



HAL
open science

On the achromatic number of signed graphs

Dimitri Lajou

► **To cite this version:**

Dimitri Lajou. On the achromatic number of signed graphs. Theoretical Computer Science, 2019. hal-02014469

HAL Id: hal-02014469

<https://hal.science/hal-02014469>

Submitted on 11 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the achromatic number of signed graphs

Dimitri Lajou*

February 11, 2019

Abstract

In this paper, we generalize the concept of complete coloring and achromatic number to 2-edge-colored graphs and signed graphs. We give some useful relationships between different possible definitions of such achromatic numbers and prove that computing any of them is NP-complete.

1 Introduction

All the graphs we consider are undirected and simple. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G , respectively. The *neighbourhood* $N_G(u)$ of a vertex u of G is the set of vertices which share an edge with u in G . A *2-edge-colored graph* is a graph where each edge of $E(G)$ can be either positive or negative. We denote by (G, C) such a graph, where G is an ordinary graph, called the *underlying graph* of (G, C) , and C is the set of negative edges, also called the *signature* of (G, C) .

A *signed graph* $[G, \Sigma]$ is an equivalence class on the set of 2-edge-colored graph (G, C) where $C \subseteq E(G)$. Two 2-edge-colored graphs (G, C) and (G, D) are *equivalent* if we can go from one to the other by a series of re-signings, where *re-signing* at a vertex v consists in inverting the sign of all the edges incident with v . A *representative* of a signed graph $[G, \Sigma]$ is a 2-edge-colored graph which belongs to $[G, \Sigma]$. A *signature* of a signed graph $[G, \Sigma]$ is a signature of one of its representatives. For the rest of this article, we will use the adjective *2-edge-colored* when the signature is fixed, and *signed* when re-signing is allowed.

We write $(G, C) \in [G, \Sigma]$ if (G, C) is one of the representatives of $[G, \Sigma]$. The *canonical representative* of $[G, \Sigma]$ is the 2-edge-colored graph (G, C) where $C = \Sigma$. Note that if $(G, C) \in [G, \Sigma]$ and $(G, C') \in [G, \Sigma]$, then $[G, \Sigma_C]$ and $[G, \Sigma_{C'}]$, where $\Sigma_C = C$ and $\Sigma_{C'} = C'$, are both equal to the signed graph $[G, \Sigma]$.

To avoid any possible confusion, signatures of 2-edge-colored graphs will be denoted by Roman letters while signatures of signed graphs will be denoted by Greek letters.

*ÉNS Lyon and Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France. Partially supported by the ANR project HOSIGRA (ANR-17-CE40-0022)

A k -(vertex-)coloring of a 2-edge-colored graph is a function $\alpha : V(G) \rightarrow \llbracket k \rrbracket$, where $\llbracket k \rrbracket$ denotes the set $\{1, \dots, k\}$, such that $\alpha(u) \neq \alpha(v)$ for every $uv \in E(G)$ and, for every two colors i and j , all the edges uv with $\alpha(u) = i$ and $\alpha(v) = j$ have the same sign. The *chromatic number* $\chi_2(G, C)$ of a 2-edge-colored graph (G, C) is the smallest k for which (G, C) admits a k -coloring. Similarly, the *chromatic number* $\chi_s[G, \Sigma]$ of a signed graph $[G, \Sigma]$ is the smallest k for which $[G, \Sigma]$ admits a representative (G, C) with $\chi_2(G, C) = k$. Alternatively, a k -(vertex-)coloring of a signed graph is a k -(vertex-)coloring of one of its representative and $\chi_s[G, \Sigma]$ is the smallest k for which a k -coloring of $[G, \Sigma]$ exists.

A (2-edge-colored) *homomorphism* of a 2-edge-colored graph (G, C) to a 2-edge-colored graph (H, D) is a function φ from $V(G)$ to $V(H)$ such that, for every pair of vertices $u, v \in V(G)$, $uv \in E(G)$ implies $\varphi(u)\varphi(v) \in E(H)$ and, for every edge $uv \in E(G)$, $uv \in C$ if and only if $\varphi(u)\varphi(v) \in D$. Similarly, a (signed) *homomorphism* of a signed graph $[G, \Sigma]$ to a signed graph $[H, \Pi]$ is a function φ from $V(G)$ to $V(H)$ which is a homomorphism of a 2-edge-colored graph (G, C) to a 2-edge-colored graph (H, D) , where (G, C) is a representative of $[G, \Sigma]$ and (H, D) is a representative of $[H, \Pi]$. As stated in [13], we can observe that we can always choose $D = \Pi$, so that re-signing is done only on $[G, \Sigma]$, if needed. A signed homomorphism of $[G, \Sigma]$ to $[H, \Pi]$ can thus be viewed as a 2-edge-colored homomorphism of (G, C) to (H, D_Π) , where (G, C) is a representative of $[G, \Sigma]$ (obtained by re-signing $[G, \Sigma]$) and $D_\Pi = \Pi$. Homomorphisms of 2-edge-colored graphs were introduced by Alon and Marshall in [1], while homomorphisms of signed graphs were introduced by Naserasr, Rollová and Sopena in [13].

A *surjective homomorphism* is a homomorphism whose co-domain is the image of its domain. With each coloring of a 2-edge-colored graph, we can associate a surjective 2-edge-colored homomorphism which identifies all vertices having the same color. Similarly, with any coloring of a signed graph, we can associate a signed homomorphism which re-signs the signed graph to get the signature for which the coloring is defined, and then identifies all vertices having the same color.

An *unbalanced path* of order $k \geq 2$ in a 2-edge-colored graph, denoted UP_k , is a path of order k having an odd number of negative edges. A *balanced path* of order $k \geq 2$ in a 2-edge-colored graph, denoted BP_k , is a path of order k having an even number of negative edges. An *unbalanced cycle* of length $k \geq 3$ in a 2-edge-colored graph, denoted UC_k , is a cycle of length k having an odd number of negative edges. An *balanced cycle* of length $k \geq 3$ in a 2-edge-colored graph, denoted BC_k , is a cycle of length k having an even number of negative edges. Note that in a signed graph, whether a cycle is balanced or unbalanced does not depend on the representative of the signed graph (*i.e.* this structure is invariant by re-signing). In fact, Zaslavsky in [17] showed that a signed graph is entirely characterized by its underlying graph and the set of its balanced cycles (or the set of its unbalanced cycles).

In what follows, a *digon* will be a UC_2 , *i.e.* two vertices linked by two edges, one positive and one negative. As our graphs are simple, we want to make sure that they contain no loops and no digons. This will become particularly impor-

tant when we construct homomorphisms, as the image graph must be simple. For a 2-edge-colored graph this means, in particular, that we cannot identify vertices which belongs to the same edge or to the same UP_3 , as identifying them would create a loop in the first case and a digon in the second case. Note that we can always identify a pair of vertices that do not belong to the same edge or to the same UP_3 . Any two such vertices are said to be *identifiable*. In a signed graph, before identifying two non-adjacent vertices u and v , we thus need to re-sign the signed graph in order to remove every UP_3 containing u and v . Note that this is not always possible. For example, in the unbalanced cycle UC_4 , we cannot identify any pair of vertices. Indeed, Naserasr, Rollová and Sopena showed in [13] that two vertices are *identifiable* if and only if they do not belong to the same edge or to the same UC_4 .

