# Fredholm and wronskian representations of solutions to the Johnson equation and the third order case

Pierre Gaillard

# Fredholm and wronskian representations of solutions to the Johnson equation and the third order case.

**P. Gaillard**,
Université de Bourgogne,
Institut de mathématiques de Bourgogne,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
E-mail : Pierre.Gaillard@u-bourgogne.fr

**Abstract**

We construct solutions to the Johnson equation (J) by means of Fredholm determinants first, then by means of wronskians of order $2N$ giving solutions of order $N$ depending on $2N-1$ parameters. We obtain $N$ order rational solutions which can be written as a quotient of two polynomials of degree $2N(N+1)$ in $x$, $t$ and $4N(N+1)$ in $y$ depending on $2N-2$ parameters.
This method gives an infinite hierarchy of solutions to the Johnson equation.
In particular, rational solutions are obtained. The solutions of order 3 with 4 parameters are constructed and studied in detail by means of their modulus in the $(x, y)$ plane in function of time $t$ and parameters $a_1$, $a_2$, $b_1$, $b_2$.

PACS numbers :
33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1 Introduction

The Johnson equation was introduced in 1980 by Johnson [1] to describe waves surfaces in shallow incompressible fluids [2, 3]. This equation was derived for internal waves in a stratified medium [4]. The Johnson equation is dissipative; it is well known that there is no solution with a linear front localized along straight lines in the $(x, y)$ plane. This Johnson equation is for example able to explain the existence of the horseshoe like solitons and multi-soliton solutions

quite naturally.
We consider the Johnson equation (J) in the following normalization

$$(u_t + 6uu_x + u_{xxx} + \frac{u}{2t})_x - 3\frac{u_{yy}}{t^2} = 0, \tag{1}$$

where as usual subscript $x$, $y$ and $t$ mean partial derivatives.
The first solutions were constructed in 1980 by Johnson [1]. Other types of solutions were found in [5]. A new approach to solve this equation was given in 1986 [6] by giving a link between solutions of the Kadomtsev-Petviashvili (KP) [9] and solutions of the Johnson equation. In 2007, another types of solutions were obtained by using the Darboux transformation [7]. More recently, in 2013, other extensions have been considered as the elliptic case [8].
Here we consider the famous Kadomtsev-Petviashvili (KPI) which can be written in the form :

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \tag{2}$$

The KPI equation first appeared in 1970 [9], in a paper written by Kadomtsev and Petviashvili. This equation is considered as a model for surface and internal water waves [10], and in nonlinear optics [11].

In the following, we will use the KPI equation to construct solutions to the Johnson equation, but in another way different from this used in [6]. Indeed, these last authors consider another representation of KPI equation given by

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{3}$$

and so the transformations between solutions of (3) and (1) are different from these we use to transform solutions of (2) in solutions to (1).
In fact, to obtain solutions to (1) from solutions to (2), we use the following transformation :

$$(x; y; t) \mapsto (x_1 = -ix - i\frac{y^2 t}{12}; \quad y_1 = yt; \quad t_1 = 4it) \tag{4}$$

In this paper, we give solutions by means of Fredholm determinants of order $2N$ depending on $2N-1$ parameters and then, by means of wronskians of order $2N$ with $2N-1$ parameters. So we construct an infinite hierarchy of solutions to the Johnson equation, depending on $2N-1$ real parameters .
New rational solutions depending a priori on $2N-2$ parameters at order $N$ are constructed, when one parameter tends to 0.
We obtain families depending on $2N-2$ parameters for the $N$-th order as a ratio of two polynomials of degree $2N(N+1)$ in $x$, $t$ and of degree $4N(N+1)$ in $y$.
In this paper, we construct only rational solutions of order 3, depending on 4 real parameters; we construct the representations of their modulus in the plane of the coordinates $(x, y)$ according to the four real parameters $a_i$, $b_i$, for $1 \leq i \leq 2$ and time $t$.

## 2 Solutions to Johnson equation expressed by means of Fredholm determinants

Some notations are given. We define first real numbers $\lambda_j$ such that $-1 < \lambda_\nu < 1$, $\nu = 1, \dots, 2N$; they depend on a parameter $\epsilon$ and can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N, \tag{5}$$

Then, we define $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$; they are functions of $\lambda_\nu$, $1 \leq \nu \leq 2N$ and are defined by the following formulas :

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}};\\ x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1, 3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2},\\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1},\\ x_{r,N+j} = -x_{r,j},, \quad \tau_{N+j} = \tau_j \quad j = 1, \dots, N. \end{array} \tag{6}$$

$e_\nu$ $1 \leq \nu \leq 2N$ are defined by :

$$\begin{array}{l} e_j = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} - i\sum_{k=1}^{1/2\,M-1} b_K (je)^{2\,k+1}\right),\\ e_{N+j} = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} + i\sum_{k=1}^{1/2\,M-1} b_k (je)^{2\,k+1}\right), \quad 1 \leq j \leq N,\\ a_k, b_k \in \mathbf{R}, \quad 1 \leq k \leq N. \end{array} \tag{7}$$

$\epsilon_\nu$, $1 \leq \nu \leq 2N$ are defined by :

$$\epsilon_j = 1, \quad \epsilon_{N+j} = 0 \quad 1 \leq j \leq N. \tag{8}$$

As usual $I$ is the unit matrix and $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ the matrix defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right) \exp(\kappa_\nu x + (\frac{\kappa_\nu y}{12} - 2\delta_\nu) y t + 4i\tau_\nu t + x_{r,\nu} + e_\nu). \tag{9}$$

Then we get :

**Theorem 2.1** *The function $v$ defined by*

$$v(x,y,t) = -2\,\frac{|n(x,y,t)|^2}{d(x,y,t)^2} \tag{10}$$

*where*

$$n(x,y,t) = \det(I + D_3(x,y,t)), \tag{11}$$

$$d(x,y,t) = \det(I + D_1(x,y,t)), \tag{12}$$

*and $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ the matrix*

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right) \exp(\kappa_\nu x + (\frac{\kappa_\nu y}{12} - 2\delta_\nu) y t + 4i\tau_\nu t + x_{r,\nu} + e_\nu). \tag{13}$$

*is a solution to (1), depending on $2N-1$ parameters $a_k$, $b_k$, $1 \leq k \leq N-1$ and $\epsilon$.*

## 3 Solutions to the Johnson equation by means of wronkians

We use the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1,3, \quad (14)$$

with

$$\Theta_{r,\nu} = \frac{-i\kappa_\nu x}{2} + i(\frac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \quad (15)$$

$W_r(w)$ is the wronskian of the functions $\phi_{r,1}, \ldots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu\in[1,\ldots,2N]}]. \quad (16)$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu\in[1,\ldots,2N]}$ defined in (13).
Then we have the following result

