimodule are again symmetric. By using the long exact cohomology sequence, it is therefore enough to prove the second part of the theorem for finite-dimensional irreducible symmetric  $\mathfrak{L}$ -bimodules. So suppose now in addition that  $M$  is irreducible and non-trivial. Then we obtain from Corollary 1.3 that  $M^{\mathfrak{L}} = 0$ , and thus Theorem 4.1 yields that  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ . Finally, suppose that  $M = \mathbb{F}$  is the trivial irreducible  $\mathfrak{L}$ -bimodule. In this case it follows from Corollary 1.6 that  $\text{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \text{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s)$  for every integer  $n \geq 1$ . Since  $\mathfrak{L}_{\text{Lie}}$  is perfect, we obtain from [17, Proposition 4.1], Corollary 1.6, and [17, Corollary 4.4 (a)] that

$$(\mathfrak{L}^*)_s \cong \text{HL}^0(\mathfrak{L}, (\mathfrak{L}^*)_s) \cong \text{HL}^1(\mathfrak{L}, \mathbb{F}) \cong \text{H}^1(\mathfrak{L}_{\text{Lie}}, \mathbb{F}) = 0.$$

Therefore another application of Theorem 4.1 yields that

$$\text{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \text{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^*)_s) = 0$$

for every integer  $n \geq 1$ . This finishes the proof for symmetric  $\mathfrak{L}$ -bimodules.

If  $M$  is anti-symmetric, then we obtain the assertion from Lemma 1.5 (b) and the statement for symmetric bimodules. Finally, if  $M$  is arbitrary, then in the short exact sequence  $0 \rightarrow M_0 \rightarrow M \rightarrow M_{\text{sym}} \rightarrow 0$  the first term is anti-symmetric and the third term is symmetric (cf. [17, Proposition 3.12 and Proposition 3.13]). Hence another application of the long exact cohomology sequence in conjunction with the statement for the anti-symmetric and the symmetric case yields  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 2$ . Now we obtain the five-term exact sequence from the long exact cohomology sequence together with [17, Corollary 4.2], [17, Corollary 4.4 (b)], and the symmetric case.  $\square$

Note that Theorem 4.2 contains [17, Theorem 7.15] as the special case  $n = 1$  and the second Whitehead lemma for Leibniz algebras as the special case  $n = 2$ . Moreover, by the same argument as for Lie algebras (see [25, p. 87]) we obtain from the second Whitehead lemma for Leibniz algebras another proof of Levi's radical splitting theorem for Leibniz algebras (see [4, Theorem 1]).

The following example shows that [32, Théorème 2.6] (and therefore also Theorem 4.2) does not hold over fields of prime characteristic.

**Example A.** Let  $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$  be the three-dimensional simple Lie algebra of traceless  $2 \times 2$  matrices over a field  $\mathbb{F}$  of characteristic  $p > 2$ . Moreover, let  $\mathbb{F}_p$  denote the field with  $p$  elements, and let  $L(n)$  ( $n \in \mathbb{F}_p$ ) denote the irreducible restricted  $\mathfrak{g}$ -module of highest weight  $n$ . (If the ground field  $\mathbb{F}$  is algebraically closed, these

modules represent all isomorphism classes of restricted irreducible  $\mathfrak{g}$ -modules.) It is well known (see [14, Theorem 4]) that  $H^\bullet(\mathfrak{g}, L(n)) = 0$  for any  $n \not\equiv p-2 \pmod{p}$  and  $H^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \cong H^2(\mathfrak{g}, L(p-2))$ . (Note that by virtue of [14, Theorem 2],  $H^\bullet(\mathfrak{g}, M) = 0$  for every non-restricted irreducible  $\mathfrak{g}$ -module.)

We obtain from [17, Corollary 4.4 (a)] that

$$\mathrm{HL}^1(\mathfrak{g}, L(p-2)_s) \cong H^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \neq 0.$$

Moreover, it follows from Proposition 2.2 that

$$0 \neq \mathbb{F}^2 \cong H^2(\mathfrak{g}, L(p-2)) \hookrightarrow \mathrm{HL}^2(\mathfrak{g}, L(p-2)_s).$$

In particular, this shows that [32, Théorème 2.6] (and therefore also Theorem 4.2) is not true over fields of prime characteristic.

**Remark.** By using more sophisticated tools one can also say something about the Leibniz cohomology of anti-symmetric irreducible  $\mathfrak{g}$ -bimodules, where again  $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$ . It follows from Lemma 1.5 (b) in conjunction with [17, Corollary 4.2 (a)] and [17, Corollary 4.4 (a)], respectively:

$$\mathrm{HL}^1(\mathfrak{g}, L(n)_a) \cong \mathrm{HL}^0(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))_s) \cong \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))^{\mathfrak{g}}$$

and

$$\mathrm{HL}^2(\mathfrak{g}, L(n)_a) \cong \mathrm{HL}^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))_s) \cong H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n))).$$

Since  $\mathfrak{g} \cong L(2)$  is a self-dual  $\mathfrak{g}$ -module, we have the following isomorphisms of  $\mathfrak{g}$ -modules:

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{g}, L(n)) \cong L(2) \otimes L(n).$$

Let us first consider the case  $p > 3$ . Then we obtain from the modular Clebsch-Gordan rule (see [7, Theorem 1.11 (a)] or Satz a) in Chapter 5 of [19]) that

$$L(2) \otimes L(2) \cong L(4) \oplus L(2) \oplus L(0)$$

and

$$L(2) \otimes L(p-4) \cong \begin{cases} L(3) \oplus L(1) & \text{if } p = 5 \\ L(p-2) \oplus L(p-4) \oplus L(p-6) & \text{if } p \geq 7 \end{cases}$$

Hence we conclude for  $p > 3$  that

$$\mathrm{HL}^1(\mathfrak{g}, L(2)_a) \cong (L(2) \otimes L(2))^{\mathfrak{g}} \cong L(0)^{\mathfrak{g}} \cong \mathbb{F} \neq 0$$

and

$$\mathrm{HL}^2(\mathfrak{g}, L(p-4)_a) \cong H^1(\mathfrak{g}, L(2) \otimes L(p-4)) \cong H^1(\mathfrak{g}, L(p-2)) \cong \mathbb{F}^2 \neq 0.$$

Let us now consider  $p = 3$ . Note that in this case  $L(2)$  is the *Steinberg module*, i.e.,  $L(2)$  is the unique projective irreducible restricted  $\mathfrak{g}$ -module. This implies that  $L(2) \otimes L(n)$  is also projective for every highest weight  $n \in \mathbb{F}_3$ . Then we obtain from the modular Clebsch-Gordan rule (cf. [7, Theorem 1.11 (b) and (c)] or Satz b) and c) in Chapter 5 of [19]) for  $p = 3$  that

$$L(2) \otimes L(n) \cong \begin{cases} L(2) & \text{if } n \equiv 0 \pmod{3} \\ P(1) & \text{if } n \equiv 1 \pmod{3}, \\ P(0) \oplus L(2) & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

where  $P(n)$  denotes the projective cover (and at the same time also the injective hull) of  $L(n)$ . As a consequence, we have that

$$(L(2) \otimes L(n))^{\mathfrak{g}} \cong \begin{cases} \mathbb{F} & \text{if } n \equiv 2 \pmod{3} \\ 0 & \text{if } n \not\equiv 2 \pmod{3} \end{cases}$$

Therefore, we obtain that

$$\mathrm{HL}^1(\mathfrak{g}, L(2)_a) \cong (L(2) \otimes L(2))^{\mathfrak{g}} \cong P(0)^{\mathfrak{g}} \cong \mathbb{F} \neq 0.$$

Moreover, by using the six-term exact sequence relating Hochschild's cohomology of a restricted Lie algebra to its Chevalley-Eilenberg cohomology (see [20, p. 575]), we also conclude that

$$\mathrm{HL}^2(\mathfrak{g}, L(2)_a) \cong \mathrm{H}^1(\mathfrak{g}, L(2) \otimes L(2)) \cong \mathbb{F}^3 \neq 0.$$

The next example shows that [32, Théorème 2.6] (and therefore also Theorem 4.2) does not hold for infinite-dimensional modules.

