

From internal to pointwise control for the 1D heat equation and minimal control time

Cyril Letrouit^{*†}

Abstract

Our goal is to study controllability and observability properties of the 1D heat equation with internal control (or observation) set $\omega_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$, in the limit $\varepsilon \rightarrow 0$, where $x_0 \in (0, 1)$. It is known that depending on arithmetic properties of x_0 , there may exist a minimal time T_0 of pointwise control at x_0 of the heat equation. Besides, for any ε fixed, the heat equation is controllable with control set ω_ε in any time $T > 0$. We relate these two phenomena. We show that the observability constant on ω_ε does not converge to 0 as $\varepsilon \rightarrow 0$ at the same speed when $T > T_0$ (in which case it is comparable to $\varepsilon^{1/2}$) or $T < T_0$ (in which case it converges faster to 0). We also describe the behavior of optimal L^2 null-controls on ω_ε in the limit $\varepsilon \rightarrow 0$.

Contents

1	Introduction and main results	1
1.1	Motivations	1
1.2	Main results	4
1.3	Perspectives and open questions	4
2	Proofs	5
2.1	Proof of Theorem 1	5
2.2	Proof of Theorem 2	10
2.3	Proof of Theorem 3	12

1 Introduction and main results

1.1 Motivations

In this paper, we consider the controlled heat equation on $(0, 1)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \partial_{xx} u(t, x) = f(t, x) & \text{in } (0, +\infty) \times (0, 1) \\ u(\cdot, 0) = u(\cdot, 1) = 0 & \text{on } (0, +\infty), \\ u(0, \cdot) = u_0 & \text{on } (0, 1), \end{cases} \quad (1)$$

where $u_0(x) \in L^2(0, 1)$ is the initial datum and $f(t, x)$ is the control. We will consider two cases in which (1) is known to be well-posed:

- either $f \in L^2((0, T) \times (0, 1))$;
- or $f(t, \cdot) = \psi(t)\delta_{x_0}$ where $\psi \in L^2(0, T)$ and $x_0 \in (0, 1)$. Here δ_{x_0} denotes the Dirac mass at x_0 .

^{*}Sorbonne Université, CNRS, Université de Paris, Inria, Laboratoire Jacques-Louis Lions (LJLL), F-75005 Paris, France (letrouit@ljl.math.upmc.fr)

[†]DMA, École normale supérieure, CNRS, PSL Research University, 75005 Paris

In the first case, well-posedness means that, for every $T > 0$, there exists a constant $C > 0$ such that for any $u_0 \in L^2(0, 1)$ and $f \in L^2((0, T) \times (0, 1))$, there exists a unique solution $u \in C^0([0, T], L^2(0, 1)) \cap L^2((0, T), H_0^1(0, 1))$ of (1), and this solution moreover satisfies

$$\|u\|_{C^0([0, T], L^2(0, 1))} + \|u\|_{L^2((0, T), H_0^1(0, 1))} \leq C(\|u_0\|_{L^2(0, 1)} + \|f\|_{L^2((0, T) \times (0, 1))}).$$

In the second case (see for example [AKBGBDT14, Proposition 6.1]), it means that, for every $T > 0$, there exists a constant $C > 0$ such that for any $u_0 \in L^2(0, 1)$ and $\psi \in L^2(0, T)$, there exists a unique solution $u \in C^0([0, T], L^2(0, 1)) \cap L^2((0, T), H_0^1(0, 1))$ of (1) with $f(t, \cdot) = \psi(t)\delta_{x_0}$, and this solution moreover satisfies

$$\|u\|_{C^0([0, T], L^2(0, 1))} + \|u\|_{L^2((0, T), H_0^1(0, 1))} \leq C(\|u_0\|_{L^2(0, 1)} + \|\psi\|_{L^2(0, T)}).$$

In this paper, what will be of interest is the case where f is concentrated only on one point $x_0 \in (0, 1)$ (in this case we speak of *pointwise* control at x_0) or on a small neighborhood of x_0 of the form $(x_0 - \varepsilon, x_0 + \varepsilon)$ for some small $\varepsilon > 0$ (in this case we speak of *internal* control). In the sequel, we fix a point $x_0 \in (0, 1)$.

Several results are known about exact observability (or, by duality, about exact controllability) of (1). In the sequel, by observability we always mean exact observability.

- By internal observability of (1) in time T on an open subset $E \subset (0, 1)$, we mean that

$$C(T, E) := \inf \left\{ \int_0^T \int_E u(t, x)^2 dx dt, \|u_0\|_{L^2(0, 1)} = 1, u \text{ solution of (1) with } f = 0 \right\} > 0.$$

The constant $C(T, E)$ is called the observability constant on E in time T .

- By pointwise observability of (1) in time T at a point $x_0 \in (0, 1)$, we mean that

$$C(T, x_0) = \inf \left\{ \int_0^T u(t, x_0)^2 dt, \|u_0\|_{L^2(0, 1)} = 1, u \text{ solution of (1) with } f = 0 \right\} > 0. \quad (2)$$

The constant $C(T, x_0)$ is called the observability constant at point x_0 in time T .

By duality (see Lemma 1), observability in time T of the heat equation on the open set E is equivalent to the property that for all $u_0 \in L^2(0, 1)$, there exists $f \in L^2((0, T) \times (0, 1))$ with support in $(0, T) \times E$ such that the solution u of (1) satisfies $u(T, \cdot) = 0$. In this case f is called a *null-control*. Similarly, pointwise observability of the heat equation at x_0 is equivalent to the property that for all $u_0 \in L^2(0, 1)$, there exists $\psi \in L^2(0, T)$ such that the solution u of (1) with $f(t, \cdot) = \psi(t)\delta_{x_0}$ satisfies $u(T, \cdot) = 0$.

Depending on the arithmetic properties of x_0 (mainly how well x_0 is approached by rational numbers), the heat equation may or may not be observable at point x_0 in time T . More precisely, we have the following result, due to [Dol73] (see also [AKBGBDT14]).

1. Given any $x_0 \in (0, 1)$, there exists $T_0 \in [0, +\infty]$ such that if $T_0 < +\infty$ and $T > T_0$, then the heat equation is pointwise observable at point x_0 in time T , and if $0 < T < T_0 \leq +\infty$, then it is not pointwise observable at point x_0 in time T .