**Theorem 3.1**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \ldots, \phi_{r,2N})(0), \quad (17)$$

*where*

$$k_r(y) = \frac{2^{2N}\exp(i\sum_{\nu=1}^{2N}\Theta_{r,\nu})}{\prod_{\nu=2}^{2N}\prod_{\mu=1}^{\nu-1}(\gamma_\nu - \gamma_\mu)}.$$

Then the solution $v$ to the Johnson equation can be rewritten as

$$v(x,y,t) = -2\frac{|\det(I + D_3(x,y,t))|^2}{(\det(I + D_1(x,y,t)))^2}.$$

With (17), the following link between Fredholm determinants and wronskians is obtained

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0)$$

and

$$\det(I + D_1) = k_1(0) \times W_1(\phi_{r,1}, \ldots, \phi_{r,2N})(0).$$

As $\Theta_{3,j}(0)$ contains $N$ terms $x_{3,j}$ $1 \leq j \leq N$ and $N$ terms $-x_{3,j}$ $1 \leq j \leq N$, we have the relation $k_3(0) = k_1(0)$, and we get :

**Theorem 3.2** *The function $v$ defined by*

$$v(x,y,t) = -2\frac{|W_3(\phi_{3,1}, \ldots, \phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1}, \ldots, \phi_{1,2N})(0))^2}$$

*is a solution of the Johnson equation which depends on $2N-1$ real parameters $a_k$, $b_k$ and $\epsilon$, with $\phi_\nu^r$ defined in (14)*

$$\phi_{r,\nu} = \sin(\frac{-i\kappa_\nu x}{2} + i(\frac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}), \quad 1 \leq \nu \leq N,$$
$$\phi_{r,\nu} = \cos(\frac{-i\kappa_\nu x}{2} + i(\frac{-\kappa_\nu y}{24} + \delta_\nu)yt - i\frac{x_{r,\nu}}{2} + 2\tau_\nu t + \gamma_\nu w - i\frac{e_\nu}{2}), \quad N+1 \leq \nu \leq 2N, \quad r = 1,3,$$

*$\kappa_\nu$, $\delta_\nu$, $x_{r,\nu}$, $\gamma_\nu$, $e_\nu$ being defined in(6), (5) and (7).*

# 4 Study of the limit case when $\epsilon$ tends to $0$

## 4.1 Rational solutions of order $N$ depending on $2N-2$ parameters

An infinite hierarchy of rational solutions to the Johnson equation depending on $2N-2$ parameters is obtained. For this, we take the limit when the parameter $\epsilon$ tends to 0.
We get the following statement :

**Theorem 4.1** *The function $v$*

$$v(x,y,t)=\lim_{\epsilon\to 0}-2\frac{|W_3(x,y,t)|^2}{(W_1(x,y,t))^2} \tag{18}$$

*is a rational solution to the Johnson equation. It is a quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters $\tilde{a_j}$ and $\tilde{b_j}$, $1\leq j\leq N-1$ of degrees $2N(N+1)$ in $x$, $t$ and $4N(N+1)$ in $y$.*

## 4.2 Families of rational solutions of order $3$ depending on $4$ parameters

Here we construct families of rational solutions to the Johnson equation of order 3 explicitly; they depend on 4 parameters.
We only give the expression without parameters and we give it in the appendix because of the length of the solutions.
We construct the patterns of the modulus of the solutions in the plane $(x,y)$ of coordinates in functions of parameters $a_i$, $b_i$, $1\leq i\leq 2$ and time $t$.
The role of the parameters $ai$ and $b_i$ for the same integer $i$ being the same one, one will be interested primarily only in parameters $a_i$.

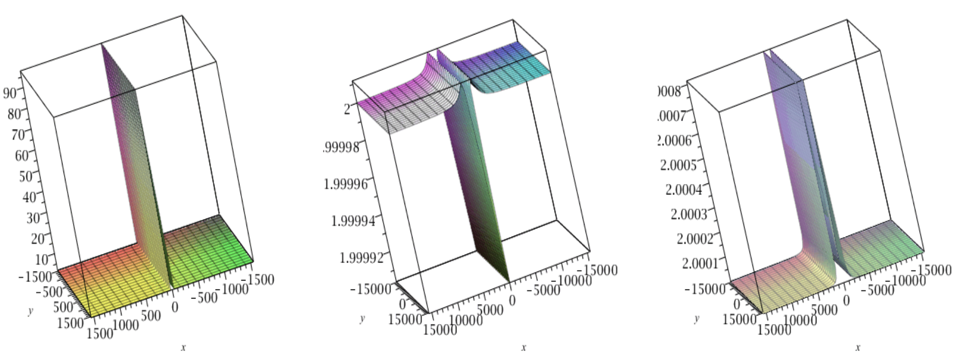

**Figure 1.** Solution of order 3 to (1), on the left for $t=0$, $a_1=0$, $b_1=0$, $a_2=0$, $b_2=0$, ; in the center for $t=0$, $a_1=10^3$, $b_1=0$, $a_2=0$, $b_2=0$; on the right for for $t=0$, $a_1=0$, $b_1=10^3$, $a_2=0$, $b_2=0$.

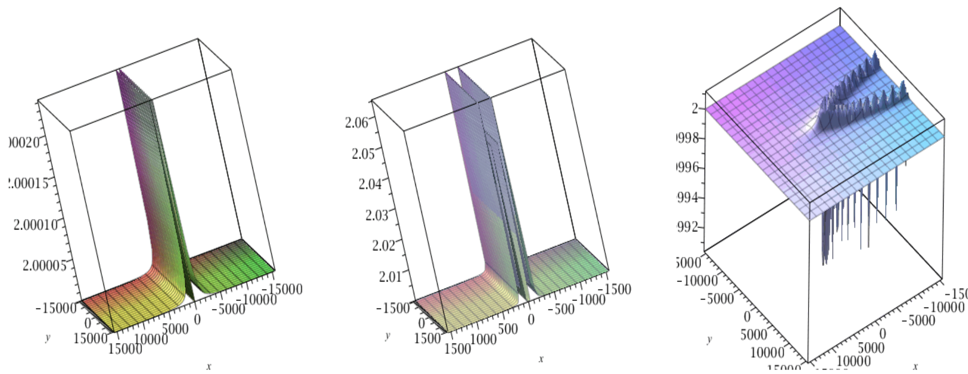

**Figure 2.** Solution of order 3 to (1), on the left for $t = 0$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$, ; in the center for $t = 0$, $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, $b_2 = 10^6$; on the right for for $t = 0,01$, $a_1 = 0$, $b_1 = 10^3$, $a_2 = 0$, $b_2 = 0$.

