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MASS-CONSERVING SELF-SIMILAR SOLUTIONS TO COAGULATION-FRAGMENTATION EQUATIONS

PHILIPPE LAURENÇOT

ABSTRACT. Existence of mass-conserving self-similar solutions with a sufficiently small total mass is proved for a specific class of homogeneous coagulation and fragmentation coefficients. The proof combines a dynamical approach to construct such solutions for a regularised coagulation-fragmentation equation in scaling variables and a compactness method.

1. INTRODUCTION

Coagulation-fragmentation equations are mean-field models describing the time evolution of the size distribution function f of a system of particles varying their sizes due to the combined effect of binary coalescence and multiple breakage. The dynamics of the size distribution function $f(t, x)$ of particles of size $x \in (0, \infty)$ at time $t > 0$ is governed by the nonlinear integral equation

$$\partial_t f(t, x) = \mathcal{C}f(t, x) + \mathcal{F}f(t, x), \quad (t, x) \in (0, \infty)^2, \quad (1.1a)$$

$$f(0, x) = f^{in}(x), \quad x \in (0, \infty), \quad (1.1b)$$

where

$$\mathcal{C}f(x) := \frac{1}{2} \int_0^x K(y, x-y) f(x-y) f(y) dy - \int_0^\infty K(x, y) f(x) f(y) dy, \quad x \in (0, \infty), \quad (1.1c)$$

and

$$\mathcal{F}f(x) := -a(x)f(x) + \int_x^\infty a(y)b(x, y)f(y) dy, \quad x \in (0, \infty), \quad (1.1d)$$

account for the coagulation and fragmentation processes, respectively. In (1.1c), the coagulation kernel K is a non-negative and symmetric function defined on $(0, \infty)^2$ and $K(x, y) = K(y, x)$ is the rate at which two particles of respective sizes x and y collide and merge. In (1.1d), $a(x)$ is the overall fragmentation rate of particles of size x and the distribution of the sizes of fragments resulting from the splitting of a particle of size y is the daughter distribution function $x \mapsto b(x, y)$. Since we discard the possibility of loss of matter during breakup, b is assumed to satisfy

$$\int_0^y xb(x, y) dx = y, \quad y > 0, \quad \text{and} \quad b(x, y) = 0, \quad x > y > 0; \quad (1.2)$$

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that is, the fragmentation of a particle of size y only produces particles of smaller sizes and no matter is lost. Coagulation being also a mass-conserving process, we expect that matter is conserved throughout time evolution; that is,

$$M_1(f(t)) := \int_0^\infty x f(t, x) \, dx = \varrho = M_1(f^{in}) := \int_0^\infty x f^{in}(x) \, dx, \quad t \geq 0. \quad (1.3)$$

Breakdown in finite time of the identity (1.3) may actually occur; that is, there is $T_l \in [0, \infty)$ such that

$$M_1(f(t)) < M_1(f^{in}), \quad t > T_l.$$

This feature is due, either to a runaway growth generated by a coagulation kernel increasing rapidly for large sizes, a phenomenon known as *gelation* [25–27], or to the appearance of dust resulting from an overall fragmentation rate a which is unbounded as $x \rightarrow 0$, a phenomenon referred to as *shattering* [13, 28]. Loosely speaking, for the coagulation and fragmentation coefficients given by

$$K(x, y) = K_0 (x^\alpha y^{\lambda-\alpha} + x^{\lambda-\alpha} y^\alpha), \quad (x, y) \in (0, \infty)^2, \quad (1.4a)$$

with $\alpha \in [0, 1]$, $\lambda \in [2\alpha, 1 + \alpha]$, and $K_0 > 0$, and

$$a(x) = a_0 x^\gamma, \quad b(x, y) = b_\nu(x, y) := (\nu + 2)x^\nu y^{-\nu-1}, \quad 0 < x < y, \quad (1.4b)$$

with $\gamma \in \mathbb{R}$, $\nu \in (-2, \infty)$, and $a_0 > 0$, gelation after a finite time occurs when $\alpha > 1/2$ in (1.4a) and $\gamma \in (0, \lambda - 1)$ in (1.4b) [10, 11, 18, 20, 25, 27], while shattering is observed when $\gamma < 0$ in (1.4b) and there is no coagulation ($K_0 = 0$) [3, 13, 28]. In contrast, mass-conserving solutions to (1.1) satisfying (1.3) for all $t \geq 0$ exist when, either $\lambda \in [0, 1]$ and $\gamma \geq 0$, or $\lambda \in (1, 2]$ and $\gamma > \lambda - 1$ [2, 4–7, 9, 10, 12, 21, 23, 34, 35, 38]. The previous discussion reveals that the value $\gamma = \lambda - 1 > 0$ is a borderline case with respect to the occurrence of the gelation phenomenon. Indeed, on the one hand, when $\lambda \in (1, 2]$, $\gamma = \lambda - 1$, and $\alpha > -\nu - 1$ in (1.4), mass-conserving solutions to (1.1) on $[0, \infty)$ exist when $M_1(f^{in})$ is sufficiently small [19], which is in accordance with numerical simulations performed in [33] for the particular choice

$$\alpha = 1, \quad \lambda = 2, \quad \gamma = 1, \quad \nu = 0. \quad (1.5)$$

On the other hand, gelation (in finite time) takes place when $\alpha = 1$, $\lambda = 2$, $\gamma = 1$, $\nu > -1$, and $M_1(f^{in})$ is large enough [7, 33, 37].

Besides, the choice $\gamma = \lambda - 1 > 0$ in (1.4) has another interesting feature. Indeed, in this case, equation (1.1a) satisfies a scale invariance which complies with the conservation of matter (1.3). More precisely, if f is a solution to (1.1a) and $r > 0$, then the function f_r defined by

$$f_r(t, x) := r^2 f(r^{1-\lambda} t, r x), \quad (t, x) \in [0, \infty) \times (0, \infty), \quad (1.6)$$

is also a solution to (1.1a) and $M_1(f_r(t)) = M_1(f(r^{1-\lambda} t))$ for $t \geq 0$. We then look for particular solutions to (1.1a) which are left invariant by the transformation (1.6), that is, $f_r = f$ for all $r > 0$; that is, according to (1.6), $r^2 f(r^{1-\lambda} t, r x) = f(t, x)$ for all $(r, t, x) \in (0, \infty)^3$. The choice $r = t^{1/(\lambda-1)}$ in the previous identity gives

$$f(t, x) = t^{2/(\lambda-1)} f(1, x t^{1/(\lambda-1)}), \quad (t, x) \in (0, \infty)^2,$$

and raises the question of the existence of mass-conserving self-similar solutions of the form

$$(t, x) \longmapsto t^{2/(\lambda-1)} \psi \left(x t^{1/(\lambda-1)} \right), \quad (t, x) \in (0, \infty)^2. \quad (1.7)$$

In (1.7), the profile ψ is yet to be determined and is requested to have a finite total mass $M_1(\psi) = \varrho \in (0, \infty)$. According to the numerical simulations performed in [33], such solutions exist for sufficiently small values of ϱ and are expected to describe the long term dynamics of mass-conserving solutions to (1.1) with the same total mass ϱ . Thus, the existence, uniqueness, and properties of mass-conserving self-similar solutions to (1.1a) of the form (1.7) are of high interest.

The purpose of this paper is to provide one step in that direction and figure out whether self-similar solutions to (1.1a) of the form (1.6) do exist when $\gamma = \lambda - 1 > 0$ in (1.4). Such a quest is not hopeless. Indeed, on the one hand, when the parameters in (1.4) are given by (1.5), their existence is supported by numerical simulations performed in [33], which indicate that there exist mass-conserving self-similar solutions to (1.1a) of the form (1.7) with $M_1(\psi) = \varrho$, provided the ratio $a_0/(\varrho K_0)$ is large enough. On the other hand, if

$$\alpha = 1, \quad \lambda = 2, \quad \gamma = 1, \quad \nu = -1, \quad (1.8)$$

then, for any $\varrho > 0$, the existence of a unique mass-conserving self-similar solution to (1.1a) of the form (1.7) with $M_1(\psi) = \varrho$ is shown in [24] and this particular solution is a global attractor for the dynamics of (1.1) when the initial condition f^{in} satisfies $M_1(f^{in}) = \varrho$. The approach developed in [24] heavily relies on the specific structure of (1.1a) for the choice of parameters (1.8), which allows us to use the Laplace transform, and is thus not likely to be adapted to the more general setting considered herein. Instead, we first construct mass-conserving self-similar solution to (1.1a) of the form (1.7) for a restricted class of daughter distribution functions b by a dynamical approach and carefully keep track of the dependence of the estimates on the various parameters involved in K , a , and b . We next use a compactness method to extend the existence result to a broader class of b .

Specifically, we consider

$$\lambda \in (1, 2], \quad \gamma := \lambda - 1 \in (0, 1], \quad \alpha \in \left[\max \left\{ \frac{1}{2}, \lambda - 1 \right\}, \frac{\lambda}{2} \right], \quad (1.9a)$$

and assume that the overall fragmentation rate a and the coagulation kernel K are given by

$$a(x) = a_0 x^{\lambda-1}, \quad x \in (0, \infty), \quad (1.9b)$$

$$K(x, y) = K_0 (x^\alpha y^{\lambda-\alpha} + x^{\lambda-\alpha} y^\alpha), \quad (x, y) \in (0, \infty)^2, \quad (1.9c)$$

for some positive constants a_0 and K_0 . We assume further that the daughter distribution function b has the scaling form

$$b(x, y) = \frac{1}{y} B \left(\frac{x}{y} \right), \quad 0 < x < y, \quad (1.9d)$$

where

$$B \geq 0 \text{ a.e. in } (0, 1), \quad B \in L^1((0, 1), z dz), \quad \int_0^1 z B(z) dz = 1, \quad (1.9e)$$

and there is $\nu \in (-2, 0]$ such that

$$\mathfrak{b}_{m,p} := \int_0^1 z^m B(z)^p dz < \infty \quad (1.9f)$$

for all $(m, p) \in \mathcal{A}_\nu$, the set \mathcal{A}_ν being defined by

$$\mathcal{A}_\nu := \{(m, p) \in (-1, \infty) \times [1, \infty) : m + p\nu > -1\} . \quad (1.9g)$$

Observe that \mathcal{A}_ν is non-empty since

$$(m, 1) \in \mathcal{A}_\nu \quad \text{for all } m > -\nu - 1 . \quad (1.10a)$$

Also, if $(m, 1) \in \mathcal{A}_\nu$, then

$$(m, p) \in \mathcal{A}_\nu \quad \text{for all } p \in \left[1, \frac{m+1}{|\nu|}\right) . \quad (1.10b)$$

We finally assume that the small size behaviour of the coagulation kernel K is related to the possible singularity of B for small sizes and require

$$-\nu - 1 < \alpha . \quad (1.11)$$

Since $(-\nu/2, 1] \in \mathcal{A}_\nu$ by (1.10), we infer from (1.9f) and the inequality

$$\int_0^1 z |\ln z| B(z) dz \leq \sup_{z \in (0,1)} \{z^{(2+\nu)/2} |\ln z|\} \int_0^1 z^{-\nu/2} B(z) dz = \frac{2\mathfrak{b}_{-\nu/2,1}}{e(\nu+2)} ,$$

that

$$\mathfrak{b}_{\ln} := \int_0^1 z |\ln z| B(z) dz < \infty . \quad (1.12)$$

We then set

$$\varrho_\star := \frac{a_0 \mathfrak{b}_{\ln}}{2K_0 \ln 2} . \quad (1.13)$$

For $m \in \mathbb{R}$, we define the weighted L^1 -space X_m and the moment $M_m(h)$ of order m of $h \in X_m$ by

$$X_m := L^1((0, \infty), x^m dx) , \quad M_m(h) := \int_0^\infty x^m h(x) dx .$$

We also denote the positive cone of X_m by X_m^+ , while $X_{m,w}$ denotes the space X_m endowed with its weak topology.

For the above described class of coagulation and fragmentation coefficients, the main result of this paper guarantees the existence of at least one mass-conserving self-similar solution to (1.1a) of the form (1.7) (up to a rescaling, see Remark 1.2 below) with a sufficiently small total mass ϱ .