In the sequel, we adopt the natural convention that if $T_0 = +\infty$, the inequality $T > T_0$ is never verified, even for $T = +\infty$. This means that if we write $T > T_0$, we also implicitly require that $T_0 < +\infty$.

It is also known that on any open sub-interval of $(0, 1)$, the heat equation is observable in any time $T > 0$ (see, e.g., [Rus78]). In particular:

2. Given any $x_0 \in (0, 1)$, any $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (0, 1)$ and any $T > 0$, the heat equation is observable on $(x_0 - \varepsilon, x_0 + \varepsilon)$ in time T .

Our goal is to understand how these two phenomena are linked, most notably by studying the limit $\varepsilon \rightarrow 0$. How does a minimal time appear when the domain of observation shrinks, i.e. when $\varepsilon \rightarrow 0$? Is it related to the size of L^2 -optimal null-controls in the limit $\varepsilon \rightarrow 0$?

The appearance of a minimal time of control at x_0 can be intuitively understood in the following way. First assume that x_0 is a rational number, $x_0 = p/q$ with $p \in \mathbb{N}$ and $q \in \mathbb{N}^*$. Then, for any time $T > 0$, the initial datum $u_0 = \sin(q\pi x)$ cannot be steered to 0 by any control of the form $\psi(t)\delta_{x_0}$ with $\psi \in L^2(0, T)$. If u denotes the solution of (1) with initial datum u_0 , the quantity $\int_0^T u(t, x_0)^2 dt$ is equal to 0, and therefore the heat equation is not pointwise observable at x_0 . In this case, $T_0 = +\infty$. If now x_0 is irrational but well approached by rational numbers, meaning that there exist sequences $(p_k), (n_k)$ of integers such that $|x_0 - p_k/n_k|$ is very small compared with $1/n_k$ (typically less than $e^{-Cn_k^2}$), then, by evaluating the quantity defining the observability constant (2) for the initial data $\sin(n_k\pi x)$, it is possible to prove that the observability constant is equal to 0 for any $T > 0$ (but the infimum defining the observability constant is not reached if $x_0 \notin \mathbb{Q}$), so that $T_0 = +\infty$.

It is also interesting to compute T_0 for x_0 an irrational algebraic number of degree m , that is a root of a polynomial of degree $m \geq 2$. Liouville's theorem on diophantine approximation states that in this case there exists a constant $c(x_0)$ such that $|x_0 - p/n| > c(x_0)/n^m$ for all integers p and n where $n > 0$. Therefore $|\sin(n\pi x_0)| \geq 2c(x_0)/n^{m-1}$ for any $n > 0$. Hence, for any $T > 0$,

$$\sum_{n=1}^{\infty} \frac{\exp(-n^2\pi^2 T)}{|\sin(n\pi x_0)|} \leq \frac{1}{2c(x_0)} \sum_{n=1}^{\infty} n^{m-1} \exp(-n^2\pi^2 T) < +\infty,$$

and according to the results of Dolecki recalled in Theorem 4 below, we get $T_0 = 0$.

In the existing literature, similar problems have been investigated. In [FP94], the authors study the convergence for the 1D wave equation of the L^2 -optimal null-controls on a spatial interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ and compute their blow-up rate. Our problem is somehow the same for the heat equation, but our situation is more intricate due to the appearance at the limit of a minimal control time. For the 1D heat equation, the cost of optimal controls on shrinking volume (i.e., at the limit $\varepsilon \rightarrow 0$) does not seem to have been studied. A different asymptotic question which has attracted much more attention is the cost of optimal controls in the limit $T \rightarrow 0$ for a fixed domain of observation, see [LL18] for recent results in this direction.

Let us also mention that the existence of a minimal time of control for parabolic equations has been studied a lot in the last few years. See for example [AKBGBdT11] or [AKBGBdT14]. However, it has apparently never been related to the blow-up of the cost of the null-controls in the limit $\varepsilon \rightarrow 0$ when the control is located in a thin domain of width ε , and this is precisely what we do in this paper for the 1D heat equation.

The specificity of our problem is that it is related to number theory, as already noted in [Dol73], since the main property which determines the cost of the optimal null-controls is how x_0 is approximated by rational numbers. The problem is tractable in dimension 1, but its extension to higher dimension is not easy. In some sense, the controllability at point x_0 of the heat equation is not a local problem but a global one: if the manifold Ω in which the heat equation evolves is deformed (even very far from x_0), the properties of controllability at point x_0 may change dramatically. Therefore, well-known methods such as Carleman estimates are not appropriate in this context since they are in some sense "local". To give an example, in [LL18, Theorem 1.15], the authors have derived a lower bound on the observability constant of the heat equation in the limit $\varepsilon \rightarrow 0$ which is uniform in x_0 , but the constants in this lower bound can probably be improved if we assume further arithmetic properties on the point x_0 .

The main method we use to address this problem is the so-called moment method, which has been widely used to deal with the 1D heat equation since the seminal work [FR71]. See for example [Lis17] for recent results and an extensive bibliography.

The paper goes as follows. In Section 1.2 we state the main results of our paper. In Section 1.3, we give some perspectives and open problems. Finally, in Section 2, we give the proofs.

Acknowledgment. We warmly thank Emmanuel Trélat for careful reading of earlier versions of this paper.

1.2 Main results

Our first main result is the following. It roughly says that the convergence of the internal observability constant to 0 in the limit $\varepsilon \rightarrow 0$ is much faster when the heat equation is not pointwise observable at the limit point x_0 in time T than when it is observable at x_0 in time T . Recall that $T_0 \in [0, +\infty]$ has been defined above following the results of Dolecki [Dol73], and that it depends only on x_0 .

