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# Restricted nonlinear model for high and low drag events in a plane channel flow

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A restricted nonlinear model, obtained by partitioning the state variables into streamwise averaged quantities and superimposed perturbations, is used in order to track the exact coherent state in plane channel flow investigated by Toh & Itano (J. Fluid Mech., vol. 481, 2003, p. 67-76). When restricting nonlinearities to quadratic interaction of the fluctuating part into the streamwise averaged component, it is shown that the coherent structure and its dynamics closely match results from direct numerical simulation, even if only a single streamwise Fourier mode is retained. In particular, both solutions exhibit long quiescent phases, spanwise shifts and bursting events. It is also shown that dynamical trajectory passes close to equilibria that exhibit either low or high drag states. When statistics are collected at time where friction velocity peaks, mean flow and root mean square profiles show essential features of wall-turbulence obtained by DNS for the same friction Reynolds number. For low drag events, mean flow profiles are related to a universal asymptotic state called maximum drag reduction (Xi & Graham Phys. Rev. Letters, vol. 108, 2012, 028301). Hence, the intermittent nature of self-sustaining processes in the buffer layer is contained in the dynamics of RNL model, organized in two exact coherent states plus an asymptotic turbulent-like attractor. We also address how closely turbulent dynamics approaches these equilibria by exploiting DNS database associated with a larger domain.

**Key words:** Wall-turbulence, exact coherent states, Restricted Non Linear system

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## 1. Introduction

The flow visualization of turbulent boundary layers realized by Kline *et al.* (1967) revealed intermittent structures in the layers closest to the wall. The near-wall region is composed of alternating spanwise streaks of lower-and-higher-speed fluid, with fairly regular spanwise spacing of about 100 wall units and amplitudes amounting to  $\pm 50\%$  of the mean velocity. Streaks are closely related to the existence of streamwise counter rotating vortices which alternatively transport high speed fluid towards the wall and low speed fluid away from it. Wallace *et al.* (1972) introduced the quadrant classification of the plane of streamwise ( $u$ ) and wall-normal ( $v$ ) velocity fluctuations. The instantaneous

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product  $uv$  was classified according to the sign of its components into four categories. They associated an upstream upwards motion ( $u < 0, v > 0$ , Q2 event) with the ejection process and attributed the opposite combination ( $u > 0, v < 0$ , Q4 event) to a sweep type motion. The two other types of motion, with positive product  $uv$ , were found to account for the 'excess' stress produced by the first two categories, which gives contributions of opposite sign. These sweep and ejection events result in transfer of wall-normal momentum and production of turbulent kinetic energy. Moreover, the low-speed streaks do not stay at the wall but migrate outward as they move downstream and then turn suddenly burst into intense, small-scale fluctuations. Kim *et al.* (1971) shown that a very high fraction of the total turbulence production does occur during these relatively short bursting times.

These pioneer work aiming to describe turbulence through coherent structures have fundamentally changed our understanding of wall-turbulence. While statistical descriptions and models for wall-turbulence fail to describe repeating patterns of organized motion, structural approaches (*i.e.* devoted to the study of their geometries, characteristics length, temporal scales and convection velocities) offered a promising route for exploring their dynamics (Robinson 1991; Jiménez 2018). In particular, since these coherent motions are seen to be responsible for the maintenance of turbulence in wall-bounded flows, their studies motivated a number of researches based on dynamical system theory in the past couple of decades.

The fact that near wall structures almost scale in inner unit, namely with the kinematic viscosity and wall shear stress, indicates they are universal features of wall-bounded flows. Jiménez & Moin (1991) used the possibilities of numerical simulations to investigate the origin of streaks. By reducing the streamwise and spanwise dimensions of the periodic domain, they could identify the dimensions of a Minimal Flow Unit (MFU) below which turbulence could not be sustained. In addition, Jiménez & Pinelli (1999) further demonstrated that the dynamics at MFU is autonomous, the self-sustaining mechanism is independent of the peculiar outer flow. In this vein, Hamilton *et al.* (1995) developed the self-sustained process (SSP) at play in wall-bounded turbulent flows (see also Panton (2001) for a review). Waleffe (1998) and Waleffe (2003) further confirmed the validity of the SSP theory by computing three-dimensional invariant states (steady or travelling waves) associated with the upper branch solution (see previous section) in plane Couette and channel flows using a continuation method. In particular, when statistics are carried out on such solutions (spanwise and streamwise averaging), mean flow and root mean square profiles exhibit strong similarities with statistics (including time averaging) provided by full DNS in large domains. It is then suggested that for wall-bounded turbulent flow, these equilibria produce a state space representation that shows the role of coherent states and their unstable manifolds in guiding turbulent dynamics (Gibson *et al.* 2008; ?). Continuation revealed that these equilibrium states arise from a saddle-node bifurcation for various wall-bounded flows (Couette and Channel flows, for instance) at some Reynolds number for a given computational box. Two solutions are found to emerge at the critical Reynolds number: one associated with a lower-branch solution that is characterized by a low skin friction drag and a upper branch solution that has a high drag. Waleffe (2003) and many others since then have shown that while the lower branch solution is associated with a separatrix between the laminar and turbulent attractor (where one unstable direction is identified), the upper branch solutions are shown to match reasonably well with wall-turbulence at low Reynolds numbers (both in terms of statistics and coherent structures). This result gave strong support that states lying on the upper branch represent saddle-points in the phase-space of the turbulent attractor. Such invariant structures are thus referenced as exact coherent states (ECSs)

by previous authors. These analyses were recently extended to high Reynolds numbers using the filtered Navier-Stokes equations by Cossu and co-workers (see Cossu & Hwang (2017) for a review).