Theorem 1.1. *Consider coagulation and fragmentation coefficients K , a , and b satisfying (1.9) and fix two auxiliary parameters*

$$m_0 \in (-\nu - 1, \alpha) \cap [0, 1) , \quad m_1 := \max\{m_0, 2 - \lambda\} \in (0, 1) . \quad (1.14)$$

Let $\varrho \in (0, \varrho_\star)$.

(a) There are $q_1 \in (1, 2)$ (defined in (2.9) below) and a non-negative profile

$$\varphi \in X_1^+ \cap L^{q_1}((0, \infty), x^{m_1} dx) \cap \bigcap_{m \geq m_0} X_m, \quad M_1(\varphi) = \varrho, \quad (1.15)$$

such that $(m_1, q_1) \in \mathcal{A}_\nu$ and

$$\begin{aligned} \int_0^\infty [\vartheta(x) - x \partial_x \vartheta(x)] \varphi(x) dx &= \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) \varphi(x) \varphi(y) dy dx \\ &\quad - \int_0^\infty a(y) N_\vartheta(y) \varphi(y) dy \end{aligned} \quad (1.16)$$

for all $\vartheta \in \Theta_1$, where

$$\Theta_1 := \{ \vartheta \in W^{1, \infty}(0, \infty) : \vartheta(0) = 0 \}, \quad (1.17)$$

and

$$\chi_\vartheta(x, y) := \vartheta(x + y) - \vartheta(x) - \vartheta(y), \quad (x, y) \in (0, \infty)^2, \quad (1.18)$$

$$N_\vartheta(y) := \vartheta(y) - \int_0^y \vartheta(x) b(x, y) dx, \quad y \in (0, \infty). \quad (1.19)$$

(b) The function F_S defined by

$$F_S(t, x) := s_\lambda(t)^2 \varphi(x s_\lambda(t)), \quad (t, x) \in [0, \infty) \times (0, \infty), \quad (1.20)$$

with $s_\lambda(t) := (1 + (\lambda - 1)t)^{1/(\lambda - 1)}$, $t \geq 0$, is a mass-conserving weak solution to (1.1) on $[0, \infty)$ with initial condition $f^{in} = \varphi$ in the following sense: for any $T > 0$,

$$F_S \in C([0, T], X_{m_1, w}) \cap C([0, T], X_{1, w}) \cap L^\infty((0, T), X_{m_0})$$

and satisfies

$$\begin{aligned} \int_0^\infty (F_S(t, x) - \varphi(x)) \vartheta(x) dx &= \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) F_S(s, x) F_S(s, y) dy dx ds \\ &\quad - \int_0^t \int_0^\infty a(x) N_\vartheta(x) F_S(s, x) dx ds, \end{aligned} \quad (1.21)$$

for all $t \in (0, \infty)$ and $\vartheta \in \Theta_{m_1}$, where $\Theta_0 := L^\infty(0, \infty)$ and

$$\Theta_m := \{ \vartheta \in C^m([0, \infty)) \cap L^\infty(0, \infty) : \vartheta(0) = 0 \}, \quad m \in (0, 1).$$

Remark 1.2. The self-similar ansatz (1.7) differs slightly from that of F_S in Theorem 1.1, see (1.20). However, they can both be mapped to each other, up to an X_1 -invariant dilation of the profile. Indeed, if $F_S(t, x) = s_\lambda(t)^2 \varphi(x s_\lambda(t))$, $(t, x) \in (0, \infty)^2$, is a mass-conserving self-similar solution to (1.1a) of the form (1.20), then it is actually well-defined for $(t, x) \in (-1/(\lambda - 1), \infty) \times (0, \infty)$. Combining this property with the autonomous character of the coagulation-fragmentation equation (1.1a) implies that $\tilde{F}_S(t, x) := F_S(t - (\lambda - 1)^{-1}, x)$, $(t, x) \in (0, \infty)^2$, is also a solution to (1.1a) and satisfies

$$\tilde{F}_S(t, x) := t^{2/(\lambda - 1)} \psi(x t^{1/(\lambda - 1)}), \quad (t, x) \in [0, \infty) \times (0, \infty),$$

with $\psi(y) = (\lambda - 1)^{-2/(\lambda-1)}\varphi(y(\lambda - 1)^{-1/(\lambda-1)})$, $y > 0$. In other words, \tilde{F}_S is a mass-conserving self-similar solution to (1.1a) of the form (1.7) and it has total mass ϱ , since $M_1(\varphi) = M_1(\psi) = \varrho$ by (1.15).

On the one hand, Theorem 1.1 and Remark 1.2 provide the existence of mass-conserving self-similar solutions to (1.1) of the form (1.7) with a sufficiently small total mass for the parameters given by (1.5), which is in perfect agreement with the numerical simulations performed in [33]. It is yet unclear whether ϱ_* is the largest value of ϱ for which a mass-conserving self-similar solution to (1.1) of the form (1.7) with total mass ϱ exists. However, Theorem 1.1 cannot be valid for any $\varrho > 0$ in general. Indeed, when the parameters in (1.9) are given by (1.5), gelation occurs for sufficiently large mass, as indicated by explicit computations performed in [33,37] and proved in [7] when $a_0/(\varrho K_0) < 1$. On the other hand, Theorem 1.1 provides the existence of mass-conserving self-similar solutions to (1.1) of the form (1.7) with a sufficiently small total mass for the parameters given by (1.8), a result which is far from optimal, since such a solution exists for any value of the total mass, according to [24]. A possible explanation for this discrepancy is that the absence of a threshold mass is due to the non-integrability as $x \rightarrow 0$ of the daughter distribution function b_{-1} , which is not really exploited in the proof of Theorem 1.1 below.

Let us now describe the approach we use in this paper to prove Theorem 1.1. Owing to the homogeneity of K , a , and B , inserting the ansatz (1.20) in (1.1a) implies that φ solves the integro-differential equation

$$y \frac{d\varphi}{dy}(y) + 2\varphi(y) = \mathcal{C}\varphi(y) + \mathcal{F}\varphi(y), \quad y \in (0, \infty). \quad (1.22)$$

Unfortunately, the equation (1.22) seems hardly tractable as an initial value problem with initial condition at $y = 0$. Indeed, on the one hand, the right hand side of (1.22) depends not only on the past $(0, y)$ of y but also on its future (y, ∞) . On the other hand, the left hand side is degenerate, as the factor y in front of $d\varphi/dy$ vanishes at $y = 0$. Assuming further that $y^2\varphi(y) \rightarrow 0$ as $y \rightarrow 0$, one can get rid of the derivative in (1.22) and show that φ also satisfies the nonlinear integral equation

$$y^2\varphi(y) = \int_y^\infty a(x)\varphi(x) \int_0^y x_* b(x_*, x) dx_* dx - \int_0^y \int_{y-x}^\infty xK(x, x_*)\varphi(x)\varphi(x_*) dx_* dx \quad (1.23)$$

for $y \in (0, \infty)$, see [13,36]. It is however unclear whether this alternative formulation is more helpful than (1.22) to investigate the existence issue, though it has been extensively used to determine the behaviour for small and large sizes of the profile of mass-conserving self-similar solutions to the coagulation equation [16,26,31,32,36]. We thus employ a different approach here, which has already proved successful for the coagulation equation [12,15,32] and the fragmentation equation [12,29]. It relies on the construction of a convex and compact subset of X_1 which is left invariant by the evolution equation associated to (1.22). This evolution equation is actually obtained from (1.1) by using the so-called scaling or self-similar variables. More precisely, recalling that $s_\lambda(t) = (1 + (\lambda - 1)t)^{1/(\lambda-1)}$, $t \geq 0$, we introduce the scaling variables

$$s := \ln s_\lambda(t), \quad y := xs_\lambda(t), \quad (t, x) \in [0, \infty) \times (0, \infty),$$

and the rescaled size distribution function

$$g(s, y) := e^{-2s} f\left(\frac{e^{(\lambda-1)s} - 1}{\lambda - 1}, ye^{-s}\right), \quad (s, y) \in [0, \infty) \times (0, \infty). \quad (1.24)$$

Equivalently,

$$f(t, x) = s_\lambda(t)^2 g(\ln s_\lambda(t), xs_\lambda(t)), \quad (t, x) \in [0, \infty) \times (0, \infty). \quad (1.25)$$

Now, if f is a solution to (1.1), then g solves

$$\partial_s g(s, y) = -y \partial_y g(s, y) - 2g(s, y) + \mathcal{C}g(s, y) + \mathcal{F}g(s, y), \quad (s, y) \in (0, \infty)^2, \quad (1.26a)$$

$$g(0, y) = f^{in}(y), \quad y \in (0, \infty), \quad (1.26b)$$

Comparing (1.22) and (1.26a), we readily see that φ is a stationary solution to (1.26a), so that proving Theorem 1.1 amounts to find a steady-state solution to (1.26a). To this end, we shall use a consequence of Schauder's fixed point theorem which guarantees the existence of a steady state for a dynamical system defined in a closed subset Y of a Banach space X which leaves invariant a convex and compact subset of Y , see [1, Proposition 22.13] and [17, Proof of Theorem 5.2] (see also [12, Theorem 1.2] for the extension of this result to a Banach space endowed with its weak topology). Applying the just mentioned result requires identifying a suitable functional framework in which, not only (1.26) is well-posed, but also leaves invariant a convex and compact subset of the chosen function space. To achieve this goal, the assumption (1.9f) for any $(m, p) \in \mathcal{A}_\nu$ does not seem to be sufficient and we first construct a family $(b_\varepsilon, B_\varepsilon)_{\varepsilon \in (0,1)}$ of approximations of (b, B) , which satisfy not only (1.9d) and (1.9e), but also (1.9f) for any $(m, p) \in \mathcal{A}_0$ and $B_\varepsilon \in W^{1,1}(0, 1)$. We then prove that the corresponding rescaled coagulation-fragmentation equation (1.26) is well-posed in X_1 for initial conditions $f^{in} \in X_{m_0}^+ \cap X_{1+\lambda}$ satisfying $M_1(f^{in}) = \varrho \in (0, \varrho_*)$. We also show the existence of an invariant convex and compact subset \mathcal{Z}_ε of X_1 for the associated dynamical system. According to the above mentioned result, this analysis guarantees the existence of a stationary solution $\varphi_\varepsilon \in X_1^+$ to (1.26a) satisfying $M_1(\varphi_\varepsilon) = \varrho$. Moreover, it turns out that there is a convex and sequentially weakly compact subset \mathcal{Z} of X_1 such that $\mathcal{Z}_\varepsilon \subset \mathcal{Z}$ for all $\varepsilon \in (0, 1)$. Consequently, $(\varphi_\varepsilon)_{\varepsilon \in (0,1)}$ is relatively sequentially weakly compact in X_1 and the information derived from \mathcal{Z} allows us to prove that cluster points in $X_{1,w}$ of $(\varphi_\varepsilon)_{\varepsilon \in (0,1)}$ as $\varepsilon \rightarrow 0$ solve (1.22), thereby completing the proof of Theorem 1.1.

Remark 1.3. *In the companion paper [19], we prove that, given an initial condition $f^{in} \in X_{m_0}^+ \cap X_{2\lambda-\alpha}$ satisfying $M_1(f^{in}) = \varrho \in (0, \varrho_*)$, the coagulation-fragmentation equation (1.1) has a unique mass-conserving weak solution on $[0, \infty)$ under the same assumptions (1.9) on the coagulation and fragmentation coefficients. This result is perfectly consistent with the numerical simulations performed in [33], as is Theorem 1.1.*

2. SELF-SIMILAR SOLUTIONS: A REGULARISED PROBLEM

In this section, we assume that K , a , and b are coagulation and fragmentation coefficients satisfying (1.9) and we fix $\varrho \in (0, \varrho_*)$.

