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MV-ALGEBRAS AND PARTIALLY CYCLICALLY ORDERED GROUPS.

G. LELOUP

ABSTRACT. We prove that there exists a functorial correspondence between MV-algebras and partially cyclically ordered groups which are wound round of lattice-ordered groups. It follows that some results about cyclically ordered groups can be stated in terms of MV-algebras. For example, the study of groups together with a cyclic order allows to get a first-order characterization of groups of unimodular complex numbers and of finite cyclic groups. We deduce a characterization of pseudofinite MV-chains and of pseudo-simple MV-chains (i.e. which share the same first-order properties as some simple ones). We can generalize these results to some non-linearly ordered MV-algebras, for example hyper-archimedean MV-algebras.

Keywords: MV-algebras, MV-chains, partially cyclically ordered abelian groups, cyclically ordered abelian groups, pseudofinite.

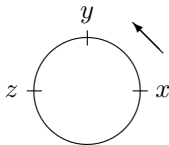
1. INTRODUCTION.

This article has been written in such a way that it can be read by someone who does not have prior knowledge of cyclically ordered groups. We list all the definitions and properties that we need. We also list all the definitions and properties about MV-algebras and logic that we need. We try to bring back whenever possible to properties partially ordered groups, lattice-ordered groups or linearly ordered groups.

Unless otherwise stated the groups are abelian groups.

Every MV-algebra can be obtained in the following way. Let (G, \leq, \wedge, u) be a lattice-ordered group (in short ℓ -group) together with a distinguished strong unit $u > 0$ (i.e. for every $x \in G$ there is a positive integer n such that $x \leq nu$); such a group will be called a unital ℓ -group. We set $[0, u] := \{x \in G \mid 0 \leq x \leq u\}$. For every x, y in $[0, u]$ we let $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. We see that the restriction of the partial order \leq to $[0, u]$ can be defined by $x \leq y \Leftrightarrow \exists z \ y = x \oplus z$.

Now, the quotient group $C = G/\mathbb{Z}u$ can be equipped with a partial cyclic order. First, we explain what is a cyclic order. On a circle C , there is no canonical linear order, but there exists a canonical cyclic order. Assume that one traverses a circle counterclockwise. We set that $R(x, y, z)$ holds if one can find x, y, z in this order starting from some point of the circle. Now, starting from another point one can find them in the order y, z, x or z, x, y . So in turn $R(y, z, x)$ and $R(z, x, y)$ hold. We say that R is cyclic. Furthermore, for any x in C the relation $y <_x z \Leftrightarrow R(x, y, z)$ is a relation of linear order on the set $C \setminus \{x\}$.



These rules give the definition of a cyclic order. We generalize this definition to a partial cyclic order by assuming that $<_x$ is a partial order relation which needs not be a linear order.

Turning to the cyclic order $R(\cdot, \cdot, \cdot)$ on the quotient group $C = G/\mathbb{Z}u$ it is defined by setting $R(x_1 + \mathbb{Z}u, x_2 + \mathbb{Z}u, x_3 + \mathbb{Z}u)$ if there exists n_2 and n_3 in \mathbb{Z} such that $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$ (see Proposition 3.16). One can prove that (C, R) satisfies for every x, y, z, v in C :

$$R(x, y, z) \Rightarrow x \neq y \neq z \neq x \text{ (} R \text{ is strict)}$$

$$R(x, y, z) \Rightarrow R(y, z, x) \text{ (} R \text{ is cyclic)}$$

by setting $y \leq_x z$ if either $R(x, y, z)$ or $x \neq y = z$ or $x = y \neq z$ or $x = y = z$, then \leq_x is a partial order relation on C .

$$R(x, y, z) \Rightarrow R(x + v, y + v, z + v) \text{ (} R \text{ is compatible).}$$

Any group equipped with a ternary relation which satisfies those properties is called a partially cyclically ordered group. If all the orders \leq_x are linear orders, then (C, R) is called a cyclically ordered group. In the case where $C = G/\mathbb{Z}u$, where G is a partially ordered group, we say that C is the wound-round of a partially ordered group.

If A is an MV-algebra, then there is a unital ℓ -group (G_A, u_A) (uniquely determined up to isomorphism) such that A is isomorphic to the MV algebra $[0, u_A]$. The group G_A is called the Chang ℓ -group of A . We show that the MV-algebra A is definable in the partially cyclically ordered group $(G_A/\mathbb{Z}u_A, R)$.

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It follows a functorial correspondence between cyclically ordered groups and MV-chains (i.e. linearly ordered MV-algebras). So some properties of MV chains can be deduced from analogous properties of cyclically ordered groups.

In [7], D. Glusschankov constructed a functor between the category of projectable MV-algebras and the category of projectable lattice partially ordered groups. The approach of the present paper is different, and it does not need to restrict to a subclass of MV-algebras.

In Section 2 we list basic properties of MV-algebras and of their Chang ℓ -groups. We also give a few basic notions of logic which we need in this paper. Section 3 is devoted to partially cyclically ordered groups. We focus on the set of elements which are non-isolated (i.e. the elements x such that there exists y satisfying $R(0, x, y)$ or $R(0, y, x)$) and on those partially cyclically ordered groups which can be seen as wound-rounds of lattice-ordered groups. We also list some results of [10] on c.o. groups which belong to the elementary class generated by the subgroups of the multiplicative group \mathbb{U} of unimodular complex numbers. From those properties it follows that there is a functor $\Theta\Xi$ from the category of MV-algebras to the category of partially cyclically ordered groups together with c-homomorphisms. In Section 4 we define a class \mathcal{AC} of partially cyclically ordered groups C in which we can define a MV-algebra $A(C)$ such that $C = \Theta\Xi(A(C))$ (Theorem 4.7). Then we prove that the wound-rounds of ℓ -groups belong to \mathcal{AC} (Theorem 4.15). Furthermore, the subgroup generated by $A(C)$ being the wound round of an ℓ -group is expressible by finitely many first-order formulas (Theorem 4.17). In the case of MV-chains, the one-to-one mapping $C \mapsto A(C)$ defines a functorial correspondence between the class of cyclically ordered groups and the class of MV-chains. In section 5 we prove that if A and A' are two MV-chains, then A and A' are elementarily equivalent if, and only if, $C(A)$ and $C(A')$ are elementarily equivalent (Proposition 5.3). We also prove that any two MV-chains are elementarily equivalent if, and only if, their Chang ℓ -groups are elementarily equivalent (Proposition 5.5). The class of pseudo-simple MV-chains is defined to be the elementary class generated by the simple chains. We define in the same way the pseudofinite MV-chains. One can prove that a pseudo-simple MV-chain is an MV-chain which is elementarily equivalent to some MV-subchain of $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, and a pseudofinite MV-chain is an MV-chain which is elementarily equivalent to some ultraproduct of finite MV-chains. We use the results of [10] on cyclically ordered groups to deduce characterizations of pseudo-simple and of pseudofinite MV-chains. Furthermore, we get necessary and sufficient conditions for such MV-chains being elementarily equivalent (Theorems 5.11, 5.12, 5.13). In Section 6, we generalize the results of Section 5 about pseudofinite and pseudo-simple MV-chains to pseudo-finite and pseudo-hyperarchimedean MV-algebras which are cartesian products of finitely many MV-chain (Theorems 6.9, 6.11).

The author would like to thank Daniele Mundici for his bibliographical advice and his suggestions.

2. MV-ALGEBRAS.

The reader can find more properties for example in [3, Chapter 1].

2.1. Definitions and basic properties.

Definition 2.1. An *MV-algebra* is a set A equipped with a binary operation \oplus , a unary operation \neg and a distinguished constant 0 satisfying the following equations. For every x, y and z :

$$\text{MV1) } x \oplus (y \oplus z) = (x \oplus y) \oplus z$$

$$\text{MV2) } x \oplus y = y \oplus x$$

$$\text{MV3) } x \oplus 0 = x$$

$$\text{MV4) } \neg\neg x = x$$

$$\text{MV5) } x \oplus \neg 0 = \neg 0$$

$$\text{MV6) } \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

If this holds, then we define $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \leq y \Leftrightarrow \exists z, x \oplus z = y$. Then \leq is a partial order called the *natural order* on A . This partial order satisfies: $x \leq y \Leftrightarrow \neg y \leq \neg x$, and A is a lattice with smallest element 0 , greatest element 1 , where $x \vee y = \neg(\neg x \oplus \neg y)$, $x \wedge y = \neg(\neg(x \oplus \neg y) \oplus \neg y) = (x \oplus \neg y) \odot y$. The operations \wedge (infimum) and \vee (supremum) are compatible with \oplus and \odot . Note that Condition MV6) can be written as $x \vee y = y \vee x$. In the case where \leq is a linear order, A is called an *MV-chain*.

Following [3], if n is a positive integer and x is an element of a group, then we denote by nx the sum $x + \dots + x$ (n times). If x belongs to an MV-algebra, then we set $n.x = x \oplus \dots \oplus x$ (n times). Further, we will set $x^n = x \odot \dots \odot x$.

If G is a partially ordered group and $u \in G$, u is said to be a *strong unit* if $u > 0$ and for every $x \in G$ there is $n \in \mathbb{N}$ such that $x \leq nu$. It follows that there exists $n' \in \mathbb{N}$ such that $-x \leq n'u$, hence $-n'u \leq x \leq nu$. A *unital ℓ -group* is an ℓ -group (i.e. a lattice-ordered group) with distinguished strong unit $u > 0$. More generally, a *unital* partially (resp. linearly) ordered group is a partially (resp. linearly) ordered group together with a distinguished strong unit.

Example 2.2. If (G, u) is a unital ℓ -group, then the set $[0, u] := \{x \in M \mid 0 \leq x \leq u\}$ together with the operations $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$, and where 0 is the identity element of G , is an MV-algebra, whose natural partial order is the restriction of the partial order on G . It is denoted $\Gamma(G, u)$. In this case, $x \odot y = (x + y - u) \vee 0$, and it follows: $(x \oplus y) + (x \odot y) = x + y$.

A *unital* homomorphism between two unital partially ordered groups (G, u) and (G', u') is an increasing group homomorphism f between G and G' such that $f(u) = u'$. An ℓ -homomorphism between two ℓ -groups G and G' is a group-homomorphism such that for every x, y in G we have that $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ (it follows that f is also an increasing homomorphism).

A *homomorphism* of MV-algebras is a function f from an MV-algebra A to an MV-algebra A' such that $f(0) = 0$ and for every x, y in A $f(x \oplus y) = f(x) \oplus f(y)$ and $f(\neg x) = \neg f(x)$.

The mapping $\Gamma : (G, u) \mapsto \Gamma(G, u)$ is a full and faithful functor from the category \mathcal{A} of unital ℓ -groups to the category \mathcal{MV} of MV-algebras. If f is a unital ℓ -homomorphism between (G, u) and (G', u') , then $\Gamma(f)$ is the restriction of f to $[0, u]$ ([3, Chapter 7]).

Now, for every MV-algebra A there exists a unital ℓ -group (G_A, u_A) , uniquely defined up to isomorphism, such that A is isomorphic to $[0, u_A]$ together with above operations; (G_A, u_A) is called the *Chang ℓ -group* of A (see [3, Chapter 2]). We will sometimes let G_A stand for (G_A, u_A) . For further purposes we describe this Chang ℓ -group at the end of this section. The mapping $\Xi : A \mapsto (G_A, u_A)$ is a full and faithful functor from the category \mathcal{MV} to the category \mathcal{A} . We do not describe here the unital ℓ -homomorphism $\Xi(f)$, where f is a homomorphism of MV-algebras. The composite functors $\Gamma\Xi$ and $\Xi\Gamma$ are naturally equivalent to the identities of the respective categories (see [3] Theorems 7.1.2 and 7.1.7).

The language of MV-algebras is $L_{MV} = (0, \oplus, \neg)$. However, since \leq, \wedge and \vee are definable in L_{MV} , we will assume that they belong to the language. We will denote by $L_o = (0, +, -, \leq)$ the language of ordered groups, by $L_{lo} = (0, +, -, \wedge, \vee)$ the language of ℓ -groups. When dealing with unital ℓ -groups, we will add a constant symbol to the language, it will be denoted by $L_{lou} = L_{lo} \cup \{u\}$, and $L_{louZu} = L_{lo} \cup \{Zu\}$ will denote the language of ℓ -groups together with a unary predicate for the subgroup generated by u .

The Chang ℓ -group G_A of A is an L_{lou} -structure where u is a constant predicate interpreted by the distinguished strong unit of G_A . Now, G_A is also a L_{louZu} -structure where Zu is a unary predicate interpreted by: $Zu(x)$ if, and only if, x belongs to the subgroup generated by the distinguished strong unit; in this case we denote it by (G_A, Zu_A) .

In the MV-algebra A , recall that $x \odot y$ stands for $\neg(\neg x \oplus \neg y)$. In $[0, u_A] \subset G_A$ we have that $x \odot y = (x + y - u_A) \vee 0$. If A is an MV-chain, then the formula $x > 0$ and $0 = x \odot x = \dots = x^n$ is equivalent to: $0 < x < 2x < \dots < n.x \leq u_A$.

Notations 2.3. In the following, if $x < y$ are elements of a partially ordered group G , then we will set $[x, y] := \{z \in G \mid x \leq z \leq y\}$, $[x, y[:= [x, y] \setminus \{y\}$, $]x, y] := [x, y] \setminus \{x\}$ and $]x, y[:= [x, y] \setminus \{x, y\}$. In the particular case where $G = \mathbb{R}$ (the group of real numbers) $x = 0$ and $y = 1$, then we let $[0, 1]_{\mathbb{R}} := \{z \in \mathbb{R} \mid 0 \leq z \leq 1\}$. The notation $[0, 1]$ will be used in the case of an MV-algebra.

Lemma 2.4. *Let G be an ℓ -group and $0 < u \in G$.*

- 1) *Either $[0, u] = \{0, u\}$,
or there exists $x \in]0, u[$ such that $[0, u] = \{0, u, x\}$
or there exists $x \in]0, u[$ such that $[0, u] = \{0, u, x, u - x\}$
or for every $x \in]0, u[$ there exists $y \in]0, u[$ such that $x < y$ or $y < x$.*
- 2) *If $]0, u[$ contains x, y such that $x < y$, then for every $z \in]0, u[$ there exists $z' \in]0, u[$ such that $z < z'$ or $z' < z$.*

Proof. 1) Assume that there exists $x \in]0, u[$, and that $[0, u] \neq \{0, u, x\}$, $[0, u] \neq \{0, u, x, u - x\}$.

If $x = u - x$ (i.e. $2x = u$), then we let $z \in]0, u[\setminus \{x\}$. By properties of ℓ -groups, if $x \wedge z = 0$, then for every $y > 0$: $(x + y) \wedge z = y \wedge z$ (see for example [6, Lemma 2.3.4]). Now, $(x + x) \wedge z = u \wedge z = z > 0$, hence $x \wedge z > 0$. If $x \wedge z < x$, then set $y = x \wedge z$. Otherwise $x < z$, hence we set $y = z$.

Now we assume that $x \neq u - x$. If $x < u - x$ or $u - x < x$, then we set $y = u - x$, otherwise, we have that $x \wedge (u - x) < x < x \vee (u - x)$. If $x \wedge (u - x) > 0$, then we let $y = x \wedge (u - x)$. Otherwise, if $x \vee (u - x) \neq u$, then we let $y = x \vee (u - x)$. Assume that $x \wedge (u - x) = 0$ and $x \vee (u - x) = u$, and let $z \in]0, u[\setminus \{x, u - x\}$. If $x < z$, then let $y = z$. Otherwise, if $z \wedge x > 0$, then we set $y = z \wedge x$. If $z \wedge x = 0$, then $z = z \wedge u = z \wedge (x \vee (u - x)) = (z \wedge x) \vee (z \wedge (u - x)) = z \wedge (u - x)$, hence $z < u - x$, and $x < u - z$, we let $y = u - z$.

2) Assume that $]0, u[$ contains x, y such that $x > y$ (so it contains at least four elements). If $[0, u] = \{0, u, x, u - x\}$, then the result holds trivially ($y = u - x$). Otherwise $[0, u]$ contains at least five elements, and the result follows from 1). \square

Corollary 2.5. *Let A be an MV-algebra. Then:
either $A = \{0, 1\}$,*

or there exists $x \in A \setminus \{0, 1\}$ such that $A = \{0, 1, x\}$

or there exists $x \in A \setminus \{0, 1\}$ such that $A = \{0, 1, x, \neg x\}$

or for every $x \in A \setminus \{0, 1\}$ there exists $y \in A \setminus \{0, 1\}$ such that $x < y$ or $y < x$.

If $A \setminus \{0, 1\}$ contains x, y such that $x < y$, then for every $z \in A \setminus \{0, 1\}$ there exists $z' \in A \setminus \{0, 1\}$ such that $z < z'$ or $z' < z$.