Theorem 1. *Fix $x_0 \in (0, 1)$ and denote by $C(T, \varepsilon)$ the observability constant in time T on the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$.*

1. *If $T > T_0$, then there exist constants $C_1, C_2 > 0$ (depending on T) such that $C_1 \varepsilon^{1/2} \leq C(T, \varepsilon) \leq C_2 \varepsilon^{1/2}$.*
2. *If $T < T_0$, then there exist a sequence $\varepsilon_k \rightarrow 0$ and constants $C_1 > 0$ and $C_2 > 1/2$ (depending on T) such that $C(T, \varepsilon_k) \leq C_1 \varepsilon_k^{C_2}$.*

Remark 1. *By duality, this theorem gives information on how, when $T > T_0$, for a fixed initial datum u_0 , the norm of the L^2 -optimal null-control ψ_ε on $(x_0 - \varepsilon, x_0 + \varepsilon)$ behaves in the limit $\varepsilon \rightarrow 0$. It says that $\|\psi_\varepsilon\|_{L^2}$ is at most of the order of $C\varepsilon^{-1/2}$. To prove our results, we will sometimes make use of this duality between controllability and observability. We refer to Lemma 1 for a precise statement on duality between controllability and observability.*

Our second result, which deals with the case $T \leq T_0$, gives a finer analysis of the behavior of the optimal null-control in the limit $\varepsilon \rightarrow 0$. Given an initial datum $u_0 \in L^2(0, 1)$ which is assumed to be not pointwise null-controllable at point x_0 in time T , we describe the behavior of the norm of the optimal control with control domain $(x_0 - \varepsilon, x_0 + \varepsilon)$ in the limit $\varepsilon \rightarrow 0$.

Theorem 2. *Let $x_0 \in [0, 1]$ and let $T \leq T_0$. We assume that $u_0 \in L^2(0, 1)$ is not pointwise null-controllable at x_0 in time T . Then the optimal L^2 null-control ψ_ε in time T with control domain $(x_0 - \varepsilon, x_0 + \varepsilon)$ of the heat equation with initial datum u_0 verifies $\varepsilon^{1/2} \|\psi_\varepsilon\|_{L^2} \rightarrow +\infty$.*

Remark 2. *Theorems 1 and 2 roughly indicate that, for a fixed initial datum $u_0 \in L^2(0, 1)$, the blow-up rate of the optimal null-controls ψ_ε in the limit $\varepsilon \rightarrow 0$ determines the controllability at point x_0 and that the key quantity for measuring this rate is $\varepsilon^{1/2} \|\psi_\varepsilon\|_{L^2}$.*

Our third result is a convergence result. Given a point x_0 , an initial datum u_0 and assuming a uniform control of the quantity $\varepsilon^{1/2} \|\psi_\varepsilon\|_{L^2}$, where ψ_ε is a null-control for u_0 in fixed time T supported in $(x_0 - \varepsilon, x_0 + \varepsilon)$, we show that ψ_ε converges in some sense to a pointwise null-control of u_0 at x_0 in time T .

Theorem 3. *Let $x_0 \in (0, 1)$ and $T > 0$. Let $\delta > 0$ be such that $(x_0 - \delta, x_0 + \delta) \subset (0, 1)$ and let $u_0 \in L^2(0, 1)$ be an initial datum. For $0 < \varepsilon < \delta$, we denote by ψ_ε a null-control in time T for u_0 of the heat equation with control domain $(x_0 - \varepsilon, x_0 + \varepsilon)$. We suppose that the quantity $\varepsilon^{1/2} \|\psi_\varepsilon\|_{L^2}$ is uniformly bounded in ε . Let $\varphi_\varepsilon(x, t) = \varepsilon \psi_\varepsilon(x_0 + \frac{\varepsilon}{\delta}x, t) \in L^2((0, T) \times (-\delta, \delta))$. Then there exists $\varphi \in L^2((0, T) \times (-\delta, \delta))$ such that up to a subsequence $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $L^2((0, T) \times (-\delta, \delta))$ and $\psi(\cdot) = \frac{1}{\delta} \int_{-\delta}^{+\delta} \varphi(\cdot, x) dx \in L^2(0, T)$ is a pointwise null-control of u_0 at x_0 in time T .*

According to Theorem 1, this result applies for example in case $T > T_0$ and ψ_ε is an optimal null-control for any $\varepsilon > 0$.

1.3 Perspectives and open questions

In this section, we gather several conjectures and open questions related to the problem addressed in this paper.

- In case $T > T_0$, we speculate that there exists a universal constant K such that we have $\varepsilon^{-1/2} C(T, \varepsilon) \rightarrow KC(T, x_0) \in (0, +\infty)$.
- In the case where $T < T_0$, we think that there exists $C > 0$ (depending on T) such that for all $\varepsilon > 0$, we have $C(T, \varepsilon) \geq C\varepsilon^{3/2}$. This exponent is the one obtained for

example if $x_0 = p/q$ is a rational number and we evaluate the observability constant at an associate eigenfunction $\sin(q\pi x)$. The moment method cannot work to prove this conjecture (the infinite series defining the scalar control does not converge). The only way we see to tackle it is to use Carleman estimates, like in [LL18, Theorem 1.15].

- It is probably true that the limit control $\frac{1}{\delta} \int_{-\delta}^{\delta} \varphi(\cdot, x) dx$ obtained in Theorem 3 is an optimal control for u_0 from point x_0 in time T . Moreover, Theorem 3 might hold without any extraction of a subsequence.
- It is of interest to extend our results to dimension > 1 , that is to understand the behavior of the observability constant of the heat equation in a manifold Ω of dimension > 1 when the domain of observation shrinks to a point or a submanifold. The moment method cannot work anymore in this context (it is restricted to dimension 1 since it requires the convergence of $\sum 1/\lambda_n$, where the λ_n denote the eigenvalues of the Laplacian) and therefore Theorem 1 cannot be easily transposed to this higher-dimensional setting, but Theorems 2 and 3 generalize well. In dimension > 1 , nodal sets play a role similar to the role of rational points in 1D and it is probable that depending on how well a measurable set E is approached by nodal sets, the heat equation may or may not be exactly observable on E in time $T > 0$.

2 Proofs

Before presenting the proofs of our results, we recall the following theorem of [Dol73], which is the starting point of our analysis.

Theorem 4. [Dol73, Theorem 1]

- (a) If the series $\sum_{n=1}^{\infty} \frac{\exp(-n^2 \pi^2 T)}{|\sin(n\pi x_0)|}$ is convergent, then the heat equation is pointwise observable at x_0 for all $T' > T$.
- (b) If this series is divergent, the heat equation is not pointwise observable at x_0 for $T' < T$.

As a corollary, we get the existence of a minimal time of control (denoted by T_0) for pointwise control at point x_0 , as already recalled in the introduction.

An important point to compute the blow-up rate of observability constants is to remark that the size observability constant is related to the one of the minimal control of the associated control problem. We recall the following lemma, for which we took the formulation of [Cor07] although it is much older (see [Rud91] for example).

