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WHITNEY CELLULATION of WHITNEY STRATIFIED SETS
and GORESKY'S HOMOLOGY CONJECTURE

C. MUROLO and D.J.A. TROTMAN

ABSTRACT. We use the proof of Goresky of triangulation of compact abstract stratified sets and the smooth version of the Whitney fibering conjecture, together with its corollary on the existence of a local Whitney wing structure, to prove that each Whitney stratified set $\mathcal{X} = (A, \Sigma)$ admits a Whitney cellulation.

We apply this result to prove the conjecture of Goresky stating that the homological representation map $R : WH_k(\mathcal{X}) \rightarrow H_k(\mathcal{X})$ between the set of the cobordism classes of Whitney (b)-regular stratified cycles of \mathcal{X} and the usual homology of \mathcal{X} is a bijection. This gives a positive answer to the extension to Whitney stratified sets of the famous Thom-Steenrod representation problem of 1954.

1. Introduction. The cellulation of a topological space is frequently a very useful tool in many mathematical applications. In singularity theory an important problem is to re-stratify a singular space \mathcal{X} in such a way that the strata of the new stratification satisfy better properties than the initial stratification.

Some classical examples occur when one finds classes of null obstructions, extensions of maps or vector fields or of a frame field. In the more interesting cases the re-stratification must be made so as to remain in the same class of equisingularity ; that is if the initial stratification \mathcal{X} satisfies certain regularity conditions such as (a), (b), (c), ... then the new strata must satisfy these conditions too. Since cells are contractible, finding a cellulation of a space is often as important as finding a triangulation.

In 1978 Goresky proved an important triangulation theorem for compact Thom-Mather stratified sets [**Go**]₃ whose proof (by induction) can be used to obtain a Whitney cellulation of a Whitney stratified set provided one knows how to obtain Whitney stratified mapping cylinders. Goresky used this idea *based on his Condition (D)* for Whitney stratifications having *only conical singularities* ([**Go**]₂, Appendix A1) for which he gave a solution of Problem 1 (below) and deduced as applications the proofs of Theorems 1 and 2 below.

In 2005 M. Shiota proved that compact semi-algebraic sets admit a Whitney triangulation [**Sh**] and more recently M. Czapla gave a new proof of this result [**Cz**] as a corollary of a more general triangulation theorem for definable sets.

An old problem posed by N. Steenrod [**Ei**] is (roughly speaking) the following : “*Given a closed n -manifold M , can every homology class $z \in H_k(M; \mathbb{Z})$ be represented by a submanifold N of M ?*” Such classes were called *realisable* (without singularities).

In his famous paper of 1954 [**Th**]₁, R. Thom answered Steenrod’s problem by proving that “*for a manifold M having $\dim M = n \geq 7$, the answer is no in general, but there exists $\lambda \in \mathbb{Z}$ such that the class λz is represented by a submanifold of M . Moreover for $k \leq \frac{n}{2}$ and $z \in H_k(M; \mathbb{Z}_2)$ the Steenrod problem has a positive solution*”.

Since the Steenrod problem does not have a positive answer in general for a manifold M (because the classes are too frequently singular spaces) one can consider its natural extension :

“For which regular stratified sets \mathcal{X} , can every class $z \in H_k(\mathcal{X}; \mathbb{Z})$ be represented by a substratified cycle satisfying the same regularity conditions as \mathcal{X} ?”

In this spirit, in his Ph.D. Thesis of 1976, M. Goresky considered Thom-Mather abstract stratified sets \mathcal{X} ([Go]₁ 2.3 and 4.1) and defined singular substratified objects \mathcal{W} to represent the geometric chains and cochains of \mathcal{X} with the aim of introducing homology and cohomology theories having many nice geometric interpretations.

Using in the definition of geometric stratified cycle a certain “condition (D)” Goresky proved that if $\mathcal{X} = M$ is a manifold every geometric stratified cycle of M is cobordant to one which is “radial” on M and then it can be represented by an abstract stratified cycle ([Go]₁ 3.7).

This result is the main step in proving his important theorem on the bijective representability of the homology of a C^1 manifold M by its geometric abstract stratified cycles and of the cohomology of an arbitrary Thom-Mather abstract stratified set ([Go]₁ Theorems 2.4 and 4.5).

In 1981, in [Go]₂ (the article which followed his thesis), Goresky redefines for a Whitney stratification $\mathcal{X} = (A, \Sigma)$ his geometric homology and cohomology theories using only Whitney (that is (b)-regular) substratified cycles and cocycles of \mathcal{X} , denoting them in this case $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ and without assuming this time the condition (D) in their definition.

With these new definitions and replacing the terminology (but for the most part not the meaning) “radial” by “with conical singularities” Goresky again proved ([Go]₂, Appendices 1, 2, 3) the bijectivity of his homology and cohomology representation maps :

Theorem 1. ([Go]₂ Theorem 3.4.) *If $\mathcal{X} = (M, \{M\})$ is the trivial stratification of a compact C^1 manifold, the homology representation map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(M)$ is a bijection.*

Theorem 2. ([Go]₂ Theorem 4.7.) *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified set, $A \subseteq \mathbb{R}^n$, the cohomology representation map $R^k : WH^k(\mathcal{X}) \rightarrow H^k(A)$ is a bijection.*

Later the first author of the present paper improved [Mu]_{1,2} the geometric theories by introducing a sum operation in $WH_k(\mathcal{X})$ and in $WH^k(\mathcal{X})$ geometrically meaning transverse union of stratified cycles and of cocycles (and called WH_* and WH^* , *Whitney homology and cohomology*).

In the revised theory of 1981 [Go]₂, condition (D) was not assumed in the definitions of the Whitney cycles, however it was once again the main tool to obtain the important representation Theorems 1 and 2, by using Condition (D) to construct Whitney cellulations of Whitney stratified sets with conical singularities which allowed one to obtain (b)-regular stratified mapping cylinders ([Go]₂, Appendices 1,2,3). We underline that in the homology case the main result “The map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(M)$ is a bijection” was established *only* when $\mathcal{X} = (M, \{M\})$ is a trivial stratification of a compact manifold M and that the complete homology statement for \mathcal{X} an arbitrary compact (b)-regular stratified set was a problem of Goresky (extending to Whitney stratified sets the Steenrod problem) which remained unsolved ([Go]₁ p.52, [Go]₂ p.178) :

Conjecture 1. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified set the homology representation map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(A)$ is a bijection.*

In this paper we use the techniques and the idea of the proof of triangulation of abstract stratified sets of Goresky [Go]₃ together with consequences of the solution of the smooth Whitney fibering conjecture [MPT] to answer positively the following :

Problem 1. *Does every compact Whitney stratified set \mathcal{X} admit a Whitney cellulation ?*

Then as an application of this result we reply affirmatively to Goresky's Conjecture 1.

This is also a first important step in a possible proof of the celebrated Thom conjecture :

Conjecture 2. *Every compact Whitney stratified set \mathcal{X} admits a Whitney triangulation.*

This Conjecture 2 will be the object of a future article of the authors of the present paper.

The content of the paper is the following :

In section 2, we begin by recalling the main definitions and properties of Thom-Mather abstract stratified sets, of Whitney stratifications (§2.1). Then we give the Thom first Isotopy Theorem [Th]₂, [Ma]_{1,2} in our horizontally- C^1 version (§2.2) : an *ad hoc* improvement which is a consequence of the solution of the smooth Whitney Fiberings Conjecture [Wh] [MPT].

The horizontally- C^1 regularity [MPT] of the stratified homeomorphisms of Thom-Mather local topological triviality $H_{x_0} : U \times \pi_X^{-1}(x_0) \rightarrow \pi_X^{-1}U$ allows us to prove (b)-regularity for some families of wings and sub-wings which are “radial” in the tubular neighbourhood $T_X(1)$ of each stratum X of \mathcal{X} (Theorem 3 and Corollary 1). We obtain similar stronger results by considering subsets W of $T_X(1)$ which are unions of wings or sub-wings parametrized by a link $L_X(x_0, 1)$ of a point $x_0 \in X$ (Theorem 4 and Corollaries 2 and 3).

In §2.3 we recall the results of Goresky on the Whitney stratified mapping cylinders (Proposition 1), the *Theorem of Whitney cellulation for Whitney stratifications having conical singularities* (Proposition 2) and some definitions, notations and properties necessary for Goresky's proof of the *Triangulation Theorem of abstract stratified sets* [Go]₃. In particular we state the *Theorem of existence of an interior d -triangulation for an abstract stratified set \mathcal{X}* (Theorem 5) that we will use in our main Theorem in section 3 for \mathcal{X} a Whitney stratification.

In section 3, we give a solution of Problem 1 above, by proving the main Theorem of this paper (Theorem 6) stating that :

“Every compact Whitney stratified set \mathcal{X} admits a Whitney cellulation $g : \mathcal{J} \rightarrow \mathcal{X}$ ”.

The proof is obtained by adapting Goresky's proof of the triangulation of abstract stratified sets \mathcal{X} to a cellular version for a Whitney stratification \mathcal{X} . It is given in four steps and requires giving details of some parts of the proof of Goresky's Theorem (that he called “*a short accessible outlined construction*”).

To prove (b)-regularity of the stratified mapping cylinders, filling in the cellulation near the singularities of \mathcal{X} , we use Theorems 3 and 4 and Corollaries 1, 2 and 3 of section 2.

As corollary of Theorem 6, we find that the cellulation of \mathcal{X} can be moreover obtained with the cells as small as desired (Corollary 4).

In section 4 we recall the basic notations, definitions and results of the geometric homology theory $WH_*(\mathcal{X})$ for Whitney stratifications $\mathcal{X} = (A, \Sigma)$ and the definition of the Goresky homology representation map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(A)$ in *Whitney Homology*, which corresponds in the homology WH_* to the Steenrod map in Thom's differentiable bordism theory of 1954.

Then, we conclude the paper by proving, as a consequence of the Whitney cellulation Theorem 6, the Goresky homology Conjecture 1 stating that R_k is a bijection for every Whitney stratification (not necessarily a manifold) of a compact set or with finitely many strata (Theorem 8).

2. Stratified Spaces and Trivialisations.

A *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y , then X is contained in the closure of Y . We write then $X < Y$ and $\partial Y = \sqcup_{X < Y} X$ so that $\bar{Y} = Y \sqcup (\sqcup_{X < Y} X) = Y \sqcup \partial Y$ and $\partial Y = \bar{Y} - Y$ ($\sqcup =$ disjoint union). The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified set* with *support* A and *stratification* Σ .

A *stratified map* $f : \mathcal{X} \rightarrow \mathcal{X}'$ between stratified sets $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \rightarrow B$ which sends each stratum X of \mathcal{X} into a unique stratum X' of \mathcal{X}' , such that the restriction $f_X : X \rightarrow X'$ is C^1 .

A *stratified submersion* is a stratified map f such that each $f_X : X \rightarrow X'$ is a C^1 submersion.

2.1. Regular Stratified Spaces.

Extra regularity conditions may be imposed on the stratification Σ , such as to be an *abstract stratified set* in the sense of Thom-Mather [Th]₂, [Ma]_{1,2} or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney [Wh]₁, or (c) of K. Bekka [Be] or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier [Ve], or (L) of Mostowski [Pa].

In this paper we will consider essentially Whitney ((b)-regular) stratifications so called because they satisfy Condition (b) of Whitney (1965, [Wh]).

Definitions 1. Let Σ be a stratification of a subset $A \subseteq \mathbb{R}^N$, $X < Y$ strata of Σ and $x \in X$.

One says that $X < Y$ is (b)-regular (or that it satisfies *Condition (b) of Whitney*) at x [Wh] if for every pair of sequences $\{y_i\}_i \subseteq Y$ and $\{x_i\}_i \subseteq X$ such that $\lim_i y_i = x \in X$ and $\lim_i x_i = x$ and moreover $\lim_i T_{y_i} Y = \tau$ and $\lim_i [y_i - x_i] = L$ in the appropriate Grassmann manifolds (where $[v]$ denotes the vector space spanned by v) then $L \subseteq \tau$.

One says that $X < Y$ is (a)-regular (or that it satisfies *Condition (a) of Whitney*) at x [Wh] if for every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and moreover $\lim_i T_{y_i} Y$ exists in the appropriate Grassmann manifold then $\lim_i T_{y_i} Y \supseteq T_x X$.

Let $\pi : T_X \rightarrow X$ be a C^1 retraction onto X induced by a C^1 tubular neighbourhood T_X of X .

One says that $X < Y$ is (b^π) -regular at x [NAT] if for every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and moreover $\lim_i T_{y_i} Y = \tau$ and $\lim_i [y_i - \pi(x_i)] = L$ in the appropriate Grassmann manifolds then $L \subseteq \tau$.

The pair $X < Y$ is called (b)- or (a)- or (b^π) -regular if it is (b)- or (a)- or (b^π) -regular at every $x \in X$. Σ is called a (b)- or (a)- or (b^π) -regular stratification if all adjacent strata $X < Y$ in Σ are (b)- or (a)- or (b^π) -regular. A (b)-regular stratification is also usually called a *Whitney stratification*.

It is well known that Condition (b) at x implies Condition (a) at x and obviously (taking $x_i = \pi_X(y_i)$) implies Condition (b^π) . Conversely, if Conditions (a) and (b^π) are satisfied at x for a retraction $\pi : T_X \rightarrow X$, then Condition (b) also holds at x [NAT].

Finally if Condition (b^π) holds at x for every retraction $\pi : T_X \rightarrow X$ as above then Conditions (a) and hence (b) hold at x .

Important properties of Whitney stratified sets follow because they are in particular abstract stratified sets [Ma]_{1,2}.

Definition 2. (Thom-Mather 1970) Let $\mathcal{X} = (A, \Sigma)$ be a stratified set.

A family $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \rightarrow X \times [0, \infty[)\}_{X \in \Sigma}$ is called a *system of control data* of \mathcal{X} if for each stratum $X \in \Sigma$ we have that:

- 1) T_X is an open neighbourhood of X in A (called *tubular neighbourhood of X*);
- 2) $\pi_X : T_X \rightarrow X$ is a continuous retraction of T_X onto X (called *projection on X*);
- 3) $\rho_X : T_X \rightarrow [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$;

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} := \pi_X|_{T_{XY}}$ and $\rho_{XY} := \rho_X|_{T_{XY}}$, on the subset $T_{XY} := T_X \cap Y$, we have that :

- 5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \rightarrow X \times]0, \infty[$ is a C^1 submersion (then $\dim X < \dim Y$);
- 6) for every stratum Z of \mathcal{X} such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :
 - i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the π -control condition)
 - ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the ρ -control condition).

In what follows $\forall \epsilon > 0$ we will pose $T_X^\epsilon := T_X(\epsilon) = \rho_X^{-1}(]0, \epsilon[)$, $S_X^\epsilon := S_X(\epsilon) = \rho_X^{-1}(\epsilon)$, and $T_{XY}^\epsilon := T_X^\epsilon \cap Y$, $S_{XY}^\epsilon := S_X^\epsilon \cap Y$ and without loss of generality will assume $T_X = T_X(1)$ [Ma]_{1,2}.

The pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set* if A is Hausdorff, locally compact and admits a countable basis for its topology. Since one usually works with a unique system of control data \mathcal{F} of \mathcal{X} , in what follows we will omit \mathcal{F} .

If \mathcal{X} is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X \in \Sigma}$ may (and will always) be chosen such that: “ $T_{XY} \neq \emptyset \Leftrightarrow X \leq Y$ ” and “ $T_X \cap T_Y \neq \emptyset \Leftrightarrow X \leq Y$ or $X \geq Y$ ” (where both implications \Leftarrow automatically hold for each $\{T_X\}_X$) as in [Ma]₁, pp. 41-46.

The notion of system of control data of \mathcal{X} , introduced by Mather, is very important because it allows one to obtain good extensions of (stratified) vector fields [Ma]_{1,2} which are the fundamental tools in showing that a stratified (controlled) submersion $f : \mathcal{X} \rightarrow M$ into a manifold, satisfies Thom’s First Isotopy Theorem : the stratified version of Ehresmann’s fibration theorem ([Th]₂, [Ma]_{1,2} [GWPL]). Moreover by applying it to the maps $\pi_X : T_X \rightarrow X$ and $\rho_X : T_X \rightarrow [0, +\infty[$ it follows in particular that \mathcal{X} has a *locally topologically trivial structure* and also a locally trivial topologically conical structure. This fundamental property allows one moreover to prove that compact abstract stratified sets are triangulable [Go]₃.

Since Whitney ((b)-regular) stratified sets are abstract stratified sets, they are locally topologically trivial and triangulable if compact.

2.2. Some consequences of the solution of the smooth Whitney Fibering Conjecture.

In proving the Whitney condition (b) in our main theorem of section 3 (Theorem 6) we need some important consequences of the smooth Whitney Fibering Conjecture proved in [MPT]. So we first recall the main results of the paper [MPT] concerning (b)-regular stratifications.