**Example B.** Let  $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$  be the three-dimensional simple complex Lie algebra of traceless  $2 \times 2$  matrices, and let  $V(\lambda)$  ( $\lambda \in \mathbb{C}$ ) denote the *Verma module* of highest weight  $\lambda$ . (Here we identify every complex multiple of the unique fundamental weight with its coefficient.) Verma modules are infinite-dimensional indecomposable  $\mathfrak{g}$ -modules (see, for example, [24, Theorem 20.2 (e)]). Furthermore, it is well known (see [24, Exercise 7 (c) in Section 7.2]) that  $V(\lambda)$  is irreducible if, and only if,  $\lambda$  is not a dominant integral weight (i.e., with our identification,  $\lambda$  is not a non-negative integer). Moreover, it follows from [37, Theorem 4.19] that

$$\mathrm{H}^n(\mathfrak{g}, V(\lambda)) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = -2 \text{ and } n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

This in conjunction with [17, Corollary 4.4 (a)] yields that

$$\mathrm{HL}^1(\mathfrak{g}, V(-2)_s) \cong \mathrm{H}^1(\mathfrak{g}, V(-2)) \cong \mathbb{C} \neq 0.$$

Furthermore, it follows from Proposition 2.2 that

$$0 \neq \mathbb{C} \cong \mathrm{H}^2(\mathfrak{g}, V(-2)) \leftrightarrow \mathrm{HL}^2(\mathfrak{g}, V(-2)_s).$$

In particular, this shows that [32, Théorème 2.6] (and therefore also Theorem 4.2) is not true for infinite-dimensional modules.

We obtain as an immediate consequence of Theorem 4.2 the following generalization of [17, Corollary 7.9] (see [17, Corollary 4.5]).

**Corollary 4.3.** *If  $\mathfrak{L}$  is a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero, then  $\mathrm{HL}^n(\mathfrak{L}, \mathbb{F}) = 0$  for every integer  $n \geq 1$ .*

**Remark.** It is well known that the analogue of Corollary 4.3 does not hold for the Chevalley-Eilenberg cohomology of Lie algebras as  $\mathrm{H}^3(\mathfrak{g}, \mathbb{F}) \neq 0$  for any finite-dimensional semi-simple Lie algebra  $\mathfrak{g}$  over a field  $\mathbb{F}$  of characteristic zero (see [11, Theorem 21.1]).

Next, we apply Theorem 4.2 to compute the cohomology of a finite-dimensional semi-simple left Leibniz algebra over a field of characteristic zero with coefficients in its adjoint bimodule and in its (anti-)symmetric counterparts (see Section 1 and [17, Example 3.16] for the definition of the (anti-)symmetric adjoint bimodule  $\mathfrak{L}_s$  resp.  $\mathfrak{L}_a$  of a Leibniz algebra  $\mathfrak{L}$ ) as well as by deriving some consequences.

**Theorem 4.4.** *For every finite-dimensional semi-simple left Leibniz algebra  $\mathfrak{L}$  over a field of characteristic zero the following statements hold:*

(a)

$$\mathrm{HL}^n(\mathfrak{L}, \mathfrak{L}_s) = \begin{cases} \mathrm{Leib}(\mathfrak{L}) & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

(b)

$$\mathrm{HL}^n(\mathfrak{L}, \mathfrak{L}_a) = \begin{cases} \mathfrak{L} & \text{if } n = 0 \\ \mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}) & \text{if } n = 1, \\ 0 & \text{if } n \geq 2 \end{cases}$$

where  $\mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell})$  denotes the vector space of endomorphisms of the left adjoint  $\mathfrak{L}$ -module  $\mathfrak{L}_{\mathrm{ad}, \ell}$ .

(c)

$$\mathrm{HL}^n(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) = \begin{cases} \mathrm{Leib}(\mathfrak{L}) & \text{if } n = 0 \\ \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L})) & \text{if } n = 1, \\ 0 & \text{if } n \geq 2 \end{cases}$$

where  $\mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L}))$  denotes the vector space of homomorphisms from the left adjoint  $\mathfrak{L}$ -module  $\mathfrak{L}_{\mathrm{ad}, \ell}$  to the Leibniz kernel  $\mathrm{Leib}(\mathfrak{L})$  considered as a left  $\mathfrak{L}$ -module.

*Proof.* (a): According to [17, Proposition 4.1] and [17, Proposition 7.5] we have that that  $\mathrm{HL}^0(\mathfrak{L}, \mathfrak{L}_s) = \mathfrak{L}_s^{\mathfrak{L}} = C_{\ell}(\mathfrak{L}) = \mathrm{Leib}(\mathfrak{L})$ . Moreover, we obtain the statement for degree  $n \geq 1$  from the second part of Theorem 4.2.

(b): It follows from [17, Corollary 4.2 (b)] that  $\mathrm{HL}^0(\mathfrak{L}, \mathfrak{L}_a) = \mathfrak{L}$ , and it follows from [17, Corollary 4.4 (b)] that  $\mathrm{HL}^1(\mathfrak{L}, \mathfrak{L}_a) = \mathrm{End}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell})$ . The remainder of the assertion is an immediate consequence of the first part of Theorem 4.2.

(c): As for the symmetric adjoint bimodule, we obtain from [17, Proposition 4.1] and [17, Proposition 7.5] that  $\mathrm{HL}^0(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) = \mathfrak{L}_{\mathrm{ad}}^{\mathfrak{L}} = C_{\ell}(\mathfrak{L}) = \mathrm{Leib}(\mathfrak{L})$ . Next, we consider the short exact sequence

$$(4.1) \quad 0 \rightarrow \mathrm{Leib}(\mathfrak{L}) \rightarrow \mathfrak{L}_{\mathrm{ad}} \rightarrow \mathfrak{L}_{\mathrm{Lie}} \rightarrow 0$$

of  $\mathfrak{L}$ -bimodules. According to [17, Example 3.11] and [17, Proposition 3.13],  $\mathfrak{L}_{\mathrm{Lie}}$  is a symmetric  $\mathfrak{L}$ -bimodule. Consequently, it follows from [17, Corollary 4.2 (a)] and [17, Proposition 7.8] resp. the second part of Theorem 4.2 that  $\mathrm{HL}^0(\mathfrak{L}, \mathfrak{L}_{\mathrm{Lie}}) = \mathfrak{L}_{\mathrm{Lie}}^{\mathfrak{L}_{\mathrm{Lie}}} = C(\mathfrak{L}_{\mathrm{Lie}}) = 0$  and  $\mathrm{HL}^1(\mathfrak{L}, \mathfrak{L}_{\mathrm{Lie}}) = 0$ . Hence the long exact cohomology sequence in conjunction with [17, Corollary 4.4 (b)] yields

$$\mathrm{HL}^1(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) \cong \mathrm{HL}^1(\mathfrak{L}, \mathrm{Leib}(\mathfrak{L})) = \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L})).$$

Finally, the assertion for degree  $n \geq 2$  is again an immediate consequence of the first part of Theorem 4.2.  $\square$

**Remark.** Note that the vanishing part of Theorem 4.4 (c) confirms a generalization of the conjecture at the end of [1]. Moreover, parts (a) and (b) of Theorem 4.4 show that the statements in Theorem 4.2 are best possible.

In particular, one can derive from Theorem 4.4 (c) that finite-dimensional semi-simple non-Lie Leibniz algebras over a field of characteristic zero have outer derivations. In this respect they behave differently than finite-dimensional semi-simple Lie algebras over a field of characteristic zero (see, for example, [24, Theorem 5.3]).

**Corollary 4.5.** *Every finite-dimensional semi-simple non-Lie Leibniz algebra over a field of characteristic zero has derivations that are not inner.*

*Proof.* If one applies the contravariant functor  $\mathrm{Hom}_{\mathbb{F}}(-, \mathrm{Leib}(\mathfrak{L}))$  to the short exact sequence (4.1) considered as a short exact sequence of left  $\mathfrak{L}$ -modules, one obtains the short exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{L})) \rightarrow \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L})) \rightarrow \mathrm{Hom}_{\mathbb{F}}(\mathrm{Leib}(\mathfrak{L}), \mathrm{Leib}(\mathfrak{L})) \rightarrow 0$$

of left  $\mathfrak{L}$ -modules. Then the long exact cohomology sequence in conjunction with Lemma 1.5 (a) yields the long exact sequence

$$\begin{aligned} 0 &\rightarrow \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{L})) \rightarrow \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L})) \rightarrow \mathrm{Hom}_{\mathfrak{L}}(\mathrm{Leib}(\mathfrak{L}), \mathrm{Leib}(\mathfrak{L})) \\ &\rightarrow \widetilde{\mathrm{HL}}^1(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{L}))) = \mathrm{HL}^1(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{\mathrm{Lie}}, \mathrm{Leib}(\mathfrak{L}))_s) \end{aligned}$$