As already mentioned, two steps are needed to prove Theorem 1.1 and this section is devoted to the first step; that is, the proof of Theorem 1.1 for a family $(b_\varepsilon)_{\varepsilon > 0}$ of approximations of the

daughter distribution function b . We begin with the construction of a suitably regularised version of the daughter distribution function b . To this end, we fix a non-negative function $\zeta \in C_0^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \zeta(z) \, dz = 1, \quad \text{supp } \zeta \subset (-1, 1),$$

and set $\zeta_\varepsilon(z) := \varepsilon^{-2} \zeta(z\varepsilon^{-2})$ for $z \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. For $\varepsilon \in (0, 1)$, we define

$$\beta_\varepsilon := \int_0^1 z \int_\varepsilon^1 \zeta_\varepsilon(z - z_*) B(z_*) \, dz_* \, dz, \quad (2.1a)$$

$$B_\varepsilon(z) := \frac{1}{\beta_\varepsilon} \int_\varepsilon^1 \zeta_\varepsilon(z - z_*) B(z_*) \, dz_*, \quad z \in (0, 1), \quad (2.1b)$$

and

$$b_\varepsilon(x, y) := \frac{1}{y} B_\varepsilon\left(\frac{x}{y}\right), \quad 0 < x < y. \quad (2.1c)$$

As we shall see below, see (2.2b), the parameter β_ε is positive for $\varepsilon > 0$ sufficiently small, so that B_ε is well-defined for such values of ε . Indeed, thanks to (1.9e), (1.9f), and the properties of ζ ,

$$B_\varepsilon \geq 0 \quad \text{a.e. in } (0, 1), \quad B_\varepsilon \in L^1((0, 1), z \, dz), \quad \int_0^1 z B_\varepsilon(z) \, dz = 1, \quad (2.2a)$$

$$\lim_{\varepsilon \rightarrow 0} \beta_\varepsilon = 1, \quad (2.2b)$$

and $B_\varepsilon \in L^p((0, \infty), z^m \, dz)$ for all $(m, p) \in \mathcal{A}_\nu$ with

$$\lim_{\varepsilon \rightarrow 0} \int_0^1 z^m |B_\varepsilon(z) - B(z)|^p \, dz = \lim_{\varepsilon \rightarrow 0} \int_0^1 z |\ln z| |B_\varepsilon(z) - B(z)| \, dz = 0. \quad (2.2c)$$

An obvious consequence of (2.2c) is that

$$\lim_{\varepsilon \rightarrow 0} \mathfrak{b}_{m,p,\varepsilon} = \mathfrak{b}_{m,p}, \quad (m, p) \in \mathcal{A}_\nu, \quad \lim_{\varepsilon \rightarrow 0} \mathfrak{b}_{\ln,\varepsilon} = \mathfrak{b}_{\ln}, \quad (2.3)$$

where

$$\mathfrak{b}_{m,p,\varepsilon} := \int_0^1 z^m B_\varepsilon(z)^p \, dz, \quad (m, p) \in \mathbb{R} \times [1, \infty), \quad \mathfrak{b}_{\ln,\varepsilon} := \int_0^1 z |\ln z| B_\varepsilon(z) \, dz.$$

Recalling that $1 + \mathfrak{b}_{1+\lambda-\alpha,1} > 2\mathfrak{b}_{1+\lambda-\alpha,1}$ due to $1 + \lambda - \alpha > 1$, it follows from (2.2b) and (2.3) that there is $\varepsilon_0 \in (0, 1)$ such that, for $\varepsilon \in (0, \varepsilon_0)$,

$$\mathfrak{b}_{m_0,1,\varepsilon} \leq 1 + \mathfrak{b}_{m_0,1}, \quad \mathfrak{b}_{1+\lambda-\alpha,1,\varepsilon} \leq \frac{1 + \mathfrak{b}_{1+\lambda-\alpha,1}}{2} < 1, \quad \mathfrak{b}_{m_1,q_1,\varepsilon} \leq 1 + \mathfrak{b}_{m_1,q_1}. \quad (2.4)$$

An immediate consequence of (2.4) is that, for $\varepsilon \in (0, \varepsilon_0)$,

$$\sup_{m \geq m_0} \{\mathfrak{b}_{m,1,\varepsilon}\} \leq 1 + \mathfrak{b}_{m_0,1}, \quad \sup_{m \geq 1+\lambda-\alpha} \{\mathfrak{b}_{m,1,\varepsilon}\} \leq \frac{1 + \mathfrak{b}_{1+\lambda-\alpha,1}}{2}. \quad (2.5)$$

Moreover,

$$B_\varepsilon(z) = 0, \quad z \in [0, \varepsilon - \varepsilon^2], \quad B_\varepsilon \in W^{1,1}(0, 1), \quad (2.6)$$

and

$$\int_0^1 B_\varepsilon(z) \, dz \leq \frac{1}{\varepsilon\beta_\varepsilon}, \quad \sup_{z \in [0,1]} \{B_\varepsilon(z)\} \leq \int_0^1 \left| \frac{dB_\varepsilon}{dz}(z) \right| \, dz \leq \frac{1}{\varepsilon^3\beta_\varepsilon} \int_{\mathbb{R}} \left| \frac{d\zeta}{dz}(z) \right| \, dz. \quad (2.7)$$

Remark 2.1. *In fact, if the function B in (1.9e) satisfies (1.9f) for any $(m, p) \in \mathcal{A}_0$, as well as $B(0) = 0$ and $B \in W^{1,1}(0, 1)$, then we may take $B_\varepsilon = B$. This is true in particular for the parabolic daughter distribution function corresponding to $B(z) = 12z(1 - z)$, $z \in (0, 1)$.*

Next, since $\varrho \in (0, \varrho_\star)$, we infer from (2.3) that there is $\varepsilon_\varrho \in (0, \varepsilon_0)$ such that

$$\varrho < \frac{\varrho + \varrho_\star}{2} \leq \varrho_{\star, \varepsilon} := \frac{a_0 \mathbf{b}_{\ln, \varepsilon}}{2K_0 \ln 2}, \quad \varepsilon \in (0, \varepsilon_\varrho). \quad (2.8)$$

Finally, since $m_1 + \lambda - 1 \in (m_0, \lambda)$ by (1.9a) and $(m_1, 1) \in \mathcal{A}_\nu$ by (1.10a), we may fix

$$q_1 \in (1, 2) \text{ such that } (m_1, q_1) \in \mathcal{A}_\nu \text{ and } \frac{m_1 + 1 + q_1(\lambda - 2)}{q_1} \in (m_0, \lambda). \quad (2.9)$$

The main result of this section is then the following:

Proposition 2.2. *Let $\varepsilon \in (0, \varepsilon_\varrho)$. There is*

$$\varphi_\varepsilon \in X_1^+ \cap L^{q_1}((0, \infty), x^{m_1} dx) \cap W^{1,1}(0, \infty) \cap \bigcap_{m \geq \lambda - 2} X_m,$$

such that $M_1(\varphi_\varepsilon) = \varrho$ and

$$\begin{aligned} \int_0^\infty [\vartheta(x) - x\partial_x \vartheta(x)] \varphi_\varepsilon(x) \, dx &= \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) \varphi_\varepsilon(x) \varphi_\varepsilon(y) \, dy dx \\ &\quad - \int_0^\infty a(x) N_{\vartheta, \varepsilon}(x) \varphi_\varepsilon(x) \, dx, \end{aligned} \quad (2.10)$$

for all $\vartheta \in \Theta_1$, where Θ_1 is defined in (1.17) and

$$N_{\vartheta, \varepsilon}(y) := \vartheta(y) - \int_0^y \vartheta(x) b_\varepsilon(x, y) \, dx, \quad y > 0.$$

Moreover,

(a) *There is $\ell > 0$ depending only on $\lambda, \alpha, K_0, a_0, B, \nu, m_0, m_1, q_1$, and ϱ such that*

$$\int_0^\infty x \ln(x) \varphi_\varepsilon(x) \, dx + \frac{3}{e(1 - m_1)} M_{m_1}(\varphi_\varepsilon) \leq \ell, \quad (2.11a)$$

$$M_{m_0}(\varphi_\varepsilon) \leq \ell, \quad (2.11b)$$

$$\int_0^\infty x^{m_1} \varphi_\varepsilon(x)^{q_1} \, dx \leq \ell. \quad (2.11c)$$

(b) *For all $m \geq 1 + \lambda$, there is $L(m) > 0$ depending only on $\lambda, \alpha, K_0, a_0, B, \nu, m_0, m_1, q_1, \varrho$, and m such that*

$$M_m(\varphi_\varepsilon) \leq L(m). \quad (2.11d)$$

The main steps in the proof of Proposition 2.2 are the derivation of (2.11a) and (2.11c). The former is inspired from [12, Lemma 4.2] and combines a differential inequality for a superlinear moment, involving here the weight $x \mapsto x \ln x$, and a differential inequality for a sublinear moment. The validity of (2.11a) requires the smallness condition $\varrho \in (0, \varrho_*)$, the value of ϱ_* being prescribed by an algebraic inequality established in [19, Lemma 2.3], see (2.20) below. As for (2.11c), it relies on the monotonicity of $x \mapsto x^{m_1} K(x, y)$ to handle the contribution of the coagulation term, similar arguments being used in [7, 8, 22, 30] to derive L^p -estimates for solutions to coagulation-fragmentation equations.

2.1. Scaling variables and well-posedness. Let $\varepsilon \in (0, \varepsilon_\varrho)$. We begin with the existence and uniqueness of a mass-conserving weak solution to

$$\partial_s g_\varepsilon(s, x) = -x \partial_x g_\varepsilon(s, x) - 2g_\varepsilon(s, x) + \mathcal{C}g_\varepsilon(s, x) + \mathcal{F}_\varepsilon g_\varepsilon(s, x) , \quad (s, x) \in (0, \infty)^2 , \quad (2.12a)$$

$$g_\varepsilon(0, x) = f^{in}(x) , \quad x \in (0, \infty) , \quad (2.12b)$$

where \mathcal{F}_ε denotes the fragmentation operator with b replaced with b_ε .

Proposition 2.3. *Consider an initial condition $f^{in} \in X_1^+ \cap X_{m_0} \cap X_{2\lambda-\alpha}$ such that*

$$M_1(f^{in}) = \varrho . \quad (2.13)$$

There is a unique mass-conserving weak solution g_ε to (1.1) on $[0, \infty)$ satisfying

$$g_\varepsilon \in C([0, T], X_{m_1, w}) \cap L^\infty((0, T), X_{m_0}) \cap L^\infty((0, T), X_{2\lambda-\alpha}) \text{ for any } T > 0 ,$$

$$M_1(g_\varepsilon(s)) = \varrho , \quad s \geq 0 , \quad (2.14)$$

and

$$\begin{aligned} \int_0^\infty (g_\varepsilon(s, x) - f^{in}(x)) \vartheta(x) \, dx &= \int_0^s \int_0^\infty [x \partial_x \vartheta(x) - \vartheta(x)] g_\varepsilon(s_*, x) \, dx ds_* \\ &\quad + \frac{1}{2} \int_0^s \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) g_\varepsilon(s_*, x) g_\varepsilon(s_*, y) \, dy dx ds_* \\ &\quad - \int_0^s \int_0^\infty a(y) N_{\vartheta, \varepsilon}(y) g_\varepsilon(s_*, y) \, dy ds_* , \end{aligned} \quad (2.15)$$

for all $s \in (0, \infty)$ and $\vartheta \in \Theta_{m_1}$, where $\Theta_0 := L^\infty(0, \infty)$ and

$$\Theta_m := \{ \vartheta \in C^m([0, \infty)) \cap L^\infty(0, \infty) : \vartheta(0) = 0 \} .$$

We recall that $N_{\vartheta, \varepsilon}$ in (2.15) is defined in Proposition 2.2,

Proof. Owing to (1.9a), (1.9b), (1.9c), (2.1c), (2.2a), and the integrability properties of B_ε , we are in a position to apply [19, Theorem 1.2], which guarantees the existence and uniqueness of a mass-conserving weak solution f_ε to the coagulation-fragmentation equation

$$\partial_t f_\varepsilon(t, x) = \mathcal{C}f_\varepsilon(t, x) + \mathcal{F}_\varepsilon f_\varepsilon(t, x) , \quad (t, x) \in (0, \infty)^2 , \quad (2.16a)$$

$$f_\varepsilon(0, x) = f^{in}(x) , \quad x \in (0, \infty) , \quad (2.16b)$$

which satisfies

$$f_\varepsilon \in C([0, T], X_{m_1, w}) \cap L^\infty((0, T), X_{m_0}) \cap L^\infty((0, T), X_{2\lambda-\alpha})$$

for any $T > 0$ and $M_1(f_\varepsilon(t)) = \varrho$ for $t \geq 0$. Setting

$$\Psi_\varepsilon(s; f^{in})(x) = g_\varepsilon(s, x) := e^{-2s} f_\varepsilon \left(\frac{e^{(\lambda-1)s} - 1}{\lambda - 1}, xe^{-s} \right), \quad (s, x) \in [0, \infty) \times (0, \infty), \quad (2.17)$$

completes the proof of Proposition 2.3. \square

The next results are devoted to the derivation of a series of estimates satisfied by the weak solutions to (2.12) provided by Proposition 2.3, except for Lemma 2.12 where the continuous dependence of $\Psi_\varepsilon(\cdot; f^{in})$ in X_1 with respect to the initial condition is established.