Lemma 1. [Cor07, Proposition 2.16] Let H_1 and H_2 be two Hilbert spaces. Let \mathcal{F} be a linear continuous map from H_1 into H_2 . Then \mathcal{F} is onto if and only if there exists $c > 0$ such that

$$\|\mathcal{F}^*(x_2)\|_{H_1} \geq c \|x_2\|_{H_2}, \quad \forall x_2 \in H_2. \quad (3)$$

Moreover, if (3) holds for some $c > 0$, there exists a linear continuous map \mathcal{G} from H_2 into H_1 such that

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(x_2) &= x_2, \quad \forall x_2 \in H_2, \\ \|\mathcal{G}(x_2)\|_{H_1} &\leq \frac{1}{c} \|x_2\|_{H_2}, \quad \forall x_2 \in H_2. \end{aligned}$$

In particular, if \mathcal{F} is the input-output map, this relates the observability constant with the controllability one.

2.1 Proof of Theorem 1

We successively prove Point 1 and Point 2 of Theorem 1. The proof of the lower bound of Point 1, which is the trickiest part, is done by modifying the Dolecki control in an x -independent way. For this, we essentially replace $\sin(n\pi x_0)$ by $\frac{1}{\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \sin(n\pi x) dx$ and the key point (Lemma 4) is that, under suitable assumptions on x_0 , $|\frac{1}{\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \sin(n\pi x) dx| \geq c |\sin(n\pi x_0)| e^{-\delta n^2}$.

Point 1. For the upper bound, we proceed as follows. Since $x_0 \notin \{0, 1\}$, we know that $\sin(\pi x_0) \neq 0$ and therefore

$$\begin{aligned} C(T, \varepsilon)^2 &\leq \frac{2}{e^{-2\pi^2 T}} \int_0^T \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{-2\pi^2 t} \sin(\pi x)^2 dx dt \\ &\leq \frac{e^{2\pi^2 T} - 1}{\pi^2} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \sin(\pi x)^2 dx \\ &\sim 2\varepsilon \frac{e^{2\pi^2 T} - 1}{\pi^2} \sin(\pi x_0)^2 \end{aligned}$$

when $\varepsilon \rightarrow 0$. Therefore, $C(T, \varepsilon) \leq C\varepsilon^{1/2}$, which proves the upper bound.

Following Remark 1 and Lemma 1, the proof of the lower bound consists roughly in proving an upper bound on the optimal null-controls ψ_ε driving a given initial datum u_0 to 0 in time T . In order to do so, we find a scalar null-control φ_ε (in the sense that $\varphi_\varepsilon = b_\varepsilon(x)f_\varepsilon(t)$ with $\text{supp } b_\varepsilon \subset [x_0 - \varepsilon, x_0 + \varepsilon]$) which is not the optimal null-control but whose L^2 norm is of the same magnitude as the one of ψ_ε in the limit $\varepsilon \rightarrow 0$. Said differently, for any $\varepsilon > 0$ and any initial datum $u_0 \in L^2(0, 1)$, we find a scalar control φ_ε on $[x_0 - \varepsilon, x_0 + \varepsilon]$ steering u_0 to 0 and whose L^2 norm is bounded above by $C\varepsilon^{-1/2}\|u_0\|_{L^2}$ for some universal constant $C > 0$ independent of ε and of u_0 .

As in [FR71], for a fixed initial datum $u_0 \in L^2(0, 1)$ with Fourier decomposition $u_0(x) = \sum \mu_n \sin(n\pi x)$, we look for φ_ε of the form $\varphi_\varepsilon = b_\varepsilon(x)f(t)$, with $b_\varepsilon(x)$ supported in $[x_0 - \varepsilon, x_0 + \varepsilon]$ and

$$f(t) = \sum_{n=0}^{\infty} \frac{e^{-n^2\pi^2 t} \mu_n}{\int_{x_0-\varepsilon}^{x_0+\varepsilon} b_\varepsilon(x) \sin(n\pi x) dx} \psi_n(t)$$

where (ψ_n) is a family of functions in $L^2(0, T)$ which is biorthogonal to the family of $L^2(0, T)$ functions $(e^{-n^2\pi^2 t})$, meaning that for $j, k \in \mathbb{N}$,

$$\int_0^T \psi_j(t) e^{-k^2\pi^2 t} dt = \delta_{jk}$$

with the Kronecker notation.

Of course, this requires that the numbers $\int_{x_0-\varepsilon}^{x_0+\varepsilon} b_\varepsilon(x) \sin(n\pi x) dx$ are not too small (and in particular non-zero), so that $f \in L^2(0, T)$. In our construction, $b_\varepsilon(x)$ will be of the form $\chi_{[x_0-\varepsilon', x_0+\varepsilon']}$ for some well-chosen $\varepsilon/2 \leq \varepsilon' \leq \varepsilon$, where the symbol χ denotes characteristic functions.

We now start the proof of the lower bound. It is based on several lemmas.

Lemma 2. *There exists a family $(\psi_n)_{n \in \mathbb{N}^*} \in L^2(0, T)$ biorthogonal to the family $e^{-n^2\pi^2 t}$ and satisfying $\|\psi_n\|_{L^2} \leq e^{Cn}$ for every $n \in \mathbb{N}^*$.*

Proof. This result follows for example from results of [FR71]. By [FR71, estimate (3.9)], we know that there exists $K > 0$ such that for all $n \in \mathbb{N}$,

$$\|\psi_n\|_{L^2(0, T)} \leq K n^2 \frac{\prod_{j=1}^{\infty} \left(1 + \frac{n^2}{j^2}\right)}{\prod_{\substack{j=1 \\ j \neq n}}^{\infty} \left(1 - \frac{n^2}{j^2}\right)}. \quad (4)$$

By [FR71, Lemma 3.1], we know that

$$\frac{\prod_{j=1}^{\infty} \left(1 + \frac{n^2}{j^2}\right)}{\prod_{\substack{j=1 \\ j \neq n}}^{\infty} \left(1 - \frac{n^2}{j^2}\right)} = \exp(Cn + o(n)) \quad (5)$$

as $n \rightarrow +\infty$. Combining (4) and (5), we get the proof of Lemma 2. \square

Lemma 3. *For all $\delta > 0$, there exist $C > 0$ and a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ tending to 0 and satisfying $\varepsilon_j \geq \varepsilon_{j+1} \geq \varepsilon_j/2$ and $\phi_n^j \geq C\varepsilon_j e^{-n^2\pi^2\delta}$ where $\phi_n^j = \inf\{|\varepsilon_j - p/n|, p \in \mathbb{Z}\}$.*

Proof. We construct $(\varepsilon_j)_{j \in \mathbb{N}}$ iteratively. First we construct $\varepsilon_0 \in (0, 1)$.