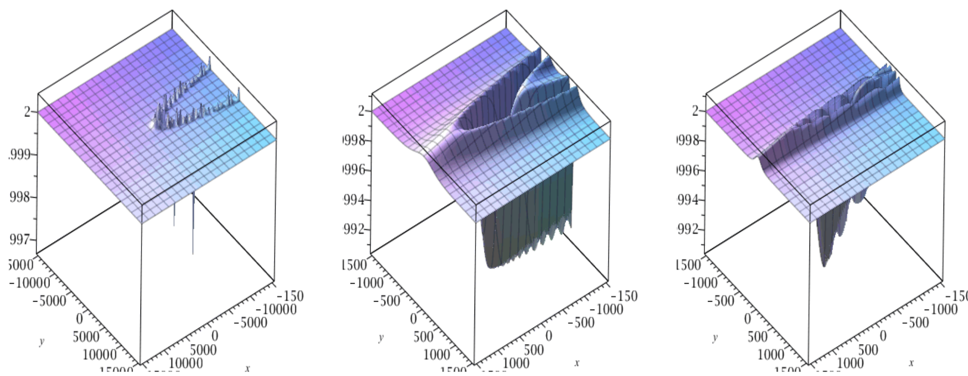

**Figure 3.** Solution of order 3 to (1), on the left for $t = 0,01$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, $b_2 = 0$, ; in the center for $t = 0,1$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$; on the right for for $t = 1$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$.

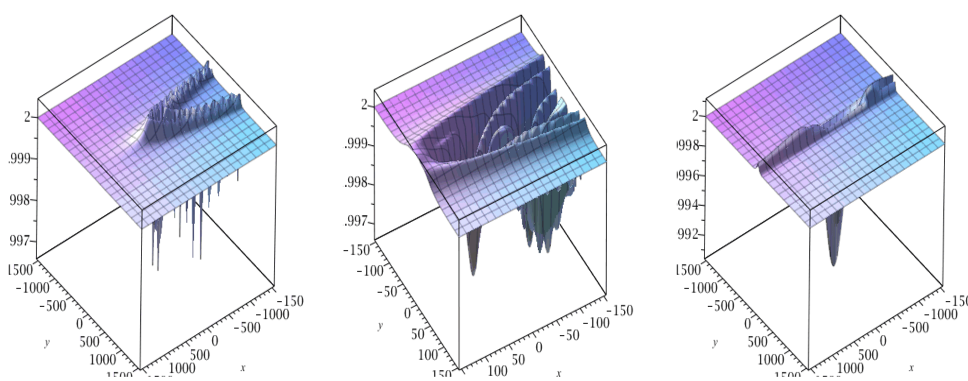

**Figure 4.** Solution of order 3 to (1), on the left for $t = 0,1$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, $b_2 = 0$, ; in the center for $t = 1$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, $b_2 = 0$; on the right for for $t = 10$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$.

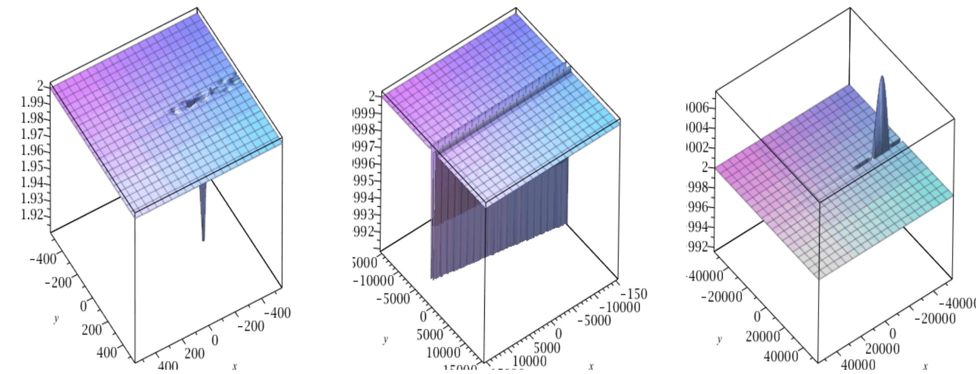

**Figure 5.** Solution of order 3 to (1), on the left for $t = 10$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, $b_2 = 0$, ; in the center for $t = 100$, $a_1 = 10^6$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$; on the right for for $t = 10^3$, $a_1 = 10^5$, $b_1 = 10^3$, $a_2 = 0$, $b_2 = 0$.

The study of these configurations makes it possible to give the following conclusions. The variation of the configuration of the module of the solutions is very fast according to time $t$. When time $t$ grows from 0 to $0,01$, one passes from a rectilinear structure with a height of 98 to a horseshoe structure with a maximum height equal to 4. The role played by the parameters $a_i$ and $b_i$ is the same one for same index $i$. When variables $x$, $y$ and time tend towards infinite, the modulus of the solutions tends towards 2 in accordance with the structure of the polynomials which will be studied in a forthcoming article.

# 5 Conclusion

We have constructed solutions to the Johnson equation, starting from the solutions of the KPI equation what make it possible to obtain rational solutions. These solutions are expressed by means of quotients of two polynomials of degree $2N(N+1)$ in $x$, $t$ and $4N(N+1)$ in $y$ depending on $2N-2$ parameters. Here we have given a new method to construct solutions to the Johnson equation related to previous results [23, 25, 51] .

We have given two types of representations of the solutions to the Johnson equation. An expression by means of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters is gen. Another expression by means of wronskians of order $2N$ depending on $2N-1$ real parameters is also constructed. Also rational solutions to the Johnson equation depending on $2N-2$ real parameters are obtained when one of parameters $(\epsilon)$ tends to zero.

The patterns of the modulus of the solutions in the plane $(x, y)$ and their evolution according to time and parameters have been studied in the figures 1, 2,

3, 4, 5.
In another study, we will give a more general representation of rational solutions to the Johnson equation. It can be written without limit at order $N$ depending on $2N-2$ real parameters. We will prove that these solutions can be written as a quotient of polynomials of degree $2N(N+1)$ in $x$, $t$ and $4N(N+1)$ in $y$.