Let \mathcal{X} be a Whitney stratified set in \mathbb{R}^m , X a stratum of \mathcal{X} , $x_0 \in X$ and $U = U_{x_0}$ a domain of a chart of X . It was proved in [MPT] that there exists a trivialization homeomorphism $H_{x_0} : U \times \pi_X^{-1}(x_0) \rightarrow \pi_X^{-1}(U)$ of X over U whose induced foliation is (a)-regular, i.e.

$$\mathcal{F}_{x_0} = \left\{ F_{z_0} = H_{x_0}(U \times \{z_0\}) \right\}_{z_0 \in \pi_X^{-1}(x_0)} \text{ satisfies : } \quad \lim_{z \rightarrow x} T_z F_z = T_x X \quad \forall x \in U,$$

(this is the smooth version of the Whitney fibering conjecture, see Theorem 7 in [MPT]) and such that the tangent space to each F_z (for each $z = H_{x_0}(t_1, \dots, t_l, z_0)$) is generated by the frame field

(w_1, \dots, w_l) where $w_i(z) = H_{x_0^*(t_1, \dots, t_l, z_0)}(E_i)$ and $\{E_1, \dots, E_l\}$ is the standard basis of $\mathbb{R}^l \times 0^{m-l}$. Moreover H_{x_0} is a *horizontally- C^1* homeomorphism rather than just a homeomorphism (see §8, Theorem 12 of [MPT]).

The (a)-regular foliation \mathcal{F}_{x_0} allows us to construct a family of wings over U :

$$\mathcal{W}_{x_0} := \left\{ W_{z_0} = H_{x_0}(U \times L_{z_0}) \right\}_{z_0 \in L_X(x_0, d)}$$

parametrized by the link $L_X(x_0, d) := \pi_X^{-1}(x_0) \cap S_X(d)$ where $L_{z_0} := \gamma_{z_0}([0, +\infty[)$ is the trajectory of z_0 via the flow of the gradient vector field $-\nabla \rho_X$.

Thus each line of the family $\{L_{z_0}\}_{z_0 \in L_X(x_0, d)}$ in the fiber $\pi_X^{-1}(x_0) \cap T_X(d)$ comes out (b^{π_X})-regular (and since $\dim L_{z_0} = 1$, equivalently (b)-regular) and each wing W_{z_0} is (b)-regular over U (see the proof of Theorem 8 in [MPT]).

In the next Theorem 3, for every point $z = \gamma_{z_0}(t) \in L_{z_0}$ we will also write $L_z := L_{z_0}$ and $W_z := W_{z_0}$ respectively for the unique trajectory and the unique wing containing $z \in \pi_X^{-1}(U)$.

Theorem 3 (of (b)-regular subwings). *Let \mathcal{X} be a (b)-regular stratified set, X a stratum of \mathcal{X} , X_0 an h -submanifold contained in a domain U of a chart for X and $x_0 \in X_0$. Then :*

$$\mathcal{W}_{x_0, X_0} := \left\{ W_{z_0, X_0} := H_{x_0}(X_0 \times L_{z_0}) \right\}_{z_0 \in L_X(x_0, d)}$$

is a family of wings (b)-regular over X_0 such that each $W_{z_0, X_0} \subseteq W_{z_0}$ is an $(h+1)$ -submanifold (sub-wing).

Proof. Let $l = \dim X$. The analysis being local (via a convenient C^1 -chart) we can suppose that $X = \mathbb{R}^l \times 0^{m-l}$, $X_0 = \mathbb{R}^h \times 0^{m-h}$ and that $(\pi_X, \rho_X) : \mathbb{R}^m \rightarrow X \times [0, +\infty[$ are the standard control data.

Since $X_0 \times L_{z_0}$ is an $(h+1)$ -submanifold of $U \times L_{z_0}$ and H_{x_0} a diffeomorphism on the strata $Z > X$, obviously each $W_{z_0, X_0} := H_{x_0}(X_0 \times L_{z_0})$ is an $(h+1)$ -submanifold (sub-wing) of W_{z_0} .

Remark also that : $W_{z_0, X_0} = H_{x_0}(X_0 \times \pi_X^{-1}(x_0)) \cap H_{x_0}(U \times L_{z_0}) = \pi_X^{-1}(X_0) \cap W_{z_0}$.

To prove that each $X_0 < W_{z_0, X_0}$ is (b)-regular we will prove that it is (a)- and (b^{π_X})-regular at each point $x \in X_0$.

a) $X_0 < W_{z_0, X_0}$ is (a)-regular at $x \in X_0$

Let (E_1, \dots, E_l) be the standard basis of $X = \mathbb{R}^l \times 0^{m-l}$.

Since the topological trivialisation H_{x_0} is horizontally- C^1 over X and $x \in X_0 \subseteq U \subseteq X$, by Theorem 8 of [MPT] :

$$\lim_{(t_1, \dots, t_l, z') \rightarrow x} H_{x_0^*(t_1, \dots, t_l, z')}(E_i) = E_i \quad \text{for every } i = 1, \dots, l \quad (z' \in \pi_X^{-1}(x_0)).$$

Since the wing $W_{z_0, X_0} = H_{x_0}(X_0 \times L_{z_0})$, is fixed (together with $z_0 \in L_X(z_0, d)$), every point $z \in W_{z_0, X_0}$ can be written as $z = H_{x_0}(t_1, \dots, t_l, z')$ with $z' \in L_{z_0} \subseteq \pi_X^{-1}(x_0)$ and $z \rightarrow X$ iff $z' \rightarrow x_0$.

Then for every sequence $\{z_n = H_{x_0}(t_1^n, \dots, t_l^n, z'_n)\}_n \subseteq W_{z_0, X_0}$ such that $\lim_n z_n = x$ and $\exists T := \lim_n T_{z_n} W_{z_0, X_0}$, since $W_{z_0, X_0} = H_{x_0}(X_0 \times L_{z_0}) \supseteq H_{x_0}(\mathbb{R}^h \times 0^{l-h} \times \{z'_n\})$, for every $x \in X_0 \subseteq U$ we find :

$$\begin{aligned}
(*) : \quad \lim_{\substack{z_n \rightarrow x \\ z_n \in W_{z_0, X_0}}} T_z W_{z_0, X_0} &\supseteq \lim_{\substack{z_n \rightarrow x \\ z_n \in W_{z_0, X_0}}} T_{z_n} H_{x_0}(X_0 \times \{z'_n\}) = \lim_{\substack{z_n \rightarrow x \\ z_n \in W_{z_0, X_0}}} T_{z_n} H_{x_0}(\mathbb{R}^h \times 0^{l-h} \times \{z'_n\}) \\
&= \lim_{(t_1^n, \dots, t_l^n, z'_n) \rightarrow x} [H_{x_0 * (t_1^n, \dots, t_l^n, z'_n)}(E_1), \dots, H_{x_0 * (t_1^n, \dots, t_l^n, z'_n)}(E_h)] = [E_1, \dots, E_h] = T_x X_0,
\end{aligned}$$

which proves the (a)-regularity at x of the pair $X_0 < W_{z_0, X_0}$.

Remark 1. Since each wing W_{z_0} of the family \mathcal{W}_{x_0} is foliated by the leaves of the “horizontal” foliation $\mathcal{F}_z = \{F_z = H_{x_0}(U \times \{z'\})\}_{z' \in \pi_X^{-1}(x_0)}$ then each (sub-)wing W_{z_0, X_0} of the family \mathcal{W}_{x_0, X_0} inherits by transverse intersection a horizontal (sub-)foliation whose leaves are :

$$F_{z, X_0} := H_{x_0}(X_0 \times L_{z'}) \cap H_{x_0}(U \times \{z'\}) = H_{x_0}(X_0 \times \{z'\}) \quad \text{for every } z \in W_{z_0, X_0}.$$

In particular, the tangent spaces which allow us to obtain above the (a)-regularity at x of the pair $X_0 < W_{z_0, X_0}$ are exactly these of the sequence $T_{z_n} F_{z_n, X_0} \xrightarrow{z_n \rightarrow x} T_x X_0$.

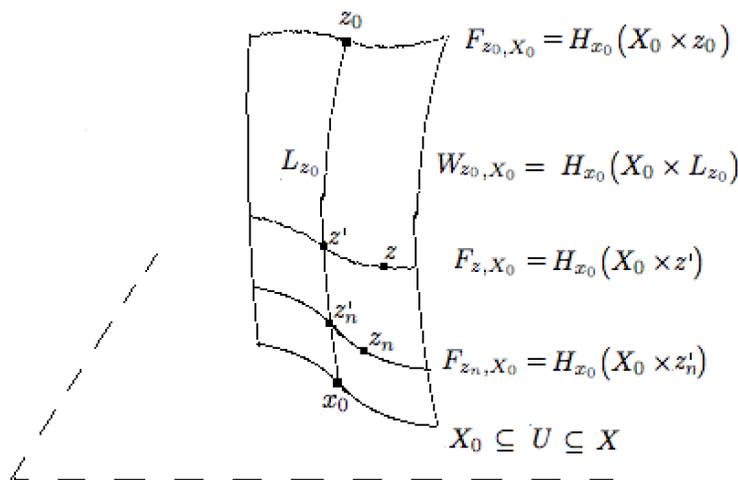


Figure 1

Remark moreover that, since \mathcal{F}_z is (a)-regular over U and so H_{x_0} is horizontally- C^1 [MPT], over the whole of U , so $\lim_{z_n \rightarrow x \in U} H_{x_0|_{F_{z_n} * z_n}} = 1_{U * x_0}$, then for every $x \in (\overline{X_0} - X_0) \cap U$, the horizontal (a)-regular (sub-)foliation $\{F_{z_0, X_0} := H_{x_0}(X_0 \times \{z'\})\}_{z' \in L_{z_0}}$ trace of W_{z_0} over \mathcal{F}_z , allows us to obtain :

$$(*) : \quad \lim_{z_n \rightarrow x} T_{z_n} W_{z_0, X_0} \supseteq \lim_{z_n \rightarrow x} T_{z_n} F_{z_0, X_0} = \lim_{z_n \rightarrow x} H_{x_0 * z_n} (T_{z_n}(X_0 \times \{z'_n\})) = \lim_{z_n \rightarrow x} T_{z_n}(X_0 \times \{z'_n\})$$

We will use this important property in the proof of Corollary 1. \square

b) $X_0 < W_{z_0, X_0}$ is (b^{π_X}) -regular at $x \in X_0$.

For the pair $X_0 < W_{z_0, X_0}$, since the trivialization H_{x_0} is π_X -controlled, one has :

$$\pi_X(W_{z_0, X_0}) = \pi_X(H_{x_0}(X_0 \times L_{z_0})) = H_{x_0}(\pi_X(X_0 \times L_{z_0})) = X_0.$$

Then one can consider as projection on X_0 the restriction $\pi_{X_0} := \pi_X| : W_{z_0, X_0} \rightarrow X_0$ of the projection $\pi_X : T_X(d) \rightarrow X$.

By definition of (b^π) -regularity at $x \in X_0 < W_{z_0, X_0}$ **[NAT]**₁, we must prove that for every sequence $\{z_n\}_n \subseteq W_{z_0, X_0}$ such that $\lim_n z_n = x$ and

$$\lim_n T_{z_n} W_{z_0, X_0} = T \in \mathbb{G}_m^{h+1} \quad \text{and} \quad \lim_n \overline{z_n \pi_X(z_n)} = L \in \mathbb{G}_m^1, \quad \text{then} \quad T \supseteq L.$$

Let $Z > X$ be the stratum of \mathcal{X} such that $z_0 \in Z$.

By hypothesis $X < Z$ is (b) -regular so we can assume that $\pi_X = \pi : \mathbb{R}^n \rightarrow \mathbb{R}^l \times 0^k$ and ρ_X is the standard distance function $\rho(t_1, \dots, t_n) = \sum_{i=l+1}^n t_i^2$, so that $-\nabla \rho_X(y) = -2(z - \pi_X(z))$ and the vectors generate the same vector space

$$(1) : \quad [\nabla \rho_X(z)] = [z - \pi_X(z)].$$

For every $n \in \mathbb{N}$, let u_n be the unit vector $u_n := \frac{z_n - x_n}{\|z_n - x_n\|}$ where $x_n = \pi_X(z_n)$.

Consider the “distance” function defined by (**[Ve]**, **[Mu]**₂ §4.2) :

$$\begin{cases} \delta(u, V) = \inf_{v \in V} \|u - v\| = \|u - p_V(u)\| & \text{for every } u \in \mathbb{R}^n \\ \text{and} \\ \delta(U, V) = \sup_{u \in U, \|u\|=1} \|u - p_V(u)\| & \text{for every subspace } U \subseteq \mathbb{R}^n. \end{cases}$$

Since $X < Z$ is (b) -regular and so (b^π) -regular at $x \in X$ then $T' := \lim_n T_{z_n} Z \supseteq L$.

Then

$$L \subseteq T' \implies \lim_n [u_n] \subseteq \lim_n T_{z_n} Z \implies \lim_n \delta([u_n], T_{z_n} Z) = 0.$$

Since ρ_{XZ} is the restriction $\rho_{X|Z}$ of ρ_X to Z , every vector $\nabla \rho_{XZ}(z_n)$ is the orthogonal projection $p_{T_{z_n} Z}(\nabla \rho_X(z_n))$ on $T_{z_n} Z$ of the vector $\nabla \rho_X(z_n)$ and, thanks to (1) above, we have:

$$T_{z_n} L_{z_n} = [\nabla \rho_{XZ}(z_n)] = p_{T_{z_n} Z}(\nabla \rho_X(z_n)) = p_{T_{z_n} Z}([z_n - x_n]) = p_{T_{z_n} Z}([u_n])$$

by which, u_n being a unit vector of the vector space $[u_n]$, one deduces that :

$$(2) : \quad \delta([u_n], T_{z_n} L_{z_n}) = \delta([u_n], p_{T_{z_n} Z}([u_n])) = \|u_n - p_{T_{z_n} Z}(u_n)\| = \delta([u_n], T_{z_n} Z).$$

On the other hand for every $n \in \mathbb{N}$, we have

$$(3) : \quad L_{z_n} \subseteq W_{z_0, X_0} \subseteq Z \quad \text{and} \quad T_{z_n} L_{z_n} \subseteq T_{z_n} W_{z_0, X_0} \subseteq T_{z_n} Z,$$

by which :

$$\delta([u_n], T_{z_n} Z) \leq \delta([u_n], T_{z_n} W_{z_0, X_0}) \leq \delta([u_n], T_{z_n} L_{z_n})$$

and by (2), one finds the equality :

$$(4) : \quad \delta([u_n], T_{z_n} L_{z_n}) = \delta([u_n], T_{z_n} W_{z_0, X_0}) = \delta([u_n], T_{z_n} Z).$$

Hence :

$$\lim_{z_n \rightarrow x_0} \delta([u_n], T_{z_n} W_{z_0, X_0}) = \lim_{z_n \rightarrow x_0} \delta([u_n], T_{z_n} L_{z_n}) = \lim_{z_n \rightarrow x_0} \delta([u_n], T_{z_n} Z) = 0,$$

so that $\lim_{z_n \rightarrow x_0} \delta([u_n], T_{z_n} W_{z_0, X_0}) = 0$ which implies :

$$L = \lim_{z_n \rightarrow x_0} \overline{z_n x_n} = \lim_{z_n \rightarrow x_0} [u_n] \subseteq \lim_{z_n \rightarrow x_0} T_{z_n} W_{z_0, X_0} = T$$

proving that $X_0 < W_{z_0, X_0}$ is (b^π) -regular at x , for every $x \in X_0$. \square

Notation. Each (sub-)wing $W_{z_0, X_0} = H_{x_0}(X_0 \times L_{z_0})$ defined in Theorem 3 defines a stratification \mathcal{W}_{z_0, X_0} with two strata given by the disjoint union :

$$\mathcal{W}_{z_0, X_0} = H_{x_0}(X_0 \times L_{z_0}) \sqcup X_0 \quad \text{which by Theorem 3 is } (b)\text{-regular.}$$

Corollary 1 below completes the analysis of the regularity adjacencies proved in Theorem 3.

Corollary 1. *Let $R < S$ be a stratification contained in a domain U of a chart of X , $R \subseteq \overline{S} \subseteq U$. Let $x_0 \in U$ and let*

$$\mathcal{W}_{x_0, R} := \{W_{z_0, R} = H_{x_0}(R \times L_{z_0})\}_{z_0 \in L(x_0, d)} \quad \text{and} \quad \mathcal{W}_{x_0, S} := \{W_{z_0, S} = H_{x_0}(S \times L_{z_0})\}_{z_0 \in L_X(x_0, d)}$$

be the families of subwings of \mathcal{W}_{x_0} constructed in Theorem 3.