According to the second part of Theorem 4.2, the fourth term is zero. Since the third term contains the identity map, it is non-zero as by hypothesis  $\mathfrak{L}$  is not a Lie algebra. Hence in this case the second term is non-zero, and we obtain from Theorem 4.4 (c) that  $\mathrm{HL}^1(\mathfrak{L}, \mathfrak{L}_{\mathrm{ad}}) \cong \mathrm{Hom}_{\mathfrak{L}}(\mathfrak{L}_{\mathrm{ad}, \ell}, \mathrm{Leib}(\mathfrak{L})) \neq 0$ .  $\square$

Let  $\mathbb{F}$  be an algebraically closed field of characteristic zero, let  $n$  be a non-negative integer, and let  $L_n(\mathbb{F}) \subseteq \mathbb{F}^{n^3}$  denote the affine variety of structure constants of the  $n$ -dimensional left Leibniz algebras over  $\mathbb{F}$  with respect to a fixed basis of  $\mathbb{F}^n$ . Then the general linear group  $\mathrm{GL}_n(\mathbb{F})$  acts on  $L_n(\mathbb{F})$ , and a point (= Leibniz multiplication law)  $\phi \in L_n(\mathbb{F})$  is called *rigid* if the orbit  $\mathrm{GL}_n(\mathbb{F}) \cdot \phi$  is open in  $L_n(\mathbb{F})$ . It follows from Corollary 2.8 that the infinite-dimensional two-sided Witt algebra over an algebraically closed field of characteristic zero is rigid as a Leibniz algebra. By using [6, Théorème 3 in Chapitre 2] in conjunction with Theorem 4.4 (c) one obtains the rigidity of any finite-dimensional semi-simple Lie algebra as a Leibniz algebra.

**Corollary 4.6.** *Every finite-dimensional semi-simple left Leibniz algebra over an algebraically closed field of characteristic zero is rigid as a Leibniz algebra.*

We conclude this section by applying the spectral sequence of Theorem 3.2 to the Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{L})$  of a Leibniz algebra  $\mathfrak{L}$  into a hemi-semidirect product of a semisimple Lie algebra  $\mathfrak{s}$  and its largest solvable ideal  $\mathrm{Rad}(\mathfrak{L})$  (see the main result of [4] or the remark after Theorem 4.2). This will give us a reduction formula for the Leibniz cohomology of a finite-dimensional Leibniz algebra, which can be seen as an analogue of the formula describing the Chevalley-Eilenberg cohomology of a finite-dimensional Lie algebra in [21, Theorem 13]. Our formula is less general as it takes only into account Leibniz cohomology with values in symmetric  $\mathfrak{s}$ -bimodules or, in other words, the radical  $\mathrm{Rad}(\mathfrak{L})$  has to act trivially on the symmetric  $\mathfrak{L}$ -bimodule  $M$ . Under this assumption, we have the following result:

**Theorem 4.7.** *Let  $\mathfrak{L}$  be a finite-dimensional left Leibniz algebra with Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{L})$ , and let  $M$  be a symmetric  $\mathfrak{L}$ -bimodule such that the radical  $\mathrm{Rad}(\mathfrak{L})$  acts trivially on  $M$ . Then for every non-negative integer  $n$  there are isomorphisms*

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \begin{cases} M^{\mathfrak{s}} & \text{if } n = 0 \\ \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathrm{Rad}(\mathfrak{L}), M^{\mathfrak{s}})_s) & \text{if } n \geq 1 \end{cases}$$

of vector spaces.

*Proof.* The given Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \text{Rad}(\mathfrak{L})$  yields for the spectral sequence in Theorem 3.2 the following  $E_2$ -term:

$$E_2^{0,q} = \text{HL}^{q-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\text{Rad}(\mathfrak{L}), M^{\mathfrak{s}})_s),$$

thanks to Theorem 4.2. The spectral sequence collapses, and we obtain

$$\text{HL}^n(\mathfrak{L}|\mathfrak{s}, M) \cong \text{HL}^n(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\text{Rad}(\mathfrak{L}), M^{\mathfrak{s}})_s).$$

On the other hand, the long exact sequence of Proposition 3.1 splits, because the extension given by the Levi decomposition splits. Hence we conclude that

$$\begin{aligned} \text{HL}^n(\mathfrak{L}, M) &\cong \text{HL}^n(\mathfrak{s}, M) \oplus \text{HL}^{n-1}(\mathfrak{L}|\mathfrak{s}, M) \\ &\cong \text{HL}^n(\mathfrak{s}, M) \oplus \text{HL}^{n-1}(\mathfrak{L}, \text{Hom}_{\mathbb{F}}(\text{Rad}(\mathfrak{L}), M^{\mathfrak{s}})_s), \end{aligned}$$

and thus the assertion follows from Theorem 4.2.  $\square$

**Remark.** There is also a theorem similar to Theorem 4.7, but applying the spectral sequence of Corollary 3.6 to the Levi decomposition. In order to use Corollary 3.6, one needs that the solvable radical  $\text{Rad}(\mathfrak{L})$  is contained in the left center of  $\mathfrak{L}$ . It seems that this condition is seldom satisfied.

In the case of the trivial bimodule Theorem 4.7 yields the following refinement of Corollary 1.6.

**Corollary 4.8.** *Let  $\mathfrak{L}$  be a finite-dimensional left Leibniz algebra over a field  $\mathbb{F}$  of characteristic zero. Then for every integer  $n \geq 1$  there are isomorphisms*

$$\text{HL}^n(\mathfrak{L}, \mathbb{F}) \cong \text{HL}^{n-1}(\mathfrak{L}, [\text{Rad}(\mathfrak{L})^*]_s)$$

*of vector spaces.*

Finally, from Theorem 4.7 we obtain immediately the following partial generalization of Theorem 4.2 to not necessarily semi-simple left Leibniz algebras.

**Corollary 4.9.** *Let  $\mathfrak{L}$  be a finite-dimensional left Leibniz algebra over a field of characteristic zero with Levi decomposition  $\mathfrak{L} = \mathfrak{s} \ltimes \text{Rad}(\mathfrak{L})$ , and let  $M$  be a symmetric  $\mathfrak{L}$ -bimodule such that the radical  $\text{Rad}(\mathfrak{L})$  acts trivially on  $M$  and  $M^{\mathfrak{s}} = 0$ . Then  $\text{HL}^n(\mathfrak{L}, M) = 0$ .*

## 5. COHOMOLOGY OF SOLVABLE LEIBNIZ ALGEBRAS

The first result in this section is the Leibniz analogue of Dixmier's vanishing theorem for the Chevalley-Eilenberg cohomology of finite-dimensional nilpotent Lie algebras (see [13, Théorème 1] for infinite ground fields and [3, Lemma 3 or Theorem 1] for arbitrary ground fields). Its proof goes exactly as the proof of Theorem 4.1 except that one replaces Whitehead's vanishing theorem by Dixmier's vanishing theorem.

**Theorem 5.1.** *Let  $\mathfrak{L}$  be a left Leibniz algebra such that the canonical Lie algebra  $\mathfrak{L}_{\text{Lie}}$  associated to  $\mathfrak{L}$  is finite-dimensional and nilpotent. If  $M$  is a finite-dimensional  $\mathfrak{L}$ -bimodule such that  $M^{\mathfrak{L}} = 0$ , then  $\text{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

Recall that a left (or right) Leibniz algebra  $\mathfrak{L}$  is called *nilpotent* if  $\mathfrak{L}_n = 0$  for some positive integer  $n$ , where  $\mathfrak{L}_n$  is defined recursively by  $\mathfrak{L}_1 := \mathfrak{L}$  and  $\mathfrak{L}_n := \sum_{k=1}^{n-1} \mathfrak{L}_k \mathfrak{L}_{n-k}$  for every integer  $n \geq 2$  (see [17, Section 5]).