Note that we can construct a surjective homomorphism of a 2-edge-colored graph or a signed graph by repeatedly identifying pairs of identifiable vertices, until no such pair exists. The image graph is then the obtained 2-edge-colored graph or signed graph.

We will only consider surjective homomorphisms in the rest of this paper, and write $(G, C) \rightarrow_2 (H, D)$ (resp. $[G, \Sigma] \rightarrow_s [H, \Pi]$) whenever there exists a surjective homomorphism of (G, C) to (H, D) (resp. of $[G, \Sigma]$ to $[H, \Pi]$).

A *2-edge-colored clique* (resp. a *signed clique*) is a 2-edge-colored graph (resp. a signed graph) which is its unique homomorphic image, up to isomorphism. Alternatively, it can be defined as a graph whose chromatic number equals its order. The chromatic number can thus be seen as the smallest order of a clique to which the graph maps by a surjective homomorphism. Note here that every signed clique is also a 2-edge-colored clique.

In Figure 1, we represent two 2-edge-colored graphs that are 2-edge-colored cliques but whose signed versions are not signed cliques. Indeed, in Figure 1a, it suffices to re-sign one of the degree 1 vertices to be able to identify them. In Figure 1b, it suffices to re-sign the vertices a and b , or c and d , to get a pair of identifiable vertices.

Harary and Hedetniemi defined in [9] the *achromatic number* $\psi(G)$ of a graph G as the largest k such that there exists a *complete k -coloring* of G , where a complete k -coloring is a coloring where each pair of colors appears on some edge (see also [4, Chapter 12]). Therefore, a complete coloring is nothing but a surjective homomorphism to a clique, where a *homomorphism* is an edge-preserving vertex mapping.

Similarly, we define the (*2-edge-colored*) *achromatic number* $\psi_2(G, C)$ of a 2-edge-colored graph (G, C) as the largest order of a 2-edge-colored clique (K, D) such that $(G, C) \rightarrow_2 (K, D)$, and the (*signed*) *achromatic number* $\psi_s[G, \Sigma]$ of a signed graph $[G, \Sigma]$ as the largest order of a signed clique $[K, \Pi]$ such that $[G, \Sigma] \rightarrow_s [K, \Pi]$ (recall that all homomorphisms we consider are surjective).

A *complete k -coloring* of a 2-edge-colored graph (G, C) , or of a signed graph $[G, \Sigma]$, can thus be defined as a k -coloring such that, for every two colors i and j , there exist an i -colored vertex and a j -colored vertex which are not identifiable. In such a case, we say that the colors i and j are *in conflict*.

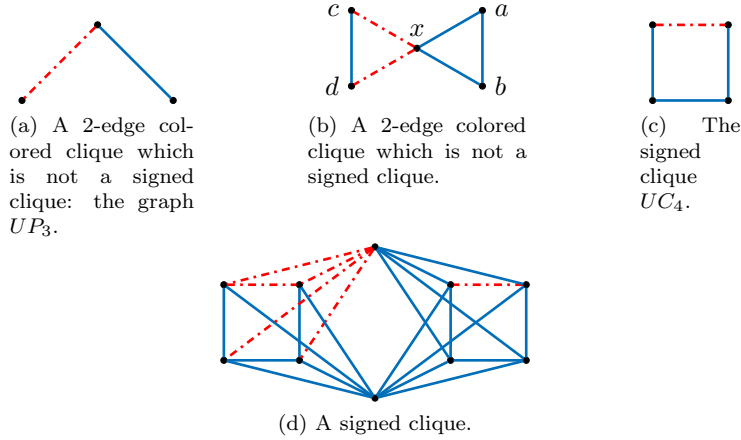


Figure 1: Two 2-edge-colored cliques and two signed cliques.

The notions of complete colorings and achromatic numbers have also been extended to digraphs in [2], [5] or [14], and to oriented graphs in [15]. However, the situation is fundamentally different since any two colored or signed edges uv and vu are identical, while any two arcs \vec{uv} and \vec{vu} are not.

In this paper, we are mainly interested in the three following questions.

1. For a given signed graph $[G, \Sigma]$, what can we say about the 2-edge-colored achromatic number of (G, C) , for any signature C being equivalent to Σ ?
2. For a given graph G , what can we say about the 2-edge-colored achromatic number of (G, C) , for any signature C ?
3. For a given graph G , what can we say about the signed achromatic number of $[G, \Sigma]$, for any signature Σ ?

To this end, we define the six following types of achromatic numbers.

Definition 1. For any graph G and any signed graph (G, Σ) , we let

- $\psi_{\max}[G, \Sigma] := \max \{\psi_2(G, C) | (G, C) \in [G, \Sigma]\}$, the *signed max-achromatic number* of $[G, \Sigma]$,
- $\psi_{\min}[G, \Sigma] := \min \{\psi_2(G, C) | (G, C) \in [G, \Sigma]\}$, the *signed min-achromatic number* of $[G, \Sigma]$,
- $\psi_{\max}(G) := \max \{\psi_2(G, C) | C \subseteq E(G)\}$, the *2-edge-colored max-achromatic number* of G ,
- $\psi_{\min}(G) := \min \{\psi_2(G, C) | C \subseteq E(G)\}$, the *2-edge-colored min-achromatic number* of G ,

- $\psi_{\max}^{\text{signed}}(G) := \max \{\psi_s[G, \Sigma] \mid \Sigma \subseteq E(G)\}$, the *signed max-achromatic number* of G ,
- $\psi_{\min}^{\text{signed}}(G) := \min \{\psi_s[G, \Sigma] \mid \Sigma \subseteq E(G)\}$, the *signed min-achromatic number* of G .

We will study the complexity status of the problem of determining each of these numbers. Our paper is organized as follows. In the next section we detail some properties of these numbers and state our main results. Section 3 is devoted to the proofs of these results and we propose directions for future research in Section 4.

2 Preliminaries and statement of results

In this section, we detail some properties of the achromatic numbers introduced in the previous section. We first compare chromatic and achromatic numbers of 2-edge-colored graphs and signed graphs.