Then, for every $z_0 \in L_X(x_0, d)$, the stratification by four strata

$$\mathcal{W}_{z_0, R \sqcup S} := \{R, S, W_{z_0, R}, W_{z_0, S}\}$$

satisfies:

- 1) If $R < S$ is (a) -regular then $\mathcal{W}_{z_0, R \sqcup S}$ is (a) -regular ;
- 2) If $R < S$ is (b^π) -regular then $\mathcal{W}_{z_0, R \sqcup S}$ is (b^π) -regular ;
- 3) If $R < S$ is (b) -regular then $\mathcal{W}_{z_0, R \sqcup S}$ is (b) -regular.

Proof. We have to prove the properties 1), 2), 3) for the following adjacency relations :

$$\begin{array}{ccccc} W_{z_0, R} & < & W_{z_0, S} & & \subseteq & T_X(d) \\ & \vee & \angle & & \vee & \\ R & < & S & & \subseteq & X. \end{array}$$

Proof of 1). First of all suppose that $R < S$ is (a) -regular.

By applying Theorem 3 for $X_0 = S$ and then for $X_0 = R$ we find that the adjacent strata $R < W_{z_0, R}$ and $S < W_{z_0, S}$ are (a)-regular.

The (a)-regularity of the adjacence $W_{z_0, R} < W_{z_0, S}$ is obtained as follows. Since $R < S$ is (a)-regular, $R \times L_{z_0} < S \times L_{z_0}$ is (a)-regular too and since H_{x_0} is a C^1 -diffeomorphism on each stratum of \mathcal{X} [Ma]₁ then

$$(1): \quad W_{z_0, R} = H_{x_0}(R \times L_{z_0}) < H_{x_0}(S \times L_{z_0}) = W_{z_0, S} \quad \text{is (a)-regular too [Tr].}$$

To prove that $R < W_{z_0, S}$ is (a)-regular, let us fix a point $r \in R$.

Since $R \subseteq U \subseteq X$ and H_{x_0} is horizontally C^1 over X at r [MPT], the (a)-regularity at r of $R < W_{z_0, S}$ follows with the same equalities (*) as in Theorem 3 (replacing X_0 by R). \square

Proof of 2). The (b^π)- (and also the (b)-) regularity of $W_{z_0, R} < W_{z_0, S}$ follows in exactly the same way as in the proof 1) for the (a)-regularity (see the equalities (1)) because these conditions are preserved by the C^1 -diffeomorphism.

To prove that $R < W_{z_0, S}$ is (b^π)-regular, let us fix a point $r \in R$.

Since $r \in R \subseteq U \cong \mathbb{R}^l$, the topological trivialisation H_r “centered at $r \equiv 0^n$ ” defined by lifting the frame field (E_1, \dots, E_l) of U on the (a)-regular foliation \mathcal{H} induced by H_{x_0} , defines the same (a)-regular foliation \mathcal{H} [MPT] and hence also the same wings :

$$W_{z_0, R} = H_{x_0}(R \times L_{z_0}) < H_{x_0}(S \times L_{z_0}) = W_{z_0, S}.$$

So it is enough to assume $x_0 = r \in R$ and $L_{z_0} \in \pi_X^{-1}(r)$ (this will simplify the notations).

We choose moreover local coordinates of $U \cong \mathbb{R}^l \times 0^{n-l}$ in which $R \cong \mathbb{R}^h \times 0^{n-h}$ ($h = \dim R$).

Let $\pi_R : T_R \subseteq U \rightarrow R$ be the canonical projection, since $T_R \subseteq U \subseteq X$ then for every $z \in T_R$ ($\pi_X(z) = z$ and so) $\pi_R(t_1, \dots, t_l, 0^{n-l}) = (t_1, \dots, t_h, 0^{n-h})$, then $\pi_R(z) = \pi_R(\pi_X(z))$

Since $R < S$ is (b^{π_R})-regular, for every sequence $\{s_n\} \subseteq S$ such that there exist both

$$L := \lim_{s_n \rightarrow x_0 \equiv 0^n} \frac{s_n - \pi_R(s_n)}{\|s_n - \pi_R(s_n)\|} \quad \text{and} \quad T := \lim_{s_n \rightarrow r} T_{s_n} S \quad \text{then} \quad L \subseteq T.$$

We will prove the (b^π)-regularity of $R < W_{z_0, S}$ with respect to the retraction :

$$\tilde{\pi}_R := \pi_R \circ \pi_X \quad : \quad \pi_X^{-1}(T_R) \xrightarrow{\pi_X} T_R \xrightarrow{\pi_R} R.$$

Let $\{z_n = H_{x_0}(s_n, z'_n)\}_n$ be a sequence in $W_{z_0, S} = H_{x_0}(S \times L_{z_0})$, where every $s_n \in S$, such that $\lim z_n = x_0$ and

$$\exists \tilde{L} := \lim_{z_n \rightarrow x_0} \frac{z_n - \tilde{\pi}_R(z_n)}{\|z_n - \tilde{\pi}_R(z_n)\|}, \quad \text{we will prove :} \quad \tilde{L} \subseteq \lim_{z_n \rightarrow x_0} T_{z_n} W_{x_0, S}.$$

Since $x_0 \in R \subseteq T_R$, for n large enough $z_n \in \pi_X^{-1}(T_R)$ and $s_n = \pi_X(z_n) \in T_R$.

Since $\tilde{\pi}_R(z_n) = \pi_R(\pi_X(z_n)) = \pi_R(s_n)$ for every n , we can write :

$$(2): \quad z_n - \tilde{\pi}_R(z_n) = z_n - \pi_R(s_n) = (z_n - \pi_X(z_n)) + (s_n - \pi_R(s_n)).$$

With the same equalities (3) and (4) as in the proof of Theorem 3, (and despite now $x_0 = r \notin S!$) we have that the lines

$$(3) : \quad L'_n := [z_n - \pi_X(z_n)] \quad \text{satisfy} : \quad L' := \lim_{z_n \rightarrow x_0} L'_n \subseteq \lim_{z_n \rightarrow x_0} T_{z_n} W_{z_0, S}.$$

On the other hand, since $R < S$ is (b^{π_R}) -regular and thanks to the property (*) in Remark 1 one finds that the lines

$$(4) : \quad L_n := [s_n - \pi_R(s_n)] \quad \text{satisfy} : \quad L := \lim_{s_n \rightarrow x_0} L_n \subseteq \lim_{s_n \rightarrow x_0} T_{s_n} S \stackrel{(*)}{\subseteq} \lim_{z_n \rightarrow x_0} T_{z_n} W_{z_0, S}.$$

Finally thanks to (2), (3) and (4) above one concludes that the lines

$$\tilde{L}_n := [z_n - \tilde{\pi}_R(z_n)] \quad \text{satisfy} : \quad \tilde{L} := \lim_{z_n \rightarrow x_0} \tilde{L}_n \subseteq \lim_{z_n \rightarrow x_0} T_{z_n} W_{z_0, S}. \quad \square$$

Proof of 3). Since (b) -regularity is equivalent to having both (a) -regularity and (b^π) -regularity it follows by 1) and 2) that $\mathcal{W}_{z_0, R \sqcup S}$ is (b) -regular. \square

Let now X_0 be an h -submanifold contained in a domain U of a chart for X , $x_0 \in X_0$ and N a p -submanifold $\subseteq L_{XY}(x_0, d)$ where $Y > X$. Let us consider

$$C_N := \sqcup_{z_0 \in N} L_{z_0} \quad \text{and respectively} \quad \tilde{C}_N := N \sqcup C_N \sqcup \{x_0\}$$

the cone union of all open and (resp.) the upper and lower closed cone union of all closed lines starting, at the time $t = 0$, from all points $z_0 \in N$. Then C_N and \tilde{C}_N are a $(p + 1)$ -submanifold of $\pi_{XY}^{-1}(X_0)$ and (resp.) a $(p + 1)$ -stratified set of $\pi_X^{-1}(X_0)$, and their images via H_{x_0} , namely :

$$W_{X_0, N} := H_{x_0}(X_0 \times C_N) \quad \text{and} \quad \tilde{W}_{X_0, N} := H_{x_0}(X_0 \times \tilde{C}_N)$$

are respectively :

$$\left\{ \begin{array}{l} W_{X_0, N} \quad \text{a } (h + p + 1)\text{-submanifold of } \pi_{XY}^{-1}(X_0) \text{ diffeomorphic to } H_{x_0}(X_0 \times C_N) \text{ and to } X_0 \times N \times]0, 1[\\ \text{and} \\ \tilde{W}_{X_0, N} \quad \text{a stratified set of } \pi_X^{-1}(X_0) \text{ homeomorphic to the mapping cylinder of } : X_0 \times N \xrightarrow{pr_1} X_0. \end{array} \right.$$

Moreover $\tilde{W}_{X_0, N} := H_{x_0}(X_0 \times \tilde{C}_N)$ is naturally stratified by :

$$\tilde{W}_{X_0, N} := H_{x_0}(X_0 \times \tilde{C}_N) = H_{x_0}(X_0 \times N) \sqcup H_{x_0}(X_0 \times C_N) \sqcup H_{x_0}(X_0 \times \{x_0\}).$$

In the same spirit and with part of the proofs of Theorem 3 and Corollary 1 we have :

Theorem 4 (of the (b) -regular pencils). *Let \mathcal{X} be a (b) -regular stratification and X a stratum of \mathcal{X} . Let X_0 be a h -submanifold contained in a domain U of a chart for X , $x_0 \in X_0$ and N a p -submanifold $\subseteq L_{XY}(x_0, d)$ where $Y > X$. Then the stratification of two strata below is (b) -regular :*

$$\mathcal{W}_{X_0, N} := \{X_0, W_{X_0, N} := H_{x_0}(X_0 \times C_N)\}.$$

Proof. As in Theorem 3, we prove separately the (a)- and (b^π)-regularity.

a) $X_0 < W_{X_0, N}$ is (a)-regular at each $x \in X_0$.

Using the notations and the (a)-regularity of Theorem 3, since $W_{X_0, N} \supseteq W_{z, X_0}$ we find :

$$\lim_{z \rightarrow x} T_z W_{X_0, N} \supseteq \lim_{z \rightarrow x} T_z W_{z, X_0} \supseteq T_x X_0.$$

b) $X_0 < W_{X_0, N}$ is (b^π)-regular at each $x \in X_0$.

Thanks to (3) of Theorem 3 (where $\forall n \in \mathbb{N}$, let u_n si the unit vector $u_n := \frac{z_n - \pi_X(z_n)}{\|z_n - \pi_X(z_n)\|}$) and thanks to the inclusions $L_{z_n} \subseteq W_{z_0, X_0} \subseteq W_{X_0, N} \subseteq Z$, we find :

$$\left\{ \begin{array}{l} T_{z_n} L_{z_n} \subseteq T_{z_n} W_{z_0, X_0} \subseteq \lim_{z \rightarrow x} T_z W_{X_0, N} \subseteq T_{z_n} Z \\ \text{and} \\ \delta([u_n], T_{z_n} Z) \leq \delta([u_n], T_{z_n} W_{X_0, N}) \leq \delta([u_n], T_{z_n} W_{z_0, X_0}) \leq \delta([u_n], T_{z_n} L_{z_n}) \end{array} \right.$$

and by (2) of Theorem 3 we have also the equality :

$$\delta([u_n], T_{z_n} L_{z_n}) = \delta([u_n], T_{z_n} W_{z_0, X_0}) = \delta([u_n], T_{z_n} W_{X_0, N}) = \delta([u_n], T_{z_n} Z),$$

and hence the equalities :

$$\left\{ \begin{array}{l} \lim_{z_n \rightarrow x} \delta([u_n], T_{z_n} W_{X_0, N}) = \lim_{z_n \rightarrow x} \delta([u_n], T_{z_n} L_{z_n}) = \lim_{z_n \rightarrow x} \delta([u_n], T_{z_n} Y) = 0 \\ L = \lim_{z_n \rightarrow x} \overline{z_n x_n} = \lim_{z_n \rightarrow x} [u_n] \subseteq \lim_{z_n \rightarrow x} T_{z_n} W_{X_0, N}. \quad \square \end{array} \right.$$

Corollary 2. Let $R < S$ and $x_0 \in U$ be as in Corollary 1 and $X < Y$ strata of \mathcal{X} . Then, for every submanifold $N \subseteq L_{XY}(x_0, d)$, the stratification by four strata

$$\mathcal{W}_{R \sqcup S, N} := \mathcal{W}_{R, N} \sqcup \mathcal{W}_{S, N} = \{R, S, W_{R, N}, W_{S, N}\}$$

having the incidence relations below :

$$\begin{array}{ccccc} W_{R, N} & < & W_{S, N} & & \subseteq T_X(d) \\ & \vee & \angle & \vee & \\ R & < & S & & \subseteq X. \end{array}$$

satisfies:

- 1) If $R < S$ is (a)-regular then $\mathcal{W}_{R \sqcup S, N}$ is (a)-regular ;
- 2) If $R < S$ is (b^π)-regular then $\mathcal{W}_{R \sqcup S, N}$ is (b^π)-regular ;
- 3) If $R < S$ is (b)-regular then $\mathcal{W}_{R \sqcup S, N}$ is (b)-regular.

Proof. The proof is completely similar to the proof of Corollary 1 using this time Theorem 4 instead of Theorem 3. \square

Corollary 3 below completes the analysis of the regularity of the adjacencies that we will use in the proof of our Whitney cellulation Theorem in section 3.

Corollary 3. *Let $R < S$ and $x_0 \in U$ be as in Corollary 1 and $X < Y < Z$ be strata of \mathcal{X} .*

Then, for every pair of adjacent submanifolds $N' < N$ of $L_X(x_0, d)$, such that $N' \subseteq L_{XY}(x_0, d)$ and $N \subseteq L_{XZ}(x_0, d)$, the stratification

$$\mathcal{W}_{R \sqcup S, N' \sqcup N} := \{W_{R, N'}, W_{S, N'} \mid W_{R, N}, W_{S, N}\}$$

whose incidence relations are as below :

$$\begin{array}{ccc} W_{R, N} & < & W_{S, N} & \subseteq & T_{XZ}(d) \\ & \vee & \angle & & \vee \\ W_{R, N'} & < & W_{S, N'} & \subseteq & T_{XY}(d) \end{array}$$

satisfies:

- 1) If $R < S$ and $N' < N$ are both (a)-regular then $\mathcal{W}_{R \sqcup S, N' \sqcup N}$ is (a)-regular ;
- 2) If $R < S$ and $N' < N$ are both (b^π) -regular then $\mathcal{W}_{R \sqcup S, N' \sqcup N}$ is (b^π) -regular ;
- 3) If $R < S$ and $N' < N$ are both (b)-regular then $\mathcal{W}_{R \sqcup S, N' \sqcup N}$ is (b)-regular.

Proof. Since $R < S$ are (a)- or (b^π) -regular, $R \times C_N < S \times C_N \subseteq T_{XZ}$ are (a)- or (b^π) -regular too and these regularity conditions are preserved by image via H_{x_0} since the restriction to T_{XZ} of H_{x_0} is a smooth diffeomorphism. So $W_{R, N} = H_{x_0}(R \times C_N) < H_{x_0}(S \times C_N) = W_{S, N}$ is (a)- or (b^π) -regular too. The same argument, applied to N' and using now that $H_{x_0|T_{XZ}}$ is a smooth diffeomorphism proves that $W_{R, N'} < W_{S, N'}$ is (a)- or (b^π) -regular.

The proof of the (a)- and (b^π) -regularity of the vertical incidence relations is similar to the proof of Corollary 1, using that since H_{x_0} is \mathcal{H} -semidifferentiable ([MPT] Theorem 12) there exists a (conical) neighbourhood V of each y in Y such that H_{x_0} coincides on $\pi_{YZ}^{-1}(V)$ with a local trivialization H_{YZ} of $\pi_{YZ}^{-1}(V)$ which is horizontally- C^1 over $V \subseteq Y$. Thus taking images via H_{x_0} , all the three vertical adjacency relations below :

$$\begin{array}{ccc} H_{x_0}(R \times C_N) & < & H_{x_0}(S \times C_N) & \subseteq & T_{XZ}(d) \\ & \vee & \angle & & \vee \\ H_{x_0}(R \times C_{N'}) & < & H_{x_0}(S \times C_{N'}) & \subseteq & T_{XY}(d) \end{array}$$

are (a)- and $(b^{\pi_{YZ}})$ -regular as in Theorem 3 and Theorem 4. \square

2.3. Goresky's results and some extension of his notions.

In 1981 Goresky redefined his geometric homology $WH_k(\mathcal{X})$ and cohomology $WH^k(\mathcal{X})$ for a Whitney stratification \mathcal{X} without asking that the substratified objects representing cycles and cocycles of \mathcal{X} satisfy condition (D) ([Go]₂ §3 and §4).