Set $C = \left(4 \sum_n (n+1) e^{-n^2\pi^2\delta}\right)^{-1}$. Define also for $n \in \mathbb{N}^*$

$$U_{0,n} = \left\{x \in [0, 1], \quad \exists p \in \mathbb{Z}, \quad \left|x - \frac{p}{n}\right| < C e^{-n^2\pi^2\delta}\right\}$$

and

$$U_0 = \bigcup_{n \in \mathbb{N}^*} U_{0,n}.$$

We search $\varepsilon_0 \in (0, 1) \setminus U_0$. Denoting by $|E|$ the Lebesgue measure of a set E , we have $|U_{0,n}| \leq 2(n+1)C e^{-n^2\pi^2\delta}$ and therefore $|U_0| \leq 2C \sum_n (n+1) e^{-n^2\pi^2\delta} = 1/2$. Hence, we can pick $\varepsilon_0 \in (0, 1) \setminus U_0$.

Let us now define ε_j (for $j \geq 0$) iteratively. Suppose that ε_j has been defined for some $j \geq 0$. We set

$$U_{j+1,n} = \left\{x \in \left(\frac{\varepsilon_j}{2}, \varepsilon_j\right), \quad \exists p \in \mathbb{Z}, \quad \left|x - \frac{p}{n}\right| < C \varepsilon_j e^{-n^2\pi^2\delta}\right\} \quad \text{for } n \in \mathbb{N}^*$$

and

$$U_{j+1} = \bigcup_{n \in \mathbb{N}^*} U_{j+1,n}.$$

We have $|U_{j+1,n}| \leq C \varepsilon_j^2 (n+1) e^{-n^2\pi^2\delta}$, and therefore $|U_{j+1}| \leq \frac{1}{4} \varepsilon_j^2 \leq \frac{\varepsilon_j}{4}$. Hence we can pick $\varepsilon_{j+1} \in (\frac{\varepsilon_j}{2}, \varepsilon_j) \setminus U_{j+1}$.

This procedure defines recursively a sequence which satisfies the statement of Lemma 3. \square

Lemma 4. *Fix $\delta > 0$. For a sequence ε_j constructed as in Lemma 3, there exists $C > 0$ such that*

$$\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right| \geq C \varepsilon_j |\sin(n\pi x_0)| e^{-n^2\pi^2\delta} \quad (6)$$

Proof. We set $\theta_n = \inf\{|x_0 - \frac{p}{n}|, p \in \mathbb{Z}\}$ and $\phi_n^j = \inf\{|\varepsilon_j - \frac{p}{n}|, p \in \mathbb{Z}\}$. We will keep these notations until the end of the proof of Theorem 1. Remark that $0 \leq \theta_n \leq \frac{1}{2n}$ and $0 \leq \phi_n^j \leq \frac{1}{2n}$. In the sequel, we fix j and n , and therefore we can write $\varepsilon_j = \frac{p}{n} \pm \phi_n^j$, omitting the dependence of p in j and n . There are two cases.

Let us first assume that $\varepsilon_j \leq \theta_n$. Then on $(x_0 - \varepsilon_j, x_0 + \varepsilon_j)$, the function $\sin(n\pi x)$ is of constant sign and therefore

$$\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right| = \left| \int_{x_0 - \varepsilon_j - p/n}^{x_0 + \varepsilon_j - p/n} \sin(n\pi x) dx \right| \geq \left| \int_{x_0 - \varepsilon_j - p/n}^{x_0 + \varepsilon_j - p/n} 2nxdx \right|$$

since $|\sin(\pi y)| \geq 2|y|$ for $|y| \leq 1/2$. Therefore

$$\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right| \geq 4\varepsilon_j n \left|x_0 - \frac{p}{n}\right| \geq \frac{4}{\pi} \varepsilon_j |\sin(n\pi x_0)|,$$

which proves that (6) holds in this case for $C = 4/\pi$.

We now assume at the contrary that $\varepsilon_j \geq \theta_n$. We set $f(\varepsilon) = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \sin(n\pi x) dx$. Then we can easily verify the following properties of f :

- If $\sin(n\pi x_0) \geq 0$, then f increases between 0 and $1/2n$ and decreases between $1/2n$ and $1/n$. Moreover $f(0) = f(1/n) = 0$.
- If $\sin(n\pi x_0) \leq 0$, then f decreases between 0 and $1/2n$ and increases between $1/2n$ and $1/n$. Moreover $f(0) = f(1/n) = 0$.

Now we write

$$\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right| = \left| \int_{x_0 - \frac{p}{n} \mp \phi_n^j}^{x_0 + \frac{p}{n} \pm \phi_n^j} \sin(n\pi x) dx \right|.$$

This last integral can be decomposed into three parts

$$\int_{x_0 - \frac{p}{n} \mp \phi_n^j}^{x_0 + \frac{p}{n} \pm \phi_n^j} = \int_{x_0 - \frac{p}{n} \mp \phi_n^j}^{x_0 - \frac{p}{n}} + \int_{x_0 - \frac{p}{n}}^{x_0 + \frac{p}{n}} + \int_{x_0 + \frac{p}{n}}^{x_0 + \frac{p}{n} \pm \phi_n^j}.$$

The middle integral equals 0 and the first one is also equal to $\int_{x_0 + \frac{p}{n} \mp \phi_n^j}^{x_0 + \frac{p}{n}} \sin(n\pi x) dx$. Finally we get

$$\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right| = \left| \int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right|.$$

Let us finally prove that

$$\left| \int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right| \geq C \phi_n^j |\sin(n\pi x_0)| \quad (7)$$

for some universal constant $C > 0$. If $|\phi_n^j| \leq \theta_n$, as in the case $\varepsilon_j \leq \theta_n$, we easily get that (7) holds for $C = 4/\pi$. If $\theta_n \leq \phi_n^j \leq 1/(2n)$, then we can suppose for example that $\sin(n\pi x_0) \geq 0$. The case $\sin(n\pi x_0) \leq 0$ can be handled similarly. The integral $\int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx$ is decomposed into

$$\int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} = \int_{x_0 - \phi_n^j}^{\frac{p}{n}} + \int_{\frac{p}{n}}^{2\frac{p}{n} - x_0 + \phi_n^j} + \int_{2\frac{p}{n} - x_0 + \phi_n^j}^{x_0 + \phi_n^j}.$$