Theorem 2. *For every signed graph $[G, \Sigma]$ and every 2-edge-colored graph $(G, C) \in [G, \Sigma]$,*

$$\chi_2(G, C) \leq \psi_2(G, C) \leq \psi_{\max}[G, \Sigma].$$

Proof. By definition, the chromatic number of a 2-edge-colored graph is at most its achromatic number. Since $\psi_{\max}[G, \Sigma]$ is the maximum value of $\psi_2(G, C')$ taken over all $(G, C') \in [G, \Sigma]$, it is at least $\psi_2(G, C)$. \square

Theorem 3. *For every signed graph $[G, \Sigma]$,*

$$\chi_s[G, \Sigma] \leq \psi_s[G, \Sigma] \leq \psi_{\max}[G, \Sigma].$$

Proof. Again, by definition, the chromatic number of a signed graph is smaller than its achromatic number. Since every signed clique is also a 2-edge-colored clique, for every signed clique $[K, \Pi]$, $[G, \Sigma] \rightarrow_s [K, \Pi]$ implies $(G, C) \rightarrow_2 (K, D_\Pi)$ for some 2-edge-colored graph $(G, C) \in [G, \Sigma]$ and $D_\Pi = \Pi$, which in turn implies $\psi_s[G, \Sigma] \leq \psi_2(G, C) \leq \psi_{\max}[G, \Sigma]$. \square

An interesting property of the achromatic number of ordinary graphs is that for every graph G and every vertex $v \in V(G)$, $\psi(G \setminus v) \leq \psi(v)$ [4]. However, this is no longer true for 2-edge-colored graphs and signed graphs. Indeed, one can check that removing the vertex v in Figures 2a and 2b increases the corresponding achromatic number by one.

This equality still holds for some particular vertices. Let (G, C) be a 2-edge-colored graph and u, v be two vertices of $V(G)$. We say that u and v are *twins* if u and v have the same colored neighborhood, *i.e.* for every vertex $w \in V(G)$, $uw \in E(G)$ if and only if $vw \in E(G)$ and $uw \in C$ if and only if $vw \in C$. In that case, we say that v is a *twin vertex* of u . Similarly, two vertices u and v of a signed graph $[G, \Sigma]$ are *twins* if they are twins in the 2-edge-colored graph (G, C_Σ) with $C_\Sigma := \Sigma$, or if they are twins in the 2-edge-colored graph obtained

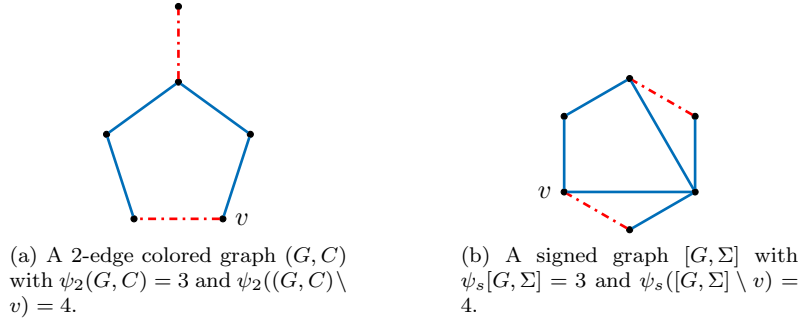


Figure 2: Deleting a vertex in a 2-edge-colored graph or in a signed graph.

from (G, C_Σ) after re-signing at u (i.e. up to re-signing one of them, they are twins in a representative of $[G, \Sigma]$).

Proposition 4. *For every 2-edge-colored graph (G, C) and every vertex $v \in V(G)$, if v has a twin vertex u , then $\psi_2((G, C) \setminus v) \leq \psi_2(G, C)$. For every signed graph $[G, \Sigma]$ and every vertex $v \in V(G)$, if v has a twin vertex u , then $\psi_s([G, \Sigma] \setminus v) \leq \psi_s[G, \Sigma]$ and $\psi_{\max}([G, \Sigma] \setminus v) \leq \psi_{\max}[G, \Sigma]$.*

Proof. Let α be a complete coloring of $(G, C) \setminus v$ using $\psi_2((G, C) \setminus v)$ colors. By setting $\alpha(v) = \alpha(u)$, α clearly extends to a complete coloring of (G, C) , which implies $\psi_2((G, C) \setminus v) \leq \psi_2(G, C)$. The same argument clearly implies the two inequalities for signed graphs, after maybe re-signing at v so that u and v are twins in the 2-edge-colored representative of $[G, \Sigma]$. \square

Before stating our results, we give some properties of the 2-edge-colored and signed cliques as they are at the center of the definition of all achromatic numbers we have introduced.

As observed before, any two vertices of a 2-edge-colored clique or of a signed clique are not identifiable.

Observation 5. *In a 2-edge-colored clique (K, D) , every two vertices u and v satisfy at least one of the following:*

- $uv \in E(G)$,
- u and v are end vertices of the same UP_3 .

This implies in particular that the diameter of K is at most 2.

Theorem 6 (Naserasr, Rollová and Sopena [13]). *In a signed clique $[K, \Pi]$, every two vertices u and v satisfy at least one of the following:*

- $uv \in E(G)$,
- u and v are antipodal vertices of the same UC_4 .

Let us remark that ψ_2 and ψ_s can be viewed as the largest order of a 2-edge-colored clique or of a signed clique obtained by the following algorithm: while there exists two vertices identifiable, identify them. In the signed case, we may need to re-sign before identifying vertices.

We can construct a signed clique from a 2-edge-colored clique by the following construction.

Lemma 7. *For a 2-edge-colored graph (K, D) , if (K', D') is the 2-edge-colored graph obtained by adding one vertex z to (K, D) and adding, for every $u \in V(K)$, a positive edge uz , then (K, D) is a 2-edge-colored clique if and only if the signed graph $[K', \Sigma']$ where $\Sigma' = D'$ is a signed clique.*

Proof. Suppose first that (K, D) is a 2-edge-colored clique and let u and v be any two vertices of K' . If $u = z$ or $v = z$, then there is an edge uv by construction, so that u and v are not identifiable. Otherwise, both u and v belong to K . If there is no edge uv in K , then u and v are the end vertices of some UP_3 of (K, D) , say uvw . Then, by construction, $uvwz$ is a UC_4 in (K', D') and thus in $[K', \Sigma']$, so that u and v are not identifiable. This implies that $[K', \Sigma']$ is a signed clique.

Suppose now that $[K', \Sigma']$ is a signed clique and let u and v be any two vertices of K , with $u \neq z$ and $v \neq z$. Since u and v are not identifiable in $[K', \Sigma']$, either uv is an edge of $[K', \Sigma']$, and thus of (K, D) , or u and v are antipodal vertices in some UC_4 of $[K', \Sigma']$, which implies that they are the end vertices of some UP_3 in (K, D) . In both cases, u and v are not identifiable in (K, D) , which implies that (K, D) is a 2-edge-colored clique. \square

The problem of deciding whether the achromatic number of a graph is at least k , for some integer k , has been shown to be NP-complete even when restricted to small classes of graphs in [16] and [3]. We recall the definition of the problem and these results below.

Problem: ACHROMATIC NUMBER
Instance: A graph G and an integer k
Question: Is $\psi(G) \geq k$?

Theorem 8 (Yannakakis and Gavril[16]). *The problem ACHROMATIC NUMBER is NP-complete even when restricted to complements of bipartite graphs.*

Theorem 9 (Bodlaender [3]). *The problem ACHROMATIC NUMBER is NP-complete even when restricted to graphs which are simultaneously connected interval graphs and co-graphs.*

We will show that a number of problems related to the achromatic numbers we have introduced are NP-complete. We first define the following problem (the name in brackets is the acronym of the problem).