Recent studies have supported that wall-turbulence exhibits different states in its dynamics. While strong turbulence activity dominates for long period of times, weak turbulence, as evidenced by a drag reduction, occurs for short time intervals. During recent decades, new numerical tools are dedicated to the computation of rare events by selecting trajectories (see among others Tailleur & Kurchan (2007); Ragone *et al.* (2017)). Given the complexity of turbulence, direct application of these methods to Navier-Stokes equations is impracticable, therefore it appears necessary to consider some simplified models.

In that respect, Park & Graham (2015) found a new family of travelling wave solutions that emerge through a saddle node bifurcation for the case of a channel flow. In particular, the authors have shown that while statistics carried out on the upper branch solution (*i.e.* associated with the higher friction velocity) strongly match with turbulent channel flow profiles, the lower branch solution (*i.e.* associated with the lower friction velocity) exhibits mean flow profile that approaches the Virk log-law (Virk *et al.* 1970). It is widely accepted that mean velocity profiles most closely approach this law in the log region of a channel flow with homogeneous additive polymer, it has only been recently shown that this state defines a universal limit for Newtonian turbulence, the so-called maximum drag reduction asymptote (MDR) (see Xi & Bai (2016) for a recent review).

More recently, Kushwaha *et al.* (2017) have shown that for low Reynolds number cases, turbulence in channel flow exhibits intermittency that is closely tied to chaotic movement of turbulence trajectories between lower and upper branch solutions found by Park & Graham (2015). By using a shooting method, Itano & Toh (2001) and Toh & Itano (2003) have tracked ECSs for a channel flow within a minimal flow framework. This solution consists of a periodic like solution localized on one side of the channel in the wall normal direction. One cycle of the ECS is characterized by two different motions. While for a very long quiescent phase, the flow state exhibits a single-streak, the latter is shifted in the spanwise direction after each bursting event where the flow structure is dominated by a double streak motion. Because previous authors only succeed to track the solution for one and half periods, none definitive statement has been made whether the solution converges on a heteroclinic cycle or a periodic orbit. Recently, Zammert & Eckhardt (2014) and Rinaldi *et al.* (2018) revisited this previous solution. Using the same shooting method, they were able to track the ECS on several cycles alternating burst and long quiet intervals. On one hand, the latter analysis gave thus strong indication that the finite amplitude solution is attracted to a period orbit. On the other hand, they show that for wider domains, the attracting state is localized in the spanwise direction. Similarly, in the asymptotic suction flat plate boundary layer, Kreilos *et al.* (2013) also found a ECS that exhibits the same dynamical activity providing further evidence that such a periodic state is a universal feature of wall-bounded flow. Hence, it may also be suggested that such periodic solutions can be representative of turbulent trajectories between low and high drag events.

In order to reduce the complexity of the dynamics, important expectations emerged from the linear analysis. The only mathematical justification for the linear assumption of non infinitesimal fluctuations is provided by the rapid distortion theory (RDT). This theory goes back to a paper of Taylor (1935) who analysed the damping of turbulent fluctuations in flow through a contraction in a wind tunnel. If the straining is sufficiently rapid, then its effect on the turbulence may be treated in a first approximation by a linearised analysis. The nonlinear interaction of the turbulence with itself is neglected.

The distinction between terms 'rapid' (linear) and 'slow' (nonlinear) are derived from the fact that only the rapid part responds immediately to a change imposed on the mean field, and the slow part will feel the change through nonlinear interactions. For further details and the various applications see the overview by Hunt & Carruthers (1990). By solving the instantaneous linear equations of the RDT, Lee & Moin (1990) have shown that a high shear rate produces structures in homogeneous turbulence similar to the streaks observed in the sublayer of wall-bounded turbulent shear flows. That numerical result have been experimentally confirmed by Diwan & Morrison (2017).

An heuristic argument for the linear dynamics of fluctuations around a nonlinear mean flow was proposed by Malkus (1956). He conjectured that the turbulent mean state is in equilibrium with the turbulent perturbations responsible for the Reynolds stresses, perturbations being in the subspace of the neutral modes of the linearized Navier-Stokes equations. Malkus tested this promising idea with a time averaged flow but failed to obtain a neutral mode. A renew of interested followed the new findings about sub-critical transition to turbulence of linearly stable wall flows. Butler & Farrell (1993) applied an optimization method to compute the most amplified streaks over an appropriate turbulent eddy turnover time and the streak spacing was found in good agreement with the expected value, around 100 wall units. Similarly, linear models based on resolvent analysis technique proposed by Hwang & Cossu (2010); McKeon & Sharma (2010) and further explored by Sharma & McKeon (2013), Sharma *et al.* (2017) have also clearly enlightened the relevancy of linear mechanism to build a modelling framework for wall-turbulent coherent motions. Nevertheless, these linear studies failed to describe self-sustained process and intermittency or bursts.

All these studies used a time averaged mean flow, extracted from DNS for example, so the mean flow is decoupled from the fluctuations. In fact, the time-averaging is too restrictive and Farrell & Ioannou (1996) had argued that non-stationarity is a fundamental ingredient in turning transient growth into sustained growth. The growth of short scales forces the temporal oscillation of long scales, and the non-stationarity of the latter continuously restarts the transient growth. Equilibrium is more a cycle rather than a permanent balance between the energy production and dissipation.

An attractive variation is the division of the flow into an infinitely long streaks and vortices and the corresponding fluctuation. The streaks are nonlinear and driven by vortices which are generated by nonlinear Reynolds stresses created by small scales. These scales evolve within the non-uniform flow field of the streak, but all nonlinear interactions among themselves are neglected. That Restricted Non Linear (RNL) model is two-dimensional in the cross-flow plane with three velocity components, it permits a reduction of the numerical cost and a simplification of the dynamics. Despite its simplicity that reduction is able to reproduce essential features of transitional and turbulent flows.