2.2. Construction of the Chang ℓ -group. The correspondence between MV-algebras and partially cyclically ordered groups relies on the good sequences defined for the construction of the Chang ℓ -groups (see [11], [3, Chapter 2]). We describe this construction and we also define an analogue of the good sequences in an ℓ -group. We start with some properties of partially ordered groups.

Remark 2.6. We know that every cancellative abelian monoid M embeds canonically in a group G generated by the image of M following the construction of \mathbb{Z} from \mathbb{N} . Now, one can deduce from properties of partially ordered groups (see for example [1], Propositions 1.1.2, 1.1.3, and also 1.2.5) that:

- (i) M is the positive cone of a compatible partial order on G if, and only if, for every x, y in M , $x + y = 0 \Rightarrow x = y = 0$, and this partial order is given by $x \leq y \Leftrightarrow \exists z \in M, y = x + z$,
- (ii) G is an ℓ -group if, and only if, for every x, y in M , $x \wedge y$ exists.

The following lemmas show that every element of the positive cone of a unital ℓ -group can be associated with a unique sequence of elements of $[0, u]$. So G is determined by its restriction to $[0, u]$. This property will give rise to the construction of the Chang ℓ -group.

Lemma 2.7. Assume that (G, u) is a unital ℓ -group. Let $0 < x \in G$ and m be a positive integer such that $x \leq mu$. Then, there exists a unique sequence x_1, \dots, x_n of elements of $[0, u]$ such that $x = x_1 + \dots + x_n$ and, for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \dots + x_n) = 0$, and $n \leq m$.

Proof. For every $y \in G$, we have that $(u - y) \wedge (x - y) = 0 \Leftrightarrow (u \wedge x) - y = 0 \Leftrightarrow y = u \wedge x$. Set $x_1 = x \wedge u$. Then x_1 is the unique element of G such that $(u - x_1) \wedge (x - x_1) = 0$. Since $0 < x$, we have that $0 \leq x_1 \leq u$, and $0 \leq x - x_1 = x - (u \wedge x) = x + ((-u) \vee (-x)) = (x - u) \vee 0 \leq (mu - u) \vee 0 = (m - 1)u$. By taking $x - x_1$ in place of x we get $x_2 \in [0, u]$ such that $(u - x_2) \wedge (x - x_1 - x_2) = 0$, and we have that $x - x_1 - x_2 \in [0, (m - 2)u]$, and so on. Hence there exists a unique sequence x_1, \dots, x_n of elements of $[0, u]$ such that $x = x_1 + \dots + x_n$ and, for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \dots + x_n) = 0$. \square

Lemma 2.8. The condition: for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \dots + x_n) = 0$ is equivalent to: for $1 \leq i < n - 1$, $(u - x_i) \wedge x_{i+1} = 0$. If this holds, then, for $1 \leq i < j \leq n$, $(u - x_i) \wedge x_j = 0$

Proof. Assume that for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \dots + x_n) = 0$. Let $i < j \leq n$. Since $0 \leq x_i$ and $0 \leq x_j \leq x_{i+1} + \dots + x_n$, it follows that $(u - x_i) \wedge x_j = 0$. Now, let y, z, z' in $[0, u]$ such that $(u - y) \wedge z = (u - z) \wedge z' = 0$, then $0 \leq (u - y) \wedge z' = (u - y) \wedge z' \wedge u = (u - y) \wedge z' \wedge (z + u - z) \leq ((u - y) \wedge z' \wedge z) + ((u - y) \wedge z' \wedge (u - z)) = 0$. Hence by induction we can prove that the condition: for $1 \leq i < n - 1$, $(u - x_i) \wedge x_{i+1} = 0$ implies for $1 \leq i < n - 1$, $(u - x_i) \wedge (x_{i+1} + \dots + x_n) = 0$. \square

Remark 2.9. By setting, for x, y in $[0, u]$, $x \oplus y = (x + y) \wedge u$, we have that $x \odot y = 0 \vee (x + y - u)$. Hence, using the fact that for every z in G we have that $z = z \vee 0 + z \wedge 0$, we get: $x + y = (x \oplus y) + (x \odot y)$. We deduce from the proof of Lemma 2.7 that $x \oplus y$ is the unique element of $[0, u]$ such that $(u - x \oplus y) \wedge (x + y - x \oplus y) = 0$, and then $x + y - x \oplus y \in [0, u]$. It follows that $x = x \oplus y \Leftrightarrow (u - x) \wedge y = 0$. Furthermore, from the equality $x + y = (x \oplus y) + (x \odot y)$ we get $x \oplus y = x \Leftrightarrow x \odot y = y$.

Now, we come to the Chang ℓ -group.

Definition 2.10. Let A be an MV-algebra. A sequence (x_i) of elements of A indexed by the natural numbers $1, 2, \dots$ is said to be a *good sequence* if, for each i , $x_i \oplus x_{i+1} = x_i$, and it contains only a finite number of nonzero terms. If $x = (x_i)$ and $y = (y_i)$ are good sequences, then we define $z = x + y$ by the rules $z_1 = x_1 \oplus y_1$, $z_2 = x_2 \oplus (x_1 \odot y_1) \oplus y_2$, and more generally, for every positive integer i :

$$z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \dots \oplus (x_1 \odot y_{i-1}) \oplus y_i.$$

We also define a partial order \leq by $x \leq y \Leftrightarrow \exists z, y = x + z$.

We see that A embeds into the monoid M_A of good sequences by $x \mapsto (x, 0, 0, \dots)$, and one can prove that M_A is cancellative, it satisfies the properties (i) and (ii) of Remark 2.6, where:

$$x \wedge y = (x_i \wedge y_i), \quad x \vee y = (x_i \vee y_i),$$

if $y = x + z$, then $z = (y_i) + (\neg x_n, \neg x_{n-1}, \dots, \neg x_1, 0, \dots)$, where x_n is the last non-zero term of (x_i) (see [3, Chapter 2]).

Consequently, M_A defines in a unique way an ℓ -group, and the image of $(1, 0, \dots)$ in this ℓ -group is a strong unit. This unital ℓ -group is the Chang ℓ -group G_A .

Remark 2.11. Let $x = (x_1, \dots, x_n, 0, \dots)$ in the positive cone of G_A . Then $x = (x_1, 0, \dots) + \dots + (x_n, 0, \dots)$. The embedding $x_i \mapsto (x_i, 0, \dots)$ of A in G_A can be considered as an inclusion. Hence we can assume that $x_i \in G_A$ and write $x = x_1 + \dots + x_n$. Hence, the good sequence defining x is the same as the sequence defined in Lemmas 2.7 and 2.8.

Proof. Let (x_i) be a good sequence, and for $k \geq 1$ let $y = (x_1, \dots, x_k, 0, \dots) + (x_{k+1}, 0, \dots)$. By Lemma 2.8 and Remark 2.9, we have, for $1 \leq i < j \leq k+1$, $x_i \oplus x_j = x_i$ and $x_i \odot x_j = x_j$. It follows that

$$y_1 = x_1 \oplus x_{k+1} = x_1,$$

$$\text{for } 2 \leq i \leq n, y_i = x_i \oplus (x_{i-1} \odot x_{k+1}) \oplus \dots \oplus (x_1 \odot 0) \oplus 0 = x_i \oplus x_{k+1} = x_i,$$

$$y_{k+1} = 0 \oplus (x_n \odot x_{k+1}) \oplus \dots \oplus (x_1 \odot 0) \oplus 0 = 0 \oplus x_{k+1} \oplus 0 = x_{k+1},$$

and for $i > k+1$, $y_i = 0$.

So by induction we get $(x_1, \dots, x_n, 0, \dots) = (x_1, 0, \dots) + \dots + (x_n, 0, \dots)$. The remainder of the proof is straightforward. \square

2.3. Elementary equivalence, interpretability. Two structures S and S' for a language L are *elementarily equivalent* if any L -sentence is true in S if, and only if, it is true in S' . We let $S \equiv S'$ stand for S and S' being elementarily equivalent. Furthermore if $S \subset S'$, then we say that S is an *elementary substructure* of S' (in short $S \prec S'$) if every existential formula with parameters in S which is true in S' is also true in S . We will need the following properties.

Theorem 2.12. ([8, Corollary 9.6.5 on p. 462], see also [4, Theorems 5.1, 5.2]) *Let L be a first-order language.*

(a) *If I is a non-empty set and for each $i \in I$, A_i and B_i are elementarily equivalent L -structures, then $\prod_I A_i \equiv \prod_I B_i$ (here \prod denotes the direct product).*

(b) *If I is a non-empty set and for each $i \in I$, A_i and B_i are L -structures with $A_i \prec B_i$, then $\prod_I A_i \prec \prod_I B_i$.*

If, for every $i \in I$, S_i is an L -structure and U is an ultrafilter on I , then the *ultraproduct* of the S_i 's is the quotient set $\left(\prod_{i \in I} S_i \right) / \sim$, where \sim is the equivalence relation: $(x_i) \sim (y_i) \Leftrightarrow \{i \in I \mid x_i = y_i\} \in U$.

If R is a unary predicate of L , then $R((x_i))$ holds in the ultraproduct if the set $\{i \in I \mid R(x_i) \text{ holds in } S_i\}$ belongs to U . Every relation symbol and every function symbol is interpreted in the same way. An *elementary class* is a class which is closed under ultraproducts.

A structure S_1 for a language L_1 is interpretable in a structure S_2 for a language L_2 if the following holds.

There is a one-to-one mapping φ from a subset T_1 of S_2 onto S_1 ,

for every L_1 -formula Φ of the form $R(\bar{x})$, $F(\bar{x}) = y$, $x = y$ or $x = c$ (where R is a relation symbol, F is a function symbol and c is a constant), there is an L_2 -formula Φ' such that for every \bar{x} in T_1 , $S_1 \models \Phi(\varphi(\bar{x})) \Leftrightarrow S_2 \models \Phi'(\bar{x})$,

(this is a particular case of the definition of interpretability p. 58 and pp. 212-214 in [8]).

Theorem 2.13. (*Reduction Theorem 5.3.2, [8]*). *If S_1, S'_1 (resp. S_2, S'_2) are structures for the language L_1 (resp. L_2) such that S_1 is interpretable in S_2 and S'_1 is interpretable in S'_2 by the same rules, then $S_2 \equiv S'_2 \Rightarrow S_1 \equiv S'_1$ and $S_2 \prec S'_2 \Rightarrow S_1 \prec S'_1$.*

Since $A = [0, u]$, $a \oplus b = (a + b) \wedge u$, $\neg a = u - a$, it follows that the L_{MV} -structure A is interpretable in the L_{lou} -structure (G_A, u_A) and in the L_{loZu} -structure $(G_A, \mathbb{Z}u_A)$. Consequently, if A, A' are MV-algebras such that $(G_A, u_A) \equiv (G_{A'}, u_{A'})$ (resp. $(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'})$), then $A \equiv A'$. The same holds with \prec instead of \equiv .

3. PARTIALLY CYCLICALLY ORDERED GROUPS.

Recall that all the groups are assumed to be abelian groups.

3.1. Basic properties.

Definitions 3.1. We say that a group C is *partially cyclically ordered* (in short a *p.c.o. group*) if it is equipped with a ternary relation R which satisfies (1), (2), (3), (4) below.

(1) R is strict i.e. for every x, y, z in C : $R(x, y, z) \Rightarrow x \neq y \neq z \neq x$.

(2) R is cyclic i.e. for every x, y, z in C : $R(x, y, z) \Rightarrow R(y, z, x)$.

(3) For every x, y, z in C set $y \leq_x z$ if either $R(x, y, z)$ or $y = z$ or $y = x$. Then for every x in C , \leq_x is a partial order relation on C . We set $y <_x z$ for $y \leq_x z$ and $y \neq z$. If y and z admit an infimum (resp. a supremum) in $(C, <_x)$, then it will be denoted by $y \wedge_x z$ (resp. $y \vee_x z$).

(4) R is compatible, i.e. for every x, y, z, v in C , $R(x, y, z) \Rightarrow R(x + v, y + v, z + v)$.

If for every $x \in C$ the order \leq_x is a linear order, then we say that C is a *cyclically ordered group* (in short a *c.o. group*).

Definition 3.2. Notation. The language $(0, +, -, R)$ of p.c.o. groups will be denoted by L_c .

Definition 3.3. A *c-homomorphism* is a group homomorphism f between two p.c.o. groups (or c.o. groups) such that for every x, y, z , if $R(x, y, z)$ holds and $f(x) \neq f(y) \neq f(z) \neq f(x)$, then $R(f(x), f(y), f(z))$ holds.

Examples 3.4. Let \mathbb{U} be the multiplicative group of unimodular complex numbers. For $e^{i\theta_j}$ ($1 \leq j \leq 3$) in \mathbb{U} , such that $0 \leq \theta_j < 2\pi$, we let $R(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ if, and only if, either $\theta_1 < \theta_2 < \theta_3$ or $\theta_2 < \theta_3 < \theta_1$ or $\theta_3 < \theta_1 < \theta_2$ (in other words, when one traverses the unit circle counterclockwise, starting from $e^{i\theta_1}$ one finds first $e^{i\theta_2}$ then $e^{i\theta_3}$). Then \mathbb{U} is a c.o. group. One sees that the group $Tor(\mathbb{U})$ of torsion elements of \mathbb{U} (that is, the roots of 1 in the field of complex numbers) is a c.o. subgroup. Now, let (C_1, R_1) and (C_2, R_2) be nontrivial c.o. groups and $C = C_1 \times C_2$ be their cartesian product. For (x_1, x_2) , (y_1, y_2) and (z_1, z_2) in C , let $R((x_1, x_2), (y_1, y_2), (z_1, z_2))$ if, and only if, $R_1(x_1, y_1, z_1)$ and $R_2(x_2, y_2, z_2)$. Then (C, R) is a p.c.o. group which is not a c.o. group.

Example 3.5. Any linearly ordered group is a c.o. group once equipped with the ternary relation: $R(x, y, z)$ iff $x < y < z$ or $y < z < x$ or $z < x < y$. In the same way, any partially ordered group is a p.c.o. group.

In ordered sets, one often uses the notation $x_1 < x_2 < \dots < x_n$. We define a similar notation for partial cyclic orders.

Definition 3.6. Notation. Let C be a p.c.o. group and x_1, \dots, x_n in C , we will denote by $R(x_1, \dots, x_n)$ the formula: $R(x_1, x_2, x_3) \& R(x_1, x_3, x_4) \& \dots \& R(x_1, x_{n-1}, x_n)$.

In the unit circle \mathbb{U} , $R(x_1, \dots, x_n)$ means that starting from x_1 one finds the elements x_2, \dots, x_n in this order.

Lemma 3.7. Let C be a p.c.o. group and x_1, \dots, x_n in C .

- a) $R(x_1, \dots, x_n) \Leftrightarrow \forall (i, j, k) \in [1, n] \times [1, n] \times [1, n], 1 \leq i < j < k \leq n \Rightarrow R(x_i, x_j, x_k)$.
- b) $\forall y \in C, (R(x_1, \dots, x_n) \Leftrightarrow R(x_1 + y, \dots, x_n + y))$.
- c) $\forall i \in [1, n - 1], (R(x_1, \dots, x_n) \Leftrightarrow R(x_{i+1}, \dots, x_n, x_1, \dots, x_i))$.

Proof. a) \Leftarrow is straightforward. Assume that $R(x_1, \dots, x_n)$ holds and let $1 \leq i < j < k \leq n$. Then $R(x_1, x_i, x_{i+1})$ and $R(x_1, x_{i+1}, x_{i+2})$ hold. Therefore, since $<_{x_1}$ is transitive, $R(x_1, x_i, x_{i+2})$ holds, and so on. Hence $R(x_1, x_i, x_j)$ holds, and in the same way $R(x_1, x_j, x_k)$ holds. It follows that $R(x_j, x_1, x_i)$ and $R(x_j, x_k, x_1)$ hold. Hence $R(x_j, x_k, x_i)$ holds which implies that $R(x_i, x_j, x_k)$ holds.

b) For every i in $[2, n - 1]$, $R(x_1, x_i, x_{i+1})$ holds. Hence $R(x_1 + y, x_i + y, x_{i+1} + y)$ holds. Consequently, $R(x_1 + y, \dots, x_n + y)$ also holds.

c) Assume that $R(x_1, \dots, x_n)$ holds. We have that $R(x_1, x_2, x_n)$ holds and by a) for every i in $[3, n - 1]$: $R(x_2, x_i, x_{i+1})$ holds. Therefore $R(x_2, \dots, x_n, x_1)$ holds. Now, c) follows by induction. \square

If C is a p.c.o. group, then by the definition $<_0$ is a partial order on the set C . Conversely, if a group is equipped with a partial order relation $<$, then we give a necessary and sufficient condition for $<$ being the order $<_0$ of some partial cyclic order.