**Theorem 5.2.** *Let  $\mathfrak{L}$  be a finite-dimensional nilpotent left Leibniz algebra, and let  $M$  be a finite-dimensional non-trivial irreducible  $\mathfrak{L}$ -bimodule. Then  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every positive integer  $n$ . Moreover, if  $M$  is symmetric, then  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

*Proof.* If  $M$  is symmetric, then the assertion follows from Corollary 1.3 and Theorem 5.1. According to [17, Theorem 3.14], we can suppose that  $M$  is anti-symmetric. We obtain from Lemma 1.5 (b) that

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s)$$

for every positive integer  $n$ . By refining the left descending central series of  $\mathfrak{L}$  (see [17, Section 5]), one can construct a composition series

$$\mathfrak{L}_{\mathrm{ad}, \ell} = \mathfrak{L}_n \supset \mathfrak{L}_{n-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

of the left adjoint  $\mathfrak{L}$ -module such that  $\mathfrak{L}_j/\mathfrak{L}_{j-1}$  is the trivial one-dimensional  $\mathfrak{L}$ -module  $\mathbb{F}$  for every integer  $1 \leq j \leq n$ . From the short exact sequences  $0 \rightarrow \mathfrak{L}_{j-1} \rightarrow \mathfrak{L}_j \rightarrow \mathbb{F} \rightarrow 0$ , we obtain by dualizing, tensoring each term with  $M$ , and symmetrizing the short exact sequences:

$$0 \rightarrow M_s \rightarrow (\mathfrak{L}_j^* \otimes M)_s \rightarrow (\mathfrak{L}_{j-1}^* \otimes M)_s \rightarrow 0$$

for every integer  $1 \leq j \leq n$ . Since  $M$  is a non-trivial irreducible left  $\mathfrak{L}$ -module, we conclude that  $M_s$  is a non-trivial irreducible symmetric  $\mathfrak{L}$ -bimodule. Hence we obtain inductively from the long exact cohomology sequence that  $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s) = 0$  for every positive integer  $n$ .  $\square$

Since the Leibniz cohomology of an abelian Lie algebra with trivial coefficients is known, in Example C we compute this cohomology for the smallest non-abelian nilpotent Lie algebra. Note that in [33, Example 1.4. iv)] the corresponding Leibniz homology has been computed. In fact, homology and cohomology of a finite-dimensional Leibniz algebra  $\mathfrak{L}$  with *trivial* coefficients are isomorphic, as we have the duality isomorphism  $\mathrm{CL}_{\bullet}(\mathfrak{L}, \mathbb{F})^* \cong \mathrm{CL}^{\bullet}(\mathfrak{L}, \mathbb{F})$  already on the level of cochain complexes. Therefore our results coincide with those of Pirashvili. We furthermore compute in Example D the Leibniz cohomology of the smallest nilpotent non-Lie Leibniz algebra with coefficients in the trivial bimodule.

**Example C.** Let  $\mathbb{F}$  denote an arbitrary field of characteristic  $\neq 2$ , and let  $\mathfrak{h} := \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}z$  be the three-dimensional Heisenberg algebra over  $\mathbb{F}$  with multiplication determined by  $xy = z = -yx$ . Then the Chevalley-Eilenberg cohomology of  $\mathfrak{h}$  with coefficients in the trivial module  $\mathbb{F}$  is well-known:

$$\dim_{\mathbb{F}} H^n(\mathfrak{h}, \mathbb{F}) = \begin{cases} 1 & \text{if } n = 0, 3 \\ 2 & \text{if } n = 1, 2 \\ 0 & \text{if } n \geq 4 \end{cases}$$

Hence it follows from [17, Corollary 4.2] that  $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{h}, \mathbb{F}) = 1$  and from [17, Corollary 4.4 (a)] that  $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{h}, \mathbb{F}) = 2$ .

As  $H^n(\mathfrak{h}, \mathbb{F}) = 0$  for every integer  $n \geq 4$ , we obtain from Proposition 2.2 the following six-term exact sequence:

$$0 \rightarrow H^2(\mathfrak{h}, \mathbb{F}) \rightarrow \mathrm{HL}^2(\mathfrak{h}, \mathbb{F}) \rightarrow H_{\mathrm{rel}}^0(\mathfrak{h}, \mathbb{F}) \rightarrow H^3(\mathfrak{h}, \mathbb{F}) \rightarrow \mathrm{HL}^3(\mathfrak{h}, \mathbb{F}) \rightarrow H_{\mathrm{rel}}^1(\mathfrak{h}, \mathbb{F}) \rightarrow 0$$

and

$$\mathrm{HL}^n(\mathfrak{h}, \mathbb{F}) \cong H_{\mathrm{rel}}^{n-2}(\mathfrak{h}, \mathbb{F})$$

for every integer  $n \geq 4$ .

Since we assume that  $\text{char}(\mathbb{F}) \neq 2$ , it follows from the remark after Proposition 2.2 that we can identify  $H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F})$  with the space of invariant symmetric bilinear forms on  $\mathfrak{h}$  and the map  $H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F}) \rightarrow H^3(\mathfrak{h}, \mathbb{F})$  with the classical Cartan-Koszul map. It is easy to see that the latter map is zero for the Heisenberg algebra, which yields the surjectivity of the map  $\text{HL}^2(\mathfrak{h}, \mathbb{F}) \rightarrow H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F})$  and the injectivity of the map  $H^3(\mathfrak{h}, \mathbb{F}) \rightarrow \text{HL}^3(\mathfrak{h}, \mathbb{F})$ . As a consequence, we obtain the following two short exact sequences:

$$\begin{aligned} 0 \rightarrow H^2(\mathfrak{h}, \mathbb{F}) \rightarrow \text{HL}^2(\mathfrak{h}, \mathbb{F}) \rightarrow H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F}) \rightarrow 0, \\ 0 \rightarrow H^3(\mathfrak{h}, \mathbb{F}) \rightarrow \text{HL}^3(\mathfrak{h}, \mathbb{F}) \rightarrow H_{\text{rel}}^1(\mathfrak{h}, \mathbb{F}) \rightarrow 0. \end{aligned}$$

In order to compute  $H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F})$  and  $H_{\text{rel}}^1(\mathfrak{h}, \mathbb{F})$ , we need the coadjoint Chevalley-Eilenberg cohomology of  $\mathfrak{h}$ . We have that  $\dim_{\mathbb{F}} H^0(\mathfrak{h}, \mathfrak{h}^*) = 2$ ,  $\dim_{\mathbb{F}} H^1(\mathfrak{h}, \mathfrak{h}^*) = 5$ ,  $\dim_{\mathbb{F}} H^2(\mathfrak{h}, \mathfrak{h}^*) = 4$ , and  $H^3(\mathfrak{h}, \mathfrak{h}^*) = 1$ . (This can be computed directly but for the complex numbers as a ground field it also follows from the main result of [31] in conjunction with [37, Theorem 3.4].)

Similar to the discussion of the consequences of Proposition 2.2 above, we obtain from Proposition 2.1 the two short exact sequences

$$\begin{aligned} 0 \rightarrow H^2(\mathfrak{h}, \mathbb{F}) \rightarrow H^1(\mathfrak{h}, \mathfrak{h}^*) \rightarrow \text{HR}^0(\mathfrak{h}) \rightarrow 0, \\ 0 \rightarrow H^3(\mathfrak{h}, \mathbb{F}) \rightarrow H^2(\mathfrak{h}, \mathfrak{h}^*) \rightarrow \text{HR}^1(\mathfrak{h}) \rightarrow 0, \end{aligned}$$

the isomorphism  $\text{HR}^2(\mathfrak{h}) \cong H^3(\mathfrak{h}, \mathfrak{h}^*)$ , and  $\text{HR}^n(\mathfrak{h}) = 0$  for every integer  $n \geq 3$ .

From these two short exact sequences and the isomorphism we derive that

$$\begin{aligned} \dim_{\mathbb{F}} \text{HR}^0(\mathfrak{h}) &= \dim_{\mathbb{F}} H^1(\mathfrak{h}, \mathfrak{h}^*) - \dim_{\mathbb{F}} H^2(\mathfrak{h}, \mathbb{F}) = 5 - 2 = 3, \\ \dim_{\mathbb{F}} \text{HR}^1(\mathfrak{h}) &= \dim_{\mathbb{F}} H^2(\mathfrak{h}, \mathfrak{h}^*) - \dim_{\mathbb{F}} H^3(\mathfrak{h}, \mathbb{F}) = 4 - 1 = 3, \end{aligned}$$

and

$$\dim_{\mathbb{F}} \text{HR}^2(\mathfrak{h}) = \dim_{\mathbb{F}} H^3(\mathfrak{h}, \mathfrak{h}^*) = 1,$$

respectively. Therefore we obtain from  $H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F}) \cong \text{HR}^0(\mathfrak{h})$  that

$$\dim_{\mathbb{F}} \text{HL}^2(\mathfrak{h}, \mathbb{F}) = \dim_{\mathbb{F}} H^2(\mathfrak{h}, \mathbb{F}) + \dim_{\mathbb{F}} H_{\text{rel}}^0(\mathfrak{h}, \mathbb{F}) = 2 + 3 = 5.$$

Now we want to apply the spectral sequence of Theorem 2.5. For this let us compute the differential

$$d_2^{0,1} : E_2^{0,1} = \text{HR}^0(\mathfrak{h}) \otimes \text{HL}^1(\mathfrak{h}, \mathbb{F}) \rightarrow E_2^{2,0} = \text{HR}^2(\mathfrak{h}) \otimes \text{HL}^0(\mathfrak{h}, \mathbb{F}).$$