## References


[1] R.E. Johnson, Water waves and Kortewegde Vries equations, J. Fluid Mech., V. 97, N. 4, 701719, 1980

[2] R.E. Johnson, A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge University Press, Cambridge, 1997

[3] M. J. Ablowitz, Nonlinear Dispersive Waves : Asymptotic Analysis and Solitons, Cambridge University Press, Cambridge, 2011

[4] V.D. Lipovskii, On the nonlinear internal wave theory in fluid of finite depth, Izv. Akad. Nauka., V. 21, N. 8, 864871, 1985

[5] V.I. Golinko, V.S. Dryuma, Yu.A. Stepanyants, Nonlinear quasicylindrical waves: Exact solutions of the cylindrical Kadomtsev- Petviashvili equation, in Proc. 2nd Int. Workshop on Nonlinear and Turbulent Processes in Physics, Kiev, Harwood Acad., Gordon and Breach, 13531360, 1984

[6] V.D. Lipovskii, V.B. Matveev, A.O. Smirnov, connection between the Kadomtsev-Petvishvili and Johnson equation, Zap. Nau. Sem., V. 150, 7075, 1986

[7] C. Klein, V.B. Matveev, A.O. Smirnov, Cylindrical Kadomtsev-Petviashvili equation: Old and new results, Theor. Math. Phys., V. 152, N. 2, 1132-1145, 2007

[8] K. R. Khusnutdinova, C. Klein, V.B. Matveev, A.O. Smirnov, On the integrable elliptic cylindrical K-P equation Chaos, V. 23, 013126-1-15, 2013

[9] B.B. Kadomtsev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, Sov. Phys. Dokl., V. 15, N. 6, 539-541, 1970

[10] M.J. Ablowitz, H. Segur On the evolution of packets of water waves, J. Fluid Mech., V. 92, 691-715, 1979

[11] D.E. Pelinovsky, Y.A. Stepanyants, Y.A. Kivshar, Self-focusing of plane dark solitons in nonlinear defocusing media, Phys. Rev. E, V. 51, 5016-5026, 1995

[12] P. Gaillard, Families of Rational Solutions of Order 5 to the KPI Equation Depending on 8 Parameters, New Hor. in Math. Phys., V. 1, N. 1, 26-31, 2017

[13] W. Wu, Some new exact solutions for the twodimensional Navier-Stokes equations, Phys. Lett. A, V. 371, 438-452, 2007

[14] M. Wang, J. Zhang, X. Li, Decay mode solutions to cylindrical KP equation, Appl. Math. Lett., V. 371, 1-7, 2016

[15] B. Konopelchenko, Introduction to Multidimensional Integrable Equations, Springer, 1992

[16] I. Anders, A. Boutet de Monvel, Asymptotic Solitons of the Johnson Equation, Jour. of Nonlin. Math. Phys., V. 7, N.3, 284-302, 2000

[17] L. Zhou, Y. Sheng, Recyclable amplification protocol for the single-photon entangled state, Las. Phys. Lett., V. 12, N.4, 2004

[18] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, Max-Planck-Institut für Mathematik, MPI 02-31, V. **161**, 2002

[19] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, Lett. Math. Phys., V. **68**, 77-90, 2004

[20] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, Lett. Math. Phys., V. **89**, 1-12, 2009

[21] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, J. Phys A : Math. Theor., V. **42**, 404409-1-16, 2009

[22] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, Eur. Phys. J. Spe. Top., V. **185**, 247-258, 2010

[23] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, J. Phys. A : Meth. Theor., V. **44**, 435204-1-15, 2011

[24] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, J. Math. Sciences : Adv. Appl., V. **13**, N. 2, 71-153, 2012

[25] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, J. Math. Phys., V. **54**, 013504-1-32, 2013

[26] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, J. Mod. Phys., V. **4**, N. 4, 246-266, 2013

[27] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, J. Math., V. **2013**, ID 645752, 1-10, 2013

[28] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, J. Math., V. **2013**, 1-111, 2013

[29] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, J. Theor. Appl. Phys., V. **7**, N. **45**, 1-6, 2013

[30] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, J. Math. Phys., V. **54**, 073519-1-22, 2013

[31] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, Phys. Rev. E, V. **88**, 042903-1-9, 2013

[32] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, Phys. Scripta, V. **89**, 015004-1-7, 2014

[33] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, Commun. Theor. Phys., V. **61**, 365-369, 2014

[34] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, J. Of Phys. : Conf. Ser., V. **482**, 012016-1-7, 2014

[35] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, Int. J. Mod. Phys. C, V. **26**, N. 2, 1550016-1-14, 2014

[36] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, J. Math. Phys., V. **5**, 093506-1-12, 2014

[37] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, J. Phys. : Conf. Ser., V. **574**, 012031-1-5, 2015

[38] P. Gaillard, Tenth Peregrine breather solution of the NLS, Ann. Phys., V. **355**, 293-298, 2015

[39] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation Phys. Lett. A, V. **379**, 1309-1313, 2015

[40] P. Gaillard, Other 2N-2 parameters solutions to the NLS equation and 2N+1 highest amplitude of the modulus of the N-th order AP breather, J. Phys. A: Math. Theor., V. **48**, 145203-1-23, 2015

[41] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, Adv. Res., V. **4**, N. **5**, 346-364, 2015

[42] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, Jour. Phys. : Conf. Ser., V. **633**, 012106-1-6, 2016

[43] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, Journal of Theoretical and Applied Physics, V. **10**,1-7, 2016

[44] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, Commun. Theor. Phys, V. **65**, N. **2**, 136-144, 2016

[45] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, Annals Of Physics, V. **367**, 1-5, 2016

[46] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, Adv. Res., V. **10**, 83-89, 2016

[47] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, Theor. And Math. Phys., V. **189**, N. **1**, 1440-1449, 2016

[48] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, Jour. of Math. Phys., V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016

[49] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, Jour. Of Bas. And Appl. Res. Int., V. **21**, N. 3, 130-139, 2017

[50] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N2$ parameters, Int. Jour. of Appl. Sci. And Math., V. **4**, N. 3, 60-70, 2017

[51] P. Gaillard, Families of Rational Solutions of Order 5 to the KPI Equation depending on 8 Parameters, New Hor. in Math. Phys., V. 1, N. 1, 26-31, 2017

[52] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, Jour. Of Bas. And Appl. Res. Int., V. **21**, N. 2, 92-98, 2017

[53] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, Int. Jour. of Adv. Res. in Phys. Sci., V. **4**, N. 11, 24-30, 2017

**Appendix** The solutions to the Johnson equation can be written as

$$v_3(x,y,t) = -2\frac{|n_3(x,y,t)|^2}{(d_3(x,y,t))^2},$$

$$n_3(x,y,t) = A_3(x,y,t) + iB_3(x,y,t), \quad d_3(x,y,t) = C_3(x,y,t)$$

with

$$\begin{aligned} A_3(x,y,t) &= \textstyle\sum_{k=0}^{12} a_k(y,t)x^k, \\ B_3(x,y,t) &= \textstyle\sum_{k=0}^{12} b_k(y,t)x^k, \\ C_3(x,y,t) &= \textstyle\sum_{k=0}^{12} c_k(y,t)x^k, \end{aligned}$$