Throughout the remainder of this section, κ and $(\kappa_i)_{i \geq 1}$ are positive constants depending only on $\lambda, \alpha, K_0, a_0, B, \nu, m_0, m_1, q_1$, and ϱ . Dependence upon additional parameters is indicated explicitly.

2.2. Moment Estimates. We begin with the derivation of estimates for moments of order $m \in [m_1, 1]$, the parameter m_1 being defined in (1.14).

Lemma 2.4. *Consider $\varepsilon \in (0, \varepsilon_\varrho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ such that $M_1(f^{in}) = \varrho$ and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). For $m \in [m_1, 1]$, there is $\kappa_1(m) > 0$ depending on m such that, for $t \geq 0$,*

$$\begin{aligned} & \int_0^\infty x \ln(x) g_\varepsilon(s, x) dx + \frac{3}{e(1-m)} M_m(g_\varepsilon(s)) \\ & \leq \max \left\{ \int_0^\infty x \ln(x) f^{in}(x) dx + \frac{3}{e(1-m)} M_m(f^{in}), \kappa_1(m) \right\}, \quad s \geq 0. \end{aligned}$$

Proof. Let $s \geq 0$ and consider $m \in [m_1, 1]$. Then

$$\chi_m(x, y) := (x + y)^m - x^m - y^m \leq 0, \quad (x, y) \in (0, \infty)^2,$$

and

$$N_{m, \varepsilon}(y) := y^m - \int_0^y x^m b_\varepsilon(x, y) dx = (1 - \mathbf{b}_{m, 1, \varepsilon}) y^m \geq -\mathbf{b}_{m, 1, \varepsilon} y^m, \quad y \in (0, \infty).$$

Consequently, we infer from (1.9b), (2.5), (2.15) (with $\vartheta(x) = x^m, x > 0$), and the non-negativity of g_ε and K that

$$\begin{aligned} \frac{d}{ds} M_m(g_\varepsilon(s)) & \leq -(1-m) M_m(g_\varepsilon(s)) + a_0 \mathbf{b}_{m, 1, \varepsilon} M_{m+\lambda-1}(g_\varepsilon(s)) \\ & \leq -(1-m) M_m(g_\varepsilon(s)) + a_0 (1 + \mathbf{b}_{m_0, 1}) M_{m+\lambda-1}(g_\varepsilon(s)). \end{aligned}$$

Observing that $m + \lambda - 1 \in [1, \lambda]$, it follows from (2.14) and Hölder's inequality that

$$\begin{aligned} M_{m+\lambda-1}(g_\varepsilon(s)) & \leq M_\lambda(g_\varepsilon(s))^{(m+\lambda-2)/(\lambda-1)} M_1(g_\varepsilon(s))^{(1-m)/(\lambda-1)} \\ & \leq \varrho^{(1-m)/(\lambda-1)} M_\lambda(g_\varepsilon(s))^{(m+\lambda-2)/(\lambda-1)}. \end{aligned}$$

We combine the previous two inequalities and use Young's inequality (since $m + \lambda - 2 < \lambda - 1$) to obtain

$$\frac{d}{ds}M_m(g_\varepsilon(s)) \leq -(1-m)M_m(g_\varepsilon(s)) + \frac{e(1-m)}{3}\delta_\varrho M_\lambda(g_\varepsilon(s)) + \frac{e(1-m)}{3}\kappa(m), \quad (2.18)$$

with

$$\delta_\varrho := \frac{K_0 \ln 2}{2}(\varrho_\star - \varrho) > 0, \quad (2.19)$$

We next set $\bar{\vartheta}(x) = x \ln x$ for $x \geq 0$ and recall the inequality

$$\chi_{\bar{\vartheta}}(x, y) = (x + y) \ln(x + y) - x \ln x - y \ln y \leq 2 \ln 2 \sqrt{xy}, \quad (x, y) \in (0, \infty)^2, \quad (2.20)$$

established in [19, Lemma 2.3], along with the following consequence of (1.9a), (1.9c), and Young's inequality

$$\begin{aligned} \sqrt{xy}K(x, y) &\leq K_0xy \left(x^{(2\alpha-1)/2}y^{(2\lambda-2\alpha-1)/2} + x^{(2\lambda-2\alpha-1)/2}y^{(2\alpha-1)/2} \right) \\ &\leq K_0xy \left(\frac{2\alpha-1}{2(\lambda-1)}x^{\lambda-1} + \frac{2\lambda-2\alpha-1}{2(\lambda-1)}y^{\lambda-1} + \frac{2\lambda-2\alpha-1}{2(\lambda-1)}x^{\lambda-1} + \frac{2\alpha-1}{2(\lambda-1)}y^{\lambda-1} \right) \\ &\leq K_0(x^\lambda y + xy^\lambda), \quad (x, y) \in (0, \infty)^2. \end{aligned}$$

Also, by (2.1c) and (2.2a),

$$N_{\bar{\vartheta}, \varepsilon}(y) = y \ln y - \int_0^1 yz \ln(yz) B_\varepsilon(z) dz = \mathfrak{b}_{\ln, \varepsilon} y, \quad y \in (0, \infty).$$

We then infer from (1.9b), (1.9c), (2.8), (2.14), and (2.15) (with $\vartheta = \bar{\vartheta}$) that

$$\begin{aligned} \frac{d}{ds} \int_0^\infty \bar{\vartheta}(x) g_\varepsilon(s, x) dx &\leq M_1(g_\varepsilon(s)) + 2K_0 \ln(2) M_1(g_\varepsilon(s)) M_\lambda(g_\varepsilon(s)) - a_0 \mathfrak{b}_{\ln, \varepsilon} M_\lambda(g_\varepsilon(s)) \\ &\leq \varrho + 2K_0 \ln 2 (\varrho - \varrho_{\star, \varepsilon}) M_\lambda(g_\varepsilon(s)) \\ &\leq \varrho - 2\delta_\varrho M_\lambda(g_\varepsilon(s)), \end{aligned}$$

the parameter δ_ϱ being defined in (2.19). Combining (2.18) and the previous inequality, we find

$$\frac{d}{ds}U_{m, \varepsilon}(s) + \frac{3}{e}M_m(g_\varepsilon(s)) + \delta_\varrho M_\lambda(g_\varepsilon(s)) \leq \kappa_2(m), \quad (2.21)$$

where

$$U_{m, \varepsilon}(s) := \int_0^\infty \bar{\vartheta}(x) g_\varepsilon(s, x) dx + \frac{3}{e(1-m)} M_m(g_\varepsilon(s)).$$

Since

$$x^{\lambda-1} \geq \ln x + \frac{1 + \ln(\lambda-1)}{\lambda-1}, \quad x \in (0, \infty),$$

there holds

$$M_\lambda(g_\varepsilon(s)) \geq \int_0^\infty \bar{\vartheta}(x) g_\varepsilon(s, x) dx + \frac{1 + \ln(\lambda-1)}{\lambda-1} M_1(g_\varepsilon(s)).$$

Consequently, setting $\kappa_3(m) := \min\{1 - m, \delta_\rho\}$ and using once more (2.14), we obtain

$$\begin{aligned} & \frac{d}{ds} U_{m,\varepsilon}(s) + \kappa_3(m) U_{m,\varepsilon}(s) \\ & \leq \frac{d}{ds} U_{m,\varepsilon}(s) + \kappa_3(m) \left[M_\lambda(g_\varepsilon(s)) - \rho \frac{1 + \ln(\lambda - 1)}{\lambda - 1} + \frac{3}{e(1 - m)} M_m(g_\varepsilon(s)) \right] \\ & \leq (\kappa_3(m) - \delta_\rho) M_\lambda(g_\varepsilon(s)) + \frac{3[\kappa_3(m) - (1 - m)]}{e(1 - m)} M_m(g_\varepsilon(s)) + \kappa_4(m) \\ & \leq \kappa_4(m) . \end{aligned}$$

Integrating with respect to s gives

$$U_{m,\varepsilon}(s) \leq e^{-\kappa_3(m)s} U_{m,\varepsilon}(0) + \frac{\kappa_4(m)}{\kappa_3(m)} (1 - e^{-\kappa_3(m)s}) \leq \max \left\{ U_{m,\varepsilon}(0), \frac{\kappa_4(m)}{\kappa_3(m)} \right\}$$

for $s \geq 0$ and Lemma 2.4 follows with $\kappa_1(m) := \kappa_4(m)/\kappa_3(m)$. \square

From now on, we assume that f^{in} satisfies

$$M_1(f^{in}) = \rho \text{ and } \int_0^\infty x \ln(x) f^{in}(x) dx + \frac{3}{e(1 - m_1)} M_{m_1}(f^{in}) \leq \kappa_1(m_1) . \quad (2.22)$$

A straightforward consequence of Lemma 2.4 is the following estimate.

Corollary 2.5. *Consider $\varepsilon \in (0, \varepsilon_\rho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). There is $\kappa_5 > 0$ such that*

$$\int_0^\infty x |\ln x| g_\varepsilon(s, x) dx + M_{m_1}(g_\varepsilon(s)) \leq \kappa_5 , \quad s \geq 0 .$$

Proof. Let $s \geq 0$. Since

$$x |\ln x| - \frac{2x^{m_1}}{e(1 - m_1)} \leq x \ln x \leq x |\ln x| , \quad x > 0 ,$$

it follows from (2.22) and Lemma 2.4 (with $m = m_1$) that

$$\begin{aligned} & \int_0^\infty x |\ln x| g_\varepsilon(s, x) dx + \frac{1}{e(1 - m_1)} M_{m_1}(g_\varepsilon(s)) \\ & \leq \int_0^\infty x \ln x g_\varepsilon(s, x) dx + \frac{3}{e(1 - m_1)} M_{m_1}(g_\varepsilon(s)) \\ & \leq \kappa_1(m_1) , \end{aligned}$$

from which Corollary 2.5 follows. \square

Thanks to Corollary 2.5, we may derive additional information on the behaviour of g_ε for large sizes.