In characteristic  $\neq 2$ , an element of  $\text{HR}^0(\mathfrak{h})$  is an invariant symmetric bilinear form  $\omega$ . It is considered as a 1-cochain with values in  $\mathfrak{h}^*$  and, as it is a representative of an element of a quotient cochain complex, it is zero in case it is skew-symmetric in all entries. Take furthermore a cocycle  $c \in \text{CL}^1(\mathfrak{h}, \mathbb{F})$  and compute for three elements  $r, s, t \in \mathfrak{h}$ :

$$\begin{aligned} d^1(\omega \otimes c)(r, s, t) &= \omega(rs, -)c(t) + \omega(s, -)c(rt) - \omega(r, -)c(st) + \\ &+ \omega(s, r-)c(t) - \omega(r, s-)c(t) + \omega(r, t-)c(s) \end{aligned}$$

Now as  $c$  is a cocycle with trivial coefficients,  $c$  vanishes on products, thus the second and third terms are zero. Furthermore, the first and fourth term cancel by the invariance of the form and skew-symmetry of the Lie product. We are left with the two last terms  $-\omega(r, s-)c(t) + \omega(r, t-)c(s)$ , which are skew-symmetric in the

three entries of the element in  $\mathrm{HR}^2(\mathfrak{h})$  and vanish therefore as well. In conclusion, the differential  $d_2^{0,1}$  is zero, and we have that

$$\mathrm{H}_{\mathrm{rel}}^1(\mathfrak{h}) = \mathrm{HR}^0(\mathfrak{h}) \otimes \mathrm{HL}^1(\mathfrak{h}, \mathbb{F}) \oplus \mathrm{HR}^1(\mathfrak{h}) \otimes \mathrm{HL}^0(\mathfrak{h}, \mathbb{F}).$$

This implies in turn

$$\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{h}, \mathbb{F}) = \dim_{\mathbb{F}} \mathrm{H}^3(\mathfrak{h}, \mathbb{F}) + \dim_{\mathbb{F}} \mathrm{H}_{\mathrm{rel}}^1(\mathfrak{h}, \mathbb{F}) = 1 + 9 = 10.$$

It seems that all differentials  $d_2$  are zero and thus that this scheme persists to yield the dimensions of the higher  $\mathrm{H}_{\mathrm{rel}}^n(\mathfrak{h}, \mathbb{F})$  and thus of  $\mathrm{HL}^n(\mathfrak{h}, \mathbb{F})$  (see the dimension formula in [33, Example 1.4 iv]).

**Remark.** As a by-product of the above computations we obtain that the space  $[S^2(\mathfrak{h})^*]^{\mathfrak{h}}$  of invariant symmetric bilinear forms on  $\mathfrak{h}$  is three-dimensional when  $\mathrm{char}(\mathbb{F}) \neq 2$ .

**Example D.** Let  $\mathbb{F}$  denote an arbitrary field, and let  $\mathfrak{N} := \mathbb{F}e \oplus \mathbb{F}f$  be the two-dimensional nilpotent left (and right) Leibniz algebra over  $\mathbb{F}$  with multiplication determined by  $ff = e$  (see [17, Example 2.4]). Then  $\mathrm{Leib}(\mathfrak{N}) = \mathbb{F}e$ , and thus  $\mathfrak{N}_{\mathrm{Lie}}$  is a one-dimensional abelian Lie algebra. Hence  $\mathrm{HL}^n(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \cong \mathbb{F}$  for every non-negative integer  $n$ .

Then it follows from [17, Corollary 4.2] that  $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = 1$  and from [17, Corollary 4.4 (a)] that  $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = 1$ .

We compute the higher cohomology with the help of the spectral sequence of Corollary 3.6. As observed in the remark after the proof of Theorem 3.5, all higher differentials are zero in our case, because the differential of the abelian Lie algebra with values in the trivial module vanishes. With the input data  $\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}, \mathbb{F}) = 1$  and  $\dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}, \mathbb{F}) = 1$ , we therefore get from the spectral sequence

$$\dim_{\mathbb{F}} \mathrm{HL}^0(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 1 \quad \text{and} \quad \dim_{\mathbb{F}} \mathrm{HL}^1(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 2.$$

In order to apply now the long exact sequence from Proposition 3.1 and deduce the dimensions of the cohomology spaces from here, we want to argue that the sequence is split. In fact, it is split, because the connecting homomorphism is surjective. This comes from the fact that the complex  $\mathrm{CL}^{\bullet}(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F})$  is one-dimensional in each degree and a generator can be hit via the connecting homomorphism which is easy to see directly (take a cochain in  $\mathrm{CL}^n(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F})$  represented by an element in  $\mathrm{CL}^{n+1}(\mathfrak{N}, \mathbb{F})$  with exactly one slot in  $e^*$  at the first place: the Leibniz product in this slot gives the only non-zero contribution). The long exact sequence from Proposition 3.1 splits into short exact sequences

$$0 \rightarrow \mathrm{HL}^n(\mathfrak{N}, \mathbb{F}) \rightarrow \mathrm{HL}^{n-1}(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \rightarrow \mathrm{HL}^{n+1}(\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) \rightarrow 0,$$

starting from  $n = 2$ , where the right-hand term is one-dimensional. These short exact sequences, together with the spectral sequence where all differentials are zero, permit to determine all relative and absolute cohomology spaces. For example, we obtain  $\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}, \mathbb{F}) = 1$ , and then  $\dim_{\mathbb{F}} \mathrm{HL}^2(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 3$ ,  $\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{N}, \mathbb{F}) = 2$ , and then  $\dim_{\mathbb{F}} \mathrm{HL}^3(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 5$ , and so on. In general, we obtain by induction that  $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}, \mathbb{F}) = 2^{n-2}$  for every integer  $n \geq 2$  and  $\dim_{\mathbb{F}} \mathrm{HL}^n(\mathfrak{N}|\mathfrak{N}_{\mathrm{Lie}}, \mathbb{F}) = 2^{n-1} + 1$  for every integer  $n \geq 1$ .

The next result is the Leibniz analogue of Barnes' vanishing theorem for the Chevalley-Eilenberg cohomology of supersolvable Lie algebras (see [3, Theorem 3]).

We say that a (left or right) Leibniz algebra  $\mathfrak{L}$  is *supersolvable* if there exists a chain

$$\mathfrak{L} = \mathfrak{L}_n \supset \mathfrak{L}_{n-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

of ideals of  $\mathfrak{L}$  such that  $\dim_{\mathbb{F}} \mathfrak{L}_j / \mathfrak{L}_{j-1} = 1$  for every integer  $1 \leq j \leq n$ . Note that as usual finite-dimensional nilpotent Leibniz algebras are supersolvable and supersolvable Leibniz algebras are solvable. Moreover, over algebraically closed fields of characteristic zero, every finite-dimensional solvable Leibniz algebra is supersolvable (see [17, Corollary 6.7]). Finally, as for Lie algebras, it is not difficult to see that subalgebras and homomorphic images of supersolvable Leibniz algebras are again supersolvable.

**Theorem 5.3.** *Let  $\mathfrak{L}$  be a finite-dimensional supersolvable left Leibniz algebra over a field  $\mathbb{F}$ , and let  $M$  be a finite-dimensional irreducible  $\mathfrak{L}$ -bimodule such that  $\dim_{\mathbb{F}} M \neq 1$ . Then  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every positive integer  $n$ . Moreover, if  $M$  is symmetric, then  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

*Proof.* First, suppose that  $M$  is symmetric. Since homomorphic images of supersolvable Leibniz algebras are supersolvable, it follows that  $\mathfrak{L}_{\mathrm{Lie}}$  is supersolvable. Since  $M$  is a symmetric  $\mathfrak{L}$ -bimodule, we obtain from [17, Lemma 3.10] that  $\mathrm{Leib}(\mathfrak{L}) \subseteq \mathrm{Ann}_{\mathfrak{L}}^{\mathrm{bi}}(M)$ . In particular,  $M$  is an irreducible symmetric  $\mathfrak{L}_{\mathrm{Lie}}$ -bimodule such that  $\dim_{\mathbb{F}} M \neq 1$ . Then [3, Theorem 3] implies that  $\mathrm{H}^n(\mathfrak{L}_{\mathrm{Lie}}, M) = 0$  for every non-negative integer  $n$ . Since  $M$  is a symmetric  $\mathfrak{L}_{\mathrm{Lie}}$ -bimodule, we thus obtain from Theorem 2.6 that  $\mathrm{HL}^n(\mathfrak{L}_{\mathrm{Lie}}, M) = 0$  for every non-negative integer  $n$ . Hence one can proceed exactly as in the proof of Theorem 4.1 to conclude that  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every integer  $n \geq 0$ .