The first two integrals compensate and therefore

$$\left| \int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right| = \left| \int_{2\frac{p}{n} - x_0 + \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right|.$$

The integral at the right-hand side has bounds $2\frac{p}{n} - x_0 + \phi_n^j$ and $x_0 + \phi_n^j$, between which $\sin(n\pi x)$ is positive. Note that for any a such that $\sin(n\pi a) > 0$ and any b such that $\sin(n\pi x)$ is positive on $(a - b, a + b)$, we have $\int_{a-b}^{a+b} \sin(n\pi x) dx \geq b \sin(n\pi a)$. Applying this with $a = 2\frac{p}{n} - x_0 + \phi_n^j$ and $b = 2(x_0 - \frac{p}{n})$, we get

$$\begin{aligned} \left| \int_{x_0 - \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right| &= \left| \int_{2\frac{p}{n} - x_0 + \phi_n^j}^{x_0 + \phi_n^j} \sin(n\pi x) dx \right| \\ &\geq \left| \sin \left(n\pi \left(\frac{p}{n} + \phi_n^j \right) \right) \right| \left| x_0 - \frac{p}{n} \right| \\ &\geq 2n\phi_n^j \left| x_0 - \frac{p}{n} \right| \text{ because } |\sin(x)| \geq \frac{2}{\pi}|x| \text{ for } |x| \leq \frac{\pi}{2} \\ &\geq \frac{2}{\pi} \phi_n^j |\sin(n\pi x_0)| \\ &\geq C\varepsilon_j |\sin(n\pi x_0)| e^{-n^2 \pi^2 \delta} \end{aligned}$$

where we have used Lemma 3. This concludes the proof of Lemma 4. \square

End of the proof of the lower bound. We first prove that there exists $C > 0$ such that for all $j \in \mathbb{N}$, we have $C\varepsilon_j^{1/2} \leq C(T, \varepsilon_j)$. It will imply the lower bound of point 1 of Theorem 1 for a particular sequence of ε , namely the sequence (ε_j) . Fix $j \in \mathbb{N}$. Following [FR71], we look for a control φ_{ε_j} in the scalar form $\varphi_{\varepsilon_j} = f(t)\chi_{[-\varepsilon_j, \varepsilon_j]}$ where χ denotes the characteristic function. We take

$$f(t) = \sum_n \frac{\mu_n e^{-n^2 \pi^2 T}}{\int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx} \psi_n(t).$$

Then

$$\|f(t)\|_{L^2(0,T)} \leq \sum_n \frac{|\mu_n| e^{-n^2 \pi^2 T}}{\left| \int_{x_0 - \varepsilon_j}^{x_0 + \varepsilon_j} \sin(n\pi x) dx \right|} \|\psi_n\|_{L^2(0,T)}.$$

Since $T > T_0$, by Theorem 4, we can pick $\delta > 0$ so that $\sum \frac{e^{-n^2 \pi^2 (T-2\delta)}}{|\sin(n\pi x_0)|} < +\infty$. This implies in particular

$$\sum_n \frac{e^{-2n^2 \pi^2 (T-2\delta)}}{|\sin(n\pi x_0)|^2} < +\infty. \quad (8)$$

For this $\delta > 0$, we take a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ as in Lemma 3. We get, following Lemma 4 and Lemma 2:

$$\|f(t)\|_{L^2(0,T)} \leq \frac{C}{\varepsilon_j} \sum_n \frac{|\mu_n| e^{-n^2 \pi^2 (T-\delta)}}{|\sin(n\pi x_0)|} \|\psi_n\|_{L^2(0,T)} \leq \frac{C}{\varepsilon_j} \sum_n \frac{|\mu_n| e^{-n^2 \pi^2 (T-2\delta)}}{|\sin(n\pi x_0)|}.$$

Using the Cauchy-Schwarz inequality, recalling that $\|u_0\|_{L^2}^2 = \sum_n |\mu_n|^2$ and $\|\varphi_{\varepsilon_j}\|_{L^2((0,T) \times (0,1))} = \sqrt{2\varepsilon_j^{1/2}} \|f\|_{L^2(0,T)}$, we finally get

$$\|\psi_{\varepsilon_j}\|_{L^2} \leq \|\varphi_{\varepsilon_j}\|_{L^2} \leq \frac{C}{\varepsilon_j^{1/2}} \left(\sum_n \frac{e^{-2n^2 \pi^2 (T-2\delta)}}{|\sin(n\pi x_0)|^2} \right)^{1/2} \|u_0\|_{L^2} \leq \frac{C}{\varepsilon_j^{1/2}} \|u_0\|_{L^2}$$

because of (8). By Lemma 1, we get that $C(T, \varepsilon_j) \geq C\varepsilon_j^{1/2}$.

We have established the lower bound of point 1 of Theorem 1 for the sequence (ε_j) , but we must now deal with all $\varepsilon \in (-\delta, \delta)$. We fix $\varepsilon \in (-\delta, \delta)$ and ε_j so that $\varepsilon/2 \leq \varepsilon_j \leq \varepsilon$ which is possible by construction of the sequence (ε_j) . Then the optimal null-control ψ_{ε_j} is equal to 0 outside $(x_0 - \varepsilon_j, x_0 + \varepsilon_j)$, and therefore it is also equal to 0 outside $(x_0 - \varepsilon, x_0 + \varepsilon)$. We denote by ψ_ε the optimal null-control on $(x_0 - \varepsilon, x_0 + \varepsilon)$. We have

$$\varepsilon \|\psi_\varepsilon\|_{L^2}^2 \leq 2\varepsilon_j \|\psi_{\varepsilon_j}\|_{L^2}^2 = 2\varepsilon_j \|\psi_{\varepsilon_j}\|_{L^2}^2 \leq 2C.$$

Therefore the lower bound for the observability constant holds with C replaced by $C/2$. \square