Proposition 3.8. Let C be a group. Then there exists a compatible partial cyclic order R on C if, and only if, there exists a partial order $<$ on the set $C \setminus \{0\}$ such that for all x and y in $C \setminus \{0\}$: $x < y \Rightarrow y - x < -x$.

If this holds, then we can set $R(x, y, z) \Leftrightarrow 0 \neq y - x < z - x$, and \leq is the restriction to $C \setminus \{0\}$ of the relation \leq_0 .

Proof. Assume that C is a p.c.o. group, and $x <_0 y$ in $C \setminus \{0\}$. Then $R(0, x, y)$ holds. Hence by compatibility: $R(-x, 0, y - x)$ holds so $R(0, y - x, -x)$ holds i.e. $y - x <_0 -x$.

Assume that $<$ is a strict partial order on $C \setminus \{0\}$ such that for all x and y in $C \setminus \{0\}$ we have that $x < y \Leftrightarrow y - x < -x$. For all x, y, z in C set $R(x, y, z)$ if, and only if, $0 \neq y - x < z - x$.

$R(x, y, z)$ implies $y - x \neq 0, z - x \neq 0$ and $y - x \neq z - x$ hence $x \neq y, x \neq z$ and $y \neq z$.

$R(x, y, z)$ implies $y - x < z - x$, hence $z - y = (z - x) - (y - x) < -(y - x) = x - y$. Therefore: $R(y, z, x)$.

Let $v \in C$ and assume that $R(x, y, z)$ and $R(x, z, v)$ hold. Then $y - x < z - x$ and $z - x < v - x$, hence: $y - x < v - x$ i.e. $R(x, y, v)$, so \leq_x is transitive. Now, if $R(x, y, z)$ holds, then $y - x < z - x$, hence $z - x \not< y - x$, i.e. $\neg R(x, z, y)$. It follows that \leq_x is a partial order.

Assume that $R(x, y, z)$ holds, then $0 \neq y - x < z - x$ hence $0 \neq (y + v) - (x + v) < (z + v) - (x + v)$ therefore $R(x + v, y + v, z + v)$. \square

The relation $<_0$ can makes easier the construction of p.c.o. groups. For example, let $C = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, -2, -1\}$, and set $1 <_0 2, 1 <_0 -1, -2 <_0 2$, and $-2 <_0 -1$. One can check that in this case, $<_0$ cannot be extended to a total order.

We know that if $<$ is a compatible partial order on a group, then $x < y \Rightarrow -y < -x$. The partial order $<_0$ on a p.c.o. group satisfies a weaker property.

Lemma 3.9. *In a p.c.o. group C , we have that $\forall x \in C \setminus \{0\}, \forall y \in C \setminus \{0\}, x <_0 y \Rightarrow \neg(-x <_0 -y)$.*

Proof. Assume that $x <_0 y$. Then: $y - x <_0 -x$, and $-y = -x - (y - x) <_0 -(y - x) = x - y$. Now, assume that $-x <_0 -y$. Then: $-y + x <_0 x$, and: $y = x - (-y + x) <_0 -(-y + x) = y - x$. By transitivity: $-y <_0 x - y <_0 x <_0 y <_0 y - x <_0 -x <_0 -y$ implies: $-y <_0 -y$, a contradiction. \square

Note that in the cyclically ordered case we deduce: $x <_0 y \Rightarrow -y <_0 -x$, since $<_0$ is a total order.

3.2. The subset of non-isolated elements. We define the set of non-isolated elements for the order $<_0$, and we prove that $<_0$ can be compatible in some cases.

Definition 3.10. Let C be a p.c.o. group, we denote by $A(C)$ the set whose elements are 0 and the $x \in C \setminus \{0\}$ such that there exists $y \in C \setminus \{0\}$ satisfying $x <_0 y$ or $y <_0 x$. The elements of $A(C)$ are called the *non-isolated* elements.

Remarks 3.11. 1) Let C be a p.c.o. group and x, y in $A(C) \setminus \{0\}$. By Proposition 3.8 since $x <_0 y \Rightarrow y - x <_0 -x$, if $x <_0 y$ in $A(C)$ and $z = y - x$, then $z \in A(C) \setminus \{0\}$. So there exists z in $A(C)$ such that $y = x + z$. We see that this is similar to condition (i) in Remark 2.6.

2) If $-x <_0 y$, then $x + y <_0 x$.

3) Assume that C and C' are p.c.o. groups. It follows from Proposition 3.8 that they are isomorphic if, and only if, there is a group isomorphism φ from C onto C' such that for every x, y in C :

$$x \in A(C) \Leftrightarrow \varphi(x) \in A(C') \ \& \ x <_0 y \Leftrightarrow \varphi(x) <_0 \varphi(y).$$

Proposition 3.12. (case of compatibility of $+$ and $<_0$). Let C be a p.c.o. group such that for every x, y in $A(C) \setminus \{0\}$ we have that $x <_0 y \Leftrightarrow -y <_0 -x$ and $x + y \in A(C) \Leftrightarrow x \leq_0 -y$ or $-y \leq_0 x$. Then for all x, y, z in $A(C) \setminus \{0\}$:

$$(x <_0 y \ \& \ 0 <_0 x + z <_0 y + z) \Leftrightarrow (x <_0 y <_0 -z \ \text{or} \ -z <_0 x <_0 y)$$

(\Leftarrow holds in every p.c.o. group).

Proof. In any p.c.o. group we have the following implications.

$$\begin{aligned} 0 <_0 x <_0 y <_0 -z &\Rightarrow R(0, x, y, -z) \\ &\Rightarrow R(0, x, y) \ \& \ R(x, y, -z) \\ &\Rightarrow R(0, x, y) \ \& \ R(x + z, y + z, 0) \\ &\Rightarrow x <_0 y \ \& \ 0 <_0 x + z <_0 y + z. \end{aligned}$$

In the same way:

$$\begin{aligned} 0 <_0 -z <_0 x <_0 y &\Rightarrow R(0, -z, x, y) \\ &\Rightarrow R(0, x, y) \ \& \ R(-z, x, y) \\ &\Rightarrow R(0, x, y) \ \& \ R(0, x + z, y + z) \\ &\Rightarrow x <_0 y \ \& \ x + z <_0 y + z. \end{aligned}$$

Assume that $x <_0 y$ and $0 <_0 x + z <_0 y + z$ hold. It follows that $x + z$ and $y + z$ belong to $A(C)$ and by hypothesis, we have that either $x <_0 -z$ or $-z <_0 x$. In the same way: either $y <_0 -z$ or $-z <_0 y$. If $-z <_0 x$, then $-z <_0 x <_0 y$. If $y <_0 -z$, then $x <_0 y <_0 -z$. It follows that $-z <_0 x \ \& \ y <_0 -z$ does not hold. Now, $R(0, x + z, y + z)$ holds. Hence $R(-z, x, y)$ holds, so $x <_0 -z <_0 y$ does not hold. \square

3.3. Wound-round p.c.o. groups. In the field \mathbb{C} of complex numbers, the multiplicative group \mathbb{U} of unimodular complex numbers is the image of the additive group \mathbb{R} of real numbers under the epimorphism $\theta \mapsto e^{i\theta}$. It follows that \mathbb{U} is isomorphic to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. Then one can define the cyclic order on $\mathbb{R}/2\pi\mathbb{Z}$ by: $R(x_1 + 2\pi\mathbb{Z}, x_2 + 2\pi\mathbb{Z}, x_3 + 2\pi\mathbb{Z})$ if, and only if, there exists x'_j in $[0, 2\pi[$ such that $x_j - x'_j \in 2\pi\mathbb{Z}$ ($1 \leq j \leq 3$) and $x'_{\sigma(1)} < x'_{\sigma(2)} < x'_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3 (in other words, $x'_1 < x'_2 < x'_3$ or $x'_2 < x'_3 < x'_1$ or $x'_3 < x'_1 < x'_2$).

More generally, if (L, u) is a unital linearly ordered group, then the quotient group $L/\mathbb{Z}u$ can be cyclically ordered by setting $R(x_1 + \mathbb{Z}u, x_2 + \mathbb{Z}u, x_3 + \mathbb{Z}u)$ if, and only if, there exists x'_j in $[0, u[$ such that $x_j - x'_j \in \mathbb{Z}u$ ($1 \leq j \leq 3$) and $x'_{\sigma(1)} < x'_{\sigma(2)} < x'_{\sigma(3)}$ for some σ in the alternating group A_3 of degree 3 ([5, p. 63]). We say that $L/\mathbb{Z}u$ is the *wound-round* of L . Now, every c.o. group can be obtained in this way as shows the following theorem.

Theorem 3.13. (Rieger, [5]). *Every c.o. group is the wound-round of a unique (up to isomorphism) unital linearly ordered group $(uw(C), u_C)$.*

Definition 3.14. Let C be a c.o. group. Then $uw(C)$ is called the *unwound* of C .

Corollary 3.15. *The wound-round mapping defines a full and faithful functor from the category of unital linearly ordered groups, together with unital increasing group homomorphisms, to the category of c.o. groups, together with c-homomorphisms. The unwound mapping defines a full and faithful functor from the category of c.o. groups, together with c-homomorphisms, to the category of unital linearly ordered groups, together with unital increasing group homomorphisms. The composites of these two functors are equivalent to the identities of respective categories.*

Now, we generalize this winding construction to partially ordered groups.

Proposition 3.16. *Let $(G, <)$ be a partially ordered group, $0 < u \in G$, C be the quotient group $C = G/\mathbb{Z}u$ and ρ be the canonical mapping from G onto C .*

(1) *For every x and y in G , there exists at most one $n \in \mathbb{Z}$ such that $x < y + nu < x + u$.*

(2) *For every x_1, x_2, x_3 in G , set $R(\rho(x_1), \rho(x_2), \rho(x_3))$ if, and only if, there exist n_2 and n_3 in \mathbb{Z} such that $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$. Then (C, R) is a p.c.o. group.*

Proof. (1) Assume that n and n' are integers such that $x < y + nu < x + u$ and $x < y + n'u < x + u$. Then $-x - u < -y - n'u < -x$. Therefore, by addition, $-u < (n - n')u < u$, hence $n - n' = 0$.

(2) Assume that $x_1 < x_2 + n_2u < x_3 + n_3u < x_1 + u$, let x'_1, x'_2, x'_3 in G such that $\rho(x'_i) = \rho(x_i)$ ($i \in \{1, 2, 3\}$), and let n'_1, n'_2, n'_3 be the integers such that $x'_i = x_i + n'_iu$ ($i \in \{1, 2, 3\}$). Then $x'_1 - n'_1u < x'_2 - n'_2u + n_2u < x'_3 - n'_3u + n_3u < x'_1 - n'_1u$. Hence $x'_1 < x'_2 + (n_2 + n'_1 - n'_2)u < x'_3 + (n_3 + n'_1 - n'_3)u < x'_1 + u$. So R is indeed a ternary relation on C .

We set $\rho(0) <_0 \rho(x) <_0 \rho(y) \Leftrightarrow R(\rho(0), \rho(x), \rho(y))$, and we prove that $<_0$ is a strict partial order relation such that $\rho(0) <_0 \rho(x) <_0 \rho(y) \Leftrightarrow \rho(y) - \rho(x) <_0 -\rho(x)$.

By the definition, $\rho(0) <_0 \rho(x) <_0 \rho(y)$ iff there exist m and n in \mathbb{Z} such that $0 < x + mu < y + nu < u$. It follows from (1) that $<_0$ is anti-reflexive and anti-symmetric. The transitivity is trivial. From $x + mu < y + nu < u$, it follows that $0 < y - x + (n - m)u < -x + (1 - m)u$. Now, we have that $-u < -y - nu < -x - mu < 0$, hence $0 < -y + (1 - n)u < -x + (1 - m)u < u$. This completes the inequality: $0 < y - x + (n - m)u < -x + (1 - m)u < u$, and consequently $\rho(y) - \rho(x) <_0 -\rho(x)$. Now, by Proposition 3.8, $G/\mathbb{Z}u$ is a p.c.o. group. \square

Example 3.17. Let G_1 be the lexicographically ordered group $\mathbb{R} \overrightarrow{\times} \mathbb{R}$, and G_2 be the group $\mathbb{R} \times \mathbb{R}$ partially ordered in the following way: $(x, y) \leq (x', y') \Leftrightarrow x = x'$ and $y \leq y'$. We also define a partial cyclic order R' on $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ by setting $R'((x_1, y_1), (x_2, y_2), (x_3, y_3))$ if, and only if, $x_1 = x_2 = x_3$ and $R(y_1, y_2, y_3)$, where R is the cyclic order of \mathbb{R}/\mathbb{Z} defined in Proposition 3.16. In G_1 and G_2 , let $u = (0, 1)$, then $G_1/\mathbb{Z}u \simeq G_2/\mathbb{Z}u \simeq \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ in the language of p.c.o. groups.

Remark 3.18. *In the proof of Proposition 3.16, we showed that the p.c.o. group $C = G/\mathbb{Z}u$ enjoys for all x, y : $0 <_0 x <_0 y \Leftrightarrow 0 <_0 -y <_0 -x$. In particular, $x \in A(C) \Leftrightarrow -x \in A(C)$.*

Remark 3.19. *If G is a partially ordered group and $0 < u \in G$, then one can easily check that the subset $H := \{x \in G \mid \exists (m, n) \in \mathbb{Z} \times \mathbb{Z}, mu \leq x \leq nu\}$ is a subgroup of G , and u is a strong unit of H . Now, if $\rho(x)$ is a non isolated element of $C = G/\mathbb{Z}u$, then there exists $n \in \mathbb{Z}$ such that $0 < x + nu < u$. In particular, $-nu < x < (1 - n)u$. Hence, $x \in H$. It follows that we can restrict ourselves to the subgroup $H/\mathbb{Z}u$, or assume that u is a strong unit of G .*

Definition 3.20. Let C be a p.c.o. group. We will say that C is a *wound-round* if there exists a unital partially ordered group (G, u) such that $C \simeq G/\mathbb{Z}u$, partially cyclically ordered as in Proposition 3.16. If G is an ℓ -group, then we say that C is the *wound-round of a lattice*. If (G, u) is uniquely defined (up to isomorphism), then it is called the *unwound* of C .

Proposition 3.21. *The wound-round mapping defines a functor Θ from the category of unital partially ordered groups, together with unital increasing group homomorphisms, to the category of p.c.o. groups, together with c-homomorphisms.*

Proof. We prove that if f is a unital increasing homomorphism between the unital partially ordered groups (G, u) and (G', u') , then we can define a c-homomorphism between $C := G/\mathbb{Z}u$ and $C' := G'/\mathbb{Z}u'$. Let ρ (resp. ρ') be the canonical epimorphism from G onto C (resp. from G' onto C'). Since $f(\mathbb{Z}u) = \mathbb{Z}u'$, we can define a group homomorphism \bar{f} between C and C' by setting for every $x \in G$ $\bar{f}(\rho(x)) = \rho'(f(x))$. Let $x < y < z$ in G such that $\bar{f}(\rho(x)) \neq \bar{f}(\rho(y)) \neq \bar{f}(\rho(z)) \neq \bar{f}(\rho(x))$. Since f is increasing, we have that $f(x) \leq f(y) \leq f(z)$. Now, $f(x) \neq f(y) \neq f(z)$, so we have that $f(x) < f(y) < f(z)$. We deduce that if $R(\rho(x), \rho(y), \rho(z))$ holds and $\bar{f}(\rho(x)) \neq \bar{f}(\rho(y)) \neq \bar{f}(\rho(z)) \neq \bar{f}(\rho(x))$, then $R(\bar{f}(\rho(x)), \bar{f}(\rho(y)), \bar{f}(\rho(z)))$ holds. Hence \bar{f} is a c-homomorphism. Now, one can check that if $f \circ g$ is the composite of two unital increasing homomorphisms, then $\bar{f} \circ \bar{g} = \bar{f} \circ \bar{g}$. \square

We turn to the first-order theory of the wound-round p.c.o. groups.