Biau & Bottaro (2009) used the RNL system associated with an optimization method to follow an optimal path to transition in a linearly stable duct flow. The model was able to reach a sustained turbulence. A parametric study on the single streamwise wavelength of the fluctuation shows a cut-off around 300 wall units, below which no transition can be observed, which corresponds to the minimal flow unit obtained with DNS. Moreover, the optimal solution from RNL system is close to the nonlinear solution sited on the laminar-turbulent separatrix, sometimes called the edge-state.

Farrell & Ioannou (2012) used the RNL system with a stochastic forcing (see also the overview in Farrell *et al.* (2017)). Their results confirmed that this RNL system supports a realistic self-sustaining process (SSP). At small forcing amplitudes, the only stable equilibrium of the model system is laminar, but, as the forcing increases, there is first a bifurcation from laminar state to steady streaks and rolls, and later another one to

an unsteady 'bursting' state. Bretheim *et al.* (2015) have shown that simulations using RNL system with appropriate chosen streamwise modes, namely the band-limited RNL model, exhibit the main characteristics of wall-turbulence, similar coherent motions and statistics even if only a few streamwise Fourier wave numbers are retained. In particular, mean velocity profiles obtained with such a band-limited RNL model follow the standard logarithmic law. A RNL model was also introduced by Willis & Kerswell (2009) for localized turbulence in pipe flow. For this purpose, the mean flow is axi-symmetric with one Fourier mode in azimuth which is able to reproduce observed behaviour such that localized 'puff' structures and 'slug' turbulence.

While previous studies have proven fruitful in gaining better understanding of RNL systems applied to sustained turbulence, the scope of this paper is to extending these approaches by capturing invariant solutions that aim to describe close path along turbulent saddle points and intermittency within a RNL framework. For that purpose, the present study is devoted to investigate the exact coherent periodic state initially found by Toh & Itano (2003) using DNSs with RNL simulations where only a single Fourier mode in the streamwise direction is retained. In particular, we will investigate whether or not this minimal representation of flow dynamics is able to capture the essential feature of the cycle such as bursting events, spanwise shifts, long quiescent phases or high and low drag events. For that purpose, both flow structures and statistics will be investigated. The paper will be organized as follows. After having presented governing equations and numerical methods in section 2, the tracked solution computed using a RNL system is discussed in the light of previous studies and statistics provided by DNS database in section 3. In particular, the connection between bursting events and near-wall turbulence will be investigated. The influence of spanwise width, Reynolds number and streamwise wave number onto the edge tracked solution will be discussed in section 4. Section 5 will be devoted to draw some conclusions and prospects.

## 2. Method

### 2.1. Governing equations

We consider pressure driven incompressible flow of a Newtonian fluid in a plane channel maintained at constant mass flux. Usual set of Cartesian coordinates  $(x, y, z)$  and velocity components  $\mathbf{u} = (u, v, w)$  are adopted in streamwise, wall-normal and spanwise directions respectively. The half-channel height  $h$  is chosen as the characteristic length scale and velocities are scaled with the laminar centreline velocity  $U_c$  for the given mass flux. The corresponding Reynolds number is defined as  $Re = U_c h / \nu$ , with  $\nu$  the kinematic viscosity. In the following, the selected value is  $Re = U_c h / \nu = 3000$ , as used in Itano & Toh (2001); Toh & Itano (2003), and corresponds to a value  $Re_b = 2000$  based on the bulk velocity.

The nondimensional Navier-Stokes equations are then given by:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -G(t)\mathbf{e}_x - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0.$$

where  $\mathbf{e}_x = (1, 0, 0)$ . The uniform pressure gradient,  $G(t)$ , that drives the flow is adjusted dynamically to maintain a constant mass flux. No-slip boundary conditions are applied on the two walls  $y = \pm 1$  and periodic boundary conditions are adopted in the homogeneous streamwise and spanwise directions. Hereafter, we note  $u_\tau$  the friction velocity and quantities in inner scaling are referenced with the + superscript. The friction Reynolds number is noted  $Re_\tau = u_\tau h / \nu$ .

In the present work, the quasi-linear restriction of the dynamics to roll/streak interactions, namely the so-called RNL system (Farrell *et al.* 2016), is used to study finite amplitude solutions of a plane channel flow. In this context, the velocity  $\mathbf{u}$  is decomposed into mean and perturbation components:

$$\mathbf{u} = \mathbf{U} + \mathbf{u}'. \quad (2.2)$$

The mean velocity is  $\mathbf{U}(y, z, t) = \langle \mathbf{u} \rangle$  with  $\langle \cdot \rangle$  the averaging operator in the streamwise direction. Next, overbar over any quantities will denote averaging along the spanwise direction. The pressure is similarly split into:  $p(x, y, z, t) = G(t)x + P(y, z, t) + p'(x, y, z, t)$ .

The fluctuation can be decomposed into a Fourier expansion. Hereafter, we will consider a minimal RNL system where only a single streamwise component is retained. The fluctuations are thus expressed as:

$$\mathbf{u}' = \hat{\mathbf{u}}e^{i\alpha x} + \hat{\mathbf{u}}^*e^{-i\alpha x}, \quad (2.3)$$

with  $\alpha$  the streamwise wave number. As a consequence, the nonlinear terms in the fluctuations equation  $\mathbf{u}' \cdot \nabla \mathbf{u}' - \langle \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle$  only contain first harmonics, namely modes with wave numbers  $\pm 2\alpha$ , hence it is neglected with the single mode restriction. The RNL model is thus derived from the Navier-Stokes equations through a Galerkin truncation procedure similar to that used to derive the Lorenz model of Rayleigh-Bénard convection. Attention must be paid to this severe truncation, indeed Curry *et al.* (1984) found that the chaos observed in the Lorenz system is in fact a product of inadequate spatial resolution. The RNL model is minimal in the sense that it assumes a separation of scales that does not exist in reality so the model does not really represent multiscale turbulence. Another issue concerns the dependence of the nonlinear mean flow with respect to the streamwise length of the computational domain. On the other hand, the model is deterministic and rigorously derived from the Navier-Stokes equations.