Point 2. By Theorem 4, since $T < T_0$, there exist $\delta > 0$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with $n_k \rightarrow +\infty$ when $k \rightarrow +\infty$ such that

$$|\sin(n_k \pi x_0)| \leq e^{-n_k^2 \pi^2 (T+\delta)}.$$

Let us recall that $\theta_{n_k} = \inf \left\{ \left| x_0 - \frac{p}{n_k} \right|, p \in \mathbb{Z} \right\}$ is the best approximation of x_0 by fractions with denominator n_k . Since $|x| \leq |\sin(\pi x)|/2$ for $x \in [-1/2, 1/2]$, we have for a p reaching the infimum in the definition of θ_{n_k} :

$$\theta_{n_k} = \frac{1}{n_k} |n_k x_0 - p| \leq \frac{1}{2n_k} |\sin(n_k \pi x_0 - p\pi)| \leq \frac{1}{2n_k} e^{-n_k^2 \pi^2 (T+\delta)} \leq e^{-n_k^2 \pi^2 (T+\delta)}.$$

Therefore,

$$n_k^2 \leq \frac{-\log \theta_{n_k}}{\pi^2 (T+\delta)}. \quad (9)$$

We set $\varepsilon_k = \theta_{n_k}$. Clearly, $\lim \varepsilon_k = 0$ when $k \rightarrow +\infty$ and we have

$$C(T, \varepsilon_k)^2 \leq \frac{e^{2n_k^2 \pi^2 T} - 1}{2n_k^2 \pi^2} \left| \int_{p/n_k}^{x_0 + \theta_{n_k}} \sin(n_k \pi x)^2 dx \right| \leq \frac{e^{2n_k^2 \pi^2 T}}{2n_k^3 \pi^2} \int_0^{2n_k \theta_{n_k}} \sin(\pi y)^2 dy.$$

Using that $|\sin(x)| \leq |x|$, we get

$$C(T, \varepsilon_k)^2 \leq \frac{e^{2n_k^2 \pi^2 T}}{2n_k^3 \pi^2} \frac{\pi^2 (2n_k \theta_{n_k})^3}{6} = \frac{2}{3} e^{2n_k^2 \pi^2 T} \theta_{n_k}^3$$

We can bound this expression by above using (9), and we get

$$C(T, \varepsilon_k)^2 \leq \frac{2}{3} e^{2\pi^2 T \left(\frac{-\log \theta_{n_k}}{\pi^2 (T+\delta)} \right)} \theta_{n_k}^3 = \frac{2}{3} \varepsilon_k^{3-2T/(T+\delta)}.$$

Finally, we have

$$C(T, \varepsilon_k)^2 \leq \sqrt{\frac{2}{3}} \varepsilon_k^{\frac{1}{2} + \frac{\delta}{T+\delta}}.$$

Setting $C_2 = \frac{1}{2} + \frac{\delta}{T+\delta}$, we get the upper bound.

2.2 Proof of Theorem 2

We proceed by contradiction and assume that there exists $C > 0$ and a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$, $\varepsilon_j \rightarrow 0$ such that $\varepsilon_j \|\psi_{\varepsilon_j}\|_{L^2}^2 \leq C$. In the sequel, we omit index j . Let $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (0, 1)$. For $x \in (x_0 - \delta, x_0 + \delta)$ and almost all $t \in (0, T)$ we set

$$\varphi_\varepsilon(x, t) = \varepsilon \psi_\varepsilon \left(x_0 + \frac{\varepsilon}{\delta} x, t \right)$$

with $\varphi_\varepsilon \in L^2((0, T) \times (-\delta, \delta))$. Then for $0 < \varepsilon < \delta$, we have

$$\int_0^T \int_{-\delta}^\delta \varphi_\varepsilon(x, t)^2 dx dt = \int_0^T \int_{-\delta}^\delta \varepsilon^2 \psi_\varepsilon \left(x_0 + \frac{\varepsilon}{\delta} x, t \right)^2 dx dt = \delta \varepsilon \int_0^T \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \psi_\varepsilon(x, t)^2 dx dt \leq C \delta.$$

Therefore, there exists $\varphi \in L^2((0, T) \times (-\delta, \delta))$ such that $\varphi_\varepsilon \rightharpoonup \varphi$ in $L^2((0, T) \times (-\delta, \delta))$.

For almost all $t \in (0, T)$, we set

$$\psi(t) = \frac{1}{\delta} \int_{-\delta}^\delta \varphi(x, t) dx \in L^2(0, T)$$

and we prove that ψ is a null control from x_0 for u_0 in time T , i.e., the function u verifying

$$\partial_t u - \Delta u = \psi(t) \delta_{x_0}, \quad u|_{t=0} = u_0$$

with Dirichlet boundary conditions also satisfies $u|_{t=T} = 0$. In other words, $\psi(t)$, which is somehow a limit of the null-controls φ_ε is also a null-control. The proof goes as follows. Fix $v_T \in L^2(0, 1)$. Let $v \in L^2((0, 1) \times (0, T))$ be a solution of the backward heat equation

$$\partial_t v + \Delta v = 0, \quad v|_{t=T} = v_T$$

with Dirichlet boundary conditions. We know that for every $\varepsilon > 0$, the solution u_ε of

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \psi_\varepsilon, \quad u|_{t=0} = u_0$$

with Dirichlet boundary conditions also satisfies $u_\varepsilon|_{t=T} = 0$, and therefore

$$(\partial_t u_\varepsilon, v) - (\Delta u_\varepsilon, v) = (\psi_\varepsilon, v)$$

where the scalar product is the $L^2((0, 1) \times (0, T))$ scalar product. Integrating by part, using the boundary conditions and the fact that v is a solution of the backward heat equation, we get

$$(u_\varepsilon(\cdot, T), v(\cdot, T)) - (u_0, v(\cdot, 0)) = \int_0^T \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_\varepsilon(x, t) v(x, t) dx dt$$

which reduces to

$$- (u_0, v(\cdot, 0)) = \int_0^T \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_\varepsilon(x, t) v(x, t) dx dt. \quad (10)$$

Similarly, we get

$$(u(\cdot, T), v(\cdot, T)) - (u_0, v(\cdot, 0)) = \int_0^T \psi(t) v(x_0, t) dt. \quad (11)$$

Let us prove that $\int_0^T \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_\varepsilon(x, t) v(x, t) dx dt \rightarrow \int_0^T \psi(t) v(x_0, t) dt$ when $\varepsilon \rightarrow 0$. We have

$$\int_0^T \int_{x_0-\varepsilon}^{x_0+\varepsilon} \psi_\varepsilon(x, t) v(x, t) dx dt = \frac{1}{\delta} \int_0^T \int_{-\delta}^{\delta} \varphi_\varepsilon(x, t) v\left(x_0 + \frac{\varepsilon}{\delta} x, t\right) dx dt = A + B \quad (12)$$

where

$$A = \frac{1}{\delta} \int_0^T \int_{-\delta}^{\delta} \varphi_\varepsilon(x, t) v(x_0, t) dx dt$$

and

$$B = \frac{1}{\delta} \int_0^T \int_{-\delta}^{\delta} \varphi_\varepsilon(x, t) \left(v\left(x_0 + \frac{\varepsilon}{\delta} x, t\right) - v(x_0, t) \right) dx dt.$$