Lemma 2.6. Consider $\varepsilon \in (0, \varepsilon_\varrho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). Assume also that $f^{in} \in X_m$ for some $m > 1 + \lambda - \alpha$. Then there is $\kappa_6(m) > 0$ depending on m such that

$$M_m(g_\varepsilon(s)) \leq \max \{M_m(f^{in}), \kappa_6(m)\} , \quad s \geq 0 .$$

Proof. Let $s \geq 0$. We infer from (2.2) and (2.15) that

$$\frac{d}{ds} M_m(g_\varepsilon(s)) = (m-1)M_m(g_\varepsilon(s)) + P_{m,\varepsilon}(s) - a_0(1 - \mathfrak{b}_{m,1,\varepsilon})M_{m+\lambda-1}(g_\varepsilon(s)) , \quad (2.23)$$

with

$$P_{m,\varepsilon}(s) := \frac{1}{2} \int_0^\infty \int_0^\infty K(y, y_*) \chi_m(y, y_*) g_\varepsilon(s, y) g_\varepsilon(s, y_*) dy_* dy .$$

On the one hand, since $\lambda > 1$, it follows from (2.14), (2.22), and Hölder's inequality that

$$M_m(g_\varepsilon(s)) \leq M_{m+\lambda-1}(g_\varepsilon(s))^{(m-1)/(m+\lambda-2)} \varrho^{(\lambda-1)/(m+\lambda-2)} .$$

Equivalently,

$$\varrho^{(1-\lambda)/(m-1)} M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} \leq M_{m+\lambda-1}(g_\varepsilon(s)) .$$

In addition, by (2.5),

$$1 - \mathfrak{b}_{m,1,\varepsilon} \geq \frac{1 - \mathfrak{b}_{1+\lambda-\alpha,1}}{2} > 0 .$$

Consequently,

$$-a_0(1 - \mathfrak{b}_{m,1,\varepsilon})M_{m+\lambda-1}(g_\varepsilon(s)) \leq -4\delta_{\varrho,m} M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} , \quad (2.24)$$

with

$$\delta_{\varrho,m} := \frac{a_0(1 - \mathfrak{b}_{1+\lambda-\alpha,1})\varrho^{(1-\lambda)/(m-1)}}{8} > 0 . \quad (2.25)$$

On the other hand, to estimate the contribution of the coagulation term, we argue as in [19, Lemma 2.6]. Since $m > 1$, there is $c_m > 0$ depending only on m such that

$$\chi_m(x, y) = (x+y)^m - x^m - y^m \leq c_m (xy^{m-1} + x^{m-1}y) , \quad (x, y) \in (0, \infty)^2 ,$$

and it follows from (1.9c) and the previous inequality that

$$\begin{aligned} P_{m,\varepsilon}(s) &\leq \frac{c_m}{2} \int_0^\infty \int_0^\infty K(x, y) (xy^{m-1} + x^{m-1}y) g_\varepsilon(s, x) g_\varepsilon(s, y) dy dx \\ &= K_0 c_m \int_0^\infty \int_0^\infty xy^{m-1} (x^\alpha y^{\lambda-\alpha} + x^{\lambda-\alpha} y^\alpha) g_\varepsilon(s, x) g_\varepsilon(s, y) dy dx \\ &= K_0 c_m [M_{1+\alpha}(g_\varepsilon(s)) M_{m+\lambda-\alpha-1}(g_\varepsilon(s)) + M_{1+\lambda-\alpha}(g_\varepsilon(s)) M_{m+\alpha-1}(g_\varepsilon(s))] . \end{aligned}$$

Owing to (1.9a) and $m > 1 + \lambda - \alpha \geq 1 + \alpha$, both $m + \lambda - \alpha - 1$ and $m + \alpha - 1$ belong to $[1, m]$ and we deduce from (2.14), (2.22), and Hölder's inequality that

$$\begin{aligned} M_{m+\lambda-\alpha-1}(g_\varepsilon(s)) &\leq \varrho^{(1+\alpha-\lambda)/(m-1)} M_m(g_\varepsilon(s))^{(m+\lambda-\alpha-2)/(m-1)} , \\ M_{m+\alpha-1}(g_\varepsilon(s)) &\leq \varrho^{(1-\alpha)/(m-1)} M_m(g_\varepsilon(s))^{(m+\alpha-2)/(m-1)} . \end{aligned}$$

Also, introducing

$$Q_\varepsilon(s, R) := \int_R^\infty y g_\varepsilon(s, y) \, dy, \quad R > 1,$$

and noticing that $1 < 1 + \alpha \leq 1 + \lambda - \alpha < m$, we infer from (2.14), (2.22), and Hölder's inequality that, for $R > 1$,

$$\begin{aligned} M_{1+\alpha}(g_\varepsilon(s)) &\leq R^\alpha \int_0^R x g_\varepsilon(s, x) \, dx \\ &\quad + \left(\int_R^\infty x^m g_\varepsilon(s, x) \, dx \right)^{\alpha/(m-1)} \left(\int_R^\infty x g_\varepsilon(s, x) \, dx \right)^{(m-1-\alpha)/(m-1)} \\ &\leq R \varrho + Q_\varepsilon(s, R)^{(m-1-\alpha)/(m-1)} M_m(g_\varepsilon(s))^{\alpha/(m-1)} \\ &\leq R \varrho + \varrho^{(\lambda-2\alpha)/(m-1)} Q_\varepsilon(s, R)^{(m+\alpha-\lambda-1)/(m-1)} M_m(g_\varepsilon(s))^{\alpha/(m-1)} \end{aligned}$$

and

$$\begin{aligned} M_{1+\lambda-\alpha}(g_\varepsilon(s)) &\leq R^{\lambda-\alpha} \int_0^R x g_\varepsilon(s, x) \, dx \\ &\quad + \left(\int_R^\infty x^m g_\varepsilon(s, x) \, dx \right)^{(\lambda-\alpha)/(m-1)} \left(\int_R^\infty x g_\varepsilon(s, x) \, dx \right)^{(m+\lambda-1-\alpha)/(m-1)} \\ &\leq R \varrho + Q_\varepsilon(s, R)^{(m+\alpha-\lambda-1)/(m-1)} M_m(g_\varepsilon(s))^{(\lambda-\alpha)/(m-1)}. \end{aligned}$$

Collecting the above estimates, we find

$$\begin{aligned} P_{m,\varepsilon}(s) &\leq \kappa_7(m) R \left[M_m(g_\varepsilon(s))^{(m+\alpha-2)/(m-1)} + M_m(g_\varepsilon(s))^{(m+\lambda-\alpha-2)/(m-1)} \right] \\ &\quad + \kappa_7(m) Q_\varepsilon(s, R)^{(m-1-\lambda+\alpha)/(m-1)} M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} \end{aligned}$$

for $R > 1$. Owing to Corollary 2.5,

$$Q_\varepsilon(s, R) \leq \frac{1}{\ln R} \int_R^\infty y |\ln y| g_\varepsilon(s, y) \, dy \leq \frac{\kappa_5}{\ln R}.$$

Introducing $R_m > 1$ defined by

$$\kappa_7(m) \left(\frac{\kappa_5}{\ln R_m} \right)^{(m-1-\lambda+\alpha)/(m-1)} = \delta_{\varrho,m}$$

and taking $R = R_m$ in the previous estimate on $P_{m,\varepsilon}(s)$ give

$$\begin{aligned} P_{m,\varepsilon}(s) &\leq \kappa_7(m) R_m \left[M_m(g_\varepsilon(s))^{(m+\alpha-2)/(m-1)} + M_m(g_\varepsilon(s))^{(m+\lambda-\alpha-2)/(m-1)} \right] \\ &\quad + \delta_{\varrho,m} M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)}. \end{aligned}$$

Since $m + \alpha - 2 < m + \lambda - 2$ and $m + \lambda - \alpha - 2 < m + \lambda - 2$, we apply Young's inequality to obtain

$$P_{m,\varepsilon}(s) \leq \kappa(m) + 2\delta_{\varrho,m} M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)}. \quad (2.26)$$

We now combine (2.23), (2.24), and (2.26) and obtain

$$\frac{d}{ds}M_m(g_\varepsilon(s)) \leq \kappa(m) + (m-1)M_m(g_\varepsilon(s)) - 2\delta_{\varrho,m}M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} .$$

Hence, using once more Young's inequality,

$$\begin{aligned} \frac{d}{ds}M_m(g_\varepsilon(s)) &\leq \kappa_8(m) - \delta_{\varrho,m}M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} \\ &= \delta_{\varrho,m} \left[\kappa_6(m)^{(m+\lambda-2)/(m-1)} - M_m(g_\varepsilon(s))^{(m+\lambda-2)/(m-1)} \right] , \end{aligned}$$

with $\kappa_6(m) := (\kappa_8(m)/\delta_{\varrho,m})^{(m-1)/(m+\lambda-2)}$. Lemma 2.6 is then a consequence of the comparison principle. \square

We finally return to the behaviour for small sizes.

Lemma 2.7. *Consider $\varepsilon \in (0, \varepsilon_\varrho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). For $m \in [m_0, m_1)$, there is $\kappa_9(m) > 0$ depending on m such that, if $f^{in} \in X_m$, then*

$$M_m(g_\varepsilon(s)) \leq \max \left\{ M_m(f^{in}), \kappa_9(m)\mathcal{M}_{1+\lambda,\varepsilon} \right\} , \quad s \geq 0 ,$$

where

$$\mathcal{M}_{1+\lambda,\varepsilon} := \sup_{s \geq 0} \{ M_{1+\lambda}(g_\varepsilon(s)) \} < \infty .$$

Proof. We first note that $\mathcal{M}_{1+\lambda,\varepsilon}$ is indeed finite according to Lemma 2.6. Next, let $s \geq 0$. As at the beginning of the proof of Lemma 2.4, we infer from (2.2), (2.5), and (2.12) that

$$\begin{aligned} \frac{d}{ds}M_m(g_\varepsilon(s)) &\leq -(1-m)M_m(g_\varepsilon(s)) + a_0\mathbf{b}_{m,1,\varepsilon}M_{m+\lambda-1}(g_\varepsilon(s)) \\ &\leq -(1-m)M_m(g_\varepsilon(s)) + a_0(1+\mathbf{b}_{m_0,1})M_{m+\lambda-1}(g_\varepsilon(s)) . \end{aligned}$$

Since $m+\lambda-1 \in (m, 1+\lambda)$, we deduce from Hölder's inequality that

$$M_{m+\lambda-1}(g_\varepsilon(s)) \leq M_{1+\lambda}(g_\varepsilon(s))^{(\lambda-1)/(1+\lambda-m)} M_m(g_\varepsilon(s))^{(2-m)/(1+\lambda-m)} .$$

Consequently,

$$\begin{aligned} \frac{d}{ds}M_m(g_\varepsilon(s)) &\leq -(1-m)M_m(g_\varepsilon(s)) + a_0(1+\mathbf{b}_{m_0,1})\mathcal{M}_{1+\lambda,\varepsilon}^{(\lambda-1)/(1+\lambda-m)} M_m(g_\varepsilon(s))^{(2-m)/(1+\lambda-m)} \\ &= (1-m)M_m(g_\varepsilon(s))^{(2-m)/(1+\lambda-m)} \left\{ [\kappa_9(m)\mathcal{M}_{1+\lambda,\varepsilon}]^{(\lambda-1)/(1+\lambda-m)} \right. \\ &\quad \left. - M_m(g_\varepsilon(s))^{(\lambda-1)/(1+\lambda-m)} \right\} , \end{aligned}$$

with $\kappa_9(m) := (a_0(1+\mathbf{b}_{m_0,1})/(1-m))^{(1+\lambda-m)/(\lambda-1)}$. Lemma 2.7 follows from the above differential inequality and the comparison principle. \square

Up to now, we have derived estimates which do not depend on $\varepsilon \in (0, \varepsilon_\varrho)$ and which will thus be of utmost importance in the next section to take the limit $\varepsilon \rightarrow 0$. However, these estimates do not provide enough control on the behaviour for small sizes for the proof of Proposition 2.2, for which the next result is required.

Lemma 2.8. Consider $\varepsilon \in (0, \varepsilon_\rho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). For $m \in (-1, 0]$, there is $\kappa_{10}(m, \varepsilon) > 0$ depending on m and ε such that

$$M_m(g_\varepsilon(s)) \leq \max \{M_m(f^{in}), \kappa_{10}(m, \varepsilon)\mathcal{M}_{1+\lambda, \varepsilon}\} \quad . \quad s \geq 0 \quad .$$

Proof. The proof is exactly the same as that of Lemma 2.7 with the only difference that $\mathbf{b}_{m,1,\varepsilon}$ cannot be bounded from above by a constant which does not depend on ε for all $m \in (-1, 0]$, though it is finite due to (2.6). \square

2.3. Weighted L^{q_1} -estimate. The last estimate which does not depend on $\varepsilon \in (0, \varepsilon_\rho)$ is the following weighted L^{q_1} -estimate, the exponent q_1 being defined in (2.9).