As stated by Waleffe & Kim (1997) and Biau & Bottaro (2009), the model may be further simplified by considering that  $\mathbf{u}'$  is of order  $\mathcal{O}(\varepsilon)$ . The roll vortices  $V$ ,  $W$  are of order  $\mathcal{O}(\varepsilon^2)$  and the streak component of the streamwise mean flow  $U$  is  $\mathcal{O}(\varepsilon^2 Re)$  as a consequence of the lift-up mechanism (Schmid & Henningson 2001). This approximation is consistent with the results given by Wang *et al.* (2007) who have shown that the travelling wave solution, associated with the lower branch, scales with Reynolds number  $Re$  in such that streaks are order one but rolls are  $\mathcal{O}(Re^{-1})$ , the fundamental fluctuating mode scales with  $\mathcal{O}(Re^{-1})$  while the harmonics are  $o(Re^{-1})$ . Hence, at the order  $\mathcal{O}(Re^{-1})$ , the dynamics of ECS lying on the lower branch is consistent with the RNL system. More recently, Hall & Sherwin (2010) and Blackburn *et al.* (2013) have developed a similar analysis to compute three-dimensional rolls/perturbation interaction equilibria of laminar plane Couette flow for high Reynolds number flow cases.

Nonetheless, the issue of capturing travelling waves characteristics of the upper branch and more complex solutions such as periodic states suffer of a lack of a rigorous justification. However, for low Reynolds numbers, Pausch *et al.* (2018) have recently shown that travelling waves states associated with the upper branch solution for both plane channel flows and Couette flows are still well approximated using assumptions that hold for lower branch solutions.



The equations governing the mean, streamwise-averaged flow read:

$$\begin{aligned}
 V_y + W_z &= 0, \\
 U_t + VU_y + WU_z &= -G(t) + Re^{-1}(U_{yy} + U_{zz}) - \partial_y(\hat{v}^*\hat{u} + \hat{v}\hat{u}^*) - \partial_z(\hat{w}^*\hat{u} + \hat{w}\hat{u}^*), \\
 V_t &= -P_y + Re^{-1}(V_{yy} + V_{zz}) - \partial_y(2\hat{v}\hat{v}^*) - \partial_z(\hat{w}^*\hat{v} + \hat{w}\hat{v}^*), \\
 W_t &= -P_z + Re^{-1}(W_{yy} + W_{zz}) - \partial_y(\hat{v}^*\hat{w} + \hat{v}\hat{w}^*) - \partial_z(2\hat{w}^*w),
 \end{aligned} \tag{2.4}$$

associated with homogeneous boundary conditions  $U = V = W = 0$  on the walls. The streamwise pressure gradient  $G(t)$  is adjusted to fix the mass flow rate. The Navier-Stokes equations, linearised around the streaky flow, are:

$$\begin{aligned}
 i\alpha\hat{u} + \hat{v}_y + \hat{w}_z &= 0, \\
 \hat{u}_t + i\alpha U\hat{u} + \hat{v}U_y + \hat{w}U_z &= -i\alpha\hat{p} + Re^{-1}(-\alpha^2\hat{u} + \hat{u}_{yy} + \hat{u}_{zz}), \\
 \hat{v}_t + i\alpha U\hat{v} &= -\hat{p}_y + Re^{-1}(-\alpha^2\hat{v} + \hat{v}_{yy} + \hat{v}_{zz}), \\
 \hat{w}_t + i\alpha U\hat{w} &= -\hat{p}_z + Re^{-1}(-\alpha^2\hat{w} + \hat{w}_{yy} + \hat{w}_{zz}),
 \end{aligned} \tag{2.5}$$

together with  $\hat{u} = \hat{v} = \hat{w} = 0$  on the walls.

In the present configuration, the system (2.4 and 2.5) is capable of yielding a self-sustained state based on linear growth of  $\mathbf{u}'$ , generation of streamwise vortices due to the fluctuation Reynolds stress forcing, production of streak by lift-up effect and regeneration of  $U$ .

## 2.2. Exact coherent state trajectory

There is typically a parameter in front of the stresses that feed back into the streaks and vortices, equivalent to the amplitude of the fluctuations, that has to be adjusted to prevent secular terms. To constraint a phase-space trajectory to stay on the edge surface, a shooting method as firstly described by Itano & Toh (2001); Toh & Itano (2003) is used by considering the coupled system of equations (2.4, 2.5). To obtain a finite amplitude solution Toh & Itano (2003) suggest to introduce a similar decomposition as the one associated with the RNL system: a quasi-two dimensional state and three-dimensional fluctuation. By using DNS, the latter authors search for a solution lying on the edge by adjusting a shooting parameter at an initial time. The procedure is then repeated for adjacent time intervals. This shooting method is well suited to track dynamical trajectory that exhibits heterocline connections (Schneider *et al.* 2008; Duguet *et al.* 2008). It further indicates the ability of this approach in dealing with the periodic solution found by Toh & Itano (2003). When considering a RNL system, the shooting method is straightforward. Let-us introduce a shooting parameter  $f_i$  such as

$$\mathbf{u}_i = \mathbf{U}_i + f_i \mathbf{u}'_i$$

where the subscript  $i$  denotes the time interval. The fitting parameter  $f_i$  is adjusted such as the amplitude of the fluctuation  $\mathbf{u}'$  is kept finite on the time interval  $i$ . Unlike previous studies based on DNS, both the mean flow and fluctuation are already provided by the RNL system. The inherent difficulty to track the edge for a long time interval due to double precision limitation is overcome by computing a distinct shooting parameter for each time sub-step. Hereafter, using a bisection method,  $f_i$  is refined up to 14 significant



digits in double precision for a given dimensionless time interval  $t_i + \delta t$ , where  $t_i$  is fixed  $\approx 350$  and  $\delta t \approx 2 \times t_i$ .