Problem: 2-EDGE-COLORED GRAPH ACHROMATIC NUMBER [2EC-AN]
Instance: A 2-edge-colored graph (G, C) and an integer k
Question: Is $\psi_2(G, C) \geq k$?

Since any graph G can be regarded as the 2-edge-colored graph (G, \emptyset) , and every complete coloring of G as a complete coloring of (G, \emptyset) , the problem 2EC-AN contains the problem ACHROMATIC NUMBER. The following theorem thus easily follows from Theorems 8 and 9.

Theorem 10. *The problem 2EC-AN is NP-complete even when restricted to graphs which are simultaneously connected interval graphs and co-graphs or to complements of bipartite graphs.*

Proof. We can verify in polynomial type whether a coloring of (G, C) on $q \geq k$ colors is a proper complete coloring of (G, C) or not. Thus 2EC-AN is in NP. The result then follows by the above remark. \square

We now define all the other decision problems that we will consider.

Problem: SIGNED GRAPH ACHROMATIC NUMBER [SIGNED-AN]

Instance: A signed graph $[G, \Sigma]$ and an integer k

Question: Is $\psi_s[G, \Sigma] \geq k$?

Problem: SIGNED GRAPH MAX-ACHROMATIC NUMBER [SIGNED-MAX-AN]

Instance: A signed graph $[G, \Sigma]$ and an integer k

Question: Is $\psi_{\max}[G, \Sigma] \geq k$?

Problem: SIGNED GRAPH MIN-ACHROMATIC NUMBER [SIGNED-MIN-AN]

Instance: A signed graph $[G, \Sigma]$ and an integer k

Question: Is $\psi_{\min}[G, \Sigma] \geq k$?

Problem: GRAPH 2-EDGE-COLORED MAX-ACHROMATIC NUMBER [MAX-2EC-AN]

Instance: A graph G and an integer k

Question: Is $\psi_{\max}(G) \geq k$?

Problem: GRAPH 2-EDGE-COLORED MIN-ACHROMATIC NUMBER [MIN-2EC-AN]

Instance: A graph G and an integer k

Question: Is $\psi_{\min}(G) \geq k$?

Problem: GRAPH SIGNED MAX-ACHROMATIC NUMBER [MAX-SIGNED-AN]

Instance: A graph G and an integer k

Question: Is $\psi_{\max}^{\text{signed}}(G) \geq k$?

Problem: GRAPH SIGNED MIN-ACHROMATIC NUMBER [MIN-SIGNED-AN]

Instance: A graph G and an integer k

Question: Is $\psi_{\min}^{\text{signed}}(G) \geq k$?

Our main results are gathered in the two following theorems, and will be proved in the next section.

Theorem 11. *The problem SIGNED-AN is NP-complete even when restricted to graphs which are simultaneously connected interval graphs and co-graphs or to complements of bipartite graphs.*

	Ordinary graphs	2-edge-colored graphs	Signed graphs
ψ	NP-complete [Th 9]	N.A.	N.A.
ψ_2	N.A.	NP-complete [Th 10]	N.A.
ψ_s	N.A.	N.A.	NP-complete [Th 11]
ψ_{\max}	NP-complete [Th 12]	N.A.	NP-complete [Th 12]
ψ_{\min}	Π_2 (complete ?) [Th 13]	N.A.	Π_2 (complete ?) [Th 13]
$\psi_{\max}^{\text{signed}}$	NP-complete [Th 12]	N.A.	N.A.
$\psi_{\min}^{\text{signed}}$	Π_2 (complete ?) [Th 13]	N.A.	N.A.

Table 1: Decision problems related to achromatic numbers.

Theorem 12. *The following problems are NP-complete:*

- SIGNED-MAX-AN, even when restricted to connected diamond-free perfect graphs
- MAX-2EC-AN, even when restricted to connected diamond-free perfect graphs
- MAX-SIGNED-AN, even when restricted to connected perfect graphs.

For the three other problems it is easy to show that:

Theorem 13. *The problems SIGNED-MIN-AN, MIN-2EC-AN and MIN-SIGNED-AN are in Π_2 .*

A natural question is thus the following.

Question 14. *Are the three problems of Theorem 13 Π_2 -Complete?*

Table 1 summarizes our results and what is known on decision problems related to achromatic numbers.

3 Proof of Theorems 11 and 12

In order to prove that all these problems are NP-complete, we need to prove that they are in NP and are NP-hard. We first prove that the four problems belong to NP.

Proof of membership to NP. Suppose that we have an instance of SIGNED-AN (resp. SIGNED-MAX-AN) consisting of a signed graph $[G, \Sigma]$ and an integer k . Assume we are given a 2-edge-colored graph (G, C) , a coloring α of (G, C) . We can verify that α is a complete coloring of (G, C) (resp. of $[G, \Sigma]$ by choosing (G, C) as the representative) using at least k colors and, that $(G, C) \in [G, \Sigma]$ in polynomial time as shown in [8]. Therefore, both problems SIGNED-AN and SIGNED-MAX-AN are in NP.

Suppose now that we have an instance of MAX-2EC-AN (resp. MAX-SIGNED-AN) consisting of an ordinary graph G and an integer k . Moreover, if we are

given a signature C and a vertex coloring α of G , we can verify in polynomial time that α is a complete coloring of (G, C) (resp. of $[G, \Sigma_C]$, the signed graph defined by $\Sigma_C := C$) using at least k colors, which implies that both problems MAX-2EC-AN and MAX-SIGNED-AN are in NP. \square

We are now ready to prove Theorem 11.

Proof of Theorem 11. We already showed that the problem is in NP. Take now an instance of ACHROMATIC NUMBER consisting of a connected graph G and an integer k . Let H be the graph obtained from G by adding a vertex z such that, for all $u \in V(G)$, $zu \in E(H)$. If G is an interval graph and a co-graph then so is H by construction. Indeed the interval corresponding to z can be chosen as the convex union of the intervals of the other vertices and there is no induced P_4 containing z . If G is a complement of a bipartite graph then so is H , we just add an isolated vertex to the complement of G , this graph is still bipartite and its complement is H . In both cases, H is in the relevant subclass. We claim that $\psi(G) \geq k$ if and only if $\psi_s[H, \emptyset] \geq k + 1$.

Suppose first that $\psi(G) = p \geq k$. This means that there exists a surjective homomorphism from G to K_p , the complete graph on p vertices. By applying this homomorphism on the copy of G in H , we get $[H, \emptyset] \rightarrow_s [K_{p+1}, \emptyset]$. Hence, $\psi_s[H, \emptyset] \geq k + 1$.

Suppose now that $\psi_s[H, \emptyset] \geq k + 1$. There exists a signed homomorphism from $[H, \emptyset]$ to $[K, \Pi]$, a signed clique on at least $k + 1$ vertices. We can re-sign $[K, \Pi]$ in such a way that, in its canonical representative, the vertex z is only incident with positive edges and has not been re-signed by the homomorphism. We want to show that, in this signature of $[K, \Pi]$, all the edges non incident with z are positive. Suppose to the contrary that there is a negative edge which is the image of the edge uv of H . Then, exactly one of u or v has been re-signed by the homomorphism, but in this case the edge linking this vertex to z would be negative, a contradiction. Thus, all the edges are positive, which gives that K is a complete graph and no vertices have been re-signed. If we take the restriction of our homomorphism to G , then we get a homomorphism from G to a complete graph of size at least k .