Lemma 2.9. Consider $\varepsilon \in (0, \varepsilon_\rho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). If f^{in} also belongs to $L^{q_1}((0, \infty), y^{m_1} dy)$, then there is $\kappa_{11} > 0$ such that

$$\int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1} dx \leq \max \left\{ \int_0^\infty x^{m_1} f^{in}(x)^{q_1} dx, \kappa_{11} \mathcal{M}_{\mu_1, \varepsilon}^{q_1} \right\} \quad ,$$

where $\mu_1 := (m_1 + 1 + q_1(\lambda - 2))/q_1 > m_0$ and

$$\mathcal{M}_{\mu_1, \varepsilon} := \sup_{s \geq 0} \{M_{\mu_1}(g_\varepsilon(s))\} \quad .$$

Proof. We first observe that, as $\mu_1 \in (m_0, \lambda)$ by (2.9), Lemma 2.6, Lemma 2.9, and Hölder's inequality imply that $\mathcal{M}_{\mu_1, \varepsilon}$ is finite. We next set

$$L_\varepsilon(s) := \frac{1}{q_1} \int_0^\infty y^{m_1} g_\varepsilon(s, y)^{q_1} dy \quad , \quad s \geq 0 \quad ,$$

and infer from (2.12) that

$$\begin{aligned} \frac{d}{ds} L_\varepsilon(s) &= -(2q_1 - m_1 - 1)L_\varepsilon(s) + \int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \mathcal{C}g_\varepsilon(s, x) dx \\ &\quad + \int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \mathcal{F}_\varepsilon g_\varepsilon(s, x) dx \quad . \end{aligned} \quad (2.27)$$

On the one hand, we use a monotonicity argument as in [8, 19, 22, 30] to estimate the contribution of the coagulation term. More precisely, thanks to the symmetry of K and the subadditivity of

$x \mapsto x^{m_1}$,

$$\begin{aligned}
R_\varepsilon(s) &:= \int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \mathcal{C} g_\varepsilon(s, x) \, dx \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty (x+y)^{m_1} K(x, y) g_\varepsilon(s, x+y)^{q_1-1} g_\varepsilon(s, x) g_\varepsilon(s, y) \, dy dx \\
&\quad - \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dy dx \\
&\leq \frac{1}{2} \int_0^\infty \int_0^\infty (x^{m_1} + y^{m_1}) K(x, y) g_\varepsilon(s, x+y)^{q_1-1} g_\varepsilon(s, x) g_\varepsilon(s, y) \, dy dx \\
&\quad - \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dy dx \\
&= \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x+y)^{q_1-1} g_\varepsilon(s, x) g_\varepsilon(s, y) \, dy dx \\
&\quad - \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dy dx .
\end{aligned}$$

We now use Young's inequality to obtain

$$\begin{aligned}
R_\varepsilon(s) &\leq \int_0^\infty \int_0^\infty x^{m_1} K(x, y) \left[\frac{q_1-1}{q_1} g_\varepsilon(s, x+y)^{q_1} + \frac{1}{q_1} g_\varepsilon(s, x)^{q_1} \right] g_\varepsilon(s, y) \, dy dx \\
&\quad - \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dy dx \\
&= \frac{q_1-1}{q_1} \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x+y)^{q_1} g_\varepsilon(s, y) \, dy dx \\
&\quad - \frac{q_1-1}{q_1} \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dy dx \\
&= \frac{q_1-1}{q_1} \int_0^\infty \int_y^\infty x^{m_1} K(x-y, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dx dy \\
&\quad - \frac{q_1-1}{q_1} \int_0^\infty \int_0^\infty x^{m_1} K(x, y) g_\varepsilon(s, x)^{q_1} g_\varepsilon(s, y) \, dx dy .
\end{aligned}$$

Owing to the monotonicity of $x \mapsto x^{m_1} K(x, y)$ for all $y \in (0, \infty)$, the right hand side of the previous inequality is non-positive. Consequently,

$$R_\varepsilon(s) = \int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \mathcal{C} g_\varepsilon(s, x) \, dx \leq 0 . \quad (2.28)$$

On the other hand, it follows from (1.9b), (2.1c), and Fubini's theorem that

$$\begin{aligned} S_\varepsilon(s) &:= \int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \int_x^\infty a(y) b_\varepsilon(x, y) g_\varepsilon(s, y) dy dx \\ &= a_0 \int_0^\infty y^{\lambda-2} g_\varepsilon(s, y) \int_0^y x^{m_1} B_\varepsilon\left(\frac{x}{y}\right) g_\varepsilon(s, x)^{q_1-1} dx dy . \end{aligned}$$

Since

$$\begin{aligned} &\int_0^y x^{m_1} B_\varepsilon\left(\frac{x}{y}\right) g_\varepsilon(s, x)^{q_1-1} dx \\ &\leq \left(\int_0^y x^{m_1} g_\varepsilon(s, x)^{q_1} dx \right)^{(q_1-1)/q_1} \left(\int_0^y x^{m_1} B_\varepsilon\left(\frac{x}{y}\right)^{q_1} dx \right)^{1/q_1} \\ &\leq q_1^{(q_1-1)/q_1} \mathfrak{b}_{m_1, q_1, \varepsilon}^{1/q_1} L_\varepsilon(s)^{(q_1-1)/q_1} y^{(m_1+1)/q_1} \\ &\leq q_1^{(q_1-1)/q_1} (1 + \mathfrak{b}_{m_1, q_1})^{1/q_1} L_\varepsilon(s)^{(q_1-1)/q_1} y^{(m_1+1)/q_1} , \end{aligned}$$

by (2.4) and Hölder's inequality, we conclude that

$$\begin{aligned} &\int_0^\infty x^{m_1} g_\varepsilon(s, x)^{q_1-1} \mathcal{F}_\varepsilon g_\varepsilon(s, x) dx \leq S_\varepsilon(s) \\ &\leq a_0 q_1^{(q_1-1)/q_1} (1 + \mathfrak{b}_{m_1, q_1})^{1/q_1} M_{\mu_1}(g_\varepsilon(s)) L_\varepsilon(s)^{(q_1-1)/q_1} . \end{aligned} \quad (2.29)$$

Collecting (2.27), (2.28), and (2.29), we end up with

$$\begin{aligned} \frac{d}{ds} L_\varepsilon(s) &\leq -(2q_1 - m_1 - 1) L_\varepsilon(s) + a_0 q_1^{(q_1-1)/q_1} (1 + \mathfrak{b}_{m_1, q_1})^{1/q_1} \mathcal{M}_{\mu_1, \varepsilon} L_\varepsilon(s)^{(q_1-1)/q_1} \\ &= \frac{2q_1 - m_1 - 1}{q_1^{1/q_1}} L_\varepsilon(s)^{(q_1-1)/q_1} \left[\kappa_{11}^{1/q_1} \mathcal{M}_{\mu_1, \varepsilon} - q_1^{1/q_1} L_\varepsilon(s)^{1/q_1} \right] \end{aligned}$$

with $\kappa_{11} = (a_0 q_1)^{q_1} (1 + \mathfrak{b}_{m_1, q_1}) / (2q_1 - m_1 - 1)^{q_1}$. Lemma 2.9 follows from the above differential inequality by the comparison principle. \square

2.4. $W^{1,1}$ -estimate. It turns out that the weighted L^{q_1} -estimate derived in Lemma 2.9, though at the heart of the proof of Theorem 1.1, is not sufficient to prove Proposition 2.2, and the final estimate needed for the proof of Proposition 2.2 is the following $W^{1,1}$ -estimate which depends strongly on $\varepsilon \in (0, \varepsilon_\rho)$.

Lemma 2.10. *Consider $\varepsilon \in (0, \varepsilon_\rho)$ and $f^{in} \in X_0^+ \cap X_{1+\lambda}$ satisfying (2.22) and let $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$ be given by (2.17). Assume also that $f^{in} \in X_{\lambda-2} \cap W^{1,1}(0, \infty)$. Then there is $\kappa_{12}(\varepsilon) > 0$ depending on ε such that*

$$\|\partial_x g_\varepsilon(s)\|_1 \leq \max \left\{ \|\partial_x f^{in}\|_1, \kappa_{12}(\varepsilon) \mathcal{M}_{\lambda-2, \varepsilon} \right\} , \quad s \geq 0 ,$$

where

$$\mathcal{M}_{\lambda-2, \varepsilon} := \sup_{s \geq 0} \{ M_{\lambda-2}(g_\varepsilon(s)) \} .$$

Proof. We first note that $\mathcal{M}_{\lambda-2,\varepsilon}$ is finite according to Lemma 2.8, as $\lambda - 2 \in (-1, 0)$ by (1.9a). Introducing $G_\varepsilon := \partial_x g_\varepsilon$, $\Sigma_\varepsilon := \text{sign}(G_\varepsilon)$, and using that $K(x, 0) = 0$, it follows from (2.12a) that G_ε solves

$$\begin{aligned}
\partial_s G_\varepsilon(s, x) &= -x \partial_x G_\varepsilon(s, x) - \left(3 + a(x) + \int_0^\infty K(x, y) g_\varepsilon(s, y) \, dy \right) G_\varepsilon(s, x) \\
&\quad + \frac{1}{2} \int_0^x K(y, x-y) g_\varepsilon(s, y) G_\varepsilon(s, x-y) \, dy \\
&\quad + \frac{1}{2} \int_0^x \partial_1 K(y, x-y) g_\varepsilon(s, y) g_\varepsilon(s, x-y) \, dy \\
&\quad - \left(\frac{da}{dx}(x) + a(x) b_\varepsilon(x, x) + \int_0^\infty \partial_1 K(x, y) g_\varepsilon(s, y) \, dy \right) g_\varepsilon(s, x) \\
&\quad + \int_x^\infty a(y) \partial_1 b_\varepsilon(x, y) g_\varepsilon(s, y) \, dy
\end{aligned} \tag{2.30}$$

for $(s, x) \in (0, \infty)^2$, where $\partial_1 K$ and $\partial_1 b_\varepsilon$ denote the partial derivatives with respect to the first variable of K and b_ε , respectively.

Let $s \geq 0$. We multiply (2.30) by Σ_ε , integrate with respect to x over $(0, \infty)$ and then infer from (1.9b), (2.1c), and Fubini's theorem that

$$\begin{aligned}
\frac{d}{ds} \|G_\varepsilon(s)\|_1 &\leq -2 \|G_\varepsilon(s)\|_1 - a_0 M_{\lambda-1}(|G_\varepsilon(s)|) \\
&\quad - \int_0^\infty \int_0^\infty K(x, y) g_\varepsilon(s, y) |G_\varepsilon(s, x)| \, dy dx \\
&\quad + \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) g_\varepsilon(s, y) |G_\varepsilon(s, x)| \, dy dx \\
&\quad + \frac{3}{2} \int_0^\infty \int_0^\infty |\partial_1 K(x, y)| g_\varepsilon(s, y) g_\varepsilon(s, x) \, dy dx \\
&\quad + a_0 \left(\lambda - 1 + B_\varepsilon(1) + \int_0^1 \left| \frac{dB_\varepsilon}{dz}(z) \right| \, dz \right) M_{\lambda-2}(g_\varepsilon(s)) .
\end{aligned}$$

Setting

$$\bar{B}_\varepsilon := 1 + B_\varepsilon(1) + \int_0^1 \left| \frac{dB_\varepsilon}{dz}(z) \right| \, dz ,$$

which is finite according to (2.6), and observing that

$$0 \leq \partial_1 K(x, y) \leq K_0 [x^{\alpha-1} y^{\lambda-\alpha} + x^\alpha y^{\lambda-\alpha-1}] , \quad (x, y) \in (0, \infty)^2 ,$$

due to (1.9a) and (1.9c), we end up with

$$\begin{aligned}
 \frac{d}{ds} \|G_\varepsilon(s)\|_1 &\leq -2\|G_\varepsilon(s)\|_1 + a_0 \bar{B}_\varepsilon M_{\lambda-2}(g_\varepsilon(s)) \\
 &\quad + \frac{3}{2} \int_0^\infty \int_0^\infty |\partial_1 K(x, y)| g_\varepsilon(s, y) g_\varepsilon(s, x) \, dy dx \\
 &\leq -2\|G_\varepsilon(s)\|_1 + a_0 \bar{B}_\varepsilon M_{\lambda-2}(g_\varepsilon(s)) \\
 &\quad + \frac{3K_0}{2} [M_\alpha(g_\varepsilon(s)) M_{\lambda-\alpha-1}(g_\varepsilon(s)) + M_{\alpha-1}(g_\varepsilon(s)) M_{\lambda-\alpha}(g_\varepsilon(s))] .
 \end{aligned}$$