### 2.3. Numerical methods

The numerical method is based on a spectral approximation of the velocity fields using Fourier expansions in both the streamwise and spanwise directions and Chebyshev polynomials in the wall-normal direction. The divergence-free is achieved by using the Uzawa method (Peyret 2002). The discrete system attached to (2.4, 2.5) is solved at each time step using a semi-implicit second order extrapolation/Backward-Differentiation scheme as described by Peyret (2002). In particular, convective terms are treated explicitly while an implicit scheme is used for viscous terms. For dealiasing, the 3/2 rule is adopted for Fourier expansions. The numerical method is further detailed in Alizard (2015) and a validation of the code is carried out in Alizard (2017). For the RNL model, the domain sizes are first identical to those used by Toh & Itano (2003):  $\pi$ , 2 and  $0.4\pi$  with  $1 \times 80 \times 32$  grid points. It gives a streamwise wave number  $\alpha = 2$ . For the direct numerical simulation, a similar spectral method is used. For DNS, the domain sizes are six times larger,  $6\pi$ , 2 and  $2.4\pi$  with  $256 \times 151 \times 256$  grid points respectively in streamwise, wall-normal and spanwise directions. The grid spacings in the streamwise and spanwise directions are respectively  $\Delta x^+ = 9.8$  and  $\Delta z^+ = 3.9$  in wall units.

## 3. Results

### 3.1. Direct numerical simulation

As a preliminary result, figure 1 shows the probability density function of the wall-shear stress fluctuations  $\tau$  normalized with its mean value  $2 \times 10^{-3}$ . The viscous scaling provides a good collapse, except for the tails (Örlü & Schlatter 2011). The PDF is well described by a lognormal distribution, as first shown by Alfredsson *et al.* (2011), given by:

$$f(\tau) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln \frac{\tau-\mu}{\sigma})^2}{2\sigma^2}},$$

with  $\mu = -0.0278$  and  $\sigma = 0.3447$  respectively the mean and rms value of  $\ln \tau$ . Lognormal variability is basically based on a variety of forces acting independently with multiplicative effects which are encountered in many field of physics (Limpert *et al.* 2001). Lognormal distributions have also been found to describe various phenomena in near-wall turbulence, e.g. the average inner normalized streak spacing (Smith & Metzler 1983), and the time interval between burst events (Rao *et al.* 1971).

The probability distribution function (PDF) of the wall shear stress (figure 1) shows the existence of high and low drag events; interestingly, the probability to observe a sub-laminar drag is about 0.17%.

### 3.2. Periodic solution of minimal RNL model

In this section, the periodic solution studied by Toh & Itano (2003) and Zammert & Eckhardt (2014) is revisited using the minimal RNL system. The same initial condition as the one given by Toh & Itano (2003) is used for the system (2.4, 2.5):

$$\mathbf{U} = \left( 1 - y^2, -\frac{\partial \Psi}{\partial z}, \frac{\partial \Psi}{\partial y} \right), \mathbf{u}' = \left( 0, 0, -\frac{\partial \Phi}{\partial y} \right), \quad (3.1)$$

where  $\Psi(y, z) = F(y) \sin(2\pi z/L_z)$ ,  $\Phi(x, y) = F(y) \sin(2\pi x/L_x)$  and  $F(y) = A (\exp(c_m(-y-1)) - 1)^2 (\exp(c_p(y-1)) - 1)^2$  with  $A = 1 \times 10^{-10}$ ,  $c_m = 1$  and

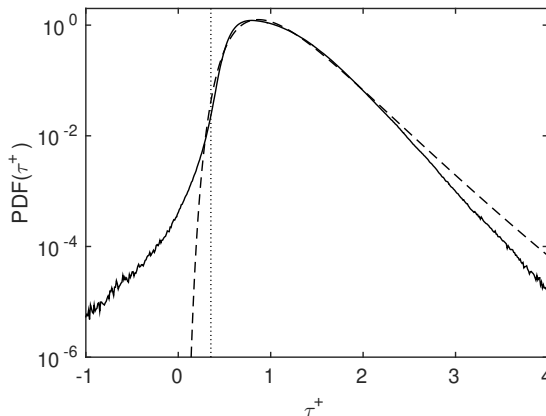


FIGURE 1. Probability distribution function of the streamwise wall shear-stress normalized with the mean value. The PDF is compared to a log-normal distribution (dashed line). The laminar value is indicated with the vertical dotted line.

$c_p = -6$ . It consists of a pair of streamwise vortices located near the lower wall superimposed to the Poiseuille flow solution. For a sufficiently high perturbation amplitude, a secondary instability occurs.

To describe the solution lying on the edge, we introduce the kinetic energy norm of the fluctuation such as:

$$E = \int_{-1}^1 \int_{-L_z/2}^{L_z/2} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}}^* dydz. \quad (3.2)$$

To characterize the number of streaks associated with the solution, we also define kinetic energy norm for the fluctuation restricted to the fundamental and its first harmonic spanwise Fourier mode as  $E_1$  and  $E_2$ . In figure 2(a),  $E(t)$  is shown. After a transient behaviour, the shooting parameter  $f_i$  is found to oscillate between 0.992 and 0.998. The fact that  $f_i$  is close to one suggests that the solution obtained by connecting each time sub-step approximates reasonably well the exact solution. The trajectory resulting from the shooting method exhibits a convergence toward a simple state. The latter is characterized by long quiescent intervals separated by quick bursting events that repeat periodically, as shown in figure 2(a). Figure 2(b) displays time evolution of  $E_1$  and  $E_2$ . The figure shows that while the flow is dominated by one pattern along the spanwise direction during quiet phases, the bursting event is characterized by a competition between the fundamental and its first harmonic spanwise Fourier mode.