$\mathbf{a}_{12} = 8916100448256, \quad \mathbf{a}_{11} = 8916100448256\,ty^2+2567836929097728\,t, \quad \mathbf{a}_{10} = 4086546038784\,t^2y^4+$
$2139864107581440\,t^2y^2 + 338954474640900096\,t^2 + 40122452017152, \quad \mathbf{a}_9 = 1135151677440\,t^3y^6 +$
$802449040343040\,t^3y^4+231105323618795520\,t^3y^2+27116357971272007680\,t^3+33435376680960\,ty^2+$
$13909116699279360\,t, \quad \mathbf{a}_8 = 212840939520\,t^4y^8+178322008965120\,t^4y^6+69545583496396800\,t^4y^4+$
$14790740711602913280\,t^4y^2+12538266255360\,t^2y^4+1464283330448688414720\,t^4+6419592322744320\,t^2y^2+$
$1964395250759761920\,t^2 - 125382662553600, \quad \mathbf{a}_7 = 28378791936\,t^5y^{10} + 26005292974080\,t^5y^8 +$
$12125896609628160\,t^5y^6+3492258223572910080\,t^5y^4+2786281390080\,t^3y^6+621211109887322357760\,t^5y^2+$
$802449040343040\,t^3y^4+56228479889229635125248\,t^5+539245755110522880\,t^3y^2+155302777471830589440\,t^3-$
$83588441702400\,ty^2-24073471210291200\,t, \quad \mathbf{a}_6 = 2759049216\,t^6y^{12}+2600529297408\,t^6y^{10}+1352275234652160\,t^6y^8+$
$467916951524474880\,t^6y^6+406332702720\,t^4y^8+112163117062988759040\,t^6y^4-104021171896320\,t^4y^6+$
$17890879964754883903488\,t^6y^2-18188844914442240\,t^4y^4+1574397436898429783506944\,t^6+25883796245305098240\,t^4y^2-$
$24379962163200\,t^2y^4+7765138873591529472000\,t^4-18456327927889920\,t^2y^2-1714031150172733440\,t^2+$
$376147987660800, \quad \mathbf{a}_5 = 197074944\,t^7y^{14}+180592312320\,t^7y^{12}+100553799499776\,t^7y^{10}+39230841972326400\,t^7y^8+$
$40633270272\,t^5y^{10}+11298482488030003200\,t^7y^6-45509262704640\,t^5y^8+2402016291564313116672\,t^7y^4-$
$24073471210291200\,t^5y^6+357817599295097678069760\,t^7y^2-3030047576335319040\,t^5y^4-4063327027200\,t^3y^6+$
$32387604416196269832142848\,t^7+776513887359152947200\,t^5y^2-5717449412444160\,t^3y^4+259417759488945816600576\,t^5-$
$722204136308736000\,t^3y^2-52692013785085378560\,t^3+188073993830400\,ty^2+78238781433446400\,t, \quad \mathbf{a}_4 =$
$10264320\,t^8y^{16}+8599633920\,t^8y^{14}+5056584744960\,t^8y^{12}+2139864107581440\,t^8y^{10}+2821754880\,t^6y^{12}+$
$697595699071549440\,t^8y^8-6501323243520\,t^6y^{10}+17748888539234959360\,t^8y^6-3845068318310400\,t^6y^8+$
$34787822153690052034560\,t^8y^4-1044253684499742720\,t^6y^6-423263232000\,t^4y^8+4907212790332768156385280\,t^8y^2-$
$109389853179563212800\,t^6y^4-947335672627200\,t^4y^6+485814066242944047482142720\,t^8+14909066637295736586240\,t^6y^2-$
$102312252643737600\,t^4y^4+5903990388369111688151040\,t^6+13866319417127731200\,t^4y^2+39182082048000\,t^2y^4-$
$249593749508299161600\,t^4+34104084214579200\,t^2y^2+9605315012906188800\,t^2+493694233804800, \quad \mathbf{a}_3 =$
$380160\,t^9y^{18} + 268738560\,t^9y^{16} + 171992678400\,t^9y^{14} + 75951966781440\,t^9y^{12} + 134369280\,t^7y^{14} +$
$27104945362698240\,t^9y^{10}-511678218240\,t^7y^{12}+7806224264457093120\,t^9y^8-288534917283840\,t^7y^{10}+$
$1814330860623290695680\,t^9y^6-100216969038397440\,t^7y^8-28217548800\,t^5y^{10}+340778665995331121971200\,t^9y^4-$
$20508457607060520960\,t^7y^6-91715095756800\,t^5y^8+44164915112994913407467520\,t^9y^2-1804470366815555420160\,t^7y^4-$
$3789342690508800\,t^5y^6 + 5182016706591403173142855680\,t^9 + 178908799647548839034880\,t^7y^2 +$
$8152882249885286400\,t^5y^4+4353564672000\,t^3y^6+90783436621156210893127680\,t^7+1358899302878517657600\,t^5y^2+$
$5934779360870400\,t^3y^4+29285666608973768294400\,t^5+1420334801407180800\,t^3y^2+526149786772124467200\,t^3+$
$164564744601600\,ty^2+119615060076134400\,t, \quad \mathbf{a}_2 = 9504\,t^{10}y^{20}+4976640\,t^{10}y^{18}+3941498880\,t^{10}y^{16}+$
$1651129712640\,t^{10}y^{14}+4199040\,t^8y^{16}+653847366205440\,t^{10}y^{12}-23648993280\,t^8y^{14}+205426954327818240\,t^{10}y^{10}-$
$10990332149760\,t^8y^{12}+54232715942544015360\,t^{10}y^8-4695812902748160\,t^8y^{10}-1175731200\,t^6y^{12}+$
$11359288866511037399040\,t^{10}y^6-1230421861859328000\,t^8y^8-5224277606400\,t^6y^{10}+2249139195569185405009920\,t^{10}y^4-$
$212000616866308423680\,t^8y^6+376147987660800\,t^6y^8+235546213935972871506493440\,t^{10}y^2-13134177751903386992640\,t^8y^4+$
$1033554363961835520\,t^6y^6+272097792000\,t^4y^8+37310520287458102846628560896\,t^{10}+1226803197583192039096320\,t^8y^2+$
$246897520732746547200\,t^6y^4+515462057164800\,t^4y^6+905380759816395724853084160\,t^8+27954499944929506099200\,t^6y^2+$
$141431643360460800\,t^4y^4+862595998300681902489600\,t^6+44295187026935808000\,t^4y^2+20570593075200\,t^2y^4+$
$13630591987036559769600\,t^4-10155995666841600\,t^2y^2+8350485326069760000\,t^2-246847116902400, \quad \mathbf{a}_1 =$
$144\,t^{11}y^{22}+41472\,t^{11}y^{20}+59719680\,t^{11}y^{18}+17199267840\,t^{11}y^{16}+77760\,t^9y^{18}+9906778275840\,t^{11}y^{14}-$
$604661760\,t^9y^{16} + 2853152143441920\,t^{11}y^{12} - 180592312320\,t^9y^{14} + 821707817311272960\,t^{11}y^{10} -$
$121358033879040\,t^9y^{12}-27993600\,t^7y^{14}+236651851385646612480\,t^{11}y^8-31028029559930880\,t^9y^{10}-$
$162855567360\,t^7y^{12}+34077866599533112197120\,t^{11}y^6-6881802969981911040\,t^9y^8+37614798766080\,t^7y^{10}+$
$9814425580665536312770560\,t^{11}y^4-1242422219774644715520\,t^9y^6+47210751873515520\,t^7y^8+9069926400\,t^5y^{10}+$
$565310913446334891615584256\,t^{11}y^2-17038933299766556098560\,t^9y^4+28592864205503201280\,t^7y^6+$
$22348298649600\,t^5y^8 + 162809543072544448785288265728\,t^{11} + 3680409592749576117288960\,t^9y^2 +$
$425233795458583756800\,t^7y^4+16216157690265600\,t^5y^6+5299789813559389608896102400\,t^9+188493199628667526840320\,t^7y^2+$
$3450530873475072000\,t^5y^4+1142810726400\,t^3y^6+10121126380061334322544640\,t^7+654028065841191321600\,t^5y^2-$
$4184646362726400\,t^3y^4+164399083009466381107200\,t^5-675561785838796800\,t^3y^2+332358343529280307200\,t^3-$
$41141186150400\,ty^2-59243308056576000\,t, \quad \mathbf{a}_0 = t^{12}y^{24}+497664\,t^{12}y^{20}+648\,t^{10}y^{20}+103195607040\,t^{12}y^{16}-$
$6635520\,t^{10}y^{18}-447897600\,t^{10}y^{16}+11412608573767680\,t^{12}y^{12}-1651129712640\,t^{10}y^{14}-291600\,t^8y^{16}-$
$113927950172160\,t^{10}y^{12}-2149908480\,t^8y^{14}+709955554156939837440\,t^{12}y^8-136951302885212160\,t^{10}y^{10}+$