Lemma 3.22. *Let (G, u) be a unital partially ordered group, C be the quotient group $C = G/\mathbb{Z}u$, ρ be the canonical mapping from G onto C and $G_u = \{x \in G \mid x \geq 0 \text{ \& } x \not\geq u\}$. Then:*

- *the restriction of ρ to the subset G_u is a one-to-one mapping onto C ,*
- *for every x, y, z in G_u , $\rho(x) + \rho(y) = \rho(z) \Leftrightarrow x + y - z \in \mathbb{Z}u$ and $R(\rho(x), \rho(y), \rho(z))$ if, and only if, either $x < y < z$ or $y < z < x$ or $z < x < y$.*

Proof. Since u is a strong unit, for every $x \in G$ there exist integers m and n such that $mu \leq x < nu$, and we can assume that m is maximal. Then $x - mu \in G_u$, hence the restriction of ρ to G_u is onto. Let x and y in G_u such that $\rho(x) = \rho(y)$, then $x - y \in \mathbb{Z}u$. Hence there exists an integer m such that $y = x + mu$, and without loss of generality we can assume that $m \geq 0$. If $m \geq 1$, then $y \geq x + u \geq u$: a contradiction. Hence $m = 0$, and $y = x$. So the restriction of ρ to G_u is one-to-one. The remainder of the proof is straightforward using properties of subsection 2.3. \square

Theorem 3.23. *Let (G, u) be a unital partially ordered group, C be the quotient group $C = G/\mathbb{Z}u$, ρ be the canonical mapping from G onto C and $G_u = \{x \in G \mid x \geq 0 \text{ \& } x \not\geq u\}$. Then:*

- *the p.c.o. group C , in the language L_c , is interpretable in $(G, \mathbb{Z}u)$ in the language $L_{l_0\mathbb{Z}u}$,*
- *if (G', u') is a unital partially ordered group, then $(G, \mathbb{Z}u) \equiv (G', \mathbb{Z}u') \Rightarrow G/\mathbb{Z}u \equiv G'/\mathbb{Z}u'$ (the same holds with $<$ instead of \equiv).*

Proof. Follows from Lemma 3.22 and properties of subsection 2.3. \square

Now we focus on the sets of non-isolated elements of wound-round p.c.o. groups and of wound-rounds of lattices.

Proposition 3.24. *Let (G, u) be a unital partially ordered group, $N = \{x \in]0, u[\mid \exists y \in]0, u[\ x < y \text{ or } y < x\}$, and C be the wound-round p.c.o. group $G/\mathbb{Z}u$. Then, $(N, <)$ and $(A(C) \setminus \{0\}, \leq_0)$ are isomorphic ordered sets.*

Proof. It follows from (1) of Proposition 3.16 that the restriction of ρ to $]0, u[$ is one-to-one. In particular, its restriction to N is one-to-one. If $x \in N$, then there exists $y \in N$ such that $0 < x < y$ or $0 < y < x$. It follows that $\rho(x) <_0 \rho(y)$ or $\rho(y) <_0 \rho(x)$. In particular, $\rho(x) \in A(C)$, and ρ is an homomorphism of ordered sets from N to $A(C)$. Now, let $x \in G$ such that $\rho(x) \in A(C)$ and $x \notin \mathbb{Z}u$. Then, there exist $y \in G$ and integers n, n' such that $0 < x + nu < y + n'u < u$ or $0 < y + n'u < x + nu < u$. In any case $x + nu \in N$. Since $\rho(x + nu) = \rho(x)$, it follows that ρ is an isomorphism of ordered sets between (N, \leq) and $(A(C) \setminus \{0\})$. \square

We saw that an MV-algebra A is isomorphic to the subset $[0, u_A]$ of its Chang ℓ -group (G_A, u_A) . We get a similar result in the case of wound-round ℓ -groups.

Corollary 3.25. *Let (G, u) be a unital ℓ -group, $C = G/\mathbb{Z}u$ and ρ be the canonical mapping from G onto C . We assume that $A(C) \neq \{0\}$. We add an element 1 to $A(C)$ and we set $x <_0 1$ for every $x \in A(C)$. Then the ordered sets $([0, u], \leq)$ and $(A(C) \cup \{1\})$ are isomorphic.*

Proof. By the definition of $A(C)$ and of \leq_0 in $C = G/\mathbb{Z}u$, if $A(C) \neq \{0\}$, then there is x, y in G such that $0 < x < y < u$ or $0 < y < x < u$. Hence, by Lemma 2.4, $N =]0, u[$. Therefore the result follows from Proposition 3.24. \square

Note that if C is the wound-round of a lattice, $C \simeq G/\mathbb{Z}u$, with $u > 0$ a strong unit of G , then for every x, y in $A(C)$, we have the following:

- $0 < x \leq_0 y \Leftrightarrow 0 > -y \leq_0 -x$,
- $x \wedge_0 y$ exists (the infimum of x and y in $(A(C), \leq_0)$),
- 0 is the smallest element,
- $x \vee_0 y$ does not exist if, and only if, $(-x) \wedge_0 (-y) = 0$, and if this holds, then for every $z \in A(C)$: $x <_0 z \Rightarrow y \not<_0 z$.

Furthermore, for every $x \in A(C)$ there exists a unique $g \in G$ such that $0 \leq g < u$ and $\rho(g) = x$. Now, if $x \in C \setminus A(C)$, then, for every $y \in C$, $x \wedge_0 y = 0$, and $x \vee_0 y$ does not exist.

Remark 3.26. *Let (G, u) be a unital ℓ -group and $C = G/\mathbb{Z}u$. Assume that $A(C)$ is not trivial. By Corollary 3.25 the ordered sets $]0, u[$ and $A(C)$ are isomorphic. Now, since G is lattice-ordered, we know that it is generated by its positive elements. We saw in Lemma 2.7 that every positive element of G is a sum of elements of $]0, u[$. It follows that the subgroup generated by $A(C)$ is equal to C .*

In general, $A(C)$ is not a subgroup of C . Now, we show that $A(C)$ is partially closed under $+$.

Proposition 3.27. *(sums of non-isolated elements). Let C be a wound-round p.c.o. group. For every x, y in $A(C)$ we have that $y - x \in A(C) \Leftrightarrow x \leq_0 y$ or $y \leq_0 x$. It follows that $x + y \in A(C) \Leftrightarrow x \leq_0 -y$ or $y \leq_0 -x$.*

Proof. Let (G, u) be a unital partially ordered group such that $C \simeq G/\mathbb{Z}u$, partially cyclically ordered as in Proposition 3.16. If $x = y$ or $x = 0$ or $y = 0$, then the result is trivial. Let $x \neq y$ in $A(C) \setminus \{0\}$. If $x <_0 y$, then we have already seen that by Proposition 3.8 we have that $y - x \in A(C)$. If $y <_0 x$, then $x - y \in A(C)$, and by Remark 3.18 we have that $y - x \in A(C)$.

Now, assume that $y - x \in A(C)$, and let g, h in G such that $x = \rho(g)$ and $y = \rho(h)$. We know that we can assume that $0 < g < u$ and $0 < h < u$. Therefore: $-u < h - g < u$. Now, $\rho(h - g) = y - x \in A(C)$, hence there exists a unique integer n such that $0 \leq h - g - nu < u$. It follows that either $-u < h - g < 0$ or $0 \leq h - g < u$. If $-u < h - g < 0$, then $h < g$, and since $0 < h$ and $g < u$, we have that $y <_0 x$. If $0 \leq h - g < u$, then $g < h$, and since $0 < g$ and $h < u$, we have that $x <_0 y$. The other assertion follows easily. \square

3.4. ℓ -c.o. groups. In [7], the lattice-cyclically-ordered groups are defined to be p.c.o. groups such that \leq_0 defines a structure of distributive lattice with first element. In the present paper we look at a larger class of groups. Indeed, we noticed after Definition 3.20 that in the case of wound-rounds of ℓ -groups for any nonzero x and y , $x \vee_0 y$ exists if, and only if, $(-x) \wedge_0 (-y) \neq 0$. This motivate the following definition.

Definitions 3.28. An ℓ -c.o. group is a p.c.o. group C such that, for every x and y in $A(C)$, $x \wedge_0 y$ exists, and $0 <_0 x <_0 y \Leftrightarrow 0 <_0 -y <_0 -x$.

An ℓ -c-homomorphism is a c-homomorphism from a ℓ -c.o. group C to a ℓ -c.o. group C' such that for every x and y in C we have that $f(x \wedge_0 y) = f(x) \wedge_0 f(y)$.

From the properties that we noticed after Corollary 3.25, it follows that the wound-round of an ℓ -group is an ℓ -c.o. group. However, the wound-round operation is not a functor from the category of ℓ -groups to the category of ℓ -c.o. groups. Indeed, let G be the ℓ -group $\mathbb{R} \times \mathbb{R}$, $u = (1, 1)$, $G' = \mathbb{R}$, $u' = 1$, and $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the natural projection onto the first component. Then, f is an ℓ -homomorphism, and $f(u) = u'$. Let $x = (\frac{1}{2}, 2)$ and $y = (\frac{1}{4}, 4)$. Both of x and y belong to $G_u \setminus [0, u[$. Hence, if ρ (resp. ρ') is the canonical epimorphism from G onto $C = G/\mathbb{Z}u$ (resp. from G' onto $C' = G'/\mathbb{Z}u'$), then $\rho(x) \notin A(C)$, $\rho(y) \notin A(C)$, so $\rho(x) \wedge_0 \rho(y) = \rho(0)$. Now, $f(x) \in [0, u'[$, $f(y) \in [0, u'[$ and $f(x) \wedge f(y) = \frac{1}{4}$. It follows that $\rho'(f(x)) \wedge_0 \rho'(f(y)) = \rho'(f(y)) \neq \rho'(0)$. Consequently, the c-homomorphism $\bar{f} : C \rightarrow C'$ induced by f (see Proposition 3.21) is not an ℓ -c-homomorphism. However, we have the following.

Proposition 3.29. Let (G, u) and (G', u') be unital ℓ -groups, f be a one-to-one unital ℓ -homomorphism from (G, u) to (G', u') and \bar{f} be the c-homomorphism defined in the proof of Proposition 3.21. Then, \bar{f} is an ℓ -c-homomorphism from $C := G/\mathbb{Z}u$ to $C' := G'/\mathbb{Z}u'$.

Proof. Let ρ (resp. ρ') be the canonical epimorphism from G onto C (resp. from G' onto C'). We recall that \bar{f} is defined by setting, for every $x \in G$, $\bar{f}(\rho(x)) = \rho'(f(x))$.

First we let $x \in G_u$, and we prove that $\rho'(f(x)) \in A(C') \Leftrightarrow \rho(x) \in A(C)$.

Recall that, since $x \in G_u$, we have that $\rho(x) \in A(C) \Leftrightarrow x \in [0, u[$. We saw in Lemmas 2.7 and 2.8 that there is a unique sequence x_1, \dots, x_n such that $x = x_1 + \dots + x_n$, $x_1 = x \wedge u$, $x_2 = (x - x_1) \wedge u$, and so on. Since f is a unital ℓ -homomorphism, we have that $f(x_1) = f(x) \wedge u'$, $f(x_2) = (f(x) - f(x_1)) \wedge u'$, and so on. Now, f is one-to-one, so, for $1 \leq i \leq n$, $f(x_i) = 0 \Leftrightarrow x_i = 0$. Hence $f(x_1), \dots, f(x_n)$ is the sequence associated with $f(x)$ as in Lemmas 2.7 and 2.8. It follows that

$$\rho(x) \in A(C) \Leftrightarrow x \in [0, u[\Leftrightarrow n = 1 \Leftrightarrow f(x) \in [0, u'[\Leftrightarrow \rho'(f(x)) \in A(C').$$

Let x, y in G_u . If x and y belong to $[0, u[$, then $f(x)$ and $f(y)$ belong to $[0, u'[$ and $f(x \wedge y) = f(x) \wedge f(y)$. Therefore

$$\bar{f}(\rho(x) \wedge_0 \rho(y)) = \bar{f}(\rho(x \wedge y)) = \rho'(f(x \wedge y)) = \rho'(f(x) \wedge f(y)) = \rho'(f(x)) \wedge_0 \rho'(f(y)) = \bar{f}(\rho(x)) \wedge_0 \bar{f}(\rho(y)).$$

If $x \notin [0, u[$, then $\rho(x) \wedge_0 \rho(y) = 0$. Now, we have proved that $f(x) \notin [0, u'[$, hence $\rho'(f(x)) \wedge_0 \rho'(f(y)) = 0$. The case where $y \notin [0, u[$ is similar. \square

3.5. Cyclically ordered groups elementarily equivalent to subgroups of \mathbb{U} . In this subsection we list some results of [10].

Definition 3.30. Let C be a c.o. group.

1) C is said to be *c-archimedean* if for every x and y in $C \setminus \{0\}$ there exists an integer $n > 0$ such that $R(0, nx, y)$ does not hold (in other words, $y \leq_0 nx$, since (C, \leq_0) is linearly ordered).

2) C is said to be *discrete* if (C, \leq_0) is a discretely ordered set.

3) C is said to be *c-regular* if for every integer $n \geq 2$ and every $0 <_0 x_1 <_0 \dots <_0 x_n$ in C there exists $x \in C$ such that $x_1 \leq_0 nx \leq_0 x_n$ and $x <_0 2x <_0 \dots <_0 (n-1)x <_0 nx$. This is equivalent to saying that its unwound is a *regular* linearly ordered group, that is, for every $n \geq 2$ and every $0 < x_1 < \dots < x_n$ in $uw(C)$ there exists $x \in uw(C)$ such that $x_1 \leq nx \leq x_n$.

- 4) C is said to be *pseudo-c-archimedean* if C belongs to the elementary class generated by the c-archimedean c.o. groups.
 5) C is said to be *pseudofinite* if C belongs to the elementary class generated by the finite c.o. groups.

Note that C is c-archimedean if, and only if, its unwound is archimedean, and C is discrete if, and only if, its unwound is a discrete linearly-ordered group.

Notations 3.31. If C is discrete, then the first positive element ε_C of C is definable, we can assume that it lies in the language. For a prime p , integers $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$, we denote by $D_{p^n, k}$ the formula: $\exists x, R(0, x, 2x, \dots, (p^n - 1)x) \& p^n x = k\varepsilon_C$.

Definition 3.32. If B is an abelian group and p is a prime, then we define the p -th *prime invariant of Zakon* of B , denoted by $[p]B$, to be the maximum number of p -incongruent elements in B . In the infinite case, we set $[p]B = \infty$, without distinguishing between infinities of different cardinalities (see [14]).

Theorem 3.33. 1) *A dense c.o. group is pseudo-c-archimedean if, and only if, it is c-regular. If this holds, then it is elementarily equivalent to some c-archimedean dense c.o. group.*

2) *Any two dense c-regular c.o. groups are elementarily equivalent if, and only if, their torsion subgroups are isomorphic and they have the same family of prime invariants of Zakon. This in turn is equivalent to: their torsion subgroups are isomorphic and their unwounds have the same family of prime invariants of Zakon.*

Theorem 3.34. 1) *Any two non-c-archimedean c-regular discrete c.o. groups are elementarily equivalent if, and only if, they satisfy the same formulas $D_{p^n, k}$.*

2) *A c.o. group is pseudofinite if, and only if, it is discrete and c-regular.*

3) *Let U be a non-principal ultrafilter on \mathbb{N}^* , C be the ultraproduct of the c.o. groups $\mathbb{Z}/n\mathbb{Z}$, p be a prime, $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$. Then C satisfies the formula $D_{p^n, k}$ if, and only if, $p^n \mathbb{N}^* - k \in U$.*

4. FROM MV-ALGEBRAS TO WOUND-ROUNDS OF LATTICES.

The correspondence between MV-algebras and p.c.o. groups is defined as follows. Let A be an MV-algebra and (G_A, u_A) be its Chang ℓ -group. We saw in Section 2 that $\Xi : A \mapsto (G_A, u_A)$ is a functor from the category of MV-algebras to the category of unital ℓ -groups, where (G_A, u_A) is the Chang ℓ -group of A . Now, the wound-round functor $\Theta : (G, u) \mapsto G/\mathbb{Z}u$ defined in Proposition 3.21 is a functor from the category of unital ℓ -groups to the category of wound-rounds of lattices, together with the c-homomorphisms. So, this gives rise to a functor $\Theta\Xi$ from the category of MV-algebras to the category of wound-rounds of lattices, together with the c-homomorphisms.

In this section, we describe the correspondence between MV-algebras and wound-round of lattices. Then, we define the converse correspondence.

Definition 4.1. Notation. We denote by $C(A)$ the wound-round of lattice $G_A/\mathbb{Z}u_A$, and by ρ the canonical epimorphism from G_A onto $C(A)$, where, for $x \in G_A$, $\rho(x) \in C(A)$ is the class of x modulo $\mathbb{Z}u_A$. Without loss of generality, we assume that $A \subset G_A$ and $1 = u_A$, we denote by φ the restriction of ρ to $[0, u_A]$.

4.1. Interpretability of A in $C(A)$. Recall that $A(C(A))$ is the set of non-isolated elements of $C(A)$ (see Definition 3.10). Assume that $A \neq \{0, 1\}$, $A \neq \{0, 1, x\}$ and $A \neq \{0, 1, x, \neg x\}$, for some x . Then by Corollary 3.25, φ is an isomorphism of ordered sets between $([0, u_A], \leq)$ and $(C(A), \leq_0)$. It follows that, for every x, y in $[0, u_A]$, $\varphi(x \wedge y) = \varphi(x) \wedge_0 \varphi(y)$, and if $x \vee y < u_A$, then $\varphi(x \vee y) = \varphi(x) \vee_0 \varphi(y)$.