The main reason for which Goresky introduced Condition (D) in 1981 was that it allows one to obtain Condition (b) for the natural stratifications on the mapping cylinder of a stratified submersion :

Proposition 1. *Let $\pi : E \rightarrow M'$ be a C^1 riemannian vector bundle and $M = S_M^\epsilon$ the ϵ -sphere bundle of E . If $\mathcal{W} \subseteq M$, $\mathcal{W}' = \pi(\mathcal{W}) \subseteq M'$ are two Whitney stratifications such that $\pi_{\mathcal{W}} : \mathcal{W} \rightarrow \mathcal{W}'$ is a stratified submersion which satisfies condition (D), then the closed stratified mapping cylinder*

$$C_{\mathcal{W}'}(\mathcal{W}) = \bigsqcup_{Y \subseteq \mathcal{W}} [(C_{\pi_{\mathcal{W}}}(Y) - \pi_{\mathcal{W}}(Y)) \sqcup \pi_{\mathcal{W}}(Y) \sqcup Y]$$

is a Whitney stratified set.

Proof. [Go]₂ Appendix A.1 or [Mu]₃ for a different proof. \square

Then Goresky proved the Proposition 2 below, a partial solution of Problem 1 (of which we give a complete solution in Theorem 6 of the present paper) which is a *relatively* synthetic amalgam of the triangulation theorem of compact abstract stratified sets in [Go]₃ and its utilization for Whitney stratifications, and of Proposition 1 in [Go]₂ App.1 (both presented in a slightly different way also in [Go]₁).

In Proposition 2 below and in the whole of this paper, a (*linear-convex*) *cellular complex* is, following [Hu], [Mun], the analogue of a simplicial complex where one replaces the simplexes by the cells and each cell is defined as the linear-convex hull of a finite set of points, not necessarily independent, of some Euclidian space \mathbb{R}^n . Thus a simplicial complex is obviously a cellular complex while a cellular complex admits a subdivision which is a simplicial complex. A cellular map $f : \mathcal{K} \rightarrow \mathcal{K}'$ between cellular complexes sends each cell C of \mathcal{K} into a cell $f(C)$ of \mathcal{K}' .

When K is a polyhedron, support of a given cellular complex [Hu], [Mun], we will denote by Σ_K its family of open cells which is of course a Whitney stratification of K .

In Proposition 2 a map $h : K - L \rightarrow X$ into a manifold X , where $K, L \subseteq \mathbb{R}^m$ are polyhedra, will be called C^1 if there exist cellular complexes Σ_K and Σ_L such that for every (open) cell $\sigma \in \Sigma_K$ and point $p \in \bar{\sigma} - L$, h is locally extendable to a C^1 -map on an open neighbourhood of p in the plane spanned by σ . We try to keep as much as possible the notations of Goresky.

Proposition 2 (Goresky [Go]₃). *Every compact Whitney stratified set $\mathcal{X} = (A, \Sigma)$ in \mathbb{R}^m with conical singularities and conical control data admits a Whitney cellulation (see Definition 3 below): a stratified homeomorphism $g : \mathcal{J} \rightarrow \mathcal{X}'$ between a cellular complex $\mathcal{J} = (J, \Sigma_J)$ and a (b)-regular refinement of Σ in (open) cells $\mathcal{X}' = (A, \Sigma')$, $\Sigma' = \{f(C)\}_{C \in \Sigma_J}$ of \mathcal{X} . Moreover for each stratum X of Σ , the restriction $g_X : g^{-1}(X) \rightarrow X$ is C^1 .*

In what follows we will consider every simplicial (or cellular) complex \mathcal{K} of support $K = |\mathcal{K}|$ as a set of open simplexes (resp. cells) $\sigma \in \mathcal{K}$ and for each closed simplex (resp. cell) we will write $\bar{\sigma} \in \mathcal{K}$. In this way the set of open simplexes of \mathcal{K} is a partition which can be considered as the stratification of K whose strata are the open simplexes (resp. cells) with the usual adjacency relations “ $\tau < \sigma \iff \tau$ is a face of σ ” and this stratification \mathcal{K} is obviously Whitney (b)-regular.

Definition 3. A C^1 -*triangulation* (resp. C^1 -*cellulation*) of a subset B of a manifold X , is a homeomorphism $f : K - L \rightarrow f(K - L) \subseteq X$ with image $B = f(K - L)$, where K and L are polyhedra (possibly $\mathcal{L} = \emptyset$) for which there exist simplicial (resp. cellular) complexes \mathcal{K} and \mathcal{L} of support $|\mathcal{K}| = K$ and $|\mathcal{L}| = L$ such that for each open simplex (resp. cell) $\sigma \in \mathcal{K}$ and every point $p \in \bar{\sigma} - L$ there is an open neighbourhood U_p of p in the affine space $[\sigma]$ generated by σ and a C^1 embedding $\tilde{f} : U_p \rightarrow X$ extending the restriction $f|_{U_p \cap \bar{\sigma}}$.

Note that the C^1 extension \tilde{f} is required for all points $p \in \bar{\sigma} - L$ but for no points $p \in \bar{\sigma} \cap L$.

Since (b) -regularity is a local C^1 -invariant [Tr] then when $\mathcal{L} = \emptyset$, $f : K \rightarrow B \subseteq X$ transforms the (b) -regular stratification in simplexes (resp. cells) of \mathcal{K} into a partition $f(\mathcal{K})$ of X which is a (b) -regular triangulation (resp. cellulation) of X .

Definition 4. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set.

A C^1 -triangulation (resp. C^1 -cellulation) of \mathcal{X} is a homeomorphism $f : K \rightarrow A$ defined on a polyhedron K such that for each stratum $X \in \Sigma$, $f^{-1}(\overline{X})$ is a subpolyhedron of K and the restriction

$$f_X : f^{-1}(X) \rightarrow X \quad \text{is a } C^1\text{-triangulation (resp. } C^1\text{-cellulation) of } X.$$

Example 1. Let $f : [0, 1] \rightarrow A$ be the map defined by $f(t) = te^{2\pi i \frac{1}{t}}$ and $f(0) = (0, 0)$ whose image $A := f([0, 1])$ is a spiral of \mathbb{R}^2 , (b) -regular at $f(0) = (0, 0)$. Of course f defines a C^0 -triangulation of the C^0 -manifold with boundary A , but nor A is a C^1 -manifold with boundary at $(0, 0)$ neither f defines a C^1 -triangulation of A in the usual meaning.

If we consider the (b) -regular stratified space $\mathcal{X} = (A, \Sigma)$ where $\Sigma := \{f(\{0\}), f(]0, 1[), \{f(\{1\})\}$ then f is a C^1 -triangulation (and a C^1 -cellulation) of $\mathcal{X} = (A, \Sigma)$ since Definitions 3 and 4 hold $\forall X \in \Sigma$: for $X = f(]0, 1[)$ taking the polyhedra $\mathcal{K} = \{\{0\},]0, 1[, \{1\}\}$ and $\mathcal{L} = \{f(0), f(1)\}$.

Remark 2. A C^1 -triangulation (or C^1 -cellulation) $f : K \rightarrow A$ of an abstract stratified set $\mathcal{X} = (A, \Sigma)$ contained in a manifold is not necessarily a (b) -regular stratification \mathcal{X} . In fact, although each stratum X of \mathcal{X} inherits a (b) -regular triangulation or cellulation of X , however if $\tau < \sigma$ are two open simplexes (or cells) such that $f(\tau) < f(\sigma)$ are contained respectively in two different adjacent strata $X < Y$ of \mathcal{X} there is no reason to have (b) -regularity at the points $x \in f(\tau) < f(\sigma)$.

Remark 3. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set [Ma]_{1,2}.

Then there exists $d > 0$ such that every chain of strata $X_1 < \dots < X_n = Y$ of \mathcal{X} satisfies the following multi-transversality property:

MT) : for every $J \subseteq \{1, \dots, n\}$, every intersection of hypersurfaces $\cap_{j \in J} S_{X_j Y}(\epsilon)$ of Y is transverse in Y to the intersection $\cap_{i \notin J} S_{X_i Y}(\epsilon')$ for every $\epsilon, \epsilon' \in]0, d[$.

Notations. For each h -stratum X of \mathcal{X} and for every $d \in]0, 1]$ one defines an h -manifold X_d^o (with corners) and its boundary ∂X_d^o by setting :

$$X_d^o := X - \bigcup_{X' < X} T_{X'}(d) \quad \partial X_d^o = X_d^o \cap \left(\bigcup_{X' < X} S_{X'}(d) \right).$$

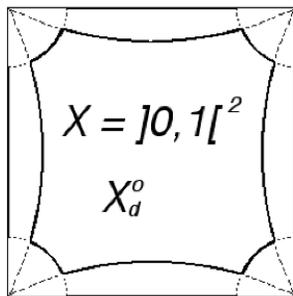


Figure 3

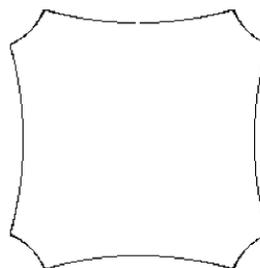


Figure 4

For $i = 0, \dots, n-1$, one denotes $T_i(d) := \bigcup_{\dim X \leq i} T_X(d)$ and $T_{-1}(d) = \emptyset$.

Remark 4. If $\dim X = i$ then $X - T_{i-1}(d) = X - \bigcup_{X' < X} T_{X'}(d) = X_d^o$. \square

Definition 5. (Goresky [Go]_{1,3}) Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set.

An *interior d -triangulation f* for \mathcal{X} is an embedding $f : K \rightarrow A$ defined on a polyhedron $K = \bigsqcup_{X \in \Sigma} K_X$ which is a disjoint union of polyhedra $\{K_X\}_{X \in \Sigma}$ such that there exists a simplicial complex $\mathcal{K} = \bigsqcup_{X \in \Sigma} \mathcal{K}_X$ such that for each stratum $X \in \Sigma$:

- 1) $|\mathcal{K}_X| = K_X$ (so $|\mathcal{K}| = K$) and $f(K_X) = X_d^o$;
- 2) the restriction $f_{K_X} : K_X \rightarrow X_d^o \subseteq X$ is a C^1 -triangulation of the subset X_d^o of X ;
- 3) if $i = \dim X$, $f^{-1}(S_X(d)) = f^{-1}(S_X(d) - T_{i-1}(d))$ is a subpolyhedron of $\sqcup_{Y > X} \partial K_Y$;
- 4) the restriction $\tilde{\pi}_X := f^{-1} \circ \pi_X \circ f| : f^{-1}(S_X(d)) \rightarrow f^{-1}(X)$ is *PL*:

$$\begin{array}{ccc} f^{-1}(S_X(d)) = f^{-1}(S_X(d) - T_{i-1}(d)) & \xrightarrow{f} & S_X(d) - T_{i-1}(d) \\ \tilde{\pi}_X \downarrow & & \downarrow \pi_X \\ f^{-1}(X) = f^{-1}(X - T_{i-1}(d)) & \xrightarrow{f} & X - T_{i-1}(d). \end{array}$$

It follows that :

Remark 5. If $f : K \rightarrow A$ is an interior d -triangulation of \mathcal{X} , then $\forall X \in \Sigma$ the restriction

$$f| : f^{-1}(\overline{S_X(d)}) \longrightarrow \overline{S_X(d)} \quad \text{is an interior } d\text{-triangulation of } \overline{S_X(d)}. \quad \square$$

Theorem 5. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set. Then :

- 1) there exists $d > 0$ small enough such that \mathcal{X} admits an interior d -triangulation f .
- 2) for every $d' \in]0, d[$ there exists an interior d' -triangulation of \mathcal{X} extending f .

Proof. [Go]₃ section 3. \square

In 1976 [Go]_{1,3} Goresky introduced the following very useful notion :

Definition 6. Let $\mathcal{X} = (A, \Sigma)$ be an abstract stratified set. A family of maps

$$\{r_X^\epsilon : T_X(1) - X \longrightarrow S_X(\epsilon)\}_{X \in \Sigma, \epsilon \in]0, 1[},$$

is said to be a *family of lines for \mathcal{X}* (with respect to a given system of control data $\{(T_X, \pi_X, \rho_X)\}$) if for every pair of strata $X < Y$, the following properties hold :

- 1) every restriction $r_{XY}^\epsilon := r_{X|Y}^\epsilon : T_{XY} \longrightarrow S_{XY}(\epsilon)$ of r_X^ϵ is a C^1 -map ;
- 2) $\pi_X \circ r_X^\epsilon = \pi_X$;
- 3) $r_X^{\epsilon'} \circ r_X^\epsilon = r_X^{\epsilon'}$;
- 4) $\pi_X \circ r_Y^\epsilon = \pi_X$;

- 5) $\rho_Y \circ r_X^\epsilon = \rho_Y$;
6) $\rho_X \circ r_Y^\epsilon = \rho_X$;
7) $r_Y^{\epsilon'} \circ r_X^\epsilon = r_X^\epsilon \circ r_Y^{\epsilon'}$.

Goresky proved the following :

Proposition 3. *Every Thom-Mather abstract stratified set \mathcal{X} admits a family of lines.*

Proof. [Go]₃ section 2. \square

Remark 6. Since (b)-regular [Ma]_{1,2} stratifications admit structures of abstract stratified sets a family of lines exists for them too.

3. Whitney cellulation of a compact Whitney stratified set

In the same spirit as in section 7 of [MPT], where in Definition 10 Example 1 and Remark 7 ii) we introduce the notion of *conical chart*, we prove the Proposition below, in which we introduce the notion of *conical trivialization of \mathcal{X}* which provides a useful tool to study global or local problems of a Whitney stratification and which we use as a starting point of the proof of our (b)-regular cellulation Theorem.

Proposition 4 (and Definition). *Let $\mathcal{X} = (A, \Sigma)$ be a compact Whitney stratified set in \mathbb{R}^m .*

There exists $\delta > 0$ such that, for every $d \in]0, \delta[$, there exists a complete family of d -small semidifferentiable conical trivializations of \mathcal{X} , $\{H_{x'_j}\}_{j, X \in \Sigma}$ of \mathcal{X} (see the proof for the definition).

Proof. Since \mathcal{X} is (b)-regular it admits a structure of an abstract stratified set and so it is locally topologically trivial via the Thom-Mather homeomorphism [Ma]₁. Also for every stratum X there exists $d_X > 0$ such that $(\pi_X, \rho_X) : T_X(d_X) \rightarrow X \times [0, d_X]$ is a proper submersion [Ma]₁.

Since \mathcal{X} has finitely many strata we can define $\delta_i := \min\{d_X : \dim X = i\}$ and $\delta := \min_i \delta_i$, so that

$$(\pi_X, \rho_X) : T_X(d) \rightarrow X \times [0, d] \quad \text{is a proper submersion} \quad \forall X \in \Sigma \quad \text{and} \quad \forall d \in]0, \delta].$$

Moreover for every i -stratum X of \mathcal{X} and $x \in X$, there exists a neighbourhood U_x of x in X and a trivializing stratified homeomorphism $H_x : U_x \times \pi_X^{-1}(x) \rightarrow \pi_X^{-1}(U_x) \cap T_X(d)$, which is smooth on each stratum.

For every $i \leq n = \dim \mathcal{X}$, since each i -stratum X_d^o is compact it admits a finite subcovering

$$(*) : \quad X_d^o \subseteq \cup_{j=0}^{r_X} U_{x'_j} \quad \text{so} : \quad \pi_X^{-1}(X_d^o) \subseteq \cup_{j=0}^{r_X} \pi_X^{-1}(U_{x'_j}) \quad \text{and}$$

$$T_i(d) - T_{i-1}(d) = \bigcup_{\dim X=i} [T_X(d) - T_{i-1}(d)] \subseteq \bigcup_{\dim X=i} \pi_X^{-1}(X_d^o) \subseteq \bigcup_{\dim X=i} \bigcup_{j=0}^{r_X} \pi_X^{-1}(U_{x'_j}).$$

We call $\{H_{x'_j} \mid j = 0, \dots, r_X\}$ a *family of d -small conical trivializations of X_d^o* .

Hence, by Remark 4 and the equalities (***) in Step 1 of the next Theorem 6, the whole of A ,

$$(**) : \quad A = \bigcup_{i=0}^n [T_i(d) - T_{i-1}(d)] \subseteq \bigcup_{i=0}^n \bigcup_{\dim X=i} \bigcup_{j=0}^{r_X} \pi_X^{-1}(U_{x'_j})$$

is completely covered by this finite family of open neighbourhoods of topological triviality of \mathcal{X} .

We call $\{H_{x'_j}\}_{j, X \in \Sigma}$ a *complete family of d -small conical trivializations of \mathcal{X}* .

Moreover, thanks to the solution of the smooth Whitney fibering conjecture [MPT], every trivialization of the family $\{H_{x'_j} : j = 0, \dots, r_X\}_{X \in \Sigma}$ may be assumed to be horizontally- C^1 (Theorem 10 of [MPT]) with respect to each stratum and moreover \mathcal{H} -semidifferentiable with respect to each pair of strata of \mathcal{X} (Theorem 12 of [MPT]).