$935210188800\,t^8y^{12}-6162808629834547200\,t^{10}y^8+594406696550400\,t^8y^{10}+125971200\,t^6y^{12}+2355462139359728715064934\,4\,t^{12}y^4-$
$3786429622170345799680\,t^{10}y^6+304663152316907520\,t^8y^8+386983526400\,t^6y^{10}+1533503996978990048870\,40\,t^{10}y^4+$
$274450410981965168640\,t^8y^6+727916013158400\,t^6y^8+325619086145088897570576531456\,t^{12}-262239832816719652454\,40\,t^8y^4-$
$47790298402652160\,t^6y^6+23808556800\,t^4y^8+1385011737943520484458181427\,2\,t^{10}+175158576526078771200\,t^6y^4-$
$208971104256000\,t^4y^6+45085017511182307436789760\,t^8+68145477097881600\,t^4y^4+73214166522434420736000\,0\,t^6-$
$1714216089600\,t^2y^4+5839742646192439296000\,t^4-629671731344179200\,t^2-30855889612800$

$\mathbf{b}_{12}=0,\quad \mathbf{b}_{11}=0,\quad \mathbf{b}_{10}=213986410758144\,ty,\quad \mathbf{b}_9=178322008965120\,t^2y^3+51356738581954560\,t^2y,\quad \mathbf{b}_8=$
$66870753361920\,t^3y^5+34237825721303040\,t^3y^3+5546527766851092480\,t^3y+802449040343040\,ty,\quad \mathbf{b}_7=$
$14860167413760\,t^4y^7+9986032502046720\,t^4y^5+2875977360589455360\,t^4y^3+354977777078469918720\,t^4y+$
$534966026895360\,t^2y^3+154070215745863680\,t^2y,\quad \mathbf{b}_6=2167107747840\,t^5y^9+1664338750341120\,t^5y^7+$
$633399775844106240\,t^5y^5+138046913308293857280\,t^5y^3+156031757844480\,t^3y^5+14909066637295736586240\,t^5y+$
$59916195012280320\,t^3y^3+12941898122652549120\,t^3y-2006122600857600\,ty,\quad \mathbf{b}_5=216710774784\,t^6y^{11}+$
$173368619827200\,t^6y^9+77035107872931840\,t^6y^7+22186111067404369920\,t^6y^5+26005292974080\,t^4y^7+$
$4141407399248815718400\,t^6y^3+7489524376535040\,t^4y^5+429381119154117213683712\,t^6y+2978690837753364480\,t^4y^3+$
$621211109887322357760\,t^4y-1003061300428800\,t^2y^3-442951870269358080\,t^2y,\quad \mathbf{b}_4=15049359360\,t^7y^{13}+$
$11557907988480\,t^7y^{11}+5646863617228800\,t^7y^9+1917318240392970240\,t^7y^7+2708884684800\,t^5y^9+$
$468373455867425587200\,t^7y^5+79515022065577261793280\,t^7y^3+727553796577 68960\,t^5y^5+8587622383082344273674240\,t^7y+$
$986049380773527552 00\,t^5y^3-208971104256000\,t^3y^5+186363332966196707328 00\,t^5y-160489808068608000\,t^3y^3-$
$17332899271409664000\,t^3y+4513775851929600\,ty\quad \mathbf{b}_3=716636160\,t^8y^{15}+481579499520\,t^8y^{13}+$
$257576235171840\,t^8y^{11}+97007172877025280\,t^8y^9+180592312320\,t^6y^{11}+27938065788583280640\,t^8y^7-$
$86684309913600\,t^6y^9+6152948136026811924480\,t^8y^5-32811249649582080\,t^6y^7+954180264786927141519360\,t^8y^3+$
$410853908655636480\,t^6y^5-23219011584000\,t^4y^7+117773106967986435753246720\,t^8y+266233332808852439040 0\,t^6y^3-$
$2273605614305280 0\,t^4y^5+35781759929509767806976 0\,t^6y-7575118940838297600\,t^4y^3+33279166601106554880 0\,t^4y+$
$1504591950643200\,t^2y^3+818498021149900800\,t^2y\quad \mathbf{b}_2=22394880\,t^9y^{17}+11466178560\,t^9y^{15}+$
$7430083706880\,t^9y^{13}+2853152143441920\,t^9y^{11}+7524679680\,t^7y^{13}+924421294475182080\,t^9y^9-$
$8668430991360\,t^7y^{11}+236651851385646612480\,t^9y^7-3358397835509760\,t^7y^9+51116799899299668295680\,t^9y^5-$
$616280862983454720\,t^7y^7-1451188224000\,t^5y^9+654295038711035754184704 0\,t^9y^3+36976851779007283200\,t^7y^5-$
$1560317578444800\,t^5y^7+105995796271187792177922048 0\,t^9y+5395662211592742764544 0\,t^7y^3-$
$805658836504412160\,t^5y^5+429381119154117213683712 0\,t^7y-801165121878491136 00\,t^5y^3+188073993830400\,t^3y^5+$
$32613583269084423782400\,t^5y+120367356051456000\,t^3y^3+340880352337723392 00\,t^3y+3949553870438400\,ty,\quad \mathbf{b}_1=$
$414720\,t^{10}y^{19}+119439360\,t^{10}y^{17}+137594142720\,t^{10}y^{15}+39627113103360\,t^{10}y^{13}+179159040\,t^8y^{15}+$
$17118912860651520\,t^{10}y^{11}-361184624640\,t^8y^{13}+4930246903867637760\,t^{10}y^9-113927950172160\,t^8y^{11}+$
$946607405542586449920\,t^{10}y^7-32811249649582080\,t^8y^9-48372940800\,t^6y^{11}+272622932796264897576960\,t^{10}y^5-$
$2875977360589455360\,t^8y^7-51081825484800\,t^6y^9+19628851161331072625541120\,t^{10}y^3+1183259256928233062 40\,t^8y^5-$
$16583946833756160\,t^6y^7+565310913446334891615584256 0\,t^{10}y+647479465391129131745280\,t^8y^3-$
$19977771308380323840\,t^6y^5+10448555212800\,t^4y^7+294432767419966089383116 80\,t^8y+20337268478454005760 00\,t^6y^3+$
$3009183901286400\,t^4y^5+670907998678308146380800\,t^6y-674057193888153600\,t^4y^3+1063084488646459392000\,t^4y+$
$658258978406400\,t^2y^3-243743896004198400\,t^2y,\quad \mathbf{b}_0=3456\,t^{11}y^{21}+1433272320\,t^{11}y^{17}+1866240\,t^9y^{17}+$
$237762678620160\,t^{11}y^{13}-5733089280\,t^9y^{15}-1031956070400\,t^9y^{13}+19720987615470551040\,t^{11}y^9-$
$475525357240320\,t^9y^{11}-671846400\,t^7y^{13}-196867497897492480\,t^9y^9-619173642240\,t^7y^{11}+8178687983887946927308 80\,t^{11}y^5+$
$39441975230941102080\,t^9y^7+664992491765760\,t^7y^9-7099555541569398374400\,t^9y^5-1711891286065152 00\,t^7y^7+$
$217678233600\,t^5y^9+1356746192271203739877402214 4\,t^{11}y+32714751935551787709235 20\,t^9y^3-$
$12233175130221576192 0\,t^7y^5-111451255603200\,t^5y^7+883298302259898268149350 40\,t^9y+35497777707846991872000\,t^7y^3+$
$222278384175022080\,t^5y^5+45238367910880206441676 80\,t^7y-77035107872931840000\,t^5y^3+27427457433600\,t^3y^5+$
$1569667358018859171840 0\,t^5y-12036735605145600\,t^3y^3-1621348286013112320 0\,t^3y-987388467609600\,ty$