In figure 3, the wall-shear stress distribution normalized with the turbulent value is shown in the plane  $(t, z)$  for a time interval that includes several bursts. The figure shows that the burst is accompanied by a strong increase of wall skin friction where the latter reaches a level close to the turbulent one. It is also interesting to notice that while the wall-shear stress has a slightly higher level than its laminar counterpart (*i.e.*  $\approx 0.35$ ) during the major part of the quiescent phase, it also exhibits sub-laminar drag events. It suggests that the minimal representation of this periodic solution is connected to close passes along the turbulent attractor that shares similarities with statistics shown in figure 1. Results are also consistent with Toh & Itano (2003), Zammert & Eckhardt (2014) and Rinaldi *et al.* (2018).

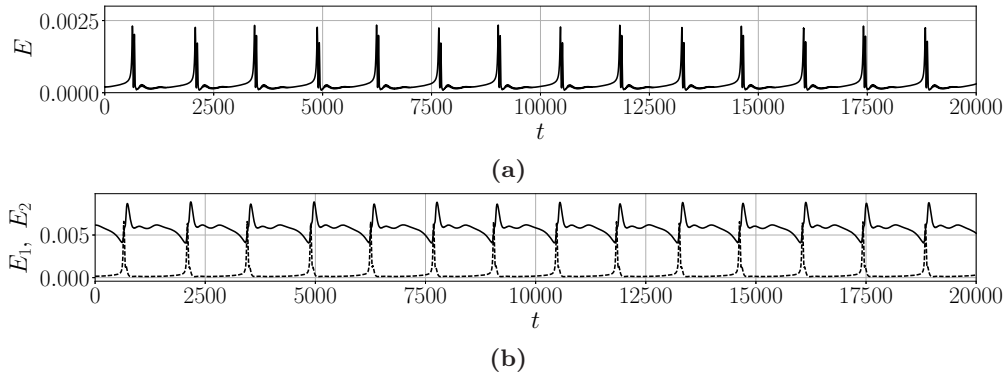


FIGURE 2. Time evolution of (a) kinetic energy of the fluctuation (b) kinetic energy of fluctuation restricted to the fundamental mode (full line) and the first harmonic (dashed lines). The transient phase before convergence to the edge state is discarded.

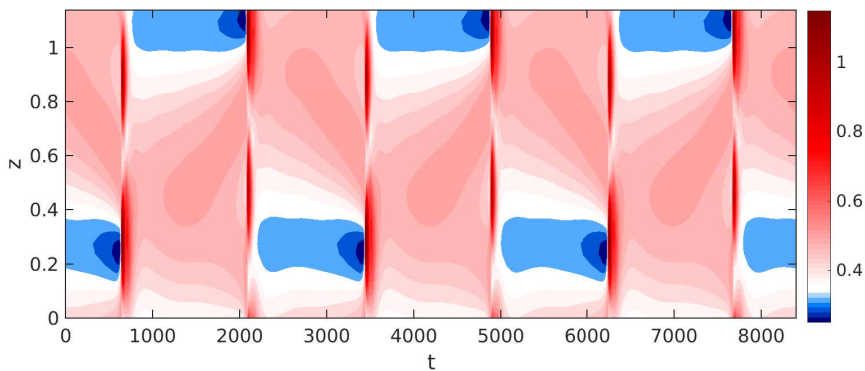


FIGURE 3. Spanwise and time evolution of the wall-shear stress, normalized with the turbulent value, associated with the periodic edge.

### 3.3. Equilibrium states and bursts

To further characterize the solution provided by the RNL restriction, we examine the evolution of the kinetic energy input rate  $\mathcal{I}$  and the dissipation  $\epsilon$  of the system :

$$\frac{d\mathcal{E}}{dt} = \mathcal{I} - \epsilon \quad (3.3)$$

with

$$\mathcal{I} = - \int_D \frac{dp}{dx} \mathbf{u} \cdot \mathbf{e}_x \, dD \quad \text{and} \quad \epsilon = \frac{1}{Re} \int_D \nabla \mathbf{u} : \nabla \mathbf{u} \, dD.$$

In figures 4(a) and (b), the energy input/dissipation rate trajectory and the corresponding time evolution of the wall shear stress for  $\bar{U}$  (*i.e.* spanwise averaged  $U$ ) are shown for the minimal RNL solution. Three characteristic points, referenced as  $P_1$ ,  $P_2$  and  $P_3$  are also depicted in figures 4(a) and (b). While the solution spends a relatively long time near  $P_1$  and  $P_2$ , it orbits only briefly along  $P_3$  where the dissipation and  $\tau^+$  are maximum. One may note that  $P_1$ ,  $P_2$  and  $P_3$  exhibit almost an equilibrium state where the production is balanced by the dissipation. The periodic solution trajectory obtained using the minimal RNL system bears a close resemblance to the one described by Toh & Itano (2003), both in amplitudes and excursions to saddle points. Nevertheless, while Toh & Itano (2003) pointed out that the phase portrait exhibits an almost equilibrium state