Note that if $A = \{0, 1\}$, then $C(A) = \{0\}$. If $A = \{0, 1, x\}$, then $C(A) \simeq \mathbb{Z}/2\mathbb{Z}$. If $A = \{0, 1, x, \neg x\}$ is not an MV-chain, then $C(A) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and in any case $A(C(A)) = \{0\}$. If $A = \{0, 1, x, \neg x\}$ is an MV-chain, then $C(A) \simeq \mathbb{Z}/4\mathbb{Z}$. In the following, we assume that $A \neq \{0, 1\}$, $A \neq \{0, 1, x\}$ and $A \neq \{0, 1, x, \neg x\}$, for some x .

Remark 4.2. *Let x in $]0, u_A[$, since $\neg x = u_A - x$, we have that $\varphi(\neg x) = -\varphi(x)$.*

Proposition 4.3. *We add an element $\mathbf{1}$ to $A(C(A))$, and we set $\varphi(u_A) = \mathbf{1}$. For every $x \in [0, u_A[$ set*

$$\varphi(x) <_0 \mathbf{1}, \neg\varphi(x) = -\varphi(x) \text{ if } x \in]0, 1[, \neg\varphi(0) = \mathbf{1} \text{ and } \neg\mathbf{1} = \varphi(0).$$

Let x, y in $[0, u_A[$, we have that $\rho(x \oplus y) = \varphi(x) \wedge_0 (\neg\varphi(y)) + \varphi(y)$.

Proof. We know that $x \oplus y = (x + y) \wedge u_A$, hence $x \oplus y = x \wedge (u_A - y) + y$. Assume that $x \wedge (u_A - y) + y < 1$ and $y \neq 0$ (the case $y = 0$ being trivial). Hence

$$\begin{aligned} \rho(x \oplus y) &= \varphi(x \oplus y) \\ &= \varphi(x \wedge (u_A - y) + y) \\ &= \varphi(x \wedge (u_A - y)) + \varphi(y) \\ &= \varphi(x) \wedge_0 \varphi(u_A - y) + \varphi(y) \\ &= \varphi(x) \wedge_0 (\neg\varphi(y)) + \varphi(y). \end{aligned}$$

If $x \wedge (u_A - y) + y = u_A$ i.e. $x \wedge (u_A - y) = u_A - y$, then $u_A - y \leq x$ and $y \neq 0$. It follows that $0 <_0 \varphi(u_A - y) = -\varphi(y) \leq_0 \varphi(x)$ and

$$\begin{aligned} \varphi(x) \wedge_0 (-\varphi(y)) + \varphi(y) &= \varphi(x) \wedge_0 (-\varphi(y)) + \varphi(y) \\ &= -\varphi(y) + \varphi(y) \\ &= 0 \\ &= \rho(u_A) \\ &= \rho(x \oplus y). \end{aligned}$$

□

Corollary 4.4. *The MV-algebra A , in the language L_{MV} , is interpretable in the L_{ι_0} -structure $A(C(A))$. In particular, if A and A' are MV-algebras such that $A(C(A)) \equiv A(C(A'))$, then $A \equiv A'$. The same holds with \prec instead of \equiv .*

Proof. For every x, y in $A(C(A)) \cup \{\mathbf{1}\}$ we set $\neg x = -x$ if $0 \neq x \neq \mathbf{1}$, $\neg 0 = \mathbf{1}$, $\neg \mathbf{1} = 0$, and $x \oplus y = x \wedge_0 \neg y + y$ if $x \wedge_0 \neg y + y \neq 0$ or $x = y = 0$, and we set $x \oplus y = \mathbf{1}$ otherwise. The remainder of the proof follows from Theorem 2.13. □

4.2. MV-algebra associated with a p.c.o. group.

Definition 4.5. Notation. Let C be a p.c.o. group. We add an element $\mathbf{1}$ to $A(C)$ and we set, for every $x \in A(C)$, $x <_0 \mathbf{1}$ and $\mathbf{1} + x = x + \mathbf{1} = x$.

Definition 4.6. Let C be a p.c.o. group. We will say that $A(C)$ defines canonically an MV-algebra if it satisfies the following.

- 1) For every x, y in $A(C) \setminus \{0\}$ we have that $x <_0 y \Leftrightarrow -y <_0 -x$.
- 2) $(A(C) \cup \{\mathbf{1}\}, \leq_0)$ is a distributive lattice.
- 3) For every x, y in $A(C)$, $x + y = x \wedge_0 y + x \vee_0 y$.
- 4) For every x, y, z in $A(C) \setminus \{0\}$, we have that

$$x - y = (x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x).$$

We will denote by \mathcal{AC} the class of p.c.o. groups C such that $A(C)$ defines canonically an MV-algebra.

Note that by Conditions 1) and 2) the elements of \mathcal{AC} are ℓ -c.o. groups.

The aim of this subsection is to prove the following theorem.

Theorem 4.7. *Let $C \in \mathcal{AC}$. Set $\neg 0 = \mathbf{1}$, $\neg \mathbf{1} = 0$ and for $x \in A(C) \setminus \{0\}$ set $\neg x = -x$. For every x, y in $A(C) \cup \{\mathbf{1}\}$ set*

$x \oplus y = x \wedge_0 (-y) + y$ if $x \wedge_0 (-y) + y \neq 0$ or $x = y = 0$, and $x \oplus y = \mathbf{1}$ otherwise.

Then $A(C) \cup \{\mathbf{1}\}$ is an MV algebra with natural partial order \leq_0 .

Corollary 4.8. *Let C be a p.c.o. group.*

- *$A(C)$ defining canonically an MV-algebra is expressible by countably many first-order formulas of the language L_C .*
- *If $A(C)$ defines canonically an MV-algebra, then the MV-algebra $A(C) \cup \{\mathbf{1}\}$ defined in Theorem 4.7 is interpretable in $C \cup \{\mathbf{1}\}$, where $\mathbf{1}$ is a new element.*

Remark 4.9. *Let A be an MV-algebra such that there exist $x < y$ in $]0, 1[$. Then the MV-algebra $A(C(A)) \cup \{\mathbf{1}\}$ (together with the operations defined in Theorem 4.7) is isomorphic to A .*

Proof. Since $]0, 1[$ contains $x < y$, we deduce from Corollary 2.5 that $A(C(A))$ is nonempty. By Corollary 3.25, the canonical epimorphism ρ from the Chang ℓ -group (G_A, u_A) of A induces an isomorphism φ between the lattices $[0, u_A]$ and $A(C(A)) \cup \{\mathbf{1}\}$. Now, for g, h in $]0, u_A[$, $\varphi(g) \wedge (-\varphi(h)) + \varphi(h) = \varphi(g) \wedge \varphi(u_A - h) + \varphi(h) = \varphi((g + h) \wedge u_A)$. Hence $\varphi(g) \wedge (-\varphi(h)) + \varphi(h) = 0$ if, and only if, either $g + h \geq u_A$ or $g + h = 0$. Consequently, by Proposition 4.3, φ is an isomorphism of MV-algebras. □

The proof of Theorem 4.7 is based on the following lemmas.

Lemma 4.10. *Let C be an ℓ -c.o. group and x, y in $A(C) \setminus \{0\}$. If $-y \neq x \wedge_0 (-y)$, then $y <_0 x \wedge_0 (-y) + y$. In particular, $x \wedge_0 (-y) + y$ belongs to $A(C)$.*

Proof. Since $-y \neq x \wedge_0 (-y)$, we have that $x \wedge_0 (-y) <_0 -y$. By hypothesis, this is equivalent to $y <_0 -(x \wedge_0 (-y))$. By Proposition 3.8, this in turn is equivalent to $-x \wedge_0 (-y) - y <_0 -y$. By hypothesis, this in turn is equivalent to $y <_0 x \wedge_0 (-y) + y$. The last assertion follows easily. □

Lemma 4.11. *Let C be a p.c.o. group such that for every x, y in $A(C) \setminus \{0\}$ we have that $x <_0 y \Leftrightarrow -y <_0 -x$. Let x, y in $A(C)$ such that the infimum $z = x \wedge_0 y$ of x and y in $(A(C), \leq_0)$ exists. Then $x - z$ and $y - z$ belong to $A(C)$, the infimum $(x - z) \wedge_0 (y - z)$ exists and is equal to 0.*

Proof. If $x \leq_0 y$, then $z = x$, $y - z = y - x <_0 -x$ (Proposition 3.8). Hence $y - z \in A(C)$, and $(y - z) \wedge_0 (x - z) = x - z = 0$. The same holds if $y \leq_0 x$. Now, assume that nor $x \leq_0 y$ nor $y \leq_0 x$. We have that $z <_0 x$, hence $x - z <_0 -z$, in particular, $x - z \in A(C)$. Let $t \in C$ such that $0 <_0 t <_0 x - z$. Then we have: $R(0, t, x - z, -z)$. Hence $R(z, t + z, x, 0)$ holds. Therefore $R(0, z, t + z, x)$ holds, i.e. $z <_0 t + z <_0 x$. In the same way, $0 <_0 t <_0 y - z \Rightarrow z <_0 t + z <_0 y$. Hence, since $z = x \wedge_0 y$, this yields a contradiction. Consequently, there is no $t \in A(C) \setminus \{0\}$ such that $t <_0 x - z$ & $t <_0 y - z$. \square

Remark 4.12. *Let C be an ℓ -c.o. group. Then, the supremum $x \vee_0 y$ exists if, and only if, $(-x) \wedge_0 (-y) \neq 0$. If this holds, then $x \vee_0 y = -((-x) \wedge_0 (-y))$. Otherwise, there is no $z \in A(C)$ such that $x \leq_0 z$ and $y \leq_0 z$. If the supremum of x and y does not exist, then we will set $x \vee_0 y = \mathbf{1}$. So $(A(C) \cup \{\mathbf{1}\}, \leq_0)$ is a lattice with smallest element 0 and greatest element $\mathbf{1}$.*

Lemma 4.13. *Let $C \in \mathcal{AC}$. Then, for every x, y in $A(C) \cup \{\mathbf{1}\}$: $x \oplus y = \mathbf{1} \Leftrightarrow (-y \leq_0 x \ \& \ (x, y) \neq (0, 0))$. In particular: $\neg x \oplus x = \mathbf{1}$.*

Proof. We have that $x \wedge_0 \neg y + y = 0$ if, and only if, $-y = x \wedge_0 (-y)$. So $x \wedge_0 \neg y + y = 0 \Leftrightarrow -y \leq_0 x$. In particular, $x \oplus y = \mathbf{1} \Leftrightarrow (-y \leq_0 x \ \& \ (x, y) \neq (0, 0))$. \square

Proof of Theorem 4.7. Note that if $x \vee_0 y$ does not exist in $A(C)$, then $x \vee_0 y = \mathbf{1}$. By Lemma 4.10, if x and y belong to $A(C) \setminus \{0\}$ and $x \oplus y \neq \mathbf{1}$, then $x \oplus y = x \wedge_0 \neg y + y \in A(C)$. Let x, y in $A(C) \cup \{\mathbf{1}\}$. If $y = 0$, then $x \oplus y = x \wedge_0 \mathbf{1} + 0 = x$. If $x = 0$, then $x \oplus y = 0 + y = y$. If $y = \mathbf{1}$, then $x \oplus y = x + 0 = x$. If $x = \mathbf{1}$, then $x \wedge_0 0 + \mathbf{1} = 0$, hence $x \oplus y = \mathbf{1}$. If $x = \mathbf{1}$, then $1 \wedge \neg y + y = \neg y + y = 0$. Hence $x \oplus y = \mathbf{1}$. It follows that in any case $x \oplus y \in A(C) \cup \{\mathbf{1}\}$.

We have to prove that \oplus and \neg satisfy the axioms of Definition 2.1.

MV4) Trivially, for every $x \in A(C) \cup \{\mathbf{1}\}$: $\neg \neg x = x$.

MV3) and MV5) have already been proved (i.e. $x \oplus 0 = x$, $x \oplus \neg 0 = \neg 0$).

MV2) ($x \oplus y = y \oplus x$) The case where $x \in \{0, \mathbf{1}\}$ or $y \in \{0, \mathbf{1}\}$ follows from above calculations. Assume that x and y belong to $A(C) \setminus \{0\}$. By Lemma 4.13, $x \oplus y = \mathbf{1} \Leftrightarrow -y \leq_0 x \Leftrightarrow -x \leq_0 y \Leftrightarrow y \oplus x = \mathbf{1}$. Otherwise, $x \oplus y - y \oplus x = x \wedge_0 (-y) + y - (y \wedge_0 (-x) + x) = x \wedge_0 (-y) - (y \wedge_0 (-x)) - (x - y) = x \wedge_0 (-y) + (-y) \vee_0 x - (x - y) = 0$, by 3) of Definition 4.6.

MV6) ($\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$) Trivially, we can assume that $x \neq y$. Since $x \vee_0 y = y \vee_0 x$, it is sufficient to prove that for all x, y in $A(C) \cup \{\mathbf{1}\}$ we have that $\neg(\neg x \oplus y) \oplus y = x \vee_0 y$. If $y = 0$, then $\neg(\neg x \oplus y) \oplus y = \neg(\neg x \oplus y) = \neg(\neg x) = x = x \vee_0 y$. If $x = 0$, then $\neg(\neg x \oplus y) \oplus y = \neg(\mathbf{1} \oplus y) \oplus y = \neg \mathbf{1} \oplus y = 0 \oplus y = y = x \vee_0 y$.

If $y = \mathbf{1}$, then $\neg(\neg x \oplus y) \oplus y = \neg(\neg x \oplus y) \oplus \mathbf{1} = \mathbf{1} = x \vee_0 y$. If $x = \mathbf{1}$ and $y \in A(C) \setminus \{0\}$, then $\neg(\neg x \oplus y) \oplus y = \neg y \oplus y = \mathbf{1} = x \vee_0 y$.

If $x <_0 y$, then, by Lemma 4.13, $\neg x \oplus y = \mathbf{1}$, and $\neg(\neg x \oplus y) \oplus y = 0 \oplus y = y = x \vee_0 y$.

Otherwise, we have that $\neg x \oplus y = (-x) \wedge_0 (-y) + y \neq 0$, and

$$\neg(\neg x \oplus y) \oplus y = -((-x) \wedge_0 (-y) + y) \oplus y = (x \vee_0 y - y) \oplus y = (x \vee_0 y - y) \wedge_0 (-y) + y.$$

Since $y <_0 x \vee_0 y$, we have that $x \vee_0 y - y <_0 -y$ (by Proposition 3.8). Hence $(x \vee_0 y - y) \wedge_0 (-y) + y = x \vee_0 y - y + y = x \vee_0 y$.

MV1) ($x \oplus (y \oplus z) = (x \oplus y) \oplus z$). This is trivial if x, y or z belongs to $\{0, \mathbf{1}\}$. We assume that x, y, z belong to $A(C) \setminus \{0\}$. Assume that $x \oplus y = \mathbf{1}$. Then, $(x \oplus y) \oplus z = \mathbf{1}$. By Lemma 4.13, we have that $-y \leq_0 x$. If $y \oplus z = \mathbf{1}$, then $(x \oplus y) \oplus z = \mathbf{1} = x \oplus (y \oplus z)$. We assume that $y \oplus z \neq \mathbf{1}$. By Lemma 4.10, $y \leq_0 y \oplus z$. Hence $-(y \oplus z) \leq_0 -y \leq_0 x$. Therefore, $x \wedge_0 (-(y \oplus z)) + (y \oplus z) = -(y \oplus z) + (y \oplus z) = 0$. It follows that $(x \oplus y) \oplus z = \mathbf{1} = x \oplus (y \oplus z)$.

Assume that $x \oplus y \neq \mathbf{1} \neq y \oplus z$, so we have that $-y \not\leq_0 x$ and $-y \not\leq_0 z$. Therefore:

$$\begin{aligned} (x \oplus y) \oplus z - x \oplus (y \oplus z) &= (x \oplus y) \oplus z - (z \oplus y) \oplus x \\ &= (x \wedge_0 (-y) + y) \wedge_0 (-z) + z - (z \wedge_0 (-y) + y) \wedge_0 (-x) - x \\ &= (x \wedge_0 (-y) + y) \wedge_0 (-z) - (z \wedge_0 (-y) + y) \wedge_0 (-x) - (x - z). \end{aligned}$$

Now, it follows from 4) of Definition 4.6 that $(x \oplus y) \oplus z - x \oplus (y \oplus z) = 0$. \square

Remark 4.14. *Let n_1 and n_2 be integers, greater than 4, C_1 be the c.o. group $\mathbb{Z}/n_1\mathbb{Z}$ and C_2 be the c.o. group $\mathbb{Z}/n_2\mathbb{Z}$. C_1 and C_2 define MV-algebras. We can define a p.c.o. group $C_1 \times C_2$ by setting $R((x_1, x_2), (y_1, y_2), (z_1, z_2)) \Leftrightarrow R(x_1, y_1, z_1) \ \& \ R(x_2, y_2, z_2)$. Then*

$$A(C_1 \times C_2) = C_1 \times C_2 \setminus [((\mathbb{Z}/n_1\mathbb{Z}) \times \{0\}) \cup (\{0\} \times (\mathbb{Z}/n_2\mathbb{Z})) \cup \{(n_1 - 1, 1), (1, n_2 - 1)\}].$$

Now, $-(3, 1) = (n_1 - 3, n_1 - 1)$ and $(1, 3)$ belong to $A(C_1 \times C_2)$, $(3, 1) \not\leq_0 (1, 3)$, $(1, 3) \not\leq_0 (3, 1)$, but $(1, 3) - (3, 1) = (n_1 - 2, 2) \in A(C_1 \times C_2)$. Hence the rule $x \in A(C), y \in A(C) \Rightarrow (x + y \in A(C) \Leftrightarrow x \leq_0 -y \ \text{or} \ -y \leq_0 x)$ does not hold. Consequently $C_1 \times C_2$ does not define canonically an MV-algebra.