If the graph G was in one of the two subclasses of the theorem then we get our result. \square

In order to prove Theorem 12, we will use a reduction from the following decision problem.

Problem: 3-PARTITION

Instance: A set $A = \{a_1, \dots, a_{3m}\} \in \mathbb{N}^{3m}$ and an integer B such that $\frac{B}{4} < a_i < \frac{B}{2}$ for every i , $1 \leq i \leq m$

Question: Is there a partition $\{P_1, \dots, P_m\}$ of A such that $|P_i| = 3$ and $\sum_{a_j \in P_i} a_j = B$ for every i , $1 \leq i \leq m$?

This problem has been shown to be strongly NP-complete in [7] by Garey and Johnson. Note that we can multiply each a_i and B by $m + 1$, and thus

assume $m < a_i$ for all i , $1 \leq i \leq 3m$. Also note that in a positive instance of 3-PARTITION, we have

$$\sum_{1 \leq i \leq 3m} a_i = mB. \quad (1)$$

We will assume that equation (1) holds in the rest of this section. Moreover, since 3-PARTITION is strongly NP-complete, the size of the instance can be taken as $O(Bm)$.

Given an instance \mathcal{I} of 3-PARTITION, we will construct the signed graph $[H(\mathcal{I}), \Sigma(\mathcal{I})]$ (see Figure 3). This graph is composed of three parts.

1. The subgraph \mathcal{S} , called the “stars”, contains $3m$ stars S_1, \dots, S_{3m} . Each star S_i has a center vertex, s_i , and a_i leaves $e_1^i, \dots, e_{a_i}^i$.
2. The subgraph \mathcal{T} , called the “target”, is a negative clique on m vertices t_1, \dots, t_m .
3. The subgraph \mathcal{G} , called the “grid”, contains $p(B + r + q)$ vertices, where $p = 2\binom{m}{2} + 2m(B + r + q) + q + 1$, $q = 6m + 2Bm$ and $r = \binom{m}{2} + qm + 1$. These vertices are denoted $x_{i,j}$, for $1 \leq i \leq B + r + q$ and $1 \leq j \leq p$. There is a positive edge between $x_{i,j}$ and $x_{k,\ell}$ if and only if $i = k$ and $j \neq \ell$. There is a negative edge between $x_{i,j}$ and $x_{k,\ell}$ if and only if $j = \ell$ and $i \neq k$. We denote the columns and rows of \mathcal{G} respectively by $C_j = \{x_{i,j} \mid 1 \leq i \leq B + r + q\}$ and $L_i = \{x_{i,j} \mid 1 \leq j \leq p\}$.

Finally we add a positive edge $t_\ell x_{i,1}$ for every ℓ , $1 \leq \ell \leq m$, and every i , $B + 1 \leq i \leq B + r + q$.

The signature of $H(\mathcal{I})$ above described (one of its many equivalent signatures) is the easiest one to work with. Note that $H(\mathcal{I})$ is a diamond-free perfect graph but is not connected. We will conduct the proof without the connectivity requirement and then explain how to modify it in such a way that the considered graph is connected.

From now on, we let $k(\mathcal{I}) := m + p(B + r + q)$.

Claim 15. *If \mathcal{I} is an instance of 3-PARTITION that admits a solution then, for the canonical representative $(H(\mathcal{I}), C_{\Sigma(\mathcal{I})})$ of $[H(\mathcal{I}), \Sigma(\mathcal{I})]$, we have:*

$$k(\mathcal{I}) \leq \psi_2(H(\mathcal{I}), C_{\Sigma(\mathcal{I})}) \leq \psi_{\max}[H(\mathcal{I}), \Sigma(\mathcal{I})] \leq \psi_{\max}(H(\mathcal{I})).$$

Proof. If there is a solution $\{P_1, \dots, P_m\}$ to \mathcal{I} then, on the 2-edge-colored graph $(H(\mathcal{I}), C_{\Sigma(\mathcal{I})})$, we assign one color to each vertex of $\mathcal{T} \cup \mathcal{G}$ (which gives $k(\mathcal{I})$ colors), then we identify t_i and s_j for $j \in P_i$ and every i , $1 \leq i \leq m$. We thus get that each t_i has B positive neighbours of degree 1 which correspond to the B leaves of the three stars S_j for $j \in P_i$. For $1 \leq i \leq m$, we identify each of these B neighbours of t_i with a unique vertex in $\{x_{1,1}, \dots, x_{B,1}\}$.

We obtain a 2-edge-colored clique on $k(\mathcal{I})$ colors. Indeed, if u and v are two vertices of the graph after the identification, we have three cases to consider. If u and v are both vertices of the target \mathcal{T} , then there is an edge uv in the graph by construction of \mathcal{T} . If u and v are both vertices of the grid \mathcal{G} , say $u = x_{i,j}$