$\mathbf{c}_{12}=8916100448256,\quad \mathbf{c}_{11}=8916100448256\,ty^2+2567836929097728\,t,\quad \mathbf{c}_{10}=4086546038784\,t^2y^4+$
$2139864107581440\,t^2y^2+338954474640900096\,t^2-13374150672384,\quad \mathbf{c}_9=1135151677440\,t^3y^6+$
$802449040343040\,t^3y^4+231105323618795520\,t^3y^2+27116357971272007680\,t^3-11145125560320\,ty^2+$
$1069932053790720\,t,\quad \mathbf{c}_8=212840939520\,t^4y^8+178322008965120\,t^4y^6+69545583496396800\,t^4y^4+$
$14790740711602913280\,t^4y^2-4179422085120\,t^2y^4+1464283330448688414720\,t^4+577763309046988800\,t^2+$
$75229597532160,\quad \mathbf{c}_7=28378791936\,t^5y^{10}+26005292974080\,t^5y^8+12125896609628160\,t^5y^6+$
$3492258223572910080\,t^5y^4-928760463360\,t^3y^6+621211109887322357760\,t^5y^2-267483013447680\,t^3y^4+$
$56228479889229635125248\,t^5+231105323618795520\,t^3y^2+66558333202213109760\,t^3+50153065021440\,ty^2+$
$14444082726174720\,t,\quad \mathbf{c}_6=2759049216\,t^6y^{12}+2600529297408\,t^6y^{10}+1352275234652160\,t^6y^8+$
$467916951524474880\,t^6y^6-135444234240\,t^4y^8+112163117062988759040\,t^6y^4-104021171896320\,t^4y^6+$
$17890879964754883903488\,t^6y^2+28888165452349440\,t^4y^4+1574397436898429783506944\,t^6+25883796245305098240\,t^4y^2+$
$14627977297920\,t^2y^4+4037872214267595325440\,t^4+7222041363087360\,t^2y^2+1521443380490403840\,t^2-$
$325994922639360\quad \mathbf{c}_5=197074944\,t^7y^{14}+180592312320\,t^7y^{12}+100553799499776\,t^7y^{10}+39230841972326400\,t^7y^8-$
$13544423424\,t^5y^{10}+11298482488030003200\,t^7y^6-19503969730560\,t^5y^8+2402016291564313116672\,t^7y^4-$
$534966026895360\,t^5y^6+357817599295097678069760\,t^7y^2+3749041916482682880\,t^5y^4+2437996216320\,t^3y^6+$
$32387604416196269832142848\,t^7+139772499724647530496 0\,t^5y^2+1504591950643200\,t^3y^4+15207247970041651317964 8\,t^5+$
$741462913276968960\,t^3y^2+102610763686745210880\,t^3-162997461319680\,ty^2-61387351586242560\,t,\quad \mathbf{c}_4=$
$10264320\,t^8y^{16}+8599633920\,t^8y^{14}+5056584744960\,t^8y^{12}+2139864107581440\,t^8y^{10}-940584960\,t^6y^{12}+$
$697595699071549440\,t^8y^8-2167107747840\,t^6y^{10}+177488888539234959360\,t^8y^6-501530650214400\,t^6y^8+$
$34787822153690052034560\,t^8y^4+22254586718846976 0\,t^6y^6+253957939200\,t^4y^8+490721279033276815638528 0\,t^8y^2+$
$16793653516299141120 0\,t^6y^4+167176883404800\,t^4y^6+4858140662429440474821427 20\,t^8+447271999118872097587 20\,t^6y^2+$