Then (in the same spirit as in Remark 7 [MPT] where we constructed *conical charts* of \mathcal{X}) we obtain, by definition, for every $d \in]0, \delta[$, a :

“*complete family $\{H_{x'_j}\}_{j, X \in \Sigma}$ of d -small semidifferentiable conical trivializations of \mathcal{X} ”.*

 \square

We now prove the Whitney Cellulation Theorem :

Theorem 6. *Every compact Whitney stratified set $\mathcal{X} = (A, \Sigma)$ in \mathbb{R}^m admits a Whitney cellulation. I.e. there exists a stratified homeomorphism $g : \mathcal{J} \rightarrow \mathcal{X}'$ between a cellular complex $\mathcal{J} = (J, \Sigma_J)$ and a (b)-regular refinement of \mathcal{X} in (open) cells $\mathcal{X}' = (A, \Sigma')$, $\Sigma' = \{g(C)\}_{C \in \Sigma_J}$ such that for each stratum X in Σ the restriction $g_X : g^{-1}(X) \rightarrow X$ is a C^1 -cellulation.*

Proof. Let $n = \dim \mathcal{X}$.

Since \mathcal{X} is a Whitney stratified set, it is an abstract stratified set for which the maps of the system of control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \rightarrow X \times [0, \infty[\}_{X \in \Sigma}$ are the restrictions to A of C^1 maps defined on open tubular neighbourhoods T_X in \mathbb{R}^m of each stratum $X \in \Sigma$ [Ma]_{1,2}.

The proof of the Whitney cellulation Theorem 6 requires a review of the triangulation theorem of abstract stratified sets of Goresky [Go]₃. We do this below, using various notations and properties of Goresky without reproving them.

Step 1: Reviewing the triangulation theorem of abstract stratified sets for Whitney stratified sets.

Let $\delta > 0$ as obtained in Proposition 4 above and of which we will preserve the notations.

Since \mathcal{X} is an abstract stratified set, for small enough $d \in]0, \delta[$ there exists an interior d -triangulation f of $\mathcal{X} = (A, \Sigma)$ [Go]₃ (see Definition 5) :

$$f : K = \sqcup_{X \in \Sigma} K_X \longrightarrow f(K) = \sqcup_{X \in \Sigma} X_d^o \subseteq A.$$

Let us recall that by definition $T_i(d) := \bigcup_{\dim X \leq i} T_X(d)$ for $i = 0, \dots, n-1$ and $T_{-1}(d) = \emptyset$.

Then the family $\{T_i(d)\}_{i=-1}^{n-1}$ defines an increasing sequence of subsets of A :

$$\emptyset =: T_{-1}(d) \subseteq T_0(d) \subseteq \dots \subseteq T_i(d) \subseteq \dots \subseteq T_{n-1}(d) = \bigcup_{\dim X \leq n-1} T_X(d) \subseteq A$$

so that by denoting $Z := \bigcup_{\dim X = n} X$,

$$Z_d^o := \bigcup_{\dim X = n} X_d^o \quad \text{and} \quad \partial Z_d^o := \bigcup_{\dim X = n} \partial X_d^o \quad \text{one has :} \quad f(K) - T_{n-1}(d) = Z_d^o$$

and so :

$$(1) : \quad A = f(K) \cup T_{n-1}(d) = Z_d^o \sqcup T_{n-1}(d) .$$

Then the family of subsets of A defined by $\{A_i := A - T_{i-1}(d)\}_{i=0}^n$ satisfies :

$$\begin{aligned} (***) : \quad A_i &:= A - T_{i-1}(d) = [A - T_{n-1}(d)] \bigsqcup [T_{n-1}(d) - T_{i-1}(d)] = \\ &= Z_d^o \bigsqcup [T_{n-1}(d) - T_{i-1}(d)] = Z_d^o \bigsqcup \bigsqcup_{r=i}^{n-1} [T_r(d) - T_{r-1}(d)] , \end{aligned}$$

and so it is a covering of A increasing for the decreasing index $i = n, \dots, 0$ with $A_n = Z_d^o$ and $A_0 = A$:

$$Z_d^o = A - T_{n-1}(d) = A_n \subseteq A_{n-1} \subseteq \dots \subseteq A_i \subseteq \dots \subseteq A_0 = A ,$$

and moreover $A_{i-1} - A_i = T_{i-1}(d) - T_{i-2}(d)$ by which one finds :

$$(2) : \quad A_{i-1} = A_i \sqcup [T_{i-1}(d) - T_{i-2}(d)] .$$

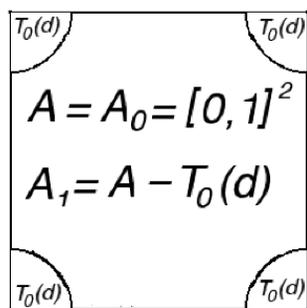


Figure 5

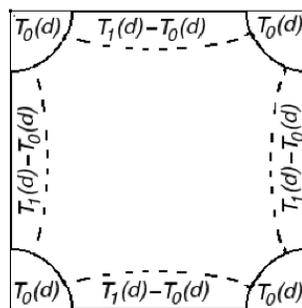


Figure 6

The Whitney cellulation $g : \mathcal{J} \rightarrow \mathcal{X}'$ of \mathcal{X} will be constructed by defining a finite sequence of Whitney cellulations $\{g_i : J_i \rightarrow f(K) \cup A_i\}_{i=n, \dots, 0}$ of the subsets $f(K) \cup A_i$ of A by extending each Whitney cellulation

$$g_i : J_i \rightarrow f(K) \cup A_i \subseteq A \quad \text{to a Whitney cellulation} \quad g_{i-1} : J_{i-1} \rightarrow f(K) \cup A_{i-1} \subseteq A$$

on a polyhedron $J_{i-1} \supset J_i$ to the new part $T_{i-1}(d) - T_{i-2}(d)$ of the image (using the equality (2)).

More precisely, we will prove by decreasing induction on i that there exists a sequence of stratified homeomorphisms $\{g_i : J_i \rightarrow f(K) \cup A_i\}_{i=n, \dots, 0}$ defined on polyhedra $K = J_n \subseteq \dots \subseteq J_0$ stratified by a sequence of cell complexes $\mathcal{J}_n, \dots, \mathcal{J}_0$:

$$\begin{array}{ccccccccc} J_n & \hookrightarrow & J_{n-1} & \cdots & J_i & \cdots & J_1 & \hookrightarrow & J_0 \\ g_n = f \downarrow & & g_{n-1} \downarrow & & g_i \downarrow & & g_1 \downarrow & & \downarrow g_0 = g \\ f(K) = f(K) \cup A_n & \hookrightarrow & f(K) \cup A_{n-1} & \cdots & f(K) \cup A_i & \cdots & f(K) \cup A_1 & \hookrightarrow & f(K) \cup A_0 = A \end{array}$$

such that for every $i = n, \dots, 0$ one has :

- 1) $|\mathcal{J}_i| = J_i$ and $g_i(J_i) = f(K) \cup A_i$;
- 2) $\forall X \in \Sigma$, the restriction $g_i| : g_i^{-1}(X) \rightarrow X$ is a C^1 -triangulation of its image and $g_i^{-1}(\bar{X})$ is a cell subcomplex of J_i ;
- 3) $\forall X' < X$, the subset $g_i^{-1}(\overline{S_{X'}(d)})$ is a cell subcomplex of J_i and the restriction $\tilde{\pi}_{X'}^i : g_i^{-1}(S_{X'}(d)) \rightarrow g_i^{-1}(X')$ is a cellular map ;
- 4) the stratification $\Sigma_{\mathcal{J}_i} = \{g_i(C)\}_{C \in \mathcal{J}_i} = \{g_i(C) \mid C \text{ is a cell of } \mathcal{J}_i\}$ induced by \mathcal{J}_i on $g_i(J_i) = A_i \cup f(K)$ is (b)-regular (i.e. g_i is a Whitney cellulation of $f(K) \cup A_i$).

In this way, we will obtain the claimed Whitney cellulation $g : J \rightarrow A$ as the last map $g = g_0$.

As $f : K = \sqcup_X K_X \rightarrow f(K) = \sqcup_X X_d^o \subseteq A$ is an interior d -triangulation of \mathcal{X} , then by the properties 1) and 2) of Definition 5, taking

$$\mathcal{J}_n = \mathcal{K} = \bigsqcup_{X \in \Sigma} \mathcal{K}_X, \quad J_n = K = \bigsqcup_{X \in \Sigma} K_X, \quad \text{and} \quad g_n = f$$

one finds :

$$g_n(J_n) = f(K) = f(K) \cup Z_d^o = f(K) \cup [A - T_{n-1}(d)] = f(K) \cup A_n$$

so the choice $f = g_n$ satisfies 1) of the inductive hypothesis above and moreover the properties 2) and 3) hold too thanks to the corresponding properties 2) and 3) of f in Definition 5.

Moreover, since for every stratum X of \mathcal{X} , the stratification in open simplexes underlying the simplicial complex \mathcal{K}_X (we denote it again \mathcal{K}_X) is obviously (b)-regular and by 2) of Definition 5 each $f_X : \mathcal{K}_X = f^{-1}(X) \rightarrow X_d^o$ is a C^1 -triangulation of X_d^o and since (b)-regularity is preserved by C^1 -diffeomorphisms, then the stratification by open simplexes $f(\mathcal{K}_X)$ induced by the simplicial complex \mathcal{K}_X on X_d^o is a (b)-regular stratification of the (cornered) manifold $X_d^o \subseteq X$ and its boundary ∂X_d^o and the property 4) above holds.

Let us suppose now that, by inductive hypothesis, there exists a sequence of stratified homeomorphisms $\{g_r : J_r \rightarrow f(K) \cup A_r\}_{r=n, \dots, i}$ defined on polyhedra $K = J_n \subseteq \dots \subseteq J_i$ and a sequence of corresponding cellular complexes $\mathcal{K} = \mathcal{J}_n \subseteq \dots \subseteq \mathcal{J}_i$ satisfying all the properties 1), ..., 4).

In order to construct the map $g_{i-1} : J_{i-1} \rightarrow f(K) \cup A_{i-1}$ and a cellular complex \mathcal{J}_{i-1} satisfying 1), ..., 4), let us consider for each $(i-1)$ -stratum X of \mathcal{X} two cellular complexes \mathcal{A}_X and \mathcal{B}_X with supports respectively $A_X := g_i^{-1}(S_X(d))$ and $B_X := g_i^{-1}(X)$ such that:

- 5) the restriction $\tilde{\pi}_X^{i-1} = g_i^{-1} \circ \pi_X \circ g_i| : \mathcal{A}_X \rightarrow \mathcal{B}_X$ is a cellular map (see also Lemma 1) :

$$\begin{array}{ccc} A_X = g_i^{-1}(S_X(d)) = g_i^{-1}(S_X(d) - T_{i-2}(d)) & \xrightarrow{g_i} & S_X(d) - T_{i-2}(d) \\ \tilde{\pi}_X^{i-1} \downarrow & & \downarrow \pi_X \\ B_X = g_i^{-1}(X) = g_i^{-1}(X - T_{i-2}(d)) & \xrightarrow{g_i} & X - T_{i-2}(d). \end{array}$$

- 6) $\forall X' < X$, the sets $g_i^{-1}(S_{X'}(d) \cap X)$ and $g_i^{-1}(S_X(d) \cap S_{X'}(d))$ are full subcomplexes of \mathcal{J}_i .

We also require that $\max\{\text{diam } g_i(\bar{\tau}) \mid \bar{\tau} \in B_X\} < d < \delta < \delta_i$ so that $\exists j \leq r_X : g_i(\bar{\tau}) \subseteq U_{x'_j}$.
Then :

γ) $\pi_X^{-1}g_i(\bar{\tau}) \subseteq \pi_X^{-1}(U_{x'_j})$ is contained in the image $\pi_X^{-1}(U_{x'_j})$ of a local trivialisation of \mathcal{X} .

Lemma 1. *If $\dim X = i - 1$, then for every $r \leq i - 1$ one has :*

a) $g_i^{-1}(S_X(d)) = g_i^{-1}(S_X(d) - T_r(d))$;

b) $g_i^{-1}(X) = g_i^{-1}(X - T_r(d))$.

Proof of a). The proof of (\supseteq) is obvious so we only have to prove (\subseteq) .

Proof of (\subseteq) . If $p \in g_i^{-1}(S_X(d))$ with $\dim X = i - 1$ then $y := g_i(p) \in S_X(d)$.

As $d > 0$ there exists a stratum $Y > X$, such that $y \in S_X(d) \cap Y$, so $y \in Y_d^\circ$.

Since $Y > X$ we have $j := \dim Y > \dim X = i - 1$, i.e. $j - 1 \geq i - 1$ and so $T_{i-1}(d) \subseteq T_{j-1}(d)$.

Hence for every $r \leq i - 1$ we have $T_r(d) \subseteq T_{j-1}(d)$ and we conclude that :

$$y \in Y_d^\circ := Y - T_{j-1}(d) \Rightarrow y \notin T_{j-1}(d) \Rightarrow y \notin T_r(d) \Rightarrow y \in S_X(d) - T_r(d). \quad \square$$

Proof of b). As in the proof of a) it suffices to prove (\subseteq) .

Proof of (\subseteq) . If $p \in g_i^{-1}(X)$, $x := g_i(p) \in X$. Then for every $r \leq i - 1 = \dim X$ one has:

$$x \in g_i(J_i) = f(K) \cup A_i \Rightarrow \begin{cases} x \in f(K) = \sqcup_{Y \in \Sigma} Y_d^\circ \Rightarrow x \in X_d^\circ \Rightarrow x \notin T_r(d) \\ \text{or} \\ x \in A_i := A - T_{i-1}(d) \Rightarrow x \notin T_{i-1}(d) \Rightarrow x \notin T_r(d). \end{cases}$$

Hence in each case we find : $x \in S_X(d) - T_r(d)$ and $p \in g_i^{-1}(S_X(d) - T_r(d))$. \square

Remark 7. The stratified set \mathcal{X} being locally topologically trivial, for every closed cell $\bar{\tau} := g_i^{-1}\pi_X g_i(\bar{\sigma}) \in \mathcal{B}_X$ the cell $g_i(\bar{\tau})$ is contained in a domain U_{x_r} of local topological triviality of the stratum X of \mathcal{X} . Then, $\pi_X^{-1}(g_i(\bar{\tau}))$ is obtained by considering the family of closed cells $\bar{\sigma} \in \mathcal{A}_X$ such that $\pi_X g_i(\bar{\sigma}) \subseteq X$, and it has a partition in open cells as follows (where below we denote $\pi := \tilde{\pi}_X^{i-1} : \mathcal{A}_X \rightarrow \mathcal{B}_X$ to write $\pi(\sigma) = \tau$) :

$$\pi_X^{-1}(g_i(\bar{\tau})) = \bigsqcup_{\pi(\sigma)=\tau} \pi_X^{-1}(g_i(\pi(\bar{\sigma}))) = \bigsqcup_{\pi(\sigma)=\tau, \sigma' \leq \sigma} \pi_X^{-1}(g_i(\pi(\sigma')))$$

where each $\sigma' \leq \sigma \in \mathcal{A}_X$ and $\tau \in \mathcal{B}_X$ are open cells and each

$$\pi_X^{-1}(g_i(\pi(\sigma'))) \text{ is homeomorphic to } g_i(\sigma') \times [0, d[\cong \sigma' \times [0, d[= \sigma' \times \{0\} \sqcup \sigma' \times]0, d[.$$

Step 2 : Whitney stratifying $T_{i-1}(d) - T_{i-2}(d)$ by open cells.

Let X be an $(i - 1)$ -stratum of \mathcal{X} . Below we will denote $l = \dim X = i - 1$, $k = m - l$, $g := g_i$.

Let x_0, \dots, x_α be the set of the vertices of the cells of X_d° . As $\pi_X : g(\mathcal{A}_X) \rightarrow g(\mathcal{B}_X)$ is a cellular map (sending 0-cells of $g(\mathcal{A}_X)$ in 0-cells of $g(\mathcal{B}_X)$), the set $\mathcal{V}(g(\mathcal{A}_X))$ of vertices of the cells of $g(\mathcal{A}_X)$ is contained in the union of the links

$$\mathcal{V}(g(\mathcal{A}_X)) \subseteq \bigsqcup_{j=0}^{\alpha} L_X(x_j, d) = \bigsqcup_{j=0}^{\alpha} \pi_X^{-1}(x_j) \cap S_X(d).$$

Since \mathcal{X} is locally trivial and X is connected, the stratified fibres $\pi_X^{-1}(x_j) \subseteq \pi_X^{-1}(X_d^o)$ are all pairwise homeomorphic and this also holds for every pair of stratified links $L_X(x_j, d) \cong L_X(x_{j'}, d)$.

For every cell $g(\tau) \in X_d^o$ ($\tau \in \mathcal{B}_X$) having maximal dimension l , let U be an open neighbourhood in X of $g(\bar{\tau}) \subseteq X_d^o$ which is a domain of a chart $\varphi : \mathbb{R}^l \times 0^k \rightarrow U$ of X centered at a vertex $x_r \in g(\bar{\tau})$.