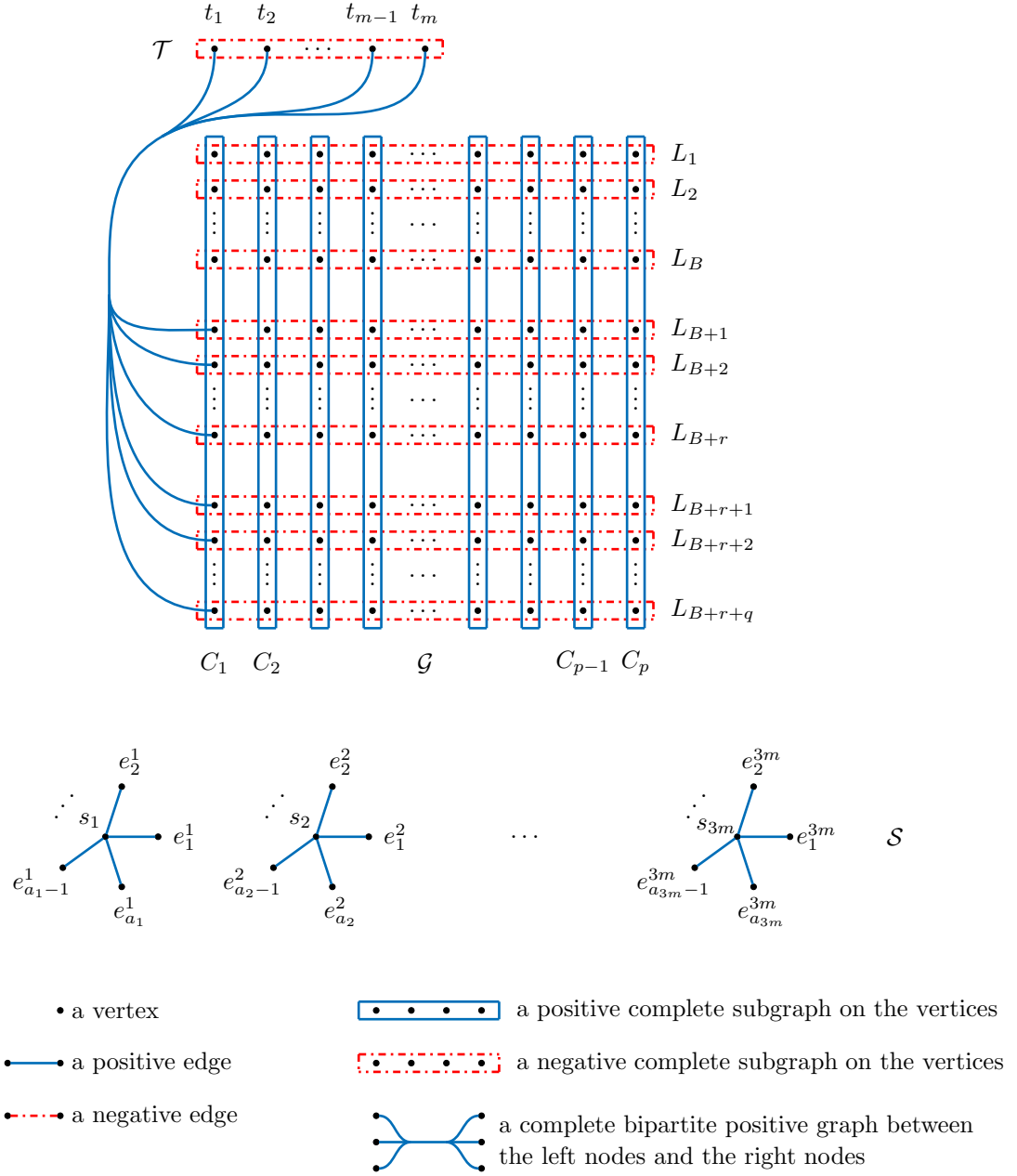


Figure 3: The signed graph $[H(\mathcal{I}), \Sigma(\mathcal{I})]$ and the legend for notation.

and $v = x_{k,\ell}$, then $ux_{k,j}v$ is a UP_3 by construction of \mathcal{G} . Otherwise, suppose that u is a vertex of the target and v is a vertex of the grid, say $u = t_i$ and $v = x_{j,k}$. If $j > B$ then, by construction of $[H(\mathcal{I}), \Sigma(\mathcal{I})]$, $ux_{j,1}v$ is a UP_3 (or uv is an edge if $k = 1$). In the other case, there is a positive edge $ux_{j,1}$ by the previous identifications and a negative edge $x_{j,1}x_{j,k}$ (if $k \neq 1$) by construction of \mathcal{G} . Hence, we cannot identify u and v , which implies that the graph is a clique on $k(\mathcal{I})$ vertices.

We also need to prove that the homomorphism is well defined, *i.e.* does not create any loop and does not create any digon. This follows from the fact that we identify edges of \mathcal{S} with non-edges of $\mathcal{T} \cup \mathcal{G}$. \square

We want to prove that if $\psi_{\max}(H(\mathcal{I}), \Sigma(\mathcal{I})) \geq k(\mathcal{I})$ then \mathcal{I} has a solution. There are two main ideas in the following result.

First, we know that if $\psi_{\max}(H(\mathcal{I}), \Sigma(\mathcal{I})) \geq k(\mathcal{I})$, then the homomorphism that reaches the signed max-achromatic number creates a clique on more than $k(\mathcal{I})$ vertices. This clique has diameter 2, as indicated in Observation 5. We want to show that constructing a “large” graph of diameter 2 from $H(\mathcal{I})$ implies that \mathcal{I} has a solution.

Secondly, in this setting, this means that the identification performed to create the “large” graph of diameter 2, is similar to the one we did in the proof of Claim 15.

Lemma 16. *Let \mathcal{I} be an instance of 3-PARTITION. If there is a surjective homomorphism φ from $H(\mathcal{I})$ to an ordinary graph K of order greater than $k(\mathcal{I})$ and diameter at most 2, then \mathcal{I} has a solution.*

The proof of Lemma 16 works as follows. We first prove that each vertex of K has a pre-image in \mathcal{T} or \mathcal{G} . We then prove that the edges in \mathcal{S} were identified in a way similar to the construction in Claim 15.

Proof. Let φ be a homomorphism of $H(\mathcal{I})$ to K , an ordinary graph of order greater than $k(\mathcal{I})$ and diameter at most 2.

Let then

$$\alpha = \varphi(\mathcal{S}) \setminus \varphi(\mathcal{G} \cup \mathcal{T}), \quad \beta = \varphi(\mathcal{T}) \setminus \varphi(\mathcal{G}) \text{ and } \gamma = \varphi(\mathcal{G}).$$

The set α represents the vertices of K that come from the identification of vertices only in \mathcal{S} . The set β represents the vertices of K that come from the identification of vertices in \mathcal{T} or \mathcal{S} . The set γ represents all other vertices. Our first goal is to prove that the set α is empty.

Note that (α, β, γ) is a partition of $\varphi(H(\mathcal{I}))$, and thus

$$|\alpha| + |\beta| + |\gamma| \geq k(\mathcal{I}) = m + p(B + r + q).$$

Moreover, we have

$$|V(H(\mathcal{I}))| - k(\mathcal{I}) = 3m + Bm.$$

Let $d = 3m + Bm$. Since the homomorphism φ can do only d identifications of vertices, there are at most $2d = q$ vertices in $H(\mathcal{I})$ which have been identified

with some other vertex. We denote by Id the set of vertices that were identified to another and by $\text{Id}_{\mathcal{G}}$ the set of vertices of \mathcal{G} that have been identified with another vertex of \mathcal{G} .

Let $\mathcal{L} = \{L_i \mid L_i \cap \text{Id}_{\mathcal{G}} = \emptyset\}$. The set \mathcal{L} is the set of lines (themselves sets of vertices) of the grid \mathcal{G} that do not contain a vertex identified with another vertex of the grid. Moreover, for every vertex $u \in K \setminus \gamma$ (*i.e.* u is a vertex that is not in the image of the grid), let

$$\mathcal{N}_u = \{L_i \in \mathcal{L} \mid \varphi(L_i) \cap N_K(u) \neq \emptyset\}.$$

The set \mathcal{N}_u is the set of lines of the grid \mathcal{G} that do not contain any vertices identified with another vertex of the grid and intersect the neighbourhood of u in K . We claim that $|\mathcal{N}_u| = |\mathcal{L}|$. By definition of \mathcal{N}_u , we have $|\mathcal{N}_u| \geq |\mathcal{L}|$.