Now suppose that  $M$  is not symmetric. Then it follows from [17, Theorem 3.14] that  $M$  is anti-symmetric. We obtain from Lemma 1.5 (b) that

$$\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}, M)_s) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s)$$

for every positive integer  $n$ . By definition of supersolvability, the left adjoint  $\mathfrak{L}$ -module has a composition series

$$\mathfrak{L}_{\mathrm{ad}, \ell} = \mathfrak{L}_n \supset \mathfrak{L}_{n-1} \supset \cdots \supset \mathfrak{L}_1 \supset \mathfrak{L}_0 = 0$$

such that  $\dim_{\mathbb{F}} \mathfrak{L}_j / \mathfrak{L}_{j-1} = 1$  for every integer  $1 \leq j \leq n$ . From the short exact sequences  $0 \rightarrow \mathfrak{L}_{j-1} \rightarrow \mathfrak{L}_j \rightarrow \mathfrak{L}_j / \mathfrak{L}_{j-1} \rightarrow 0$ , we obtain by dualizing, tensoring each term with  $M$ , and symmetrizing the short exact sequences:

$$0 \rightarrow [(\mathfrak{L}_j / \mathfrak{L}_{j-1})^* \otimes M]_s \rightarrow (\mathfrak{L}_j^* \otimes M)_s \rightarrow (\mathfrak{L}_{j-1}^* \otimes M)_s \rightarrow 0$$

for every integer  $1 \leq j \leq n$ . Since  $M$  is irreducible and  $\dim_{\mathbb{F}} \mathfrak{L}_j / \mathfrak{L}_{j-1} = 1$ , we conclude that  $[(\mathfrak{L}_j / \mathfrak{L}_{j-1})^* \otimes M]_s$  is an irreducible symmetric  $\mathfrak{L}$ -bimodule. Moreover, we have that  $\dim_{\mathbb{F}} [(\mathfrak{L}_j / \mathfrak{L}_{j-1})^* \otimes M]_s \neq 1$  as  $\dim_{\mathbb{F}} M \neq 1$ . Hence we obtain inductively from the long exact cohomology sequence that  $\mathrm{HL}^n(\mathfrak{L}, M) \cong \mathrm{HL}^{n-1}(\mathfrak{L}, (\mathfrak{L}^* \otimes M)_s) = 0$  for every positive integer  $n$ .  $\square$

**Remark.** It follows from Lie's theorem for Leibniz algebras that every finite-dimensional irreducible bimodule of a finite-dimensional solvable left Leibniz algebra over an algebraically closed field of characteristic zero is one-dimensional (see [17, Corollary 6.5 (a)]). Consequently, in this case the hypothesis of Theorem 5.3 is never satisfied, and thus this result is only applicable over non-algebraically closed fields of characteristic zero or over fields of prime characteristic.

The following two examples give a first indication what to expect for the Leibniz cohomology of supersolvable Leibniz algebras with coefficients in one-dimensional bimodules, and thereby complementing Theorem 5.3. We begin by computing this Leibniz cohomology for the smallest non-nilpotent Lie algebra. Note that for a ground field of characteristic 2 the Leibniz cohomology of this Lie algebra is far more complicated than for a field of characteristic  $\neq 2$ .

**Example E.** Let  $\mathbb{F}$  denote an arbitrary field, and let  $\mathfrak{a} := \mathbb{F}h \oplus \mathbb{F}e$  be the non-abelian two-dimensional Lie algebra over  $\mathbb{F}$  with multiplication determined by  $he = e = -eh$ . For any scalar  $\lambda \in \mathbb{F}$  one can define a one-dimensional left  $\mathfrak{a}$ -module  $F_\lambda := \mathbb{F}1_\lambda$  with  $\mathfrak{a}$ -action defined by  $h \cdot 1_\lambda := \lambda 1_\lambda$  and  $e \cdot 1_\lambda := 0$ . Then the Chevalley-Eilenberg cohomology of  $\mathfrak{a}$  with coefficients in  $F_\lambda$  is as follows:

$$H^n(\mathfrak{a}, F_\lambda) \cong \begin{cases} \mathbb{F} & \text{if } \lambda = 0 \text{ and } n = 0, 1 \text{ or } \lambda = 1 \text{ and } n = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

In particular, if  $\lambda \neq 0, 1$ , then  $H^\bullet(\mathfrak{a}, F_\lambda) = 0$ .

First, let us consider  $F_\lambda$  as a symmetric  $\mathfrak{a}$ -bimodule  $(F_\lambda)_s$ . Then it follows from Theorem 2.6 that  $HL^\bullet(\mathfrak{a}, (F_\lambda)_s) = 0$  for  $\lambda \neq 0, 1$ .

In order to be able to compute the Leibniz cohomology for  $\lambda = 0, 1$ , and for the anti-symmetric  $\mathfrak{a}$ -bimodules  $(F_\lambda)_a$ , let  $M$  be an arbitrary left  $\mathfrak{a}$ -module considered as a symmetric  $\mathfrak{a}$ -bimodule  $M_s$ . Since  $H^n(\mathfrak{a}, M) = 0$  for every integer  $n \geq 3$ , we obtain from Proposition 2.2 the short exact sequence

$$(5.1) \quad 0 \rightarrow H^2(\mathfrak{a}, M) \rightarrow HL^2(\mathfrak{a}, M_s) \rightarrow H_{\text{rel}}^0(\mathfrak{a}, M) \rightarrow 0$$

and the isomorphisms

$$(5.2) \quad HL^n(\mathfrak{a}, M_s) \cong H_{\text{rel}}^{n-2}(\mathfrak{a}, M)$$

for every integer  $n \geq 3$ . Moreover, it follows from [17, Corollary 4.2 (a)] and [17, Corollary 4.4 (a)], respectively, that  $HL^0(\mathfrak{a}, M_s) \cong M^a$  and  $HL^1(\mathfrak{a}, M_s) \cong H^1(\mathfrak{a}, M)$ .

For the computation of the relative cohomology spaces  $H_{\text{rel}}^n(\mathfrak{a}, M)$  we need the coadjoint Chevalley-Eilenberg cohomology of  $\mathfrak{a}$ . It is easy to verify that

$$\dim_{\mathbb{F}} H^0(\mathfrak{a}, \mathfrak{a}^*) = 1,$$

$$\dim_{\mathbb{F}} H^1(\mathfrak{a}, \mathfrak{a}^*) = \begin{cases} 2 & \text{if } \text{char}(\mathbb{F}) = 2 \\ 1 & \text{if } \text{char}(\mathbb{F}) \neq 2, \end{cases}$$

and

$$\dim_{\mathbb{F}} H^2(\mathfrak{a}, \mathfrak{a}^*) = \begin{cases} 1 & \text{if } \text{char}(\mathbb{F}) = 2 \\ 0 & \text{if } \text{char}(\mathbb{F}) \neq 2. \end{cases}$$

Consequently, we have to consider the cases  $\text{char}(\mathbb{F}) = 2$  and  $\text{char}(\mathbb{F}) \neq 2$  differently.

Let us first assume that  $\text{char}(\mathbb{F}) \neq 2$ . Then it follows from Proposition 2.1 that  $HR^0(\mathfrak{a}) \cong H^1(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}$  and  $HR^n(\mathfrak{a}) = 0$  for every integer  $n \geq 1$ . Hence the spectral sequence of Theorem 2.5 implies that  $H_{\text{rel}}^n(\mathfrak{a}, M) \cong HL^n(\mathfrak{a}, M_s)$  for every non-negative integer  $n$ . In conclusion, we obtain that

$$(5.3) \quad HL^2(\mathfrak{a}, M_s) \cong M^a \oplus H^2(\mathfrak{a}, M)$$

and

$$(5.4) \quad HL^n(\mathfrak{a}, M_s) \cong HL^{n-2}(\mathfrak{a}, M_s) \text{ for every integer } n \geq 3.$$

As an immediate consequence, we deduce when  $\text{char}(\mathbb{F}) \neq 2$  that  $\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, \mathbb{F}) = 1$  for every non-negative integer  $n$  and

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_1)_s) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$$

In summary, we have for the Leibniz cohomology of  $\mathfrak{a}$  over a field  $\mathbb{F}$  of characteristic  $\neq 2$  with coefficients in a one-dimensional symmetric bimodule that