It is not restrictive to suppose that U is one of the $U_{x_j'}$ covering X_d^o in the formula (*) at begin of the proof of the Theorem and changing (possibly) the origin $x_r = x_0$ and so to suppose $U = U_{x_0}$.

By Theorem 7 in [MPT] there exists a local topological trivialization H_{x_0} of X over $U \cong \mathbb{R}^l \times 0^k$ defined in the local coordinates of U by :

$$H_{x_0} : U \times \pi_X^{-1}(x_0) \longrightarrow \pi_X^{-1}(U), \quad H_{x_0}(t_1, \dots, t_l, z_0) = \phi_l(t_l, \dots, \phi_1(t_1, z_0) \dots)$$

and satisfying the smooth version of the Whitney fibering conjecture.

That is, H_{x_0} induces an (a)-regular foliation:

$$\mathcal{F}_{x_0} = \left\{ F_{z_0} = H_{x_0}(U \times \{z_0\}) \right\}_{z_0 \in \pi_X^{-1}(x_0)} \quad \text{satisfying} \quad (3) : \quad \lim_{z \rightarrow x} T_z F_z = T_x X \quad \forall x \in U,$$

whose tangent spaces to each F_z (for each $z = H_{x_0}(t_1, \dots, t_l, z_0)$) are generated by the frame fields (w_1, \dots, w_l) where $w_i(z) = H_{x_0 *}(t_1, \dots, t_l, z_0)(E_i)$.

Moreover, by Theorem 12, §8 in [MPT]), H_{x_0} is a *horizontally- C^1* and a semidifferentiable homeomorphism rather than just a homeomorphism (see also §2.2 for more recalls). Then, for every $Y > X$ and $y \in Y$ there exists a neighbourhood V of y in $\pi_{XY}^{-1}(U)$ such that the foliation

$$\mathcal{F}_{x_0, Y} = \left\{ F_{z_0, Y} = H_{x_0}(V \times \{z_0\}) \right\}_{z_0 \in \pi_{XY}^{-1}(x_0)} \quad \text{satisfies} \quad (4) : \quad \lim_{z \rightarrow y} T_z F_{z, Y} = T_y Y, \quad \forall y \in \pi_{XY}^{-1}(U).$$

Also for every x_j vertex of the cell $g(\bar{\tau}) \in g(\mathcal{B}_X)$, H_{x_0} induces by restriction a stratified homeomorphism H_{x_0, x_j} between the fibers and by restriction between the links :

$$\begin{array}{ccc} \pi_X^{-1}(x_0) & \xrightarrow{H_{x_0, x_j}} & \pi_X^{-1}(x_j) \\ \cup & & \cup \\ L_X(x_0, d) & \xrightarrow{H_{x_0, x_j}} & L_X(x_j, d) \end{array} \quad \text{defined by:} \quad H_{x_0, x_j}(t_1^j, \dots, t_l^j, z_0) = \phi_l(t_l^j, \dots, \phi_1(t_1^j, z_0) \dots)$$

where $(t_1^j, \dots, t_l^j) \equiv x_j$ are the local coordinates over U of x_j and $z_0 \in L_X(x_0, d) \subseteq \pi_X^{-1}(x_0)$.

Let be $S_U(d) := S_X(d) \cup \pi_X^{-1}(U)$ and $\psi := \psi_U : S_U(d) \times [0, +\infty[\rightarrow T_X(d)$ the restriction of the flow of $-\nabla \rho_X(z)$. Then, for every $x \in X$, $\lim_{z \rightarrow x \in X} -\nabla \rho_X(z) = 0$ and $\lim_{(z, t) \rightarrow (x, +\infty)} \psi(z, t) = x$.

We re-parametrize now the flow $\psi : S_U(d) \times [0, +\infty[\rightarrow T_X(d)$ of the vector field $-\nabla \rho_X$ by a map $r_U : S_U(d) \times]0, 1[\rightarrow T_X(d)$ asking that $r_U(z, t) = L_z \cap S_X(t)$.

For every fixed point $z \in S_U(d)$ and $t \in]0, d[$ there is a unique time $s \in [0, +\infty[$ in which the trajectory $L_z = \psi_z([0, +\infty[$ meets $S_X(t)$ and it satisfies $\rho_X(\psi_z(s)) = \rho_X(\psi_s(z)) = t$.

There is hence a unique decreasing diffeomorphism $\gamma :]0, d[\rightarrow [0, +\infty[$, $\gamma(t) = s$ making the following diagram commutative

$$\begin{array}{ccccc} S_U(d) \times \{t\} & \hookrightarrow & S_U(d) \times]0, d[& \xrightarrow{r_U^t} & S_U(t) \\ \downarrow \mathbb{1}_{S_U(d)} \times \gamma & & \downarrow \mathbb{1}_{S_U(d)} \times \gamma & \nearrow \psi_{\gamma(t)} & \\ S_U(d) \times \{\gamma(t)\} & \hookrightarrow & S_U(d) \times [0, +\infty[& & \end{array}$$

and which allows us to re-interpret r_U^t as a level preserving stratified homeomorphism :

$$r_U^t : S_U(d) \times \{t\} \longrightarrow S_U(t), \quad r_U^t(z) = \psi(z, \gamma(t)) = \psi_z(\gamma(t))$$

where γ satisfies $\gamma(d) = 0$ (since $\psi(z, 0) = z$ and $z \in S_U(d)$ implies $r_U^d(z) = z$).

Since $\pi_X : g(\mathcal{A}_X) \rightarrow g(\mathcal{B}_X)$ is a cellular map the link $L_X(x_0, d)$ is a cellular sub-complex of $g(\mathcal{A}_X)$ (see also Remark 5) and recall moreover by (1) of step 1) that $Z = \cup_{X \in \Sigma, \dim X = n} X$ ($n = \dim \mathcal{X}$) so $L_X(x_0, d) \subseteq S_X(d) \cap Z \subseteq Z$

Then for every $\sigma \in \mathcal{A}_X$ and open cell $g(\sigma) \subseteq L_X(x_0, d) \subseteq Z$ of maximal dimension ($n - l - 1$) of the stratified link $L_X(x_0, d) = \sqcup_{Y > X} L_{XY}(x_0, d)$, the *open cone over $g(\sigma)$* defined (using the notations of §2.2) by

$$C_{g(\sigma)} := \bigsqcup_{z_0 \in g(\sigma)} L_{z_0} = \psi(g(\sigma) \times]0, +\infty[) \cong g(\sigma) \times]0, +\infty[\cong g(\sigma) \times]0, d[$$

is an open $(n - l)$ -cell of the fiber $\pi_{XZ}^{-1}(x_0)$ vertically foliated by the (b^π) -regular trajectories of the vector field $-\nabla \rho_X$ (see §2.2).

Similarly we have the following partition in open cells :

$$C_{g(\bar{\sigma})} := \bigsqcup_{z_0 \in g(\bar{\sigma})} L_{z_0} = \bigsqcup_{g(\sigma') \leq g(\sigma), z_0 \in g(\sigma')} L_{z_0} = \bigsqcup_{g(\sigma') \leq g(\sigma)} \bigsqcup_{z_0 \in g(\sigma')} L_{z_0} = \bigsqcup_{g(\sigma') \leq g(\sigma)} C_{g(\sigma')}$$

where for every $\sigma' \leq \sigma$, $\dim \sigma' = h$, the cone $C_{g(\sigma')}$ over $g(\sigma')$ is an $(h + 1)$ open cell, such that :

i) $C_{g(\sigma')} \subset C_{g(\sigma)}$ for every $\sigma' < \sigma$;

ii) $C_{g(\sigma')} \subseteq \pi_{XY}^{-1}(x_0) \cap T_{XY}(d)$, for every $g(\sigma') \subseteq L_{XY}(x_0, d)$ with $Y > X$.

iii) For every open cell $\sigma \in \mathcal{A}_X$ and $z_0 \in g(\sigma)$, let $\overline{L_{z_0}} = \{z_0\} \sqcup L_{z_0} \sqcup \{x_0\}$ be the closure of L_{z_0} in A . Then for every $\sigma' \leq \sigma$ the *upper and lower closed cylinder over σ'* , defined by

$$\tilde{C}_{g(\sigma')} := \psi(g(\sigma') \times [0, +\infty[) \sqcup \pi_X(g(\sigma')) = \bigsqcup_{z_0 \in g(\sigma')} \overline{L_{z_0}} = \bigsqcup_{z_0 \in g(\sigma')} \{z_0\} \sqcup L_{z_0} \sqcup \{x_0\}$$

is naturally stratified by :

$$\begin{array}{ccccc} \tilde{C}_{g(\sigma')} & = & g(\sigma') & \sqcup & C_{g(\sigma')} & \sqcup & \{\pi_X(g(\sigma'))\} = \{x_0\} \\ & & \cap & & \cap & & \cap \\ & & S_X(d) & \sqcup & T_X(d) - X & \sqcup & X \end{array}$$

By using the corresponding notations for $\tilde{C}_{g(\bar{\sigma})} := \bigsqcup_{z_0 \in g(\bar{\sigma})} \{z_0\} \sqcup L_{z_0} \sqcup \{x_0\}$ one finds that the product $g(\bar{\sigma}) \times \tilde{C}_{g(\bar{\sigma})}$ is a closed n -cell contained in $U \times \pi_X^{-1}(x_0)$, and then by image of the stratified homeomorphism H_{x_0} , one obtains that :

$$H_{x_0}(g(\bar{\sigma}) \times \tilde{C}_{g(\bar{\sigma})}) \subseteq \pi_X^{-1}(U) \quad \text{is a closed } n\text{-cell}$$

admitting the following natural stratification in open cells (since $\pi_X(g(\sigma)) = \pi_X(g(\sigma')) = \{x_0\}$):

$$\begin{aligned}
(5) : \quad H_{x_0}(g(\bar{\tau}) \times \tilde{C}_{g(\bar{\sigma})}) &= \bigsqcup_{g(\tau') \leq g(\tau) \subseteq X} \bigsqcup_{g(\sigma') \leq g(\sigma) \subseteq Lk(x_0, d)} H_{x_0}(g(\tau') \times \tilde{C}_{g(\sigma')}) \\
&= \bigsqcup_{g(\tau') \leq g(\tau) \subseteq X} \bigsqcup_{g(\sigma') \leq g(\sigma) \subseteq Lk(x_0, d)} H_{x_0}\left(g(\tau') \times \left(g(\sigma') \sqcup C_{g(\sigma')} \sqcup \{x_0\}\right)\right) \\
&= \bigsqcup_{\substack{g(\tau') \leq g(\tau) \subseteq X \\ g(\sigma') \leq g(\sigma) \subseteq Lk(x_0, d)}} H_{x_0}\left(g(\tau') \times g(\sigma')\right) \sqcup H_{x_0}\left(g(\tau') \times C_{g(\sigma')}\right) \sqcup H_{x_0}\left(g(\tau') \times \{x_0\}\right).
\end{aligned}$$

which is a refinement of the stratification of $\pi_X^{-1}(g(\bar{\tau})) = \sqcup_{Y \geq X} \pi_{XY}^{-1}(g(\bar{\tau}))$. We call each cell $H_{x_0}(g(\tau') \times C_{g(\sigma')})$ the *open cylinder determined by $g(\tau') \subseteq g(\bar{\tau})$ and by $C_{g(\sigma')} \subseteq \pi_X^{-1}(x_0)$* .

Lemma 2. *If $\{x_0, \dots, x_r\}$ are the vertices of $g(\bar{\tau})$ and $g(\sigma'_j) := H_{x_0}(x_j, g(\sigma'))$, then*

$$H_{x_0}(g(\tau') \times C_{g(\sigma')}) = H_{x_j}([g(\tau') - x_j] \times C_{g(\sigma'_j)}) \quad \text{for every } x_j = (t_1^j, \dots, t_l^j) \in U.$$

Thus each open cylinder $H_{x_j}(g(\tau') \times C_{g(\sigma')})$ does not depend on the vertex x_j chosen as origin of the topological trivialisation H_{x_j} to define it.

Proof. Thanks to the solution of the Whitney fibering conjecture [MPT] the frame fields (w_1, \dots, w_l) whose flows (ϕ_1, \dots, ϕ_l) define H_{x_0} , by $H_{x_0}(t_1, \dots, t_l, z_0) = \phi_l(t_l, \dots, \phi_1(t_1, z_0))$ and H_{x_j} by the same formula, satisfy $[w_i, w_j] = 0$ and so the flows ϕ_1, \dots, ϕ_l commute [MPT].

It follows that for every $t = (t_1, \dots, t_l) \in U$ one has

$$H_{x_0}(t, z) = H_{x_0}(x_j + (t - x_j), z) = H_{x_j}(t - x_j, z_j) \quad \text{where } z_j := H_{x_0}(x_j, z) \in \pi_X^{-1}(x_j)$$

i.e. the diagram below topological trivializations H_{x_0} and H_{x_j}

$$\begin{array}{ccc}
U \times \pi_X^{-1}(x_0) & \xrightarrow{H_{x_0}} & \pi_X^{-1}(U) \\
H_j \downarrow & & \downarrow id \\
U \times \pi_X^{-1}(x_j) & \xrightarrow{H_{x_j}} & \pi_X^{-1}(U).
\end{array}$$

is commutative where $H_j(t, z) := H_{x_0}(t - x_j, z)$.

Hence, since all vertices $\{x_0, \dots, x_r\}$ of $g(\bar{\tau})$ are contained in the same domain $U \subseteq X$ of topological triviality, one easily find :

- i) if $z \in \pi_X^{-1}(x_0)$ then $z_j := H_{x_0}(x_j, z) \in \pi_X^{-1}(x_j)$;
- ii) if $g(\sigma') \subseteq L_X(x_0, d)$ then $g(\sigma'_j) := H_{x_0}(x_j, g(\sigma')) \subseteq L_X(x_j, d)$;
- iii) if $z_0 \in L_X(x_0, d)$ then $L_{z_0}^j := H_{x_0}(\{x_j\} \times L_{z_0})$ is a line of $-\nabla \rho_X$ starting from z_0^j .
- iv) $C_{g(\sigma'_j)} = H_{x_0}(\{x_j\} \times C_{g(\sigma')})$ is the open cone over $g(\sigma'_j)$.
- v) $H_{x_0}(g(\tau') \times C_{g(\sigma')}) = H_{x_j}((g(\tau') - x_j) \times C_{g(\sigma'_j)})$. \square

Step 3. The stratification of each closed cell $H_{x_0}(g(\bar{\tau}) \times \tilde{C}_{g(\bar{\sigma})})$ is (b)-regular.

The closed cell $H_{x_0}(g(\bar{\tau}) \times \tilde{C}_{g(\bar{\sigma})})$ has three different types of strata (open cells) :

- 1) $H_{x_0}(g(\tau') \times g(\sigma')) \subseteq S_X(d)$;
- 2) $H_{x_0}(g(\tau') \times C_{g(\sigma')}) \subseteq T_X(d) - X$;
- 3) $H_{x_0}(g(\tau') \times \{x_0\}) \subseteq X$,

where $g(\tau') \leq g(\tau) \subseteq X_d^o$ and $g(\sigma') \leq g(\sigma) \subseteq L(x_0, d)$.

Two cells of type 1) are both contained in the stratification $\sqcup_{Y>X} S_{XY}(d) - T_{i-2}(d) \subseteq g(A_X)$ so they satisfy (b)-regularity thanks to the inductive hypothesis on $g = g_i$. The same reason proves (b)-regularity of any two cells of type 3) contained this time in $X_d^o \subseteq g(B_X)$.

So it remains to prove (b)-regularity of two adjacent cells one at least of which is of type 2) and this reduces the proof of (b)-regularity to the case where the bigger stratum is $H_{x_0}(g(\tau) \times C_{g(\sigma)})$.

We have then the following five cases corresponding via H_{x_0} to the following adjacencies ;

$$(6) : \begin{array}{ccc} g(\tau) \times g(\sigma) & & g(\tau') \times g(\sigma') \\ \wedge & \nearrow & \\ g(\tau) \times C_{g(\sigma)} & > & g(\tau') \times C_{g(\sigma')} \\ \vee & \searrow & \\ g(\tau) \times \{x_0\} & & g(\tau') \times \{x_0\} , \end{array}$$

and since the restriction $H_{x_0|U} = id$, is the identity over U , corresponding to the adjacencies in the Figure 7 below.