Suppose to the contrary that there exists $u \in K \setminus \gamma$ such that $|\mathcal{N}_u| < |\mathcal{L}|$. Therefore, there exists $L_i \in \mathcal{L}$ with $\varphi(L_i) \cap N_K(u) = \emptyset$. Since K has diameter at most 2, for every vertex $v \in \varphi(L_i)$, there exists a neighbour w of u that is a neighbour of v (recall that there is no edge uv). There are at least $p - q$ vertices of L_i belonging to some C_j with $C_j \cap \text{Id} = \emptyset$ with $1 \leq j \leq p$ (*i.e.* columns where we did not identify any vertices). Among these columns, there are at most $|N_K(u)|$ C_j 's which contain a neighbour of u in $\varphi(H(\mathcal{I}))$. Thus, there are at least $p - q - |N_K(u)|$ vertices of L_i such that the column C_j they belong to intersects neither Id nor $\varphi^{-1}(N_K(u))$. These vertices correspond to $p - q - |N_K(u)|$ vertices in $\varphi(L_i)$ as they do not belong to $\text{Id}_{\mathcal{G}}$.

Moreover,

$$|N(u)| \leq |E(H(\mathcal{I})) \setminus E(\mathcal{G})| \leq \binom{m}{2} + m(B + r + q).$$

Therefore, there are at least $p - q - \binom{m}{2} - m(B + r + q)$ vertices in L_i not belonging to a column whose image by φ intersects Id or contains a neighbour of u . It follows that every such vertex, say $x_{i,j}$, has no neighbour in $\varphi(L_i) \cup \varphi(C_j)$ which is a neighbour of u . As L_i does not intersect $\text{Id}_{\mathcal{G}}$ and C_j does not intersect Id , in K , $\varphi(L_i) \cup \varphi(C_j)$ contains all the neighbours of $x_{i,j}$ among the vertices of \mathcal{G} . These remarks imply that $x_{i,j}$ is linked to u by a path of length 2 whose interior vertex is in $K \setminus \gamma$ (*i.e.* $x_{i,j}$ is linked to u by a path which is not induced in the grid).

There are at most $\binom{m}{2} + m(B + r + q)$ edges not in γ , and $p - q - \binom{m}{2} - m(B + r + q) \geq \binom{m}{2} + m(B + r + q) + 1$ vertices to which u must be linked in $\varphi(L_i)$ by a path of length 2 whose interior vertex is not in γ . This is not possible since we do not have enough edges that can be used for such paths. Therefore, $|\mathcal{N}_u| \geq |\mathcal{L}| \geq B + r + q - |\text{Id}_{\mathcal{G}}| \geq B + r$.

Moreover,

$$|E(\varphi(H(\mathcal{I})) \setminus \gamma)| \leq Bm + \binom{m}{2} + (r + q)m - (r + q)(m - |\beta|).$$

The last term comes from the fact that the edges between vertices of \mathcal{G} and $\mathcal{T} \setminus \varphi^{-1}(\beta)$ have images by φ in $\varphi(\mathcal{G})$ and do not contribute to $|E(\varphi(H(\mathcal{I})) \setminus \gamma)|$.

This number of edges must be greater than

$$\sum_{u \in \alpha \cup \beta} |\mathcal{N}_u| \geq (B+r)(|\alpha| + |\beta|).$$

But $|\alpha| + |\beta| \geq k(\mathcal{I}) - |\gamma| \geq m$ by the choice of $k(\mathcal{I})$. We then get

$$Bm + \binom{m}{2} + (r+q)|\beta| \geq Bm + r(|\alpha| + |\beta|),$$

and thus, since $|\beta| \leq m$,

$$\binom{m}{2} + qm \geq r|\alpha|.$$

If $|\alpha| > 0$, then $\binom{m}{2} + qm + 1 = r \leq \binom{m}{2} + qm$, a contradiction, and thus $\alpha = \emptyset$.

Because $\alpha = \emptyset$, every vertex of \mathcal{S} is identified with a vertex of $\mathcal{G} \cup \mathcal{T}$, and since there are d vertices in \mathcal{S} , this accounts for all the identifications. We then get $\beta = \mathcal{T}$ and $\gamma = \mathcal{G}$, which implies $\text{Id}_{\mathcal{G}} = \emptyset$ and $|\mathcal{N}_u| = B+q+r$. The number of edges that can contribute to $\sum_{u \in \beta} |\mathcal{N}_u|$ is limited: Bm edges in \mathcal{S} and $(r+q)m$ edges between \mathcal{T} and \mathcal{G} , which gives $\sum_{u \in \beta} |\mathcal{N}_u| \leq m(B+r+q)$. Therefore, there is a one-to-one correspondence between the pairs in $\mathcal{T} \times \mathcal{L}$ and the edges that can contribute to the sum.

Suppose now that some s_i was identified with a vertex in \mathcal{G} . Since $m < a_i$, and no two leaves of a star can be identified with each other, the leaves of this star cannot all be identified with the vertices of \mathcal{T} as \mathcal{T} is of order m . So at least one leaf e_j^i is identified with a vertex of \mathcal{G} , but this means that the edge $s_i e_j^i$ does not contribute to the sum $\sum_{u \in \beta} |\mathcal{N}_u|$, a contradiction.

Now, note that for any $u \in \beta$, $|\mathcal{N}_u| \leq B+r+q$. If at most two star centers are identified with some vertex u of \mathcal{T} , then, since these two stars have less than B leaves between them, we have $|\mathcal{N}_u| < B+r+q$ and thus $\sum_{u \in \beta} |\mathcal{N}_u| < m(B+r+q)$. Hence, we finally get that each vertex $u \in \mathcal{T}$ was identified with three s_i 's whose sum of subscripts equals $B+r+q - (q+r) = B$. This gives us a partition of the set A which is a solution of 3-PARTITION. \square

We can now prove that MAX-2EC-AN and SIGNED-MAX-AN are NP-complete.

Proof that MAX-2EC-AN (resp. SIGNED-MAX-AN) is NP-complete.

We already proved that both these problems are in NP. If \mathcal{I} is an instance of 3-PARTITION, then we construct the graph $H(\mathcal{I})$ (resp. the signed graph $[H(\mathcal{I}), \Sigma(\mathcal{I})]$) in polynomial time.

By Claim 15, if \mathcal{I} has a solution, then $k(\mathcal{I}) \leq \psi_{\max}(H(\mathcal{I}))$ (resp. $k(\mathcal{I}) \leq \psi_{\max}[H(\mathcal{I}), \Sigma(\mathcal{I})]$).

If $k(\mathcal{I}) \leq \psi_{\max}(H(\mathcal{I}))$ (resp. $k(\mathcal{I}) \leq \psi_{\max}[H(\mathcal{I}), \Sigma(\mathcal{I})]$), then there exists a signature C and a surjective 2-edge-colored homomorphism φ such that $(H(\mathcal{I}), C) \rightarrow_2 (K, D)$ by φ , where (K, D) is a 2-edge-colored clique of order

greater than $k(\mathcal{I})$ and, in the signed case, $C \in \Sigma(\mathcal{I})$. As (K, D) has diameter 2, by Lemma 16, we get that \mathcal{I} has a solution since φ is also a surjective homomorphism from $H(\mathcal{I})$ to K .