$138422459459174400\, t^4 y^4 + 37570847925985256197324480\, t^6 + 32354745306631372800\, t^4 y^2 - 33957804441600\, t^2 y^4 + 4409489574646618521600\, t^4 - 18055103407718400\, t^2 y^2 - 5705412676839014400\, t^2 + 117546246144000, \quad \mathbf{c}_3 = 380160\, t^9 y^{18} + 268738560\, t^9 y^{16} + 171992678400\, t^9 y^{14} + 75951966781440\, t^9 y^{12} - 44789760\, t^7 y^{14} + 27104945362698240\, t^9 y^{10} - 150493593600\, t^7 y^{12} + 7806224264457093120\, t^9 y^8 - 55725627801600\, t^7 y^{10} + 1814330860623290695680\, t^9 y^6 + 1069932053790720\, t^7 y^8 + 16930529280\, t^5 y^{10} + 340778665995331121971200\, t^9 y^4 + 8662169907489669120\, t^7 y^6 + 10448555212800\, t^5 y^8 + 44164915112994913407467520\, t^9 y^2 + 3756848140747139973120\, t^7 y^4 + 12259638116352000\, t^5 y^6 + 5182016706591403173142855680\, t^9 + 894543998237744195174400\, t^7 y^2 + 8666449635704832000\, t^5 y^4 - 3773089382400\, t^3 y^6 + 61340159879159601954816000\, t^7 + 2495937495082991616 00\, t^5 y^2 - 1755357275750400\, t^3 y^4 + 11874006643274818781184 0\, t^5 - 24073471210291200\, t^3 y^2 - 33356201708979486720 0\, t^3 + 39182082048000\, t y^2 - 12789031580467200\, t, \quad \mathbf{c}_2 = 9504\, t^{10} y^{20} + 4976640\, t^{10} y^{18} + 3941498880\, t^{10} y^{16} + 1651129712640\, t^{10} y^{14} - 1399680\, t^8 y^{16} + 653847366205440\, t^{10} y^{12} - 6449725440\, t^8 y^{14} + 205426954327818240\, t^{10} y^{10} - 2941074800640\, t^8 y^{12} + 54232715942544015360\, t^{10} y^8 - 416084687585280\, t^8 y^{10} + 705438720\, t^6 y^{12} + 11359288866511037399040\, t^{10} y^6 + 181888449144422400\, t^8 y^8 + 348285173760\, t^6 y^{10} + 2249139195569185405009920\, t^{10} y^4 + 14297716021216149504 0\, t^8 y^6 + 487599243264000\, t^6 y^8 + 235546213935972871506493440\, t^{10} y^2 + 42242355472337920327680\, t^8 y^4 + 930840886797926400\, t^6 y^6 - 235818086400\, t^4 y^8 + 37310520287458102846628560896\, t^{10} + 1104122877824872835186688 0\, t^8 y^2 + 2561417336774983680 00\, t^6 y^4 - 41794220851200\, t^4 y^6 + 64039126913842624440827904 0\, t^8 - 18902566629428523171840\, t^6 y^2 + 57174494124441600\, t^4 y^4 + 19360487961859749366988 80\, t^6 + 35050974082183987200\, t^4 y^2 + 4897760256000\, t^2 y^4 - 1116238713078782361600 0\, t^4 + 1880739938304000\, t^2 y^2 - 2193695064037785600\, t^2 - 105791621529600, \quad \mathbf{c}_1 = 144\, t^{11} y^{22} + 41472\, t^{11} y^{20} + 59719680\, t^{11} y^{18} + 17199267840\, t^{11} y^{16} - 25920\, t^9 y^{18} + 9906778275840\, t^{11} y^{14} - 156764160\, t^9 y^{16} + 2853152143441920\, t^{11} y^{12} - 77396705280\, t^9 y^{14} + 821707817311272960\, t^{11} y^{10} - 12383472844800\, t^9 y^{12} + 16796160\, t^7 y^{14} + 236651851385646612480\, t^{11} y^8 - 1069932053790720\, t^9 y^{10} + 4837294080\, t^7 y^{12} + 34077866599533112197120\, t^{11} y^6 + 174612911178645504 0\, t^9 y^8 + 4179422085120\, t^7 y^{10} + 981442558066553631277056 0\, t^{11} y^4 + 1360748145467468021760\, t^9 y^6 + 247421787439104 00\, t^7 y^8 - 7860602880\, t^5 y^{10} + 565310913446334891615584256\, t^{11} y^2 + 187428266297432117084160\, t^9 y^4 + 2376533077879947264 0\, t^7 y^6 + 2612138803200\, t^5 y^8 + 16280954307254444878528826572 8\, t^{11} + 7728860144774109846306816 0\, t^9 y^2 + 317631156781672562688 0\, t^7 y^4 - 3176360784691200\, t^5 y^6 + 3886512529943552379857141760\, t^9 - 50158359901187799515136 0\, t^7 y^2 + 5568996339980697600\, t^5 y^4 + 272097792000\, t^3 y^6 + 174819455655604865571225 60\, t^7 + 404434316332892160000\, t^5 y^2 + 579894814310400\, t^3 y^4 - 1886263162950719530598 40\, t^5 - 844076084310835200\, t^3 y^2 - 113674931054995046400\, t^3 - 17631936921600\, t y^2 - 16362437463244800\, t, \quad \mathbf{c}_0 = t^{12} y^{24} + 497664\, t^{12} y^{20} - 216\, t^{10} y^{20} + 103195607040\, t^{12} y^{16} - 1658880\, t^{10} y^{18} - 806215680\, t^{10} y^{16} + 11412608573767680\, t^{12} y^{12} + 174960\, t^8 y^{16} - 173368619827200\, t^{10} y^{12} + 709955554156939837440\, t^{12} y^8 + 68475651442606080\, t^{10} y^{10} - 148343685120\, t^8 y^{12} - 11093055533702184960\, t^{10} y^8 - 534966026895360\, t^8 y^{10} - 109175040\, t^6 y^{12} + 23554621393597287150649344\, t^{12} y^4 + 757285924434069159936 0\, t^{10} y^6 + 666300186498170880\, t^8 y^8 + 116095057920\, t^6 y^{10} - 51116799899299668295680\, t^{10} y^4 + 12161275696206839808 0\, t^8 y^6 - 400527949824000\, t^6 y^8 + 325619086145088897570576531456\, t^{12} + 235546213935972871506493440\, t^{10} y^2 + 1175863886572431605760 0\, t^8 y^4 + 164056248247910400\, t^6 y^6 + 5668704000\, t^4 y^8 + 10458251898757195494888308736\, t^{10} - 3680409592749576117288960\, t^8 y^2 + 1993283416212111360 0\, t^6 y^4 + 27862813900800\, t^4 y^6 + 671674750676797641405235 20\, t^8 - 61233666546036060979 20\, t^6 y^2 + 25013841179443200\, t^4 y^4 - 1242244730886105480560640\, t^6 - 34280623003454668800\, t^4 y^2 - 734664038400\, t^2 y^4 - 497165327434933862400\, t^4 + 1880739938304000\, t^2 y^2 - 1126187075056435200\, t^2 + 4407984230400$