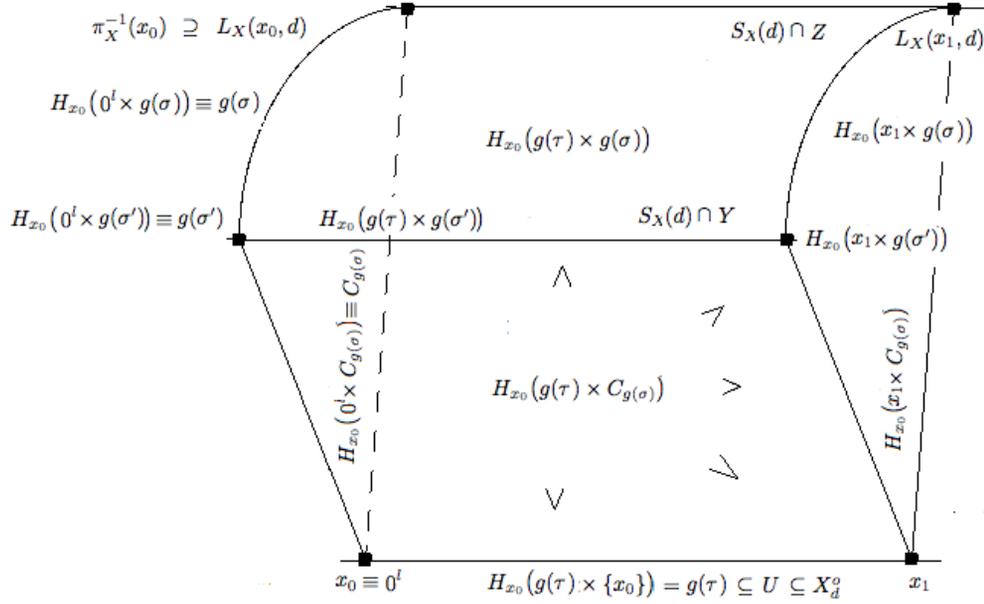


Figure 7

Figure 7 represents a closed stratified mapping cylinder over a cell $g(\bar{\tau})$ in the three strata cases $X^{(1)} < Y^{(2)} < Z^{(3)} \subseteq \mathbb{R}^3$ where $g(\bar{\tau})$ is stratified by $g(\tau)$ and two vertices $x_0, x_1 = g(\tau')$ and $\dim L(x_0, d) = 1$ and it is stratified by an arc $g(\sigma)$ and two points one of which is $g(\sigma')$.

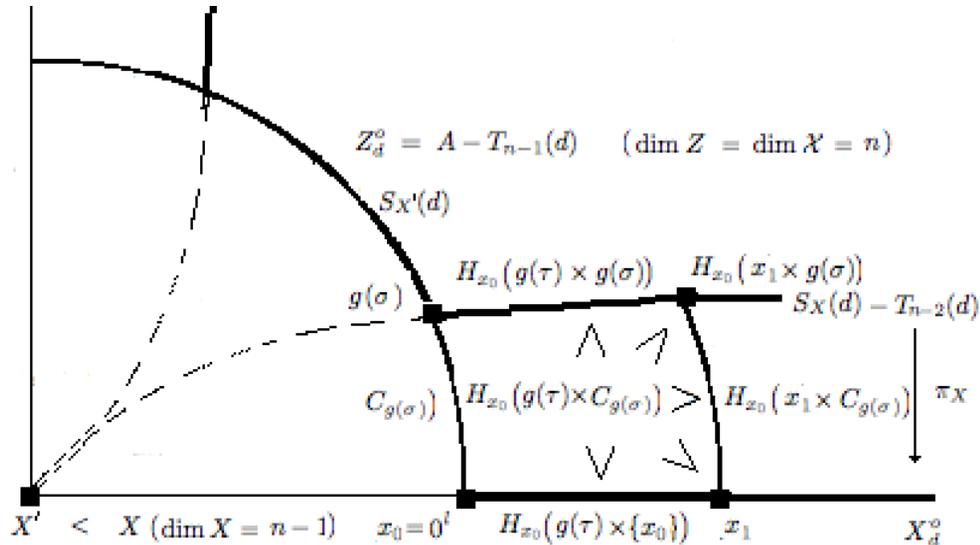


Figure 8

Figure 8 represents a closed stratified mapping cylinder over a cell $g(\bar{\tau})$ in the two strata cases $X^{(1)} < Z^{(2)} \subseteq \mathbb{R}^2$ where $g(\bar{\tau})$ is stratified by $g(\tau)$ and two vertices $x_0, x_1 = g(\tau')$ and $\dim L(x_0, d) = 0$ and it is stratified by a point $g(\sigma)$.

Now the (b)-regularity of the incidence relation of the image via H_{x_0} of the two lower lines of diagram (6) :

$$\begin{array}{ccc} H_{x_0}(g(\tau) \times C_{g(\sigma)}) & > & H_{x_0}(g(\tau') \times C_{g(\sigma')}) \\ \vee & \Delta & \\ H_{x_0}(g(\tau) \times \{x_0\}) & > & H_{x_0}(g(\tau') \times \{x_0\}) \end{array}$$

follows (using the solution of the Whitney fibering conjecture [MPT]) by Corollary 3 of §2.2 by taking $R = g(\tau')$, $S = g(\tau)$ and $N' = g(\sigma')$ and $N = g(\sigma)$.

The (b)-regularity of the incidence relation of the image via H_{x_0} of the two upper lines of diagram (6) :

$$\begin{array}{ccc} H_{x_0}(g(\tau) \times g(\sigma)) & > & H_{x_0}(g(\tau') \times g(\sigma')) \\ \wedge & \nabla & \\ H_{x_0}(g(\tau) \times C_{g(\sigma)}) & > & H_{x_0}(g(\tau') \times C_{g(\sigma')}) \end{array}$$

is obtained as follows.

Since $g(\sigma) \sqcup C_{g(\sigma)}$ is a C^1 manifold with boundary $g(\sigma)$ then $g(\sigma) < C_{g(\sigma)}$ is (b) -regular and so $g(\tau) \times g(\sigma) < g(\tau) \times C_{g(\sigma)}$ is too. Moreover the restriction of H_{x_0} to the stratum of \mathcal{X} containing $g(\sigma) \sqcup C_{g(\sigma)}$ being a C^1 -diffeomorphism then

$$H_{x_0}(g(\tau) \times g(\sigma)) < H_{x_0}(g(\tau) \times C_{g(\sigma)})$$

is also (b) -regular by [Tr].

The proof that $H_{x_0}(g(\tau) \times C_{g(\sigma)}) > H_{x_0}(g(\tau') \times g(\sigma'))$ is (b) -regular is similar and easier than the (b) -regularity of $H_{x_0}(g(\tau) \times C_{g(\sigma)}) > H_{x_0}(g(\tau') \times g(\sigma'))$. \square

Step 4) : Definition of a cellulation $G : \mathcal{C} \longrightarrow \overline{T_{i-1}(d)} - \overline{T_{i-2}(d)}$.

Let us consider the closed linear *cellular cone* $\mathcal{C}_{\bar{\sigma}} := \{t \cdot q \mid q \in \bar{\sigma}, t \in [0, d]\}$ over $\bar{\sigma}$ and the *cylinder* of $\mathcal{C}_{\bar{\sigma}}$ over $\bar{\tau}$, defined by $\mathcal{C}_{\bar{\tau}, \bar{\sigma}} := \bar{\tau} \times \mathcal{C}_{\bar{\sigma}}$ with their natural stratifications in open cells.

Let us denote moreover (with obvious meaning of the symbols) by $C_{\bar{\sigma}}, C_{\bar{\tau}, \bar{\sigma}}$ their supports and by $C_{\sigma}, C_{\tau, \sigma}$ the supports of their corresponding open cells.

Then the map

$$G_{\bar{\tau}, \bar{\sigma}} : \mathcal{C}_{\bar{\tau}, \bar{\sigma}} := \bar{\tau} \times \mathcal{C}_{\bar{\sigma}} \longrightarrow H_{x_0}(g(\bar{\tau}) \times C_{g(\bar{\sigma})}) \subseteq \pi_X^{-1}(g(\bar{\tau})), \quad G_{\bar{\tau}, \bar{\sigma}}(p, t \cdot q) = H_{x_0}(g(p), r_U^t(g(q)))$$

is a stratified homeomorphism which is a cellulation of $H_{x_0}(g(\bar{\tau}) \times C_{g(\bar{\sigma})})$.

Moreover since $\pi_X^{-1}(x_0) = \cup_{\bar{\sigma} \in L(x_0, d)} C_{g(\bar{\sigma})}$, using the same trivialization H_{x_0} , one has

$$\pi_X^{-1}(g(\bar{\tau})) = H_{x_0}(g(\bar{\tau}) \times \pi_X^{-1}(x_0)) = \bigcup_{g(\bar{\sigma}) \subseteq L_X(x_0, d)} H_{x_0}(g(\bar{\tau}) \times C_{g(\bar{\sigma})})$$

and by considering the cellular complex union : $\mathcal{C}_{\bar{\tau}} = \bigcup_{\bar{\sigma}} \mathcal{C}_{\bar{\tau}, \bar{\sigma}}$ (where $g(\bar{\sigma}) \subseteq L_X(x_0, d)$) we find a cellulation $G_{\bar{\tau}} : \mathcal{C}_{\bar{\tau}} \longrightarrow \pi_X^{-1}(g(\bar{\tau}))$ of $\pi_X^{-1}(g(\bar{\tau}))$.

Then all linear closed cell complexes of the family $\{\mathcal{C}_{\bar{\tau}} = \bigcup_{g(\bar{\sigma}) \subseteq L_X(x_0, d)} \mathcal{C}_{\bar{\tau}, \bar{\sigma}}\}_{\bar{\tau} \subseteq B_X, \dim \bar{\tau} = l}$, each of which defines a cellulation of $\pi_X^{-1}(g(\bar{\tau}))$, glue together into a unique (abstract) cell complex by the equivalence relation identifying the points having the same image via some H_{x_j} $j = 0, \dots, \alpha$ in A . We obtain thus the cell complex $\mathcal{C}_X := \bigcup_{\bar{\tau}} \mathcal{C}_{\bar{\tau}} / \equiv$ (where $g(\bar{\tau}) \subseteq X_d^o$, and $\dim g(\bar{\tau}) = l$), where the equivalence \equiv for every $(p, q) \in \mathcal{C}_{\bar{\tau}}$ and every $(p', q') \in \mathcal{C}_{\bar{\tau}'}$ is defined by :

$$(p, q) \equiv (p', q') \iff \exists i, j \in \{0, \dots, \alpha\} : \begin{cases} (p, q) \in \mathcal{C}_{\bar{\tau}, \bar{\sigma}} & \text{with } g(\bar{\tau}) \subseteq U_{x_i}, \\ (p', q') \in \mathcal{C}_{\bar{\tau}', \bar{\sigma}'} & \text{with } g(\bar{\tau}') \subseteq U_{x_j}, \\ G_{\bar{\tau}, \bar{\sigma}}(p, q) = G_{\bar{\tau}', \bar{\sigma}'}(p', q') & \text{in } \pi_X^{-1}(U_{x_i} \cap U_{x_j}). \end{cases}$$

In this way, all maps $G_{\bar{\tau}, \bar{\sigma}}$ (or equivalently all $G_{\bar{\tau}}$) glue together defining a map

$$G_X : \mathcal{C}_X \longrightarrow \bigcup_{g(\bar{\tau}) \subseteq X_d^o, \dim g(\bar{\tau}) = l} \pi_X^{-1}(g(\bar{\tau}))$$

which is a stratified homeomorphism and a cellulation of

$$\bigcup_{g(\bar{\tau}) \subseteq X_d^o, \dim g(\bar{\tau})=l} \pi_X^{-1}(g(\bar{\tau})) = \pi_X^{-1}(X_d^o) = \overline{T_X(d) - T_{i-2}(d)}.$$

Remark 8. Since

$$\begin{cases} g(\bar{\sigma}) \subseteq L_X(x_0, d) \iff \bar{\sigma} \subseteq g^{-1}(L_X(x_0, d)) \subseteq A_X \iff C_{\bar{\sigma}} = (\bar{\sigma} \times [0, d])_{/(q,0) \equiv \pi(q)} \\ g(\bar{\tau}) \subseteq X_d^o \iff \bar{\tau} \subseteq g^{-1}(X_d^o) = B_X \iff \tau \in \mathcal{B}_X, \end{cases}$$

then one has the homeomorphism of cellular complexes :

$$\mathcal{C}_X := \bigcup_{\bar{\tau} \in \mathcal{B}_X} \mathcal{C}_{\bar{\tau}} = \bigcup_{\tau \in \mathcal{B}_X} \bigcup_{\pi(\sigma) \leq \tau} \bar{\tau} \times C_{\bar{\sigma}} = \bigcup_{\tau \in \mathcal{B}_X} \bigcup_{\pi(\sigma) \leq \tau} (\bar{\sigma} \times [0, d])_{/(q,0) \equiv \pi(q)} \cong (\mathcal{A}_X \times [0, d] \cup \mathcal{B}_X)_{/(q,0) \equiv \pi(q)}.$$

Finally taking X such that $\dim X = l = i - 1$, the union map defined on $\mathcal{C} := \bigsqcup_{\dim X = i-1} \mathcal{C}_X$:

$$G := \bigsqcup_{\dim X = i-1} G_X : \mathcal{C} \longrightarrow \overline{T_{i-1}(d) - T_{i-2}(d)}$$

fills by open cells the closed subset

$$\bigsqcup_{\dim X = i-1} \left(\overline{T_X(d) - T_{i-2}(d)} \right) = \overline{T_{i-1}(d) - T_{i-2}(d)}$$

and defines a cellular homeomorphism which is C^1 on each stratum of \mathcal{X} and induces by image the cellular stratification $G(\mathcal{C})$ of $\overline{T_{i-1}(d) - T_{i-2}(d)}$. \square

In analogy with the notations of Goresky, by setting $\mathcal{W} := g(\mathcal{A}_X)$ and $\mathcal{W}' := \pi_X(\mathcal{W}) = g(\mathcal{B}_X)$, the cellular complex $G(\mathcal{C})$ coincides with the stratified mapping cylinder $C_{\mathcal{W}'}(\mathcal{W})$ whose largest dimensional closed cells are those of the union below :

$$C_{\mathcal{W}'}(\mathcal{W}) := G(\mathcal{C}) = \bigsqcup_{\dim X = i-1} \bigcup_{\substack{\dim g(\bar{\tau})=l \\ g(\bar{\tau}) \subseteq U_{x_j} \subseteq X_d^o}} \bigcup_{\substack{\dim g(\bar{\sigma})=n-l-1 \\ g(\bar{\sigma}) \subseteq L_X(x_j, d)}} H_{x_j}(g(\bar{\tau}) \times C_{g(\bar{\sigma})})$$

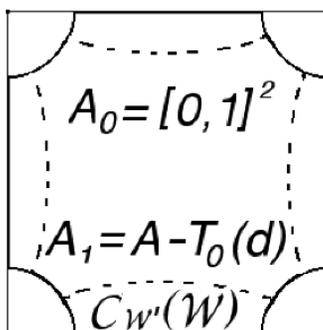


Figure 8

Remark 9. Since the cellular complex \mathcal{J}_i has a (b) -regular natural stratification in open cells and by property 2) of the inductive hypothesis the map $g_i = g : J_i \rightarrow f(K) \cup A_i \subseteq A$ is a C^1 -embedding on each stratum of \mathcal{X} , then g_i induces on its image $f(K) \cup A_i$ a (b) -regular stratification in cells. Hence, the partitions in cells \mathcal{W} and \mathcal{W}' below $\subseteq f(K) \cup A_i$ inherit two Whitney cellulations:

$$\left\{ \begin{array}{l} \mathcal{W} \quad \text{a } (b)\text{-regular cellulation of} \quad \bigsqcup_{\dim X=i-1} S_X(d) - T_{i-2}(d) \\ \text{and of its projection} \\ \mathcal{W}' \quad \text{a } (b)\text{-regular cellulation of} \quad \bigsqcup_{\dim X=i-1} X - T_{i-2}(d) = \bigsqcup_{\dim X=i-1} X_d^o . \end{array} \right.$$

Remark 10. Every closed cell $H_{x_j}(g(\bar{\tau}) \times C_{g(\bar{\sigma})})$ with $g(\bar{\tau}) \subseteq U_{x_j} \subseteq X_d^o$ is the image of the cellular complex $\bar{\tau} \times C_{\bar{\sigma}}$ via the composition map

$$\begin{aligned} H_{x_j} \circ F_{\bar{\tau}, \bar{\sigma}} : \bar{\tau} \times C_{\bar{\sigma}} &\xrightarrow{F_{\bar{\tau}, \bar{\sigma}}} g(\bar{\tau}) \times C_{g(\bar{\sigma})} \xrightarrow{H_{x_j}} H_{x_j}(g(\bar{\tau}) \times C_{g(\bar{\sigma})}) \subseteq \pi_X^{-1}(g(\bar{\tau})) \subseteq \pi_X^{-1}(U_{x_j}) \\ (p, t \cdot q) &\longrightarrow (g(p), r_{U_{x_j}}^t(g(q))) \longrightarrow H_{x_j}(g(p), r_{U_{x_j}}^t(g(q))) . \end{aligned}$$

Since the cellular complex $\bar{\tau} \times C_{\bar{\sigma}}$ with its natural stratification in open cells is obviously (b) -regular, also the map $g = g_i$ is (by induction) a C^1 -embedding, $r_{U_{x_j}}^t$ and H_{x_j} are C^1 on each stratum and H_{x_j} is horizontally- C^1 near each adjacency relation, it follows that the closed cell $H_{x_j}(g(\bar{\tau}) \times C_{g(\bar{\sigma})})$ with its induced stratification is (b) -regular (as we saw in Step 3).