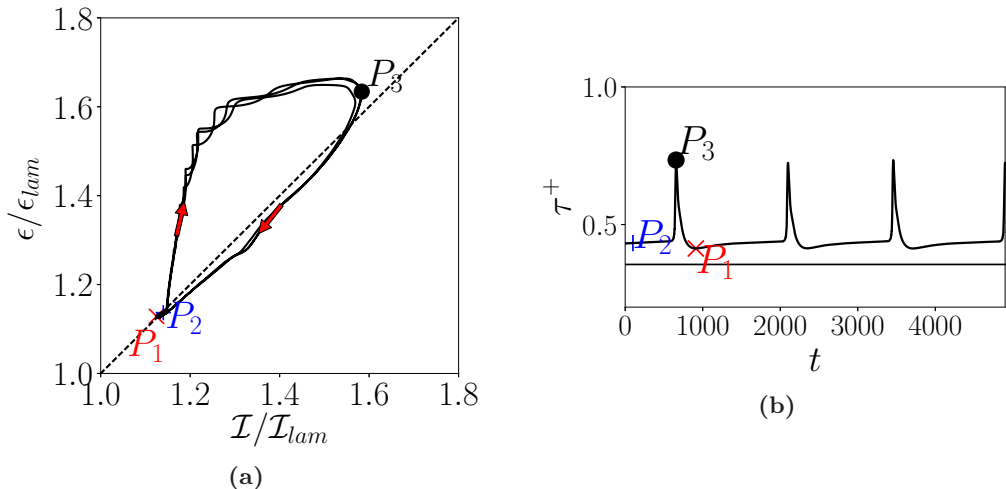


FIGURE 4. (a) Energy input/dissipation rate for the periodic solution. Values are scaled by their laminar Poiseuille flow solution counterparts ( $\epsilon_{lam}$  and  $\mathcal{I}_{lam}$ , respectively). Dashed line corresponds to  $\mathcal{I}/\mathcal{I}_{lam} = \epsilon/\epsilon_{lam}$  with  $\mathcal{I}_{lam} = \epsilon_{lam} = 8/3 Re^{-1}$ . The direction of time is indicated by the arrows; (b) time evolution of the wall-shear stress associated with the streamwise and spanwise averaged quantity normalized with the turbulent value:  $\tau^+$ . The laminar value is indicated with the horizontal line. Time intervals are identical in (a) and (b).

for  $\mathcal{P}/\mathcal{P}_{lam} \approx 1.25$  at which the authors argue that the solution corresponds to the two streaks mode, here we found that the latter case coincides with the solution that orbits close to the maximum of dissipation (*i.e.* when the wall-shear stress peaks). For instance, a visual representation of the mean flow  $U$  in the cross-section plane ( $y, z$ ) is provided in figure 5(c) for  $P_3$ . The figure shows that for this specific point, the streamwise averaged flow consists of a pair of low-speed streak generated by quasi-streamwise vortices.

As shown in figures 5(a) and (b), although the streak/vortex pattern does not really change for  $P_1$  and  $P_2$  (*i.e.* both are mainly associated with a single low-speed streak flanked by a pair of streamwise vortices), it appears that the wall normal position of the eddies slightly changes. In particular, the vortex centers are seen to be displaced away from the lower wall for the  $P_1$  case in comparison to  $P_2$  case. It has for consequence to displace the low-speed streak away from the wall and then to reduce the friction exerted along the lower wall. For comparison, the vortex/streak pattern shown in figure 5(c), where the dissipation rate peaks, is pushed toward the lower wall.

### 3.4. Two exact coherent states?

In the phase plane,  $P_1$  and  $P_2$  exhibit characteristics of equilibrium states driven by a self-sustained mechanism (Waleffe & Kim 1997). A linear temporal stability analysis is carried out for streamwise averaged flows  $\mathbf{U}$  that orbits close to  $P_1$  and  $P_2$ , assuming a steady solution (the code is detailed in Alizard (2015)). In figure 6, we show the spectra for  $P_1$  and  $P_2$  flow cases in the  $(c, \sigma)$  plane, where  $c$  is the phase velocity and  $\sigma$  the temporal amplification rate. The figure shows that spectra exhibit a quasi neutral mode for  $c \approx 0.72$  and  $c \approx 0.83$  for  $P_2$  and  $P_1$ , respectively. We investigate now the rolls/streaks regeneration by the nonlinear self-interaction of that neutral instability mode and the lift-up effect for both solutions. The spanwise components of the neutral modes are shown in figures 7(a) and (b), for  $P_1$  and  $P_2$ , respectively. Both modes exhibit a characteristic sinuous pattern with the energy mainly concentrated near the critical layer, similarly as lower branch states found by Wang *et al.* (2007) and Hall & Sherwin (2010) in a Couette