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or if } \lambda = 1 \text{ and } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Next, let us assume that  $\text{char}(\mathbb{F}) = 2$ . Then it follows from Proposition 2.1 that  $\text{HR}^0(\mathfrak{a}) \cong \text{H}^1(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}^2$ ,  $\text{HR}^1(\mathfrak{a}) \cong \text{H}^2(\mathfrak{a}, \mathfrak{a}^*) \cong \mathbb{F}$ , and  $\text{HR}^n(\mathfrak{a}) = 0$  for every integer  $n \geq 2$ . Hence in the spectral sequence of Theorem 2.5, we have only two non-zero columns, namely the  $p = 0$  and the  $p = 1$  column. In the  $p = 0$  column, we have spaces  $\mathbb{F}^2 \otimes \text{HL}^q(\mathfrak{a}, M_s) \cong \text{HL}^q(\mathfrak{a}, M_s) \oplus \text{HL}^q(\mathfrak{a}, M_s)$ , while in the  $p = 1$  column, we have just  $\text{HL}^q(\mathfrak{a}, M_s)$  for every integer  $q \geq 0$ . Therefore, the spectral sequence degenerates at the term  $E_2$ , and for every integer  $n \geq 1$  we obtain that

$$(5.5) \quad \text{H}_{\text{rel}}^n(\mathfrak{a}, M) \cong \text{HL}^n(\mathfrak{a}, M_s) \oplus \text{HL}^n(\mathfrak{a}, M_s) \oplus \text{HL}^{n-1}(\mathfrak{a}, M_s),$$

and

$$(5.6) \quad \text{H}_{\text{rel}}^0(\mathfrak{a}, M) \cong E_2^{0,0} \cong \text{HL}^0(\mathfrak{a}, M_s) \oplus \text{HL}^0(\mathfrak{a}, M_s) \cong M^{\mathfrak{a}} \oplus M^{\mathfrak{a}}.$$

This, together with (5.1), (5.2), and induction yields the recursive relation

$$(5.7) \quad \text{HL}^n(\mathfrak{a}, M_s) \cong \text{HL}^{n-1}(\mathfrak{a}, M_s) \oplus \text{HL}^{n-2}(\mathfrak{a}, M_s)$$

for every integer  $n \geq 2$ .

As a consequence, we obtain for the Leibniz cohomology of  $\mathfrak{a}$  over a field  $\mathbb{F}$  of characteristic 2 with coefficients in a one-dimensional symmetric bimodule that

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_\lambda)_s) = \begin{cases} f_{n+1} & \text{if } \lambda = 0 \\ f_n & \text{if } \lambda = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every non-negative integer  $n$ , where  $f_n$  denotes the  $n^{\text{th}}$  term of the standard Fibonacci sequence given by  $f_0 := 0$ ,  $f_1 := 1$ , and  $f_n := f_{n-1} + f_{n-2}$  for every integer  $n \geq 2$ . In particular, we have that

$$\text{HL}^n(\mathfrak{a}, (F_1)_s) \cong \text{HL}^{n-1}(\mathfrak{a}, \mathbb{F})$$

for every integer  $n \geq 1$ .

Finally, let us consider  $F_\lambda$  as an anti-symmetric Leibniz  $\mathfrak{a}$ -bimodule  $(F_\lambda)_a$  with the same left  $\mathfrak{a}$ -action as above and with the trivial right  $\mathfrak{a}$ -action (see Section 1). Then we conclude from [17, Corollary 4.2 (b)] that

$$\dim_{\mathbb{F}} \text{HL}^0(\mathfrak{a}, (F_\lambda)_a) = 1 \text{ for every } \lambda \in \mathbb{F}.$$

Let us now compute  $\text{HL}^n(\mathfrak{a}, (F_\lambda)_a)$  for any integer  $n \geq 1$ . It follows from Lemma 1.5 (b) that

$$(5.8) \quad \text{HL}^n(\mathfrak{a}, (F_\lambda)_a) \cong \text{HL}^{n-1}(\mathfrak{a}, \text{Hom}_{\mathbb{F}}(\mathfrak{a}, F_\lambda)_s) \cong \text{HL}^{n-1}(\mathfrak{a}, (\mathfrak{a}^* \otimes F_\lambda)_s).$$

A straightforward computation shows that

$$0 \rightarrow F_\lambda \rightarrow \mathfrak{a}^* \otimes F_\lambda \rightarrow F_{\lambda-1} \rightarrow 0$$

is a short exact sequence of left  $\mathfrak{a}$ -modules. Then we obtain from the long exact cohomology sequence and another straightforward computation in the case  $\lambda = 1$ :

$$\dim_{\mathbb{F}}(\mathfrak{a}^* \otimes F_{\lambda})^{\mathfrak{a}} = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim_{\mathbb{F}} H^1(\mathfrak{a}, \mathfrak{a}^* \otimes F_{\lambda}) = \begin{cases} 2 & \text{if } \lambda = 0 \text{ and } \text{char}(\mathbb{F}) = 2 \\ 1 & \text{if } \lambda = 0, 2 \text{ and } \text{char}(\mathbb{F}) \neq 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim_{\mathbb{F}} H^2(\mathfrak{a}, \mathfrak{a}^* \otimes F_{\lambda}) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } \text{char}(\mathbb{F}) = 2 \text{ or } \lambda = 2 \text{ and } \text{char}(\mathbb{F}) \neq 2 \\ 0 & \text{otherwise.} \end{cases}$$

If  $\text{char}(\mathbb{F}) \neq 2$ , then we conclude by applying (5.3) in conjunction with [17, Corollary 4.2 (a)], [17, Corollary 4.4 (a)], and (5.8), respectively, to the symmetric  $\mathfrak{a}$ -bimodule  $M_s := (\mathfrak{a}^* \otimes F_{\lambda})_s$  that

$$\dim_{\mathbb{F}} \text{HL}^1(\mathfrak{a}, (F_{\lambda})_a) = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim_{\mathbb{F}} \text{HL}^3(\mathfrak{a}, (F_{\lambda})_a) = \dim_{\mathbb{F}} \text{HL}^2(\mathfrak{a}, (F_{\lambda})_a) = \begin{cases} 1 & \text{if } \lambda = 0, 2 \\ 0 & \text{otherwise,} \end{cases}$$

Finally, we use (5.4) to deduce for every integer  $n \geq 2$ :

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_{\lambda})_a) = \begin{cases} 1 & \text{if } \lambda = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

In summary, we have for the Leibniz cohomology of  $\mathfrak{a}$  over a field  $\mathbb{F}$  of characteristic  $\neq 2$  with coefficients in a one-dimensional anti-symmetric bimodule that

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_{\lambda})_a) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or } \lambda = 2 \text{ and } n \geq 2 \\ & \text{or } n = 0 \text{ and } \lambda \text{ is arbitrary} \\ 0 & \text{otherwise.} \end{cases}$$

If  $\text{char}(\mathbb{F}) = 2$ , then we obtain by applying (5.7) in conjunction with [17, Corollary 4.2 (a)], [17, Corollary 4.4 (a)], and (5.8), respectively:

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{a}, (F_{\lambda})_a) = \begin{cases} 1 & \text{if } n = 0 \text{ and } \lambda \text{ is arbitrary} \\ f_{n+1} & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise.} \end{cases}$$

**Remark.** Since every invariant symmetric bilinear form on  $\mathfrak{a}$  is a multiple of the Killing form, we have that  $[S^2(\mathfrak{a})^*]^{\mathfrak{a}} \cong \mathbb{F}$ . On the other hand, from the computations in Example E we obtain that

$$H_{\text{rel}}^0(\mathfrak{a}, \mathbb{F}) \cong \text{HR}^0(\mathfrak{a}) \cong H^1(\mathfrak{a}, \mathfrak{a}^*) \cong \begin{cases} \mathbb{F}^2 & \text{if } \text{char}(\mathbb{F}) = 2 \\ \mathbb{F} & \text{if } \text{char}(\mathbb{F}) \neq 2, \end{cases}$$

This shows that, in general,  $\text{HR}^0(\mathfrak{a}) \not\cong [S^2(\mathfrak{a})^*]^{\mathfrak{a}}$  and  $H_{\text{rel}}^0(\mathfrak{a}, \mathbb{F}) \not\cong [S^2(\mathfrak{a})^*]^{\mathfrak{a}}$  when  $\text{char}(\mathbb{F}) = 2$ .

In the next example we compute the Leibniz cohomology of the smallest non-nilpotent supersolvable non-Lie left Leibniz algebra with coefficients in one-dimensional bimodules. Note that contrary to the semidirect product of two one-dimensional Lie algebras in Example E the Leibniz algebra in Example F is the hemisemidirect product of two one-dimensional Lie algebras. It turns out that this somewhat simplifies matters.