We can define another partial cyclic order on $C_1 \times C_2$ by setting $(x_1, x_2) \leq_0 (y_1, y_2) \Leftrightarrow (x_1 \leq_0 y_1 \ \& \ x_2 \leq_0 y_2)$. In this case $A(C_1 \times C_2) = C_1 \times C_2$, and we conclude in the same way that $C_1 \times C_2$ does not define canonically an MV-algebra.

Now, by Theorem 4.15 the p.c.o. group $(\mathbb{Z} \times \mathbb{Z})/\mathbb{Z}(n_1, n_2)$ defines canonically an MV-algebra.

4.3. Wound-rounds of lattices.

Theorem 4.15. *Let C be the wound-round of a lattice. Then $C \in \mathcal{AC}$. Furthermore, if $C = G/\mathbb{Z}u$, then $A(C) \cup \{\mathbf{1}\}$ is isomorphic to the MV-algebra $\Gamma(G, u)$.*

Proof. We have to prove that C satisfies conditions 1), 2), 3), 4) of Definition 4.6.

1) has been proved in Remark 3.18.

2) Let (G, u) be a unital ℓ -group such that $C \simeq G/\mathbb{Z}u$ and ρ be the natural mapping from G onto C . By Lemma 3.22 the restriction of ρ is a one-to-one mapping from $G_u = \{g \in G \mid 0 \leq g \ \& \ g \not\leq 0\}$ onto C . We saw in Proposition 3.24 that $A(C)$ can be identified with a subset of $[0, u[$. Let g, h in $[0, u[$ such that $\rho(g) \in A(C)$ and $\rho(h) \in A(C)$. We have that $g < h \Leftrightarrow \rho(g) <_0 \rho(h)$. It follows that $\rho(g \wedge h) \in A(C)$, $\rho(g \wedge h) = \rho(g) \wedge_0 \rho(h)$, and if $g \vee h \neq u$, then $\rho(g \vee h) \in A(C)$, $\rho(g \vee h) = \rho(g) \vee_0 \rho(h)$. By setting $\rho(u) = \mathbf{1}$, we have that $g \vee h = u \Leftrightarrow \rho(g) \vee_0 \rho(h) = \mathbf{1}$, hence $A(C) \cup \{\mathbf{1}\}$ embeds into a sublattice of $[0, u]$, so it is a distributive lattice, with smallest element 0 and greatest element $\mathbf{1}$. Note that by Corollary 3.25, if $A(C) \neq \{0\}$, then $A(C) \cup \{\mathbf{1}\}$ is isomorphic to the lattice $[0, u]$.

3) Let x, y in $A(C)$, and g, h be the elements of $[0, u[$ such that $\rho(g) = x$ and $\rho(h) = y$. Since G is an ℓ -group, we have that $g + h = g \wedge h + g \vee h$, with $0 \leq g \wedge h < u$ and $0 \leq g \vee h \leq u$. We saw in Corollary 3.25 that ρ induces an isomorphism of ordered sets between $[0, u[$ and $A(C)$. Hence $\rho(g \wedge h) = x \wedge_0 y$, and if $g \vee h < u$, then $\rho(g \vee h) = x \vee_0 y$. If $g + h \in G_u$, then $g \wedge h + g \vee h \in G_u$. Hence $g \vee h < u$ and $x + y = \rho(g + h) = \rho(g \wedge h + g \vee h) = \rho(g \wedge h) + \rho(g \vee h) = x \wedge_0 y + x \vee_0 y$. Assume that $g + h \notin G_u$, then $g + h - u \in G_u$. If $g \vee h < u$, then we have that

$$\begin{aligned} x + y &= \rho(g) + \rho(h) \\ &= \rho(g + h - u) \\ &= \rho(g \wedge h + g \vee h - u) \\ &= \rho(g \wedge h) + \rho(g \vee h) \\ &= x \wedge_0 y + x \vee_0 y. \end{aligned}$$

If $g \vee h = u$, then $x \vee_0 y = \mathbf{1}$. Hence $x \wedge_0 y + x \vee_0 y = x \wedge_0 y$ (see Notation 4.5). Then:

$$\begin{aligned} x + y &= \rho(g) + \rho(h) \\ &= \rho(g + h - u) \\ &= \rho(g \wedge h + g \vee h - u) \\ &= \rho(g \wedge h) \\ &= x \wedge_0 y. \end{aligned}$$

4) Let x, y, z in $A(C) \setminus \{0\}$ and g, h, k in $]0, u[$ such that $\rho(g) = x$, $\rho(h) = y$ and $\rho(k) = z$.

$$\begin{aligned} &(x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x) = \\ &(\rho(g) \wedge_0 \rho(u - k) + \rho(k)) \wedge_0 \rho(u - h) - (\rho(h) \wedge_0 \rho(u - k) + \rho(k)) \wedge_0 \rho(u - g) = \\ &(\rho(g \wedge (u - k)) + \rho(k)) \wedge_0 \rho(u - h) - (\rho((h \wedge (u - k)) + \rho(k)) \wedge_0 \rho(u - g) = \\ &\rho(g \wedge (u - k) + k) \wedge_0 \rho(u - h) - \rho(h \wedge (u - k) + k) \wedge_0 \rho(u - g). \end{aligned}$$

$g \wedge (u - k) < u - k$, hence $g \wedge (u - k) + k < u - k + k = u$, hence $\rho(g \wedge (u - k) + k) \in A(C)$, and in the same way $\rho(h \wedge (u - k) + k) \in A(C)$, so:

$$\begin{aligned} &(x \wedge_0 (-z) + z) \wedge_0 (-y) - (y \wedge_0 (-z) + z) \wedge_0 (-x) = \\ &\rho((g \wedge (u - k) + k) \wedge (u - h)) - \rho((h \wedge (u - k) + k) \wedge (u - g)) = \\ &\rho((g + k) \wedge u \wedge (u - h)) - \rho((h + k) \wedge u \wedge (u - g)) = \\ &\rho((g + k) \wedge (u - h) - (h + k) \wedge (u - g)) = \rho((g + k + h) \wedge u - h - (h + k + g) \wedge u + g) = \\ &\rho(g - h) = \rho(g) - \rho(h) = x - y. \end{aligned}$$

The last assertion follows from Proposition 4.3 and from the definition of the MV-algebra $A(C) \cup \{\mathbf{1}\}$ given in Theorem 4.7. \square

One can wonder if being the wound-round of a lattice can be characterized by first-order sentences. We will see that this holds if the p.c.o. group C is generated by $A(C)$. This characterization relies on good sequences.

Definition 4.16. Notation. Let $C \in \mathcal{AC}$. We denote by $\langle A(C) \rangle$ the subgroup of C generated by $A(C)$.

Theorem 4.17. *Let C be a p.c.o. group.*

$\langle A(C) \rangle$ is the wound-round of a lattice if, and only if, it is isomorphic to the wound-round of the Chang ℓ -group of the MV-algebra $A(C(A)) \cup \{\mathbf{1}\}$.

$\langle A(C) \rangle$ being the wound-round of a lattice is expressible by countably many first-order formulas of the language L_c .

Proof. Recall that the MV-algebra $A(C) \cup \{\mathbf{1}\}$ is first-order definable in C , by the rules defined in Theorem 4.7. In particular, we can assume that \oplus belongs to the language. Trivially, if $\langle A(C) \rangle$ is isomorphic to the wound-round of the Chang ℓ -group of the MV-algebra $A(C(A)) \cup \{\mathbf{1}\}$, then it is the wound-round of a lattice.

Assume that $\langle A(C) \rangle = G/\mathbb{Z}u$, where (G, u) is a unital ℓ -group. Let A be the MV-algebra $A(C) \cup \{\mathbf{1}\}$, (G_A, u_A) be the Chang ℓ -group of A and C' be the p.c.o. group $G_A/\mathbb{Z}u_A$.

1 We know that $A \simeq \Gamma(G_A, u_A)$, and, by Theorem 4.15, $A \simeq \Gamma(G, u)$. By uniqueness of the Chang ℓ -group, it follows that there is a unital ℓ -isomorphism between (G, u) and (G_A, u_A) . Hence the p.c.o. groups $\langle A(C) \rangle$ and C' are isomorphic.

2 By Remark 2.11, every element x of the positive cone of G_A is a sum of elements x_1, \dots, x_n of A satisfying the conditions of Lemmas 2.7 and 2.8, where $x \leq nu$. Furthermore, if $x \not\leq u_A$, then the x_i 's are different from u_A . Let call the sequence $(x_1, \dots, x_n, 0, \dots)$ the *good sequence* associated with x . By Lemma 3.22, the canonical epimorphism $\rho : G_A \rightarrow G_A/\mathbb{Z}u_A$ induces a one-to-one mapping between $G_{u_A} = \{x \in G_A \mid x \geq 0 \ \& \ x \not\leq u_A\}$ and $\langle A(C) \rangle$. It follows that every element x of $\langle A(C) \rangle$ can be represented by a unique good sequence of elements of $A(C)$. Furthermore, by Lemma 2.7, if x is a sum of n elements of $A(C)$, then the good sequence associated with x contains at most n elements different from 0. So C satisfies the following family of first-order formulas. For every $n \in \mathbb{N}^*$,

$$\begin{aligned} & \forall (x_1, \dots, x_n) \in A(C)^n \exists (y_1, \dots, y_n) \in A(C)^n \bigwedge_{1 \leq i < n} (y_{i+1} \wedge_0 -y_i = 0 \ \& \ y_i = 0 \Rightarrow y_{i+1} = 0) \\ & \ \& \ x_1 + \dots + x_n = y_1 + \dots + y_n \ \& \ (\forall (z_1, \dots, z_n) \in A(C)^n \bigwedge_{1 \leq i < n} (z_{i+1} \wedge_0 -z_i = 0 \ \& \ z_i = 0 \Rightarrow z_{i+1} = 0) \\ & \ \& \ x_1 + \dots + x_n = z_1 + \dots + z_n) \Rightarrow z_1 = y_1, \dots, z_n = y_n \end{aligned}$$

Every element x of the positive cone of G_A is equivalent modulo $\mathbb{Z}u_A$ to an element x' of G_{u_A} , and the good sequence associated with x' is obtained by dropping the u_A 's from the good sequence associated with x . Hence so is the good sequence associated with $\rho(x)$. Now, if x and y belong to G_{u_A} , then the good sequence associated with $z = x + y$ is obtained by the rules $z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \dots \oplus (x_1 \odot y_{i-1}) \oplus y_i$. The good sequence associated with $\rho(x)$ is obtained by dropping the u_A 's from the good sequence associated with z . Consequently, C satisfies the following family of first-order formulas. For every $n \in \mathbb{N}^*$,

$$\begin{aligned} & \forall (x_1, \dots, x_n) \in A(C)^n \forall (y_1, \dots, y_n) \in A(C)^n \forall (z_1, \dots, z_n) \in A(C)^n \bigwedge_{1 \leq i < n} (x_{i+1} \wedge_0 -x_i = 0 \\ & \ \& \ x_i = 0 \Rightarrow x_{i+1} = 0) \ \& \ \bigwedge_{1 \leq i < n} (y_{i+1} \wedge_0 -y_i = 0 \ \& \ y_i = 0 \Rightarrow y_{i+1} = 0) \ \& \ \bigwedge_{1 \leq i < n} (z_{i+1} \wedge_0 -z_i = 0 \\ & \ \& \ z_i = 0 \Rightarrow z_{i+1} = 0) \ \& \ x_1 + \dots + x_n + y_1 + \dots + y_n = z_1 + \dots + z_n \Rightarrow \bigcup_{1 \leq i_0 < n} \\ & \bigcup_{1 \leq i < i_0} (x_i \oplus (x_{i-1} \odot y_1) \oplus \dots \oplus (x_1 \odot y_{i-1}) \oplus y_i = 1) \ \& \ x_{i_0} \oplus (x_{i_0-1} \odot y_1) \oplus \dots \oplus (x_1 \odot y_{i_0-1}) \oplus y_{i_0} \neq 1 \\ & \ \& \ \bigcup_{i_0 \leq i \leq n} z_i = x_i \oplus (x_{i-1} \odot y_1) \oplus \dots \oplus (x_1 \odot y_{i-1}) \oplus y_i \end{aligned}$$

Conversely, assume that $C \in \mathcal{AC}$ and that C satisfies above families of formulas (recall that by Corollary 4.8, $C \in \mathcal{AC}$ is expressible by countably many first-order formulas). We prove that $\langle A(C) \rangle$ is isomorphic to the wound-round of $G_A/\mathbb{Z}u_A$, where A is the MV-algebra $A(C) \cup \{\mathbf{1}\}$. The group operation on $\langle A(C) \rangle$ is determined by $A(C)$ and by above formulas, which are also satisfied by the group $C' = G_A/\mathbb{Z}u_A$. It follows that the groups $\langle A(C) \rangle$ and C' are isomorphic. Furthermore, the ordered sets $[0, u]$ and $(A(C) \cup \{\mathbf{1}\}, \leq_0)$ are isomorphic. By Corollary 3.25 they are isomorphic to $(A(C') \cup \{\mathbf{1}\}, \leq_0)$. By Remarks 3.11, the p.c.o. groups C' and $\langle A(C) \rangle$ are isomorphic. This proves that being the wound-round of a lattice is expressible by countably many first-order formulas. \square

Remark 4.18. *If $\langle A(C) \rangle$ is the wound-round of a lattice and is not linearly ordered (when equipped with the partial order \leq_0), then it is infinite.*

Proof. Let (G, u) be a unital ℓ -group such that $\langle A(C) \rangle \simeq G/\mathbb{Z}u$. If $\langle A(C) \rangle$ is not linearly ordered, then G is not linearly ordered. Hence there exist $x > 0$ and $y > 0$ in G such that $x \not\leq y$ and $y \not\leq x$. So $x \wedge y < x$ and $x \wedge y < y$. By taking $x - x \wedge y$ instead of x , and $y - x \wedge y$ instead of y , we can assume that $x > 0$ and $y > 0$ and $x \wedge y = 0$. By properties of ℓ -groups, for every positive integer n we have $nx \wedge y = 0 = x \wedge ny$ (this follows for example from 1.2.24 on p. 22 of [1]). In particular, x and y are not strong units. It follows that, for every $n \in \mathbb{N}^*$, $nx \not\sim u$, hence $x, 2x, \dots, nx, \dots$ belong to different classes modulo $\mathbb{Z}u$, therefore $G/\mathbb{Z}u$ is infinite. \square

5. CASE OF MV-CHAINS.

We know that every c.o. group C is the wound-round of a unique (up to isomorphism) unital linearly ordered group $(uw(C), u_C)$ (see Theorem 3.13). So there is a one-to-one correspondence between c.o. groups and unital linearly ordered groups. In fact, this correspondence is a functorial one (see Corollary 3.15).

We construct $(uw(C), u_C)$. The linearly ordered group $uw(C)$ is isomorphic to $\mathbb{Z} \times C$. The partial order \leq is the lexicographic order of $(\mathbb{Z}, \leq) \times (C, \leq_0)$, $u_C = (1, 0)$ and $(m, x) + (n, y) = (m + n, x + y)$ if $x = y = 0$ or $\min_0(x, y) <_0 x + y$, and $(m, x) + (n, y) = (m + n + 1, x + y)$ otherwise.

There is also a one-to-one correspondence between unital linearly ordered groups and MV-chains: a unital linearly ordered group (G, u) is associated with the MV-chain $\Gamma(G, u) = [0, u]$ (see [2, Lemma 6]). Conversely, an MV-chain A is associated with its Chang ℓ -group G_A . Furthermore, this correspondence is a functorial one (see Section 2). It follows a functorial one-to-one correspondence between MV-chains and c.o. groups. If A is an MV-chain, then $C(A) = G_A/\mathbb{Z}u_A$ is a c.o. group. Note that if C is a c.o. group with at least three elements, then $A(C) = C$. It follows that if C contains at least three elements, then the unital linearly ordered groups $(uw(C), u_C)$ and $\Gamma(A(C) \cup \{\mathbf{1}\}, \mathbf{1})$ are isomorphic.