Integrating the weak convergence $\varphi_\varepsilon \rightharpoonup \varphi$, which holds in $L^2((-\delta, \delta) \times (0, T))$, against $\frac{1}{\delta} \mathbf{1}_{(-\delta, \delta) \times (0, T)} v(x_0, t)$, we get

$$A \rightarrow \int_0^T \psi(t) v(x_0, t) dt. \quad (13)$$

For B , we prove that $B \rightarrow 0$. The proof goes as follows. We write that

$$B^2 \leq \left(\frac{1}{\delta} \int_0^T \int_{-\delta}^{\delta} \varphi_\varepsilon(x, t)^2 dx dt \right) \left(\frac{1}{\delta} \int_0^T \int_{-\delta}^{\delta} \left| v\left(x_0 + \frac{\varepsilon}{\delta} x, t\right) - v(x_0, t) \right|^2 dx dt \right)$$

and since the first integral is bounded above by a constant C , we just have to prove that the second one converges to 0. We decompose v , writing $v(x, t) = \sum a_j \sin(j\pi x) e^{-j^2 \pi^2 t}$, and we get

$$\begin{aligned} \int_0^T \int_{-\delta}^{\delta} \left| v\left(x_0 + \frac{\varepsilon}{\delta} x, t\right) - v(x_0, t) \right|^2 dx dt &= \frac{\delta}{\varepsilon} \int_0^T \int_{-\varepsilon}^{\varepsilon} |v(x_0 + y, t) - v(x_0, t)|^2 dy dt \\ &\leq \frac{2\delta \|v\|_\infty}{\varepsilon} \int_0^T \int_{-\varepsilon}^{\varepsilon} |v(x_0 + y, t) - v(x_0, t)| dy dt \\ &= \frac{2\delta \|v\|_\infty}{\varepsilon} \int_0^T \int_{-\varepsilon}^{\varepsilon} \sum |a_j| e^{-j^2 \pi^2 t} |\sin(j\pi(x_0 + y)) - \sin(j\pi x_0)| dy dt \\ &\leq \frac{2\delta \|v\|_\infty}{\varepsilon} \int_0^T \int_{-\varepsilon}^{\varepsilon} \sum |a_j| j\pi y e^{-j^2 \pi^2 t} dy dt \\ &\leq 2\varepsilon \delta \|v\|_\infty \int_0^T \sum |a_j| j\pi e^{-j^2 \pi^2 t} dt \end{aligned}$$

which goes to 0 in the limit $\varepsilon \rightarrow 0$. Therefore we have obtained

$$B \rightarrow 0. \quad (14)$$

Combining (10), (11), (12), (13) and (14), we finally get $(u(\cdot, T), v_T) = 0$. Since this is true for all v_T , we get that $u_T = 0$, which means that $\psi(t)\delta_{x_0}$ is a null-control for u_0 . This is a contradiction. It finishes the proof of Theorem 2.

2.3 Proof of Theorem 3

Theorem 3 follows from the computations done in the proof of Theorem 2. As in the proof of Theorem 2, if we set

$$\varphi_\varepsilon(x, t) = \varepsilon \psi_\varepsilon \left(x_0 + \frac{\varepsilon}{\delta} x, t \right), \quad \varphi_\varepsilon \in L^2((-\delta, \delta) \times (0, T))$$

then for $0 < \varepsilon < \delta$, we have

$$\int_0^T \int_{-\delta}^{\delta} \varphi_\varepsilon(x, t)^2 dx dt \leq C\delta$$

and therefore, there exists $\varphi \in L^2((-\delta, \delta) \times (0, T))$ such that $\varphi_\varepsilon \rightharpoonup \varphi$ in $L^2((0, T) \times (-\delta, \delta))$.

For almost all $t \in (0, T)$, we finally set

$$\psi(t) = \frac{1}{\delta} \int_{-\delta}^{\delta} \varphi(x, t) dx, \quad \psi \in L^2(0, T)$$

and the proof of Theorem 2 shows that ψ is a null-control from x_0 for u_0 in time T .

As a side remark, note that if $T > T_0$, Theorem 1 ensures that the quantity $\varepsilon^{1/2} \|\psi_\varepsilon\|_{L^2}$ is uniformly bounded in ε when ψ_ε is the optimal null-control steering u_0 to 0 in time T and with control domain $(x_0 - \varepsilon, x_0 + \varepsilon)$. Therefore, Theorem 3 applies in this case.

References

- [AKBGBdT11] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. The Kalman condition for the boundary controllability of coupled parabolic systems. bounds on biorthogonal families to complex matrix exponentials. *Journal de Mathématiques Pures et Appliquées*, 96(6):555–590, 2011.
- [AKBGBDT14] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz De Teresa. Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences. *Journal of Functional Analysis*, 267(7):2077–2151, 2014.
- [Cor07] Jean-Michel Coron. *Control and nonlinearity*. Number 136. American Mathematical Soc., 2007.
- [Dol73] Simon Dolecki. Observability for the one-dimensional heat equation. *Studia Mathematica*, 48, 1973.
- [FP94] Caroline Fabre and Jean-Pierre Puel. Pointwise controllability as limit of internal controllability for the wave equation in one space dimension. *Portugaliae Mathematica*, 51(3):335–350, 1994.
- [FR71] Hector O. Fattorini and David L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Archive for Rational Mechanics and Analysis*, 43(4):272–292, 1971.
- [Lis17] Pierre Lissy. The cost of the control in the case of a minimal time of control: the example of the one-dimensional heat equation. *Journal of Mathematical Analysis and Applications*, 451(1):497–507, 2017.
- [LL18] Camille Laurent and Matthieu Léautaud. Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller. *arXiv preprint arXiv:1806.00969*, 2018.
- [Rud91] Walter Rudin. Functional analysis. International Series in Pure and Applied Mathematics, 1991.
- [Rus78] David L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *Siam Review*, 20(4):639–739, 1978.