The problems MAX-2EC-AN and SIGNED-MAX-AN are thus NP-complete even when restricted to diamond-free perfect graphs. To make the graph $H(\mathcal{I})$ connected, it suffices to increase q by one and r by $3m$, and to add an edge joining the vertices $x_{B+r+q,1}$ and s_i for every i , $1 \leq i \leq 3m$. The graph is now clearly connected and the same arguments as in Lemma 16 works, since the $3m$ new edges cannot be used to create conflicts between vertices of \mathcal{T} and \mathcal{G} that did not already exist. \square

We now consider the case of signed graphs. Let $H'(\mathcal{I})$ be the graph obtained from $H(\mathcal{I})$ (the underlying graph of the signed graph previously defined, see Figure 3) by adding a new vertex z such that, for every vertex $v \in V(H(\mathcal{I}))$, zv is an edge. We also define $k'(\mathcal{I}) = k(\mathcal{I}) + 1$.

We are left to prove that MAX-SIGNED-AN is NP-complete.

Proof that MAX-SIGNED-AN is NP-complete.

We already proved that this problem is in NP. If \mathcal{I} is an instance of 3-PARTITION, then we construct the graph $H'(\mathcal{I})$ in polynomial time. Note that $H'(\mathcal{I})$ is a connected perfect graph.

By Claim 15, if \mathcal{I} has a solution, then $(H(\mathcal{I}), C_{\Sigma(\mathcal{I})}) \rightarrow_2 (K, D)$, where (K, D) is a 2-edge-colored clique of order greater than $k(\mathcal{I})$ and $C_{\Sigma(\mathcal{I})} = \Sigma(\mathcal{I})$. Thus, $(H'(\mathcal{I}), C_{\Sigma(\mathcal{I})}) \rightarrow_2 (K', D')$, where (K', D') is obtained from (K, D) by adding one vertex z that is a positive neighbour of every vertex of (K, D) . By Lemma 7, $[K', \Sigma']$, where $\Sigma' = D'$, is a signed clique. Hence $k'(\mathcal{I}) \leq \psi_{\max}^{\text{signed}}(H'(\mathcal{I}))$.

If $k'(\mathcal{I}) \leq \psi_{\max}^{\text{signed}}(H'(\mathcal{I}))$, then there exists a signature Σ_1 and a surjective signed homomorphism φ' such that $[H'(\mathcal{I}), \Sigma_1] \rightarrow_s [K', \Pi']$ by φ' , where $[K', \Pi']$ is a signed clique of order greater than $k'(\mathcal{I})$. Up to re-signing $[K', \Pi']$, we can assume that z is a positive neighbour of all the other vertices. Let K be the graph obtained from K' by removing the image of z . Note that z was not identified by φ' . By Lemma 7, (K, D) is a 2-edge-colored clique, where D is Σ_1 from which we removed the edges incident to z . Let φ be the restriction of φ' to $H(\mathcal{I})$. Then, by φ , $H(\mathcal{I}) \rightarrow_2 K$. As $[K', \Pi']$ is a signed clique, K has diameter 2 and, by Lemma 16, we get that \mathcal{I} has a solution. \square

4 Discussion

In this paper, we introduced and study achromatic numbers of 2-edge-colored graphs and of signed graphs. In particular, Theorems 10, 11 and 12 state that computing the achromatic number of a 2-edge-colored graph or of a signed graph is NP-complete.

The two following results allow to conclude that the problem of computing the achromatic number of an ordinary graph is FPT. Recall that a *reducing congruence class* (an *r.c. class* for short) on a graph G is an equivalence class of

the relation \equiv_G defined by $u \equiv_G v$ if and only if $N_G(u) = N_G(v)$, where $N_G(u)$ denotes the neighborhood of the vertex u in G . In other words, $u \equiv_G v$ if and only if u and v are twins in G .

Theorem 17 (Hell and Miller[10], Hoffman [11] and Máté [12]). *There is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every integer k , if a graph G has more than $f(k)$ r.c. classes then $\psi(G) \geq k$.*

Theorem 18 (Farber, Hahn, Hell and Miller [6]). *For a fixed integer k , there is an algorithm that, given a graph G , determines whether $\psi(G) \geq k$ or not in time $O(|E(G)|)$.*

The following question is thus natural when considering these two results.

Question 19. *Is it possible to determine if one of our parameters is greater than some integer k in FPT time where k is the parameter?*

Theorem 17 can be generalized to 2-edge-colored graphs and to signed graphs but we were not able to generalize Theorem 18 using the same techniques as in [6].

Acknowledgment. I would like to thank Hervé Hocquard and Éric Sopena for their advice and for their help in the writing of this article. Also thanks to Pascal Ochem for helpful discussions.

References

- [1] N. Alon and T.H. Marshall. Homomorphisms of edge-colored graphs and coxeter groups. *Journal of Algebraic Combinatorics*, 8(1):5–13, Jul 1998.
- [2] Gabriela Araujo-Pardo, Juan José Montellano-Ballesteros, Mika Olsen, and Christian Rubio-Montiel. The diachromatic number of digraphs. *arXiv:1712.00495 [math.CO]*, 2017.
- [3] Hans L. Bodlaender. Achromatic number is NP-complete for cographs and interval graphs. *Information Processing Letters*, 31(3):135–138, 1989.
- [4] Gary Chartrand and Ping Zhang. *Chromatic Graph Theory*. Chapman & Hall/CRC, 1st edition, 2008.
- [5] Keith J. Edwards. Harmonious chromatic number of directed graphs. *Discrete Applied Mathematics*, 161(3):369 – 376, 2013.
- [6] Martin Farber, Geña Hahn, Pavol Hell, and Donald Miller. Concerning the achromatic number of graphs. *Journal of Combinatorial Theory, Series B*, 40(1):21–39, 1986.
- [7] Michael R. Garey and David S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1990.

- [8] Frank Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2(2):143–146, 1953.
- [9] Frank Harary and Stephen Hedetniemi. The achromatic number of a graph. *Journal of Combinatorial Theory*, 8(2):154–161, 1970.
- [10] Pavol Hell and Donald J. Miller. Graph with given achromatic number. *Discrete Mathematics*, 16(3):195–207, 1976.
- [11] A.J. Hoffman. Eigenvalues and partitionings of the edges of a graph. *Linear Algebra and its Applications*, 5(2):137–146, 1972.
- [12] Attila Máté. A lower estimate for the achromatic number of irreducible graphs. *Discrete Mathematics*, 33(2):171–183, 1981.
- [13] Reza Naserasr, Edita Rollová, and Éric Sopena. Homomorphisms of signed graphs. *Journal of Graph Theory*, 79(3):178–212, 2015.
- [14] V Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265 – 270, 1982.
- [15] Éric Sopena. Complete oriented colourings and the oriented achromatic number. *Discrete Applied Mathematics*, 173:102–112, 2014.
- [16] M. Yannakakis and F. Gavril. Edge dominating sets in graphs. *SIAM Journal on Applied Mathematics*, 38(3):364–372, 1980.
- [17] Thomas Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4(1):47–74, 1982.