Step 5) : End of the induction and of the proof of Theorem 6 of Whitney cellulation.

Let us denote by $\tilde{\pi}_{i-1}$ the disjoint union of the cellular maps $\{\tilde{\pi}_X^{i-1} : \mathcal{A}_X \rightarrow \mathcal{B}_X\}_{\dim X=i-1}$:

$$\tilde{\pi}_{i-1} := \bigsqcup_{\dim X=i-1} \tilde{\pi}_X^{i-1} : \bigsqcup_{\dim X=i-1} \mathcal{A}_X \longrightarrow \bigsqcup_{\dim X=i-1} \mathcal{B}_X$$

By gluing the two polyhedra $J_i = |J_i|$ and $C = |C|$ and their corresponding cellular complexes \mathcal{J}_i and \mathcal{C} by identifying their points and subpolyhedra $p \in P \subseteq J_i$ and $p' \in P' \subseteq C$ and their sub-complexes $\mathcal{P} \subseteq \mathcal{J}_i$ and $\mathcal{P}' \subseteq \mathcal{C}$ having the same images via g_i and G :

$$p \equiv p' \iff g_i(p) = G(p') \quad \text{in} \quad \bigsqcup_{\dim X=i-1} (S_X(d) \sqcup X_d^o)$$

one finds a new polyhedron $J_{i-1} := J_i \sqcup_{P \equiv P'} C$ and a new cellular complex $\mathcal{J}_{i-1} := \mathcal{J}_i \sqcup_{\mathcal{P} \equiv \mathcal{P}'} \mathcal{C}$ and one defines a cellulation g_{i-1} extending g_i by :

$$g_{i-1} := g_i \sqcup_{P \equiv P'} G : J_{i-1} := J_i \sqcup_{P \equiv P'} C \longrightarrow g_i(J_i) \cup G(C) .$$

By the inductive hypothesis and the equality (2) in Step 1 the image of g_{i-1} is :

$$g_{i-1}(J_i \sqcup_{P \equiv P'} C) = g_i(J_i) \cup G(C) = [f(K) \cup A_i] \cup [T_{i-1}(d) - T_{i-2}(d)] = f(K) \cup A_{i-1}$$

and hence the induced stratification in open cells on A_{i-1} satisfies 1) of the inductive hypothesis, moreover 2) and 3) are satisfied by the construction of the polyhedron \mathcal{J}_{i-1} and finally the cell stratification $\Sigma_{\mathcal{J}_{i-1}} := \{g_{i-1}(C) \mid C \in \mathcal{J}_{i-1}\}$ has all incidence relations which are (b)-regular thanks to Step 3 and so g_{i-1} satisfies all the inductive hypotheses 1), ..., 4) and defines a Whitney cellulation of $f(K) \cup A_{i-1}$.

This completes our proof by induction and so the final map $g_0 : J_0 \rightarrow A$ defines the desired Whitney cellulation $\mathcal{X}' := g_0(\mathcal{J}_0)$ of \mathcal{X} . \square

By the proof of Theorem 6 one can deduce the following Corollaries :

Corollary 4. *Let $\mathcal{X} = (A, \Sigma)$ be a compact Whitney stratified set in \mathbb{R}^m .*

1) *There exists $\epsilon > 0$, such that $\forall \delta \in]0, \epsilon]$, \mathcal{X} has a Whitney cellulation \mathcal{X}' having radius $< \delta$.*

2) *For every open covering $\mathcal{U} = \{U_i\}_i$ of A , there exists a Whitney cellulation \mathcal{X}' of \mathcal{X} subordinated to \mathcal{U} : every (open) cell of \mathcal{X}' is contained in some open set U_i of \mathcal{U} .*

Remark 11. When $\mathcal{X} = (A, \Sigma)$ admits 0-dimensional strata in the last step of the induction each simplex $g(\bar{\tau}) \subseteq \mathcal{X}^0$ reduces to a simple point x_0 and so after smoothly triangulating each cell $g(\bar{\sigma}) \subseteq L_A(x_0, d)$, the mapping cylinder $\mathcal{C}_{g(\bar{\sigma})}$, for every $\sigma' < \sigma$ becomes (more than a cellular complex also) a simplicial complex.

Thus in the special case of stratifications having only isolated singularities one finds a Whitney triangulation. Remark that in this case (b)-regularity of \mathcal{X} reduces to (b^π) -regularity.

Corollary 5. *Each Whitney stratification having only 0-dimensional isolated singularities admits a Whitney triangulation.* \square

4. Whitney Homology and the Goresky representation conjecture.

4.1. Whitney homology.

In this section we recall the problem posed by Goresky in his thesis [Go]₁ *Geometric cohomology and homology of stratified objects* (1976) and later in his paper *Whitney stratified chains and cochains* (1981) [Go]₂.

In these works Goresky defines for every Whitney stratified set $\mathcal{X} = (A, \Sigma)$, a homology theory of sets $WH_*(\mathcal{X})$ whose cycles are Whitney substratified sets of \mathcal{X} and whose homologies are cobordisms of Whitney cycles in $\mathcal{X} \times [0, 1]$ and defines a *representation map* $R_k : WH_k(\mathcal{X}) \rightarrow H_k(\mathcal{X})$ corresponding to the Thom-Steenrod representation map between the differential bordism and the singular homology of a space. Then in a main Theorem (Theorem 3.4. [Go]₂) he proves that the map R_k is a bijection if the stratification \mathcal{X} is reduced to a single smooth closed manifold.

Despite the depth of his work Goresky does not obtain the bijectivity of the representation map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(\mathcal{X})$ in the case where \mathcal{X} is an arbitrary Whitney stratification ; so he poses the conjecture “*Theor. 3.4. may even be true if $\mathcal{X} \dots$ ” (1981, p.174) or again “*it is almost certainly true . . . that the map R_k is a bijection for an arbitrary Whitney stratification \mathcal{X} ”* (1976, p. 52).*

It is useful also to recall that :

i) Goresky introduces and develops a corresponding geometric cohomology theory of sets $WH^k(\mathcal{X})$ and maps $R^k : WH^k(\mathcal{X}) \rightarrow H^k(\mathcal{X})$, and in this case he proves the bijectivity of the cohomological representation R^k .

ii) Both Goresky's theories were reconsidered and slightly improved by the first author of the present paper in **[Mu]**_{1,2} through the introduction of a sum operation geometrically meaning "transverse union" of stratified cycles and cocycles with which the Goresky sets $WH_k(\mathcal{X})$ (when \mathcal{X} is a manifold) and $WH^k(\mathcal{X})$ (for every Whitney stratification \mathcal{X}) become groups and the representation maps R_k et R^k become group isomorphisms. Starting from **[Mu]**₁ one says *Whitney homology* to refer to the sets $WH_k(\mathcal{X})$ and uses *Goresky Representation* for the maps R_k .

The formal description below of Goresky's theory comes from **[Mu]**₁ (§2.1.).

§4.2. Goresky's representation conjecture.

The Goresky representation. A stratified k -subspace of a Whitney stratified set $\mathcal{X} = (A, \Sigma)$ is a Whitney stratified set $\mathcal{V} = (V, \Sigma_V)$ of dimension k which has every stratum contained in some stratum of \mathcal{X} .

A k -orientation of \mathcal{V} is an element $z = \sum_{j \in J_k} n_j V_j^k$ of the free abelian group $C_k(\mathcal{V})$ on \mathbb{Z} generated by the set of the oriented k -strata V_j^k of \mathcal{V} in which one identifies the elements with opposite orientations and multiplicity in \mathbb{Z} . With these hypotheses *the reduction of ξ* is defined as the new chain $\xi_j := (V_{/z}, z)$ obtained by restricting the support of \mathcal{V} to its essential part: i.e. by considering only the strata V_j^k adjacent to some V_j with maximal dimension k (hence $A \subseteq \overline{V_j}$) and multiplicity $n_j \neq 0$ in Z . Explicitly $V_{/z} := \bigcup_{n_j \neq 0} \overline{V_j} = \overline{\bigcup_{n_j \neq 0} V_j}$ with the obvious induced stratification. A k -cycle $\xi = (\mathcal{V}, z)$ is a chain whose boundary $\partial \xi$ is zero, where $\partial \xi$ is defined by the reduction $\partial \xi := (V_{k-1}, \partial z)_j$ and ∂z is given by the homological boundary operator through the natural isomorphisms ψ_k, ψ_{k-1} as in the diagram

$$\partial : C_k(V) \xrightarrow{\psi_k} H_k(V_k, V_{k-1}) \xrightarrow{\partial_k} H_{k-1}(V_{k-1}, V_{k-2}) \xrightarrow{\psi_{k-1}^{-1}} C_{k-1}(V_{k-1}).$$

Two Whitney k -cycles ξ, ξ' of \mathcal{X} are called *cobordant* if there exists a $(k+1)$ -chain ζ of $\mathcal{X} \times [0, 1]$ such that $\partial \zeta = \xi' \times \{1\} - \xi \times \{0\}$ and one writes : $\zeta : \xi \equiv \xi'$. This defines an equivalence relation (the *stratified cobordism*) on the class of all Whitney stratified k -cycles of \mathcal{X} .

Goresky introduces in **[Go]**_{1,2} the quotient set $WH_k(\mathcal{X})$ of the class of all k -cycles modulo stratified cobordism.

Fix now a k -cycle $\xi = (V, z)$; we have $\partial z = 0$ and therefore $\partial_k \psi_k(z) = 0$. Thus by the exactness of the pair (V_k, V_{k-1}) and looking at the diagram

$$\begin{array}{ccccccc} & & & & \partial_k & & \\ & & & & \longmapsto & & \\ & & & & \downarrow & & \\ H_k(V_{k-1}) & \rightarrow & H_k(V_k) & \xrightarrow{i_*} & H_k(V_k, V_{k-1}) & \xrightarrow{\partial} & H_{k-1}(V_{k-1}) & \xrightarrow{j_*} & H_{k-1}(V_{k-1}, V_{k-2}) \\ & & I_* \downarrow & & \psi_k \uparrow & & & & \uparrow \psi_{k-1} \\ & & H_k(X) & & C_k(V_k) & \xrightarrow{\partial} & C_{k-1}(V_{k-1}) & & \end{array}$$

we get $0 = \partial_k \psi_k(z) = j_* \partial \psi_k(z)$ and $\partial \psi_k(z) = 0$.

Hence $\psi_k(z) \in \text{Ker} \partial = \text{Im} i_*$ so it comes from a unique element $i_*^{-1}(\psi_k(z))$ whose image $R_k([\xi]) = I_* i_*^{-1}(\psi_k(z))$ in $H_k(A)$ depends only on the cobordism class $[\xi]_{\mathcal{X}} \in WH_k(\mathcal{X})$ [Go]_{1,2} and is called the *fundamental class* of ξ in $H_k(A)$.

The map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(A)$ is the analogue of the Steenrod map in the differentiable Thom cobordism theory, thus we call it the *Goresky representation map*.

Goresky [Go]_{1,2} proved the following:

Theorem 7. *If \mathcal{X} is the trivial stratification $\Sigma = \{M\}$ of a manifold M , the representation map*

$$R_k : WH_k(\mathcal{X}) \rightarrow H_k(A) \quad \text{is a bijection.} \quad \square$$

Thanks to our cellulation Theorem 6 we can now prove Goresky's Conjecture :

Theorem 8. *For every Whitney stratification $\mathcal{X} = (A, \Sigma)$ having finitely many strata the Goresky homology representation map $R_k : WH_k(\mathcal{X}) \rightarrow H_k(\mathcal{X})$ is a bijection for every $k \leq \dim \mathcal{X}$.*

Proof. The proof is similar to that of Goresky (Theorem 3.4., [Go]₂) for a manifold M using a smooth (so (b)-regular) triangulation of M . We will just give some slight modifications.

Since we consider a Whitney stratification \mathcal{X} instead of a smooth manifold M we will consider a Whitney cellulation \mathcal{X}' of \mathcal{X} , which exists thanks to Theorem 6, instead of a smooth triangulation of M and we use the cellular homology of \mathcal{X} instead of the simplicial homology of M .

Surjectivity. The cellular homology being isomorphic to the singular homology $H_k(A)$ of A , every $\alpha \in H_k(A)$ can be represented by a cellular stratified cycle ξ of \mathcal{X}' . Since the cellulation \mathcal{X}' is (b)-regular then the cobordism class $[\xi]_{\mathcal{X}}$ defines an element of $WH_k(\mathcal{X})$ such that $R_k([\xi]_{\mathcal{X}}) = \alpha$.

Injectivity. Let $\xi = (\mathcal{V}, z)$ and $\xi' = (\mathcal{V}', z')$ be two Whitney cycles of \mathcal{X} whose classes $[\xi]_{\mathcal{X}}$ and $[\xi']_{\mathcal{X}}$ represent the same homology class via R_k in $H_k(\mathcal{X})$: that is $R_k([\xi]_{\mathcal{X}}) = R_k([\xi']_{\mathcal{X}})$.

By considering the restratification \mathcal{H} of $A \times [0, 1]$ whose strata are those of the partition :

$$\left\{ \begin{array}{ll} \left(\mathcal{X} \times \{0\} - \mathcal{V} \times \{0\} \right) \sqcup \mathcal{V} \times \{0\} & \text{in } \mathcal{X} \times \{0\} \\ \mathcal{X} \times]0, 1[& \text{in } \mathcal{X} \times]0, 1[\\ \left(\mathcal{X} \times \{1\} - \mathcal{V}' \times \{1\} \right) \sqcup \mathcal{V}' \times \{1\} & \text{in } \mathcal{X} \times \{1\}, \end{array} \right.$$

it is easy to see that \mathcal{H} is a Whitney stratification and a refinement of $\mathcal{X} \times [0, 1]$.

Thanks to the Whitney cellulation Theorem 6, \mathcal{H} admits a Whitney cellulation \mathcal{H}' inducing on $\mathcal{V} \times \{0\}$ and $\mathcal{V}' \times \{1\}$ two refinements $\mathcal{W} \times \{0\}$ and $\mathcal{W}' \times \{1\}$.

Hence the Whitney substratified object \mathcal{W} and \mathcal{W}' of \mathcal{X} define two cycles $\eta = (\mathcal{W}, w)$ and $\eta' = (\mathcal{W}', w')$ of \mathcal{X} such that $[\eta]_{\mathcal{X}} = [\xi]_{\mathcal{X}}$ and $[\eta']_{\mathcal{X}} = [\xi']_{\mathcal{X}} \in WH_k(\mathcal{X})$ so that :

$$R_k([\eta]_{\mathcal{X}}) = R_k([\xi]_{\mathcal{X}}) = R_k([\xi']_{\mathcal{X}}) = R_k([\eta']_{\mathcal{X}}).$$

Since $R_k([\eta]_{\mathcal{X}}) = R_k([\eta']_{\mathcal{X}})$, the cellular homology cycles η et η' represent the same cellular homology class of \mathcal{X} . Then there exists a cellular homology \mathcal{Z} between \mathcal{W} et \mathcal{W}' which is a cellular subcomplex of \mathcal{H}' and since \mathcal{H}' is a (b)-regular cellulation, then \mathcal{Z} is (b)-regular too and it defines a substratified Whitney $(k+1)$ -chain $\zeta = (\mathcal{Z}, u)$ of $\mathcal{H}' \times \{1\}$ such that $\partial \zeta = \eta' \times \{1\} - \eta \times \{1\}$.

By embedding all these cycles and chains in $\mathcal{X} \times [0, 1]$, the $(k+1)$ -chain ζ' of $\mathcal{X} \times [0, 1]$ obtained by adding to ζ the stratified chain $\eta \times [0, 1]$, satisfies :

$$\partial\zeta' = \partial\zeta + \partial(\eta \times [0, 1]) = (\eta' \times \{1\} - \eta \times \{1\}) + (\eta \times \{1\} - \eta \times \{0\}) = \eta' \times \{1\} - \eta \times \{0\}$$

(modulo reduction) and it defines thus a Whitney cobordism $\zeta' : \eta \equiv \eta'$.

Hence we conclude that :

$$[\eta]_{\mathcal{X}} = [\eta']_{\mathcal{X}} \text{ in } WH_k(\mathcal{X}) \quad \text{and so :} \quad [\xi]_{\mathcal{X}} = [\xi']_{\mathcal{X}} \text{ in } WH_k(\mathcal{X}). \quad \square$$

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