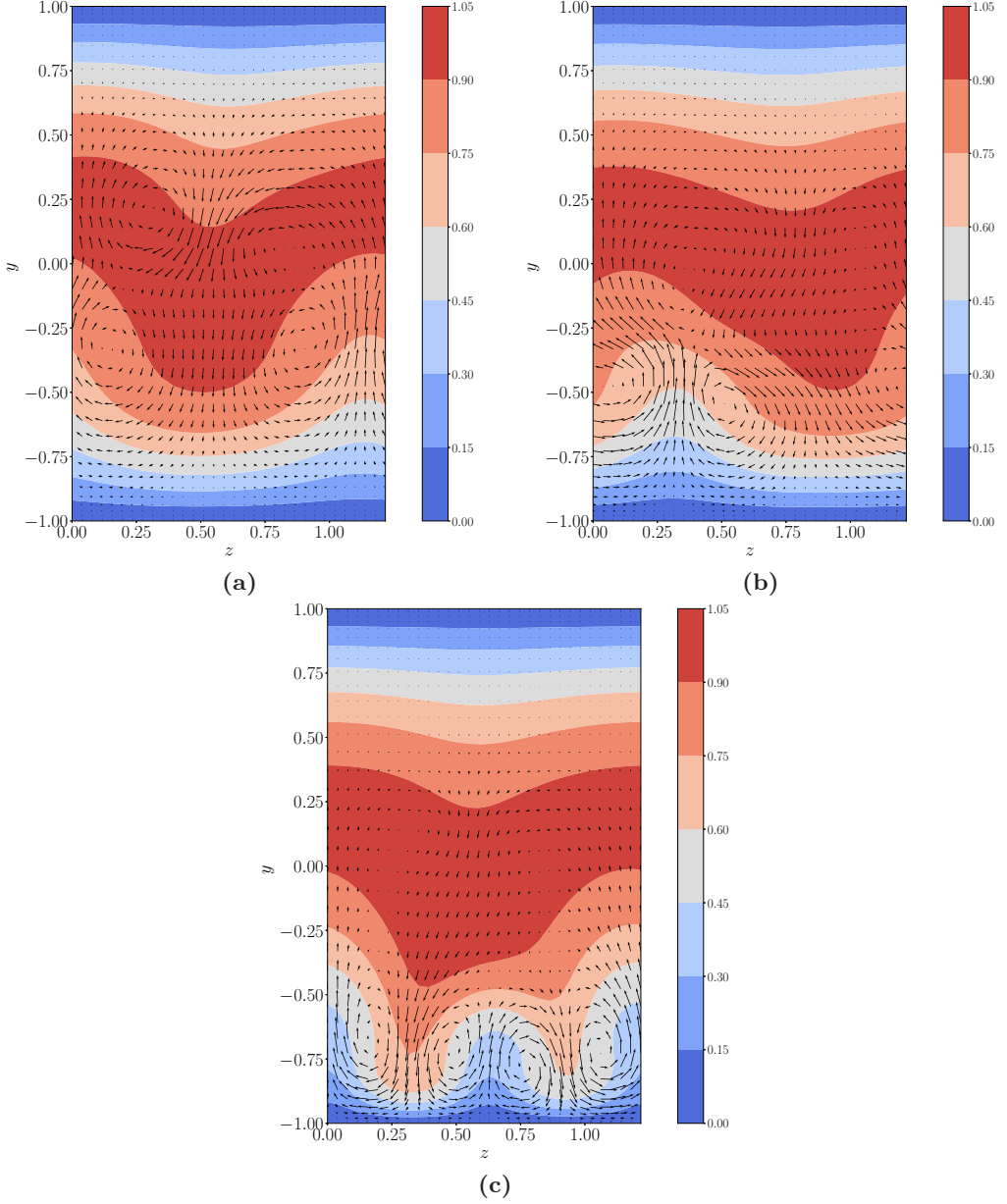


FIGURE 5. Streamwise averaged flow  $\mathbf{U}$ . Mean streamwise velocity contours with superposed mean cross-stream vectors for (a)  $P_1$ , (b)  $P_2$  and (c)  $P_3$  (see figure 4).

flow. Rolls are given by equations (3.4):

$$V_y + W_z = 0,$$

$$Re^{-1} (V_{yy} + V_{zz}) = -P_y - \partial_y (2\hat{v}\hat{v}^*) - \partial_z (\hat{w}^*\hat{v} + \hat{w}\hat{v}^*), \quad (3.4)$$

$$Re^{-1} (W_{yy} + W_{zz}) = -P_z - \partial_y (\hat{v}^*\hat{w} + \hat{v}\hat{w}^*) - \partial_z (2\hat{w}^*w).$$

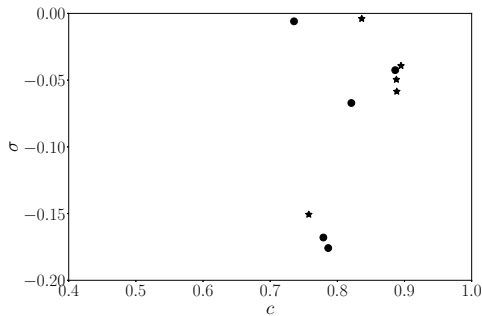


FIGURE 6. Spectra of the linear stability problem based on the mean flow velocity at  $P_1$  (stars) and  $P_2$  (full circles). The phase velocity is  $c$  and the temporal growth rate is  $\sigma$ .

For the streak component, we solve iteratively the following equations:

$$Re^{-1} (U_{yy}^c + U_{zz}^c) = G^n - VU_y^n - WU_z^n - \partial_y (\hat{v}^* \hat{u} + \hat{v} \hat{u}^*) - \partial_z (\hat{w}^* \hat{u} + \hat{w} \hat{u}^*), \quad (3.5)$$

$$U^{n+1} = \gamma U^c + (1 - \gamma) U^n,$$

with  $\gamma$  a relaxation parameters ( $0 \leq \gamma \leq 1$ ) and  $G$  a pressure gradient coefficient that is adjusted to fix the mass flow rate. The amplitude of the Reynolds stress part is provided by the RNL simulation for either  $P_1$  or  $P_2$  events. The numerical implementation of the system (3.4, 3.5) only requires minor modifications of the RNL solver.

The solutions obtained by solving (3.4) and (3.5) are illustrated in figures 7(c) and (d). Interestingly, the forcing terms in (3.4) create streamwise vortices that exhibit strong similarities with rolls provided by the unsteady RNL system shown in figures 4(a) and (b). In addition, under the action of the lift-up effect, the streaky motion derived from the system (3.5) closely resembles the time-dependent solution for both  $P_1$  and  $P_2$  events. Hence, near  $P_1$  and  $P_2$ , the finite amplitude solution orbits near quasi-equilibria states where the roll, low-speed streak and streak instability mode have the right pattern and amplitude to almost stay in mutually self-sustained travelling wave state (Waleffe 2003). Finally, one may pointed out that the point  $P_1$  is not identified by Toh & Itano (2003). However, this discrepancy may be well due to the relatively limited time integration carried out by the previous authors.

Aiming to connect turbulent coherent structures near the wall with close passes along the orbit  $P_3$ , statistics obtained by averaging the instantaneous flow (*i.e.*  $\mathbf{U} + \mathbf{u}'$ ) with respect to the spanwise and to the streamwise direction are evaluated at  $P_3$ . For comparison purposes, the state  $P_1$  is also considered. In figure 8, we show  $\bar{U}$  in inner units for  $P_1$  and  $P_3$  events. The standard law of the wall, compounding  $\bar{U}^+ = y$  with  $\bar{U}^+ = 1/\kappa \ln y^+ + C$  where  $\kappa = 0.41$  and  $C = 5.2$  and the mean velocity associated with an asymptotic state called the maximum drag reduction, the so-called Virk log-law:  $\bar{U}^+ = 11.7 \ln y^+ - 17.0$  (Virk *et al.* 1970; Xi & Graham 2012) are also represented. Figure 8 also shows