The following lemma shows that the construction of $\Gamma(A(C) \cup \{\mathbf{1}\}, \mathbf{1})$, in the linearly ordered case, is similar to the construction of the unwound of a c.o. group.

Lemma 5.1. ([2, Lemmas 5 and 6]) *Let A be an MV-chain. Then G_A is isomorphic to $\mathbb{Z} \times (A \setminus \{1\})$ lexicographically ordered and with the rules: $(m, x) + (n, y) = (m + n, x \oplus y)$ if $x \oplus y < 1$ and $(m, x) + (n, y) = (m + n + 1, x \odot y)$ otherwise.*

We will also need the following fact.

Fact 5.2. *If ρ is the natural mapping from $uw(C)$ onto $C \simeq uw(C)/\mathbb{Z}u_C$, then for g, h in $[0, u_C[$ we have that $\rho(g) <_0 \rho(h) \Leftrightarrow g < h$ and if $g \leq h \neq 0$, then $g < g + h < g + u_C$. So, $\rho(g + h) = \rho(g) + \rho(h)$ if, and only if, $g + h < u_C$, which in turn is equivalent to: $\rho(g) <_0 \rho(g) + \rho(h)$. Otherwise, we have that $\rho(g) + \rho(h) = \rho(g + h - u_C)$.*

Now, we prove that this correspondence between MV-chains and c.o. groups also preserves elementary equivalence.

Proposition 5.3. *Let A be an MV-chain.*

The c.o. group $C(A)$, in the language L_c , is interpretable in the L_{MV} -structure A .

The L_{MV} -structure A is interpretable in the L_c structure $C(A) \cup \{\mathbf{1}\}$.

If A and A' are MV-chains, then:

$A \equiv A' \Leftrightarrow C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\} \Leftrightarrow C(A) \equiv C(A')$, and

$A \prec A' \Leftrightarrow C(A) \cup \{\mathbf{1}\} \prec C(A') \cup \{\mathbf{1}\} \Leftrightarrow C(A) \prec C(A')$.

Proof. In the MV-chain A , the set $C(A)$ is interpreted by $A \setminus \{1\}$, the cyclic order is given by $R(x, y, z) \Leftrightarrow x < y < z$ or $y < z < x$ or $z < x < y$, the addition is given by $x + y = x \oplus y$ if $x \odot y = 0$ and $x + y = x \odot y$ otherwise. Indeed, we saw in Section 2 that in G_A $x \oplus y = (x + y) \wedge 1$, and $x \odot y = (x + y - 1) \vee 0$. Since G_A is linearly ordered, $x \oplus y = \min(x + y, 1)$, and $x \odot y = \max(x + y, 1) - 1$. Let $z \in [0, 1[$ such that $x + y - z \in \mathbb{Z} \cdot 1$, then $z = x + y$ if $x + y < 1$ and $z = x + y - 1$ otherwise.

In $C(A) \cup \{\mathbf{1}\}$, the set A is interpreted by $C(A) \cup \{\mathbf{1}\}$, $\neg x$ is interpreted by $-x$ if $x \notin \{0, \mathbf{1}\}$, $\neg 0 = \mathbf{1}$ and $\neg \mathbf{1} = 0$. \oplus is interpreted by $\mathbf{1} \oplus x = \mathbf{1}$ and for x, y in $A(C)$ $x \oplus y = x + y$ if $x + y \neq 0$ and $\min_0(x, y) <_0 x + y$, $x \oplus y = \mathbf{1}$ if $x \neq 0 \neq y$ and $x + y \leq_0 \min_0(x, y)$, and $0 \oplus 0 = 0$. Indeed, we have seen in Fact 5.2 that if g, h are the elements of $[0, u_C[\subset uw(C)$ such that $\rho(g) = x$ and $\rho(h) = y$, then $g + h < u_C \Leftrightarrow \min_0(x, y) <_0 x + y$.

It follows from Theorem 2.13 that $A \equiv A' \Rightarrow C(A) \equiv C(A')$ and $C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\} \Rightarrow A \equiv A'$. Now we see that $C(A) \equiv C(A') \Rightarrow C(A) \cup \{\mathbf{1}\} \equiv C(A') \cup \{\mathbf{1}\}$. The last proposition can be proved in the same way. \square

Definition 5.4. Notation. We consider the language $L_{oMV} = (0, +, -, \leq, \oplus, \neg)$. The L_o -structure \mathbb{Z} will be seen as a L_{oMV} -structure where $x \oplus y = z \Leftrightarrow x = y = z = 0$ and $\neg x = y \Leftrightarrow x = y = 0$. If A is an MV-chain, then it will be seen as a L_{oMV} -structure, where $x + y = z \Leftrightarrow x = y = z = 0$ and $x - y = z \Leftrightarrow x = y = z = 0$.

Proposition 5.5. *Let A be an MV-chain.*

The L_{MV} -structure A is interpretable in the L_{loZu} -structure $(G_A, \mathbb{Z}u_A)$ (resp. in the L_{lou} -structure (G_A, u_A)).

The L_{loZu} -structure $(G_A, \mathbb{Z}u_A)$ (resp. the L_{lou} -structure (G_A, u_A)) is interpretable in the L_{oMV} -structure $\mathbb{Z} \times A$.

If A and A' are MV-chains, then:

$(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'}) \Leftrightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Leftrightarrow A \equiv A'$, and

$(G_A, u_A) \equiv (G_{A'}, u_{A'}) \Leftrightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Leftrightarrow A \equiv A'$.

The same holds with \prec instead of \equiv .

Proof. In $(G_A, \mathbb{Z}u_A)$ (resp. in (G_A, u_A)), 1 is the smallest positive element of $\mathbb{Z}u$ (resp. $1 = u$), the set A is interpreted by $\{x \in G_A \mid 0 \leq x \leq u_A\}$, $x \oplus y = \min(x + y, u_A)$, $\neg x = u_A - x$.

In $\mathbb{Z} \times A$, G_A is interpreted by $\mathbb{Z} \times (A \setminus \{u\})$, $\mathbb{Z}u$ is interpreted by $\mathbb{Z} \times \{0\}$ (resp. $u = (1, 0)$). The order relation is the lexicographic order: $(m, x) \leq (n, y) \Leftrightarrow m < n$ or $(m = n$ and $x \leq y)$. The sum is defined by $(m, x) + (n, y) = (m + n, x \oplus y)$ if $x \oplus y < 1$, and $(m, x) + (n, y) = (m + n + 1, x \odot y)$ if $x \oplus y = 1$.

It follows from Theorem 2.13 that $(G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'}) \Rightarrow A \equiv A'$, $(G_A, u_A) \equiv (G_{A'}, u_{A'}) \Rightarrow A \equiv A'$, $\mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Rightarrow (G_A, \mathbb{Z}u_A) \equiv (G_{A'}, \mathbb{Z}u_{A'})$ and $\mathbb{Z} \times A \equiv \mathbb{Z} \times A' \Rightarrow (G_A, u_A) \equiv (G_{A'}, u_{A'})$ (the same holds with \prec). Now, we deduce from Theorem 2.12 that in the language L_{oMV} : $A \equiv A' \Rightarrow \mathbb{Z} \times A \equiv \mathbb{Z} \times A'$. Now, clearly, if $A \equiv A'$ in L_{MV} , then $A \equiv A'$ in L_{oMV} (the same holds with \prec). \square

Thanks to this transfert principle, we deduce from [10] similar results in the case of MV-chains. In particular we characterize pseudofinite and pseudo-hyperarchimedean MV-chains.

Definitions 5.6. ([3, Chapter 6])

- 1) In an ordered set, by an *atom* we mean an element x such that $x > 0$ and whenever $y \leq x$ then either $y = 0$ or $y = x$ ([3, Definitions 6.4.2 and 6.7.1]).
- 2) An ℓ -group is *hyperarchimedean* if for every positive x and y there exists $n \in \mathbb{N}^*$ such that $nx \wedge y = (n + 1)x \wedge y$ (see [1, Theorem 14.1.2]).
- 3) An MV-algebra is *atomic* if for each $x \neq 0$ there is an atom y with $y \leq x$. It is *atomless* if no element is an atom ([3, Definition 6.7.1]).
- 4) An element x of an MV-algebra is *archimedean* if there exists $n \in \mathbb{N}^*$ such that $\neg x \vee n.x = 1$. This is equivalent to saying that there exists $n \in \mathbb{N}$ such that $n.x = (n + 1).x$ ([3, Corollary 6.2.4]).
- 5) An MV-algebra is *hyperarchimedean* if all its elements are archimedean ([3, Definition 6.3.1]).
- 6) An MV-algebra is *simple* if it embeds in $[0, 1]_{\mathbb{R}}$ ([3, Theorem 3.5.1]).

Note that if an MV-chain A is atomic, then it contains only one atom, and the underlying ordered set is discretely ordered. If it is atomless, then the underlying ordered set is densely ordered.

Saying that an MV-chain is hyperarchimedean is equivalent to saying that it is simple.

Recall the notations, for x in an MV-algebra, $2.x = x \oplus x$, $x^2 = x \odot x$ and so on.

Definitions 5.7. Let A be an MV-chain.

- 1) We will say that A is *regular* if for every integer $n \geq 2$ and every $0 < x_1 < \dots < x_n$ in A there exists $x \in A$ such that $x_1 \leq n.x \leq x_n$, and $0 < x < 2.x < \dots < (n - 1).x < n.x$.
- 2) We will say that A is *pseudo-simple* if A belongs to the elementary class generated by the simple MV-chains.
- 3) We will say that A is *pseudofinite* if A belongs to the elementary class generated by the finite MV-chains.

Let A be an MV-chain, it is easy to see that A is regular if, and only if, $C(A)$ is c-regular, and since the unwound of $C(A)$ is isomorphic to the Chang ℓ -group G_A of A , this is equivalent to saying that G_A is regular (see Definition 3.30). One can also see that A is atomic if, and only if, $C(A)$ is discrete. Moreover, A is simple if, and only if, G_A is archimedean. Note that a linearly ordered group is hyperarchimedean if, and only if, it is archimedean.

In the MV-chain $A(C)$, the formula $R(0, x, 2x, \dots, (p^n - 1)x)$ can be reformulated as $0 < x < 2.x < \dots < (p^n - 1).x$, (which is equivalent to $x \neq 0$ and $0 = x^2 = \dots = x^{p^n - 1}$, since $x^2 = 0 \Leftrightarrow 2.x \leq 1$) hence we can define formulas $D_{p^n, k}$ in MV-chains.

Definition 5.8. If A is an atomic and not simple MV-chain, then the atom ε_A of A (which is the smallest positive element) is definable, we can assume that it lies in the language. For a prime p , for $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$, we denote by $D_{p^n, k}$ the formula: $\exists x, 0 < x < 2.x < \dots < (p^n - 1).x \wedge p^n.x = k.\varepsilon_A$.

In the same way, the torsion subgroup has an analogue in MV-chains.

Definition 5.9. Let x be an element of an MV-chain. We will say that x is a *torsion element* if there exists $n \in \mathbb{N}^*$, such that $n.x = 1$ and $x^n = 0$.

Note that $x \in A \setminus \{1\}$ is a torsion element in the MV-chain A if, and only if, it is a torsion element in the group $C(A)$.

Lemma 5.10. *Let A and $(A_i)_{i \in \mathbb{N}^*}$ be MV-chains, U be an ultrafilter on \mathbb{N}^* and ΠA_i be the ultraproduct of $(A_i)_{i \in \mathbb{N}^*}$. Then $A \equiv \Pi A_i \Leftrightarrow C(A) \equiv \Pi C(A_i)$.*

Proof. Let Φ be a L_{MV} -sentence and Φ_c be the corresponding L_c -sentence. Then: $A \models \Phi \Leftrightarrow C(A) \models \Phi_c$, and for every i in \mathbb{N}^* , $A_i \models \Phi \Leftrightarrow C(A_i) \models \Phi_c$. Hence $\{i \in \mathbb{N}^* \mid A_i \models \Phi\} \in U \Leftrightarrow \{i \in \mathbb{N}^* \mid C(A_i) \models \Phi_c\} \in U$. The equivalence follows. \square

So various theorems proved in [10], can be expressed in terms of MV-chains.

Theorem 5.11. *An atomless MV-chain is pseudo-simple if, and only if, it is regular.*

Theorem 5.12. *1) Any atomless regular MV-chain is elementarily equivalent to some simple atomless MV-chain.*

2) Any two atomless regular MV-chains are elementarily equivalent if, and only if, their subchain of torsion elements are isomorphic and their Chang ℓ -groups have the same family of prime invariants of Zakon.

Theorem 5.13. *1) Any two infinite atomic regular MV-chains are elementarily equivalent if, and only if, they satisfy the same formulas $D_{p^n, k}$.*

2) An infinite MV-chain is pseudofinite if, and only if, it is atomic and regular.

3) Let U be a non principal ultrafilter on \mathbb{N}^ , A be the ultraproduct of the MV-chains $[0, n]$, p be a prime, $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$. Then A satisfies the formula $D_{p^n, k}$ if, and only if, $p^n \mathbb{N}^* - k \in U$.*

6. NON-LINEARLY ORDERED CASE.

First we list some properties of abelian ℓ -groups (see [1]). The aim is to get a sufficient condition for an ℓ -group being a cartesian product of finitely many linearly ordered groups. We let G be an ℓ -group. We know that, for every $x \in G$, there exists a unique pair x_+ , x_- of non-negative elements such that $x = x_+ + x_-$ and $x_+ \wedge x_- = 0$. We let $|x| := x_+ + x_-$.

Two elements x, y of G are said to be *orthogonal* if $|x| \wedge |y| = 0$. This is equivalent to: $x_+ \wedge y_+ = x_+ \wedge y_- = x_- \wedge y_+ = x_- \wedge y_- = 0$. A subset A of G is said to be *orthogonal* if its elements are pairwise orthogonal. Every orthogonal subset is contained in a maximal orthogonal subset.

If $A \subset G$, then the *polar* of A is the set $A^\perp := \{y \in G \mid \forall x \in A, |x| \wedge |y| = 0\}$; if $A = \{x\}$, then we let $x^\perp := \{x\}^\perp$. The set $A^{\perp\perp}$ is called a *bipolar*. Every polar of G is a convex ℓ -subgroup of G . A polar A^\perp is said to be *principal* if $A^\perp = x^{\perp\perp}$ for some $x \in G$ (see [1, Chapter 3]).

An element x of G_+ is said to be *basic* if $x^{\perp\perp}$ is a linearly ordered group, which is equivalent to saying that the set $[0, x]$ is linearly ordered. If x and y are basic elements, then either $x \leq y$ or $y < x$ or $x \wedge y = 0$. If $x \wedge y > 0$, then $x^\perp = y^\perp$, hence $x^{\perp\perp} = y^{\perp\perp}$ (see [1, pp. 133-135]).

The group G is said to be *projectable* if, for every $x \in G$, G is the direct sum of x^\perp and $x^{\perp\perp}$. Note that being projectable is a first-order property. Let x, y in G_+ . Then $y \in x^\perp \Leftrightarrow x \wedge y = 0$, and $y \in x^{\perp\perp} \Leftrightarrow \forall z (x \wedge z = 0 \Rightarrow y \wedge z = 0)$. Hence $y \in x^\perp$ and $y \in x^{\perp\perp}$ are first-order properties.

Lemma 6.1. *If $\{x_1, \dots, x_n\}$ is a maximal orthogonal set of an ℓ -group G whose elements are basic elements, then:*

- for every $x \in G$ there is some $i \in \{1, \dots, n\}$ such that $x_i^{\perp\perp} \subset x^{\perp\perp}$,
- $x > 0$ is basic if, and only if, there exists $i \in \{1, \dots, n\}$ such that $x_i^{\perp\perp} = x^{\perp\perp}$,
- the minimal principal polars of G are $x_1^{\perp\perp}, \dots, x_n^{\perp\perp}$.

Proof. Let $0 < x \in G$. Since $\{x_1, \dots, x_n\}$ is maximal orthogonal, there is some $i \in \{1, \dots, n\}$ such that $x \wedge x_i > 0$. Now, $[0, x_i]$ is linearly ordered, hence for every $y \in G_+$, we have that either $x_i \wedge x \leq x_i \wedge y$ or $x_i \wedge y \leq x_i \wedge x$. If $x_i \wedge y \leq x_i \wedge x$, then $x_i \wedge y \leq x_i \wedge (x \wedge y)$. Hence $y \in x^\perp \Rightarrow y \in x_i^\perp$. If $x_i \wedge x < x_i \wedge y$, then $x_i \wedge x \leq x_i \wedge (x \wedge y)$. Hence $y \wedge x > 0$. It follows that $x^\perp \subset x_i^\perp$. Therefore $x_i^{\perp\perp} \subset x^{\perp\perp}$.