**Example F.** Let  $\mathbb{F}$  denote an arbitrary field, and let  $\mathfrak{A} := \mathbb{F}h \oplus \mathbb{F}e$  be the two-dimensional supersolvable left Leibniz algebra over  $\mathbb{F}$  with multiplication determined by  $he = e$  (see [17, Example 2.3]). For any scalar  $\lambda \in \mathbb{F}$  one can define a one-dimensional left  $\mathfrak{A}$ -module  $F_\lambda := \mathbb{F}1_\lambda$  with  $\mathfrak{A}$ -action defined by  $h \cdot 1_\lambda := \lambda 1_\lambda$  and  $e \cdot 1_\lambda := 0$ . Note that  $\text{Leib}(\mathfrak{A}) = \mathbb{F}e$ , and thus  $\mathfrak{A}_{\text{Lie}}$  is a one-dimensional abelian Lie algebra. Then we obtain from Theorem 5.2 and [17, Corollary 4.2 (b)] that

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{A}_{\text{Lie}}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{A}_{\text{Lie}}, (F_\lambda)_a) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary or if } \lambda \neq 0 \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In order to be able to apply the spectral sequence of Theorem 3.5, we first compute  $\text{HL}^\bullet(\mathfrak{A}_{\text{Lie}}, [\text{Leib}(\mathfrak{A})^*]_s)$ . Observe that the module  $\text{Leib}(\mathfrak{A})^* = \mathbb{F}e^* \cong F_{-1}$  is non-trivial irreducible and furthermore viewed as a symmetric  $\mathfrak{A}_{\text{Lie}}$ -bimodule. Hence from the above it follows that  $\text{HL}^n(\mathfrak{A}_{\text{Lie}}, [\text{Leib}(\mathfrak{A})^*]_s) = 0$  for every non-negative integer  $n$ . This implies in turn that the spectral sequence of Theorem 3.5 collapses at the  $E_2$ -term and that

$$\begin{aligned} \text{HL}^n(\mathfrak{A}|\mathfrak{A}_{\text{Lie}}, (F_\lambda)_a) &= \text{HL}^n(\mathfrak{A}_{\text{Lie}}, \text{Hom}_{\mathbb{F}}(\text{Leib}(\mathfrak{A}), \text{HL}^0(\mathfrak{A}, (F_\lambda)_a))_s) \\ &= \text{HL}^n(\mathfrak{A}_{\text{Lie}}, \text{Hom}_{\mathbb{F}}(\text{Leib}(\mathfrak{A}), F_\lambda)_s) \end{aligned}$$

for all non-negative integers  $n$ , while  $\text{HL}^n(\mathfrak{A}|\mathfrak{A}_{\text{Lie}}, (F_\lambda)_s) = 0$  for all  $n \geq 0$  by Corollary 3.6. Notice that as an  $\mathfrak{A}$ -bimodule  $\text{Hom}_{\mathbb{F}}(\text{Leib}(\mathfrak{A}), F_\lambda)_s \cong [F_{\lambda-1}]_s$ . We have already remarked that the long exact sequence of Proposition 3.1 splits, and therefore we conclude from Proposition 3.1 that

$$\text{HL}^n(\mathfrak{A}, (F_\lambda)_a) \cong \text{HL}^n(\mathfrak{A}_{\text{Lie}}, [F_\lambda]_a) \oplus \text{HL}^n(\mathfrak{A}_{\text{Lie}}, [F_{\lambda-1}]_s)$$

for all  $\lambda$  and all non-negative integers  $n$ . Consequently, we obtain that

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{A}, (F_\lambda)_s) = \begin{cases} 1 & \text{if } \lambda = 0 \text{ and } n \text{ is arbitrary} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{A}, (F_\lambda)_a) = \begin{cases} 1 & \text{if } \lambda = 0, 1 \text{ and } n \text{ is arbitrary or if } \lambda \neq 0, 1 \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

**Remark.** In particular, we have that  $\dim_{\mathbb{F}} \text{HL}^n(\mathfrak{A}, \mathbb{F}) = 1$  for every non-negative integer  $n$ . Note that this follows as well from the scheme of proof of Proposition 4.3 in [33] (using also the isomorphism between Leibniz homology and cohomology with trivial coefficients). Indeed, the characteristic element  $\text{ch}(\mathfrak{A}) \in \text{HL}^2(\mathfrak{A}_{\text{Lie}}, \text{Leib}(\mathfrak{A}))$  of  $\mathfrak{A}$  is zero as  $\text{Leib}(\mathfrak{A}) = \mathbb{F}e \cong F_1$ . Since also  $\text{HL}^\bullet(\mathfrak{A}_{\text{Lie}}, [\text{Leib}(\mathfrak{A})^*]_s)$  is zero, we can reason in the same way as Pirashvili does.

The last general vanishing result is the Leibniz analogue of Barnes' vanishing theorem for the Chevalley-Eilenberg cohomology of finite-dimensional solvable Lie algebras (see [3, Theorem 2]). Recall that a left (or right) Leibniz algebra  $\mathfrak{L}$  over a field  $\mathbb{F}$  is called *solvable* if  $\mathfrak{L}^{(n)} = 0$  for some non-negative integer  $n$ , where  $\mathfrak{L}^{(n)}$  is defined recursively by  $\mathfrak{L}^{(0)} := \mathfrak{L}$  and  $\mathfrak{L}^{(n)} := \langle xy \mid x, y \in \mathfrak{L}^{(n-1)} \rangle_{\mathbb{F}}$  for every positive integer  $n$  (see [17, Section 1]).

**Proposition 5.4.** *Let  $\mathfrak{L}$  be a finite-dimensional solvable left Leibniz algebra, and let  $M$  be a finite-dimensional faithful irreducible symmetric  $\mathfrak{L}$ -bimodule. Then  $\mathrm{HL}^n(\mathfrak{L}, M) = 0$  for every non-negative integer  $n$ .*

*Proof.* It follows from [17, Proposition 1.1] (see also [17, Proposition 6.1]) that  $\mathfrak{L}_{\mathrm{Lie}}$  is solvable. Since by hypothesis  $M$  is a faithful symmetric  $\mathfrak{L}$ -bimodule, we obtain from Lemma 1.4 that  $\mathrm{Ann}_{\mathfrak{L}}^{\mathrm{bi}}(M) = \mathrm{Leib}(\mathfrak{L})$ . In particular,  $M$  is a faithful irreducible symmetric  $\mathfrak{L}_{\mathrm{Lie}}$ -bimodule. Then [3, Theorem 2] implies that  $\mathrm{H}^n(\mathfrak{L}_{\mathrm{Lie}}, M) = 0$  for every non-negative integer  $n$ . As  $M$  is a symmetric  $\mathfrak{L}_{\mathrm{Lie}}$ -bimodule, we conclude from Theorem 2.6 that  $\mathrm{HL}^n(\mathfrak{L}_{\mathrm{Lie}}, M) = 0$  for every non-negative integer  $n$ , and one can proceed exactly as in the proof of Theorem 4.1.  $\square$

**Remark.** The proof of Proposition 5.4 shows that it is enough to assume that  $\mathfrak{L}$  is a left Leibniz algebra such that the canonical Lie algebra  $\mathfrak{L}_{\mathrm{Lie}}$  associated to  $\mathfrak{L}$  is finite-dimensional and solvable.

At the moment we are not able to decide whether Proposition 5.4 generalizes to faithful irreducible anti-symmetric bimodules. The above proof shows it would be enough to show for a finite-dimensional faithful irreducible anti-symmetric  $\mathfrak{g}$ -bimodule over a finite-dimensional solvable Lie algebra  $\mathfrak{g}$  that  $\mathrm{HL}^n(\mathfrak{g}, M) = 0$  for every positive integer  $n$ .

It is well known that the Chevalley-Eilenberg cohomology of the non-abelian two-dimensional Lie algebra with coefficients in the adjoint module vanishes. According to Theorem 2.6, this implies that the corresponding Leibniz cohomology vanishes as well. Similarly, by applying Theorem 2.6 in conjunction with [26, Proposition 5.6] one obtains the following more general result in characteristic zero. (Note that the adjoint bimodule of a Lie algebra is symmetric.)

**Proposition 5.5.** *Let  $\mathfrak{b}$  be the Borel subalgebra of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero. Then  $\mathrm{HL}^n(\mathfrak{b}, \mathfrak{b}_{\mathrm{ad}}) = 0$  for every non-negative integer  $n$ . In particular, Borel subalgebras of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero are rigid as Leibniz algebras.*

**Remark.** It follows from Theorem 2.6 in conjunction with [36, Theorem 1] (see also [27, Section 1]) that the conclusion of Proposition 5.5 holds more generally for every parabolic subalgebra of a finite-dimensional semi-simple Lie algebra over a field of characteristic zero (cf. also [33, Proposition 2.3] for the rigidity of parabolic subalgebras).

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