We next infer from (1.9a) and Hölder's inequality that

$$\begin{aligned}
 M_\alpha(g_\varepsilon(s)) &\leq M_1(g_\varepsilon(s))^{(\alpha+2-\lambda)/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{(1-\alpha)/(3-\lambda)} , \\
 M_{\lambda-\alpha-1}(g_\varepsilon(s)) &\leq M_1(g_\varepsilon(s))^{(1-\alpha)/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{(\alpha+2-\lambda)/(3-\lambda)} , \\
 M_{\alpha-1}(g_\varepsilon(s)) &\leq M_1(g_\varepsilon(s))^{(\alpha+1-\lambda)/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{(2-\alpha)/(3-\lambda)} , \\
 M_{\lambda-\alpha}(g_\varepsilon(s)) &\leq M_1(g_\varepsilon(s))^{(2-\alpha)/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{(\alpha+1-\alpha)/(3-\lambda)} ,
 \end{aligned}$$

so that, by (2.14) and (2.22),

$$M_\alpha(g_\varepsilon(s)) M_{\lambda-\alpha-1}(g_\varepsilon(s)) + M_{\alpha-1}(g_\varepsilon(s)) M_{\lambda-\alpha}(g_\varepsilon(s)) \leq 2\varrho M_{\lambda-2}(g_\varepsilon(s)) .$$

Collecting the above inequalities and using (2.7), we conclude that

$$\frac{d}{ds} \|G_\varepsilon(s)\|_1 + 2\|G_\varepsilon(s)\|_1 \leq 2\kappa_{12}(\varepsilon) \mathcal{M}_{\lambda-2, \varepsilon} ,$$

with $\kappa_{12}(\varepsilon) := (a_0 \bar{B}_\varepsilon + 3\varrho K_0)/2$. Integrating the previous differential inequality gives Lemma 2.10. \square

2.5. Invariant Set. The analysis performed in the previous three sections now allows us to construct a compact and convex subset of X_1 which is left invariant by (2.12). Let us first recall that, owing to (2.9), the parameter μ_1 (defined in Lemma 2.9) satisfies

$$1 + \lambda > \mu_1 = \frac{m_1 + 1 + q_1(\lambda - 2)}{q_1} > m_0 > -\nu - 1 . \quad (2.31)$$

For $\varepsilon \in (0, \varepsilon_\rho)$, we define the subset \mathcal{Z}_ε of X_1^+ as follows: $h \in \mathcal{Z}_\varepsilon$ if and only if h satisfies the following conditions:

$$h \in X_1^+ \cap \bigcap_{m \geq \lambda - 2} X_m \cap W^{1,1}(0, \infty), \quad M_1(h) = \rho, \quad (2.32a)$$

$$\int_0^\infty x \ln(x) h(x) dx + \frac{3}{e(1-m_1)} M_{m_1}(h) \leq \kappa_1(m_1), \quad (2.32b)$$

$$M_m(h) \leq \kappa_6(m), \quad m \geq 1 + \lambda, \quad (2.32c)$$

$$M_{m_0}(h) \leq \kappa_9(m_0) \kappa_6(1 + \lambda), \quad (2.32d)$$

$$M_{\mu_1}(h) \leq \kappa_9(m_0)^{(1+\lambda-\mu_1)/(1+\lambda-m_0)} \kappa_6(1 + \lambda), \quad (2.32e)$$

$$\int_0^\infty x^{m_1} h(x)^{q_1} dx \leq \kappa_{11} \kappa_9(m_0)^{q_1(1+\lambda-\mu_1)/(1+\lambda-m_0)} \kappa_6(1 + \lambda)^{q_1}, \quad (2.32f)$$

$$M_{\lambda-2}(h) \leq \kappa_{10}(\lambda - 2, \varepsilon) \kappa_6(1 + \lambda), \quad (2.32g)$$

$$\|\partial_x h\|_1 \leq \kappa_{12}(\varepsilon) \kappa_{10}(\lambda - 2, \varepsilon) \kappa_6(1 + \lambda). \quad (2.32h)$$

Note that we may assume that $E_\rho : x \mapsto \rho e^{-x}$ belongs to \mathcal{Z}_ε , after possibly taking larger constants in (2.32) without changing their dependence with respect to the involved parameters. In particular, \mathcal{Z}_ε is non-empty.

As we shall see now, the outcome of the analysis performed in the previous sections provides the invariance of \mathcal{Z}_ε for the dynamics of (2.12) when $\varepsilon \in (0, \varepsilon_\rho)$.

Lemma 2.11. *Consider $\varepsilon \in (0, \varepsilon_\rho)$ and $f^{in} \in \mathcal{Z}_\varepsilon$. Then $\Psi_\varepsilon(s; f^{in}) \in \mathcal{Z}_\varepsilon$ for all $s \geq 0$. Furthermore, \mathcal{Z}_ε is a non-empty, convex, and compact subset of X_1 .*

Proof. Let $f^{in} \in \mathcal{Z}_\varepsilon$. Setting $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$, see (2.17), it satisfies (2.14) by Lemma 2.4, from which we readily obtain that $g_\varepsilon(s) \in X_1^+$ and $M_1(g_\varepsilon(s)) = \rho$ for all $s \geq 0$.

Next, let $s \geq 0$. We infer from (2.32b) and Lemma 2.4 (with $m = m_1$) that $g_\varepsilon(s)$ satisfies (2.32b). Also, since f^{in} satisfies (2.22) according to (2.32b), we are in a position to apply Lemma 2.6 for $m \geq 1 + \lambda > 1 + \lambda - \alpha$ and deduce from (2.32c) for f^{in} that (2.32c) is satisfied by $g_\varepsilon(s)$ for any $m \geq 1 + \lambda$. This property (with $m = 1 + \lambda$) along with Lemma 2.7 (with $m = m_0$) guarantees that $g_\varepsilon(s)$ satisfies (2.32d). We further use (2.32c) (with $m = 1 + \lambda$) and (2.32d) that we just established for g_ε together with (2.31) and Hölder's inequality to obtain

$$\begin{aligned} M_{\mu_1}(g_\varepsilon(s)) &\leq M_{1+\lambda}(g_\varepsilon(s))^{(\mu_1-m_0)/(1+\lambda-m_0)} M_{m_0}(g_\varepsilon(s))^{(1+\lambda-\mu_1)/(1+\lambda-m_0)} \\ &\leq \kappa_6(1 + \lambda)^{(\mu_1-m_0)/(1+\lambda-m_0)} [\kappa_6(1 + \lambda) \kappa_9(m_0)]^{(1+\lambda-\mu_1)/(1+\lambda-m_0)} \\ &\leq \kappa_9(m_0)^{(1+\lambda-\mu_1)/(1+\lambda-m_0)} \kappa_6(1 + \lambda). \end{aligned}$$

Hence, $g_\varepsilon(s)$ satisfies (2.32e) for $s \geq 0$. We now combine the just established property (2.32e) for g_ε with Lemma 2.10 and realize that $g_\varepsilon(s)$ satisfies (2.32f) for $s \geq 0$. Finally, since f^{in} satisfies (2.32g) and (2.32h), it follows at once from the already proved property (2.32c) for g_ε (for $m = 1 + \lambda$),

Lemma 2.8, and Lemma 2.10 that $g_\varepsilon(s)$ also satisfies (2.32g) and (2.32h). Summarizing, we have shown that $g_\varepsilon(s) \in \mathcal{Z}_\varepsilon$ for all $s \geq 0$.

Next, the set \mathcal{Z}_ε is convex and its compactness in X_1 follows from its boundedness in $X_{\lambda-2} \cap X_{1+\lambda}$, the compactness of the embedding of $W^{1,1}(1/R, R)$ in $L^1(1/R, R)$, which holds true for all $R > 1$, and Vitali's theorem [14, Theorem 2.24]. \square

To complete the proof of Proposition 2.2, the missing tile is the continuity of weak solutions to (2.12) with respect to the initial condition which we establish now.

Lemma 2.12. *Let $\varepsilon \in (0, \varepsilon_\rho)$.*

- (a) *For $s \geq 0$, the map $f^{in} \mapsto \Psi_\varepsilon(s; f^{in})$, defined in (2.17), is continuous from \mathcal{Z}_ε endowed with the norm topology of X_1 to itself.*
- (b) *For $f^{in} \in \mathcal{Z}_\varepsilon$, the map $s \mapsto \Psi_\varepsilon(s; f^{in})$ belongs to $C([0, \infty), X_1)$.*

In other words, $\Psi_\varepsilon : [0, \infty) \times \mathcal{Z}_\varepsilon \rightarrow \mathcal{Z}_\varepsilon$ is a dynamical system for the norm topology of X_1 .

Proof of Lemma 2.12 (a). Consider $(f_1^{in}, f_2^{in}) \in \mathcal{Z}_\varepsilon^2$ and put $g_{i,\varepsilon} := \Psi_\varepsilon(\cdot; f_i^{in})$, $i = 1, 2$. Arguing as in the proof of [19, Theorem 1.2 (c)], it follows from (2.12) that, for $s \geq 0$,

$$\begin{aligned} & \frac{d}{ds} \int_0^\infty W(x) |g_{1,\varepsilon}(s, x) - g_{2,\varepsilon}(s, x)| \, dx \\ & \leq \int_0^\infty \left[x \frac{dW}{dx}(x) - W(x) \right] |g_{1,\varepsilon}(s, x) - g_{2,\varepsilon}(s, x)| \, dx \\ & \quad + [9K_0 v_\varepsilon(s) + a_0 \mathbf{b}_{\alpha,1,\varepsilon}] \int_0^\infty W(x) |g_{1,\varepsilon}(s, x) - g_{2,\varepsilon}(s, x)| \, dx , \end{aligned}$$

where $W(x) = x^\alpha + x^\lambda$, $x \geq 0$, and

$$v_\varepsilon(s) := M_\alpha(g_{1,\varepsilon}(s)) + M_\alpha(g_{2,\varepsilon}(s)) + M_{2\lambda-\alpha}(g_{1,\varepsilon}(s)) + M_{2\lambda-\alpha}(g_{2,\varepsilon}(s)) .$$

Since both f_1^{in} and f_2^{in} belong to \mathcal{Z}_ε , so do $g_{1,\varepsilon}(s)$ and $g_{2,\varepsilon}(s)$ for all $s \geq 0$ by Lemma 2.11. Consequently, as $m_0 < \alpha < 2\lambda - \alpha \leq 1 + \lambda$ by (1.9a) and (1.14),

$$V_\varepsilon := \sup_{s \geq 0} \{v_\varepsilon(s)\} < \infty .$$

In addition,

$$xW'(x) - W(x) = (\alpha - 1)x^\alpha + (\lambda - 1)x^\lambda \leq x^\lambda \leq W(x) , \quad x \in (0, \infty) ,$$

by (1.9a) and we infer from (2.5) and the previous differential inequality that, for $s \geq 0$,

$$\int_0^\infty W(x) |g_{1,\varepsilon}(s, x) - g_{2,\varepsilon}(s, x)| \, dx \leq e^{\kappa_{13}(\varepsilon)s} \int_0^\infty W(x) |f_1^{in}(x) - f_2^{in}(x)| \, dx , \quad (2.33)$$

with $\kappa_{13}(\varepsilon) := 1 + 9K_0 V_\varepsilon + a_0 \mathbf{b}_{\alpha,1,\varepsilon}$.