Let $x > 0$ and $i \in \{1, \dots, n\}$ such that $x^\perp = x_i^\perp$. If x is basic, then we have that $x_i \leq x$ or $x \leq x_i$. Assume that $x_i \leq x$. For every $y > 0$ we have that $y \wedge x_i = y \wedge x_i \wedge x = y \wedge x \wedge x_i = \min(y \wedge x, x_i)$, since $[0, x]$ is linearly ordered. Therefore: $y \wedge x_i = 0 \Leftrightarrow y \wedge x = 0$, hence $x^\perp = x_i^\perp$. The case $x \leq x_i$ is similar.

The last assertion follows trivially. \square

By [9, Théorème 6, Chapitre II], we know that an ℓ -group G is a direct sum of linearly ordered groups if, and only if, the following holds:

- for every $x^{\perp\perp}$ which is minimal, G is the direct sum of x^\perp and $x^{\perp\perp}$,
- every $x^{\perp\perp}$ contains some $y^{\perp\perp}$ which is minimal,
- there is at most a finite number of minimal $y^{\perp\perp}$.

It follows that G is the direct sum of n linearly ordered groups if, and only if, it contains a maximal orthogonal set $\{x_1, \dots, x_n\}$ whose elements are basic elements and, for every $i \in \{1, \dots, n\}$, G is the

direct sum of x_i^\perp and $x_i^{\perp\perp}$. This is equivalent to saying that G is projectable and contains a maximal orthogonal set of n elements which are basic.

Now, we have also the following result.

Proposition 6.2. *Let G be an ℓ -group together with a distinguished strong unit u . Then G is the product of n linearly ordered groups if, and only if, G_+ contains a maximal orthogonal set $\{u_1, \dots, u_n\}$, whose elements are basic, such that $u = u_1 + \dots + u_n$.*

Proof. \Rightarrow is straightforward. Assume that G_+ contains a maximal orthogonal set $\{u_1, \dots, u_n\}$, whose elements are basic, such that $u = u_1 + \dots + u_n$. We know that if x, y, z in G_+ satisfy $x \wedge y = 0$, then $x + y = x \vee y$ and $(x + z) \wedge y = z \wedge y$ (see, for example, [6, Lemma 2.3.4]). Let $x \in G_+$, and $p \in \mathbb{N}^*$ such that $x \leq pu$. We have that $pu = pu_1 + \dots + pu_n$, where the pu_i 's are pairwise orthogonal. Hence $x = x \wedge pu = x \wedge pu_1 + \dots + x \wedge pu_n \in u_1^{\perp\perp} + \dots + u_n^{\perp\perp}$. Note that, since $\{u_1, \dots, u_n\}$ is a orthogonal, we have that $x = (x \wedge pu_1) \vee \dots \vee (x \wedge pu_n)$. Assume that $x = x_1 + \dots + x_n = x_1 \vee \dots \vee x_n$ with $x_i \in u_i^{\perp\perp}$ ($1 \leq i \leq n$). Then $x_i = x \wedge pu_i$, which proves the uniqueness of the decomposition. It follows that G is the direct sum of $u_1^{\perp\perp}, \dots, u_n^{\perp\perp}$. \square

Now, we turn to MV-algebras. From [6, Lemma 2.3.4], which we recalled in the proof of Proposition 6.2, one deduces by induction that for every orthogonal family $\{x_1, \dots, x_n\}$ in the positive cone of an ℓ -group we have $x_1 + \dots + x_n = x_1 \vee \dots \vee x_n$. Now, in an MV-algebra if $\{x_1, \dots, x_n\}$ is an orthogonal family, then $x_1 \oplus \dots \oplus x_n = x_1 \vee \dots \vee x_n$. The following proposition is similar to Lemma 6.4.5 in [3].

Proposition 6.3. *Let A be an MV-algebra. Then, the Chang ℓ -group of A is isomorphic to a product of n linearly ordered groups if, and only if, there exist non zero elements u_1, \dots, u_n of A such that:*

- $1 = u_1 \oplus \dots \oplus u_n$,
- for all i, j in $\{1, \dots, n\}$: $i \neq j \Rightarrow u_i \wedge u_j = 0$,
- for all x, y in A , if $x \leq u_i$ and $y \leq u_i$, then $x \leq y$ or $y \leq x$.

If this holds, then the Chang ℓ -group of A is interpretable in $(\mathbb{Z} \times [0, u_1]) \times \dots \times (\mathbb{Z} \times [0, u_n])$, where $(p_1, x_1, \dots, p_n, x_n) \leq (q_1, y_1, \dots, q_n, y_n)$ if, and only if, for every $i \in \{1, \dots, n\}$, $p_i < q_i$ or $(p_i = q_i$ and $x_i \leq y_i)$. The addition is defined componentwise, $(p_i, x_i) + (q_i, y_i) = (p_i + q_i, x_i \oplus y_i)$ if $x_i \oplus y_i < u_i$, and $(p_i, x_i) + (q_i, y_i) = (p_i + q_i + 1, x_i \odot y_i)$ if $x_i \oplus y_i = u_i$.

Proof. The equivalence follows from Proposition 6.2. Now let $x \in G_{A+}$. We know that there exists a good sequence (x_1, \dots, x_p) of elements of $[0, u]$ such that $x = x_1 + \dots + x_p$, where, for $1 \leq k \leq p-1$, $(u - x_k) \wedge x_{k+1} = 0$ (see Section 2). For $j \in \{1, \dots, n\}$ let $x_j = x_{1,j} + \dots + x_{n,j}$, with $x_{i,j} \in u_i^{\perp\perp}$ ($1 \leq i \leq n$). We have that $u - x_j = (u_1 - x_{1,j}) + \dots + (u_n - x_{n,j})$, hence $u_i - x_{i,j} > 0 \Rightarrow x_{i,j+1} = 0$ i.e. $x_{i,j} \neq u_i \Rightarrow x_{i,j+1} = 0$. Therefore we can write x as $x = k_1 u_1 + x_1 + \dots + k_n u_n + x_n$, with $0 \leq k_i \leq p$ and $x_i \in [0, u_i[$ ($1 \leq i \leq n$). So, every element of G_A can be written in a unique way as $x = k_1 u_1 + x_1 + \dots + k_n u_n + x_n$, with $k_i \in \mathbb{Z}$ and $x_i \in [0, u_i[$ ($1 \leq i \leq n$). Let $x = k_1 u_1 + x_1 + \dots + k_n u_n + x_n$, and $y = l_1 u_1 + y_1 + \dots + l_n u_n + y_n$ in G_A .

Trivially, $x \leq y$ if, and only if, for every $i \in \{1, \dots, n\}$, $k_i < l_i$ or $(k_i = l_i$ and $x_i \leq y_i)$.

Set $x + y = z = m_1 u_1 + z_1 + \dots + m_n u_n + z_n$. Since $k_i u_i + x_i + l_i u_i + y_i \in u_i^{\perp\perp}$, we have that $m_i u_i + z_i = (k_i + l_i) u_i + x_i + y_i$, i.e. $x_i + y_i - z_i = (m_i - k_i - l_i) u_i$. If $x_i + y_i < u_i$, then $-u_i < x_i + y_i - z_i < u_i$, hence $x_i + y_i - z_i = 0$ and $z_i = x_i + y_i = x_i \oplus y_i$ and $m_i = k_i + l_i$. Otherwise, in the same way we prove that $z_i = x_i + y_i - u_i$ and $m_i = k_i + l_i + 1$. Now, $x_i \oplus y_i = (x_i + y_i) \wedge u = (x_i + y_i) \wedge u_i = u_i$ and $x_i \odot y_i = u - [(2u_i - x_i - y_i) \wedge u] = (x_i + y_i - u) \vee 0 = x_i + y_i - u_i$. \square

Remarks 6.4. *Since in G_A we have that $A = [0, u_A]$, saying that G_A is isomorphic to a product of n linearly ordered groups is equivalent to saying that A is isomorphic to a product of n MV-chains. It follows from Proposition 6.3 that being isomorphic to a product of n MV-chains is a first-order property.*

Proposition 6.5. ([3, Proposition 3.6.5]). *Let A be a finite MV-algebra. Then A is isomorphic to a product of finite MV-chains and its Chang ℓ -group G_A is isomorphic to some $\mathbb{Z} \times \dots \times \mathbb{Z}$.*

Since an MV-algebra embeds in the positive cone of its Chang ℓ -group, we have for every a : $|a| = a$. Hence we can define orthogonal elements and polars in the following way.

Definition 6.6. Let A be an MV-algebra. Two elements a, b are *orthogonal* if $a \wedge b = 0$. The polar of a subset B of A is $B^\perp = \{a \in A \mid \forall b \in B \ a \wedge b = 0\}$. The MV-algebra A is said to be *projectable* if, for every a, b in A , b can be written in a unique way as $b = b_1 \oplus b_2$, with $b_1 \in a^\perp$ and $b_2 \in a^{\perp\perp}$.

One can also prove that an MV-algebra is projectable if, and only if, its Chang ℓ -group is projectable.

We see that every finite MV-algebra is projectable, and that every minimal principal polar is discrete. Consequently, every pseudofinite MV-algebra is projectable, and its minimal principal polars are discrete and regular.

Turning to first-order theory, we consider the language $L_{MVn} = (0, \oplus, \neg, \underline{1}, \dots, \underline{1}_n)$, with n new

constant symbols. Let $(A_1, 1_1), \dots, (A_n, 1_n), (A'_1, 1'_1), \dots, (A'_n, 1'_n)$ be MV-chains. For $i \in \{1, \dots, n\}$, we assume that $(A, 1_i)$ is a L_{MVn} -structure, by setting, for $j \in \{1, \dots, n\}$, $x = 1_j$ if either $i = j$ and $x = 1_i$, or $i \neq j$ and $x = 0$. We define in the same way the L_{MVn} -structures $(A'_1, 1'_1), \dots, (A'_n, 1'_n)$. Let A be the L_{MVn} -structure $A_1 \times \dots \times A_n$ and A' be the L_{MVn} -structure $A'_1 \times \dots \times A'_n$.

Remark 6.7. *With the same notations, we deduce from Theorem 2.12 that in the language L_{MVn} $(A, 1_1, \dots, 1_n) \equiv (A', 1'_1, \dots, 1'_n) \Leftrightarrow (A_1, 1_1) \equiv (A'_1, 1'_1), \dots, (A_n, 1_n) \equiv (A'_n, 1'_n)$.*

Now, we consider families of MV-chains, $(A_{1,\alpha_1}, 1_{1,\alpha_1})_{\alpha_1 \in I_1}, \dots, (A_{n,\alpha_n}, 1_{n,\alpha_n})_{\alpha_n \in I_n}$ (we can do the same thing with families of linearly ordered groups $(T_{1,\alpha_1})_{\alpha_1 \in I_1}, \dots, (T_{n,\alpha_n})_{\alpha_n \in I_n}$). For every $(\alpha_1, \dots, \alpha_n)$ in $I_1 \times \dots \times I_n$ we set $(A_{(\alpha_1, \dots, \alpha_n)}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n}) = (A_{1,\alpha_1} \times \dots \times A_{n,\alpha_n}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n})$. We let U be an ultrafilter on $I_1 \times \dots \times I_n$ and $(A, 1_1, \dots, 1_n)$ be the ultraproduct of the family $(A_{(\alpha_1, \dots, \alpha_n)}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n})$. We know that for $i \in \{1, \dots, n\}$, the canonical projection $p_i(U)$ on I_i is an ultrafilter on I_i . Denote by A_i the ultraproduct of the family (A_{i,α_i}) . Then one can prove that $(A, 1_1, \dots, 1_n) \simeq (A_1 \times \dots \times A_n, 1_1, \dots, 1_n)$, where 1_i is the greatest element of A_i . Note that the maximal element of A is $1 = 1_1 + \dots + 1_n$.

In order to characterize some pseudofinite MV-algebras, we fix $n \in \mathbb{N}^*$, and we restrict to the finite MV-algebras A which are isomorphic to a product of n MV-chains $[0, 1_1], \dots, [0, 1_n]$. In this case, saying that A is hyperarchimedean is equivalent to saying that each of $[0, 1_1], \dots, [0, 1_n]$ is simple.

Definition 6.8. We will say that an MV-algebra is *n-pseudofinite* if it is elementarily equivalent to some ultraproduct of a family of finite MV-algebras which are isomorphic to products of n MV-chains.

We deduce the following.

Theorem 6.9. *Let $(A, 1)$ and $(A', 1')$ be MV-algebras.*

1) *$(A, 1)$ is n-pseudofinite if, and only if, A is projectable, it is isomorphic to a product of n MV-chains $[0, 1_1] \times \dots \times [0, 1_n]$ and, for every $i \in \{1, \dots, n\}$, the MV-chain $[0, 1_i]$ is either finite or infinite discrete regular.*

2) *If $(A, 1)$ and $(A', 1')$ are n-pseudofinite, then $(A, 1_1, \dots, 1_n) \equiv (A', 1'_1, \dots, 1'_n)$ if, and only if, for every $i \in \{1, \dots, n\}$ either the MV-chains $[0, 1_i], [0, 1'_i]$ are finite and isomorphic, or they are infinite regular and satisfy the same formulas $D_{p^m, k}$.*

Proof. 1) \Rightarrow . Let $(A_{1,\alpha_1}, 1_{1,\alpha_1})_{\alpha_1 \in I_1}, \dots, (A_{n,\alpha_n}, 1_{n,\alpha_n})_{\alpha_n \in I_n}$ be families of MV-chains. For every $(\alpha_1, \dots, \alpha_n)$ in $I_1 \times \dots \times I_n$ we set $(A_{(\alpha_1, \dots, \alpha_n)}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n}) = (A_{1,\alpha_1} \times \dots \times A_{n,\alpha_n}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n})$. We let U be an ultrafilter on $I_1 \times \dots \times I_n$ and for every i let A_i be the ultraproduct of the family (A_{i,α_i}) (associated with $p_i(U)$). If A is the ultraproduct of the family $(A_{(\alpha_1, \dots, \alpha_n)}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n})$, then $(A, 1_1, \dots, 1_n) \simeq (A_1 \times \dots \times A_n, 1_1, \dots, 1_n)$. Now, by Theorem 5.13 every A_i is an MV-chain which is either finite or infinite discrete regular.

\Leftarrow If this holds, then in the language L_{MVn} A is isomorphic to $[0, 1_1] \times \dots \times [0, 1_n]$. By Theorem 5.13, every $[0, 1_i]$ is isomorphic to an ultraproduct of a family of finite MV-chains $(A_{i,\alpha_i}, 1_{i,\alpha_i})_{\alpha_i \in I_i}$. Hence, A is isomorphic to the ultraproduct of the family $(A_{1,\alpha_1} \times \dots \times A_{n,\alpha_n}, 1_{1,\alpha_1}, \dots, 1_{n,\alpha_n})$.

2) follows from Remark 6.7 and Theorem 5.13. \square

Now we turn to hyperarchimedean MV-algebras. By [3, Corollary 6.5.6], being hyperarchimedean is equivalent to being a boolean product of simple MV-algebras. We will restrict ourselves to MV-algebras which are isomorphic to finite products of simple MV-algebras. One can prove that if it is hyperarchimedean, then every sub-MV-algebra is projectable, hyperarchimedean and is isomorphic to a finite product of simple MV-algebras.

Definition 6.10. We will say that an MV-algebra is *n-pseudo-hyperarchimedean* if it is elementarily equivalent to some ultraproduct of a family of hyperarchimedean MV-algebras which are isomorphic to products of n simple MV-algebras.

In the same way as Theorem 6.9, and by using Theorem 5.12, one can prove the following.

Theorem 6.11. *Let $(A, 1)$ and $(A', 1')$ be MV-algebras.*

1) *$(A, 1)$ is n-pseudo-hyperarchimedean if, and only if, A is projectable, it is isomorphic to a product of n MV-chains $[0, 1_1] \times \dots \times [0, 1_n]$ and, for every $i \in \{1, \dots, n\}$, the MV-chain $[0, 1_i]$ is either finite or infinite and regular.*

2) *If $(A, 1)$ and $(A', 1')$ are n-pseudo-hyperarchimedean, then $(A, 1_1, \dots, 1_n) \equiv (A', 1'_1, \dots, 1'_n)$ if, and only if, for every $i \in \{1, \dots, n\}$ either the MV-chains $[0, 1_i], [0, 1'_i]$ are finite and isomorphic, or they are both infinite regular and satisfy the same formulas $D_{p^m, k}$, or they are infinite dense regular and their Chang ℓ -groups have the same prime invariants of Zakon.*

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