Now, $W(x) \geq x$ for $x \geq 0$ as $\alpha \leq 1 < \lambda$, while, for $R > 1$, it follows from (1.9a) and (1.14) that

$$\begin{aligned}
\int_0^\infty W(x)|f_1^{in}(x) - f_2^{in}(x)| \, dy &\leq \int_0^{1/R} W(x)[f_1^{in}(x) + f_2^{in}(x)] \, dx \\
&\quad + \int_{1/R}^R W(x)|f_1^{in}(x) - f_2^{in}(x)| \, dx \\
&\quad + \int_R^\infty W(x)[f_1^{in}(x) + f_2^{in}(x)] \, dx \\
&\leq (R^{m_0-\alpha} + R^{m_0-\lambda}) [M_{m_0}(f_1^{in}) + M_{m_0}(f_2^{in})] \\
&\quad + (R^{1-\alpha} + R^{\lambda-1}) \int_{1/R}^R x|f_1^{in}(x) - f_2^{in}(x)| \, dx \\
&\quad + (R^{\alpha-1-\lambda} + R^{-1}) [M_{1+\lambda}(f_1^{in}) + M_{1+\lambda}(f_2^{in})] \\
&\leq \kappa_{14} \left[R^{m_0-\alpha} + R^{-1} + R^{1-\alpha} \int_0^\infty x|f_1^{in}(x) - f_2^{in}(x)| \, dx \right],
\end{aligned}$$

the last inequality relying on the property $f_i^{in} \in \mathcal{Z}_\varepsilon$, $i = 1, 2$. Combining (2.33) and the previous inequalities gives, for $s \geq 0$,

$$\begin{aligned}
&\int_0^\infty x|g_{1,\varepsilon}(s, x) - g_{2,\varepsilon}(s, x)| \, dx \\
&\leq \kappa_{14} e^{\kappa_{13}(\varepsilon)s} \omega \left(\int_0^\infty x|f_1^{in}(x) - f_2^{in}(x)| \, dx \right),
\end{aligned}$$

with

$$\omega(r) := \inf_{R>1} \{ R^{m_0-\alpha} + R^{-1} + R^{1-\alpha}r \}, \quad r > 0.$$

Since $\omega(r) \rightarrow 0$ as $r \rightarrow 0$, the claimed continuity follows. \square

Proof of Lemma 2.12 (b). Set $g_\varepsilon = \Psi_\varepsilon(\cdot; f^{in})$. Let $s \geq 0$. We infer from (1.9a), (1.9b), (1.9c), (1.14), (2.1c), (2.7), (2.12a), (2.14), (2.22), and Hölder's inequality that

$$\begin{aligned}
\int_0^\infty \frac{|\partial_s g_\varepsilon(s, x)|}{1+x} \, dx &\leq \|\partial_x g_\varepsilon(s)\|_1 + 2M_0(g_\varepsilon(s)) + 3K_0 M_\alpha(g_\varepsilon(s)) M_{\lambda-\alpha}(g_\varepsilon(s)) \\
&\quad + a_0 (1 + \mathbf{b}_{0,1,\varepsilon}) M_{\lambda-1}(g_\varepsilon(s)) \\
&\leq \|\partial_x g_\varepsilon(s)\|_1 + 2\varrho^{(2-\lambda)/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{1/(3-\lambda)} \\
&\quad + 3K_0 \varrho^{(\lambda-2m_0)/(1-m_0)} M_{m_0}(g_\varepsilon(s))^{(2-\lambda)/(1-m_0)} \\
&\quad + a_0 (1 + \mathbf{b}_{0,1,\varepsilon}) \varrho^{1/(3-\lambda)} M_{\lambda-2}(g_\varepsilon(s))^{(2-\lambda)/(3-\lambda)}.
\end{aligned}$$

Since $g_\varepsilon(s) \in \mathcal{Z}_\varepsilon$ by Lemma 2.11, we further obtain

$$\int_0^\infty \frac{|\partial_s g_\varepsilon(s, x)|}{1+x} dx \leq \kappa_{15}(\varepsilon), \quad s \geq 0.$$

Hence, for $s_2 > s_1 \geq 0$ and $R \geq 1$,

$$\begin{aligned} \int_0^\infty x |g_\varepsilon(s_2, x) - g_\varepsilon(s_1, x)| dy &\leq R(1+R) \int_0^R \frac{|g_\varepsilon(s_2, x) - g_\varepsilon(s_1, x)|}{1+x} dx \\ &\quad + R^{-\lambda} \int_R^\infty x^{1+\lambda} [g_\varepsilon(s_2, x) + g_\varepsilon(s_1, x)] dx \\ &\leq 2R^2 \int_{s_1}^{s_2} \int_0^\infty \frac{|\partial_s g_\varepsilon(s, x)|}{1+x} dx ds + 2R^{-\lambda} \sup_{s \geq 0} \{M_{1+\lambda}(g_\varepsilon(s))\} \\ &\leq 2R^2 \kappa_{15}(\varepsilon)(s_2 - s_1) + 2R^{-\lambda} \kappa_6(1+\lambda). \end{aligned}$$

Choosing $R = (s_2 - s_1)^{-1/(\lambda+2)}$ if $s_2 - s_1 < 1$ and $R = 1$ otherwise in the previous inequality, we are led to

$$\int_0^\infty x |g_\varepsilon(s_2, x) - g_\varepsilon(s_1, x)| dx \leq 2 [\kappa_{15}(\varepsilon) + \kappa_6(1+\lambda)] ((s_2 - s_1)^{\lambda/(\lambda+2)} + s_2 - s_1),$$

which provides the claimed continuity. \square

We have now established all the properties required to prove Proposition 2.2.

Proof of Proposition 2.2. Let $\varepsilon \in (0, \varepsilon_\rho)$. Owing to Lemma 2.11 and Lemma 2.12, Ψ_ε is a dynamical system on \mathcal{Z}_ε endowed with the norm topology of X_1 and \mathcal{Z}_ε is a non-empty, convex, and compact subset of X_1 , which is additionally left positively invariant by Ψ_ε . A consequence of Schauder's fixed point theorem, see [1, Proposition 22.13] or [17, Proof of Theorem 5.2], implies that there is $\varphi_\varepsilon \in \mathcal{Z}_\varepsilon$ such that $\Psi_\varepsilon(s; \varphi_\varepsilon) = \varphi_\varepsilon$ for all $s \geq 0$. In other words, φ_ε is a stationary solution to (2.12a), from which we deduce that it satisfies (2.10). Also, since φ_ε lies in \mathcal{Z}_ε , it has the properties (2.11) due to (2.32b), (2.32c), (2.32d), and (2.32f). \square

3. SELF-SIMILAR SOLUTIONS

In this section, we assume that K , a , and b are coagulation and fragmentation coefficients satisfying (1.9) and we fix $\varrho \in (0, \varrho_\star)$. For $\varepsilon \in (0, \varepsilon_\varrho)$, it follows from Proposition 2.2 that there is

$$\varphi_\varepsilon \in X_1^+ \cap L^{q_1}((0, \infty), x^{m_1} dx) \cap W^{1,1}(0, \infty) \cap \bigcap_{m \geq \lambda-2} X_m$$

satisfying (2.10),

$$M_1(\varphi_\varepsilon) = \varrho, \tag{3.1}$$

$$\sup_{\varepsilon \in (0, \varepsilon_\varrho)} \{M_{m_0}(\varphi_\varepsilon)\} + \sup_{\varepsilon \in (0, \varepsilon_\varrho)} \left\{ \int_0^\infty x^{m_1} \varphi_\varepsilon(x)^{q_1} dx \right\} < \infty, \tag{3.2}$$

and

$$\sup_{\varepsilon \in (0, \varepsilon_\varrho)} \{M_m(\varphi_\varepsilon)\} < \infty \quad (3.3)$$

for all $m \geq 1 + \lambda$. Since $q_1 > 1$ and $m_1 < 1$, we infer from (3.1), (3.2), the reflexivity of $L^{q_1}((0, \infty), x^{m_1} dx)$, and Dunford-Pettis' theorem that there are $\varphi \in X_{m_1} \cap L^{q_1}((0, \infty), x^{m_1} dx)$ and a subsequence $(\varphi_{\varepsilon_n})_{n \geq 1}$ of $(\varphi_\varepsilon)_{\varepsilon \in (0, \varepsilon_\varrho)}$ such that

$$\varphi_{\varepsilon_n} \rightharpoonup \varphi \text{ in } X_{m_1} \text{ and in } L^{q_1}((0, \infty), x^{m_1} dx) . \quad (3.4)$$

Combining (3.2), (3.3), and (3.4), we further obtain that $\varphi \in X_{m_0}$ and

$$\varphi \in X_m \text{ and } \varphi_{\varepsilon_n} \rightharpoonup \varphi \text{ in } X_m , \quad m > m_0 . \quad (3.5)$$

Since the positive cone X_1^+ of X_1 is weakly closed in X_1 , we infer from (3.1) and (3.5) (with $m = 1$) that

$$\varphi \in X_1^+ \text{ and } M_1(\varphi) = \varrho . \quad (3.6)$$

We are left with taking the limit $\varepsilon \rightarrow 0$ in (2.10). To this end, consider $\vartheta \in \Theta_1$, the space Θ_1 being defined in (1.17), and note that

$$|\vartheta(x)| \leq \|\partial_x \vartheta\|_\infty x , \quad x \in [0, \infty) . \quad (3.7)$$

Then $x \mapsto \vartheta(x)/x$ belongs to $L^\infty(0, \infty)$ and it readily follows from (3.5) (with $m = 1$) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty [\vartheta(x) - x \partial_x \vartheta(x)] \varphi_{\varepsilon_n}(x) dx &= \lim_{n \rightarrow \infty} \int_0^\infty x \left[\frac{\vartheta(x)}{x} - \partial_x \vartheta(x) \right] \varphi_{\varepsilon_n}(x) dx \\ &= \int_0^\infty [\vartheta(x) - x \partial_x \vartheta(x)] \varphi(x) dx . \end{aligned} \quad (3.8)$$

Similarly, $\chi_\vartheta \in L^\infty((0, \infty)^2)$ and we argue as in [34], see also [7], to deduce from (1.9a), (1.9c), (1.14), and (3.5) (with $m = \alpha$ and $m = \lambda - \alpha$) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) \varphi_{\varepsilon_n}(x) \varphi_{\varepsilon_n}(y) dy dx \\ = \int_0^\infty \int_0^\infty K(x, y) \chi_\vartheta(x, y) \varphi(x) \varphi(y) dy dx . \end{aligned} \quad (3.9)$$

Finally, by (1.9a), (1.9b), and (3.5) (with $m = \lambda$),

$$[y \mapsto ya(y) \varphi_{\varepsilon_n}(y)] \rightharpoonup [y \mapsto ya(y) \varphi(y)] \text{ in } L^1(0, \infty) , \quad (3.10)$$

while (2.1), (2.2a), and (3.7) entail, for $y \in (0, \infty)$,

$$\begin{aligned} \left| \frac{N_{\vartheta, \varepsilon_n}(y)}{y} \right| &\leq \frac{|\vartheta(y)|}{y} + \frac{1}{y} \int_0^1 |\vartheta(yz)| B_{\varepsilon_n}(z) dz \\ &\leq \|\partial_x \vartheta\|_\infty \left(1 + \int_0^1 z B_{\varepsilon_n}(z) dz \right) = 2 \|\partial_x \vartheta\|_\infty . \end{aligned} \quad (3.11)$$

Using once more (3.7), we obtain, for $y \in (0, \infty)$,

$$\begin{aligned} \left| \int_0^y \vartheta(x)b_{\varepsilon_n}(x, y) \, dx - \int_0^y \vartheta(x)b(x, y) \, dx \right| &= \left| \int_0^1 \vartheta(yz)[B_{\varepsilon_n}(z) - B(z)] \, dz \right| \\ &\leq y \|\partial_x \vartheta\|_\infty \int_0^1 z |B_{\varepsilon_n}(z) - B(z)| \, dz . \end{aligned}$$

Hence, thanks to (2.2c) (with $(m, p) = (1, 1)$),

$$\lim_{n \rightarrow \infty} \frac{1}{y} \int_0^y \vartheta(x)b_{\varepsilon_n}(x, y) \, dx = \frac{1}{y} \int_0^y \vartheta(x)b(x, y) \, dx ,$$

which implies, in turn,

$$\lim_{n \rightarrow \infty} \frac{N_{\vartheta, \varepsilon_n}(y)}{y} = \frac{N_\vartheta(y)}{y} , \quad y \in (0, \infty) . \quad (3.12)$$

Due to (3.10), (3.11), and (3.12), we are in a position to apply [14, Proposition 2.61] (which is a consequence of Dunford-Pettis' and Egorov's theorems) and conclude that

$$\lim_{n \rightarrow \infty} \int_0^\infty a(y)N_{\vartheta, \varepsilon_n}(y)\varphi_{\varepsilon_n}(y) \, dy = \int_0^\infty a(y)N_\vartheta(y)\varphi(y) \, dy . \quad (3.13)$$

Having established (3.8), (3.9), and (3.13), we may take the limit $\varepsilon \rightarrow 0$ in (2.10) and deduce that φ satisfies (1.16), thereby completing the proof of Theorem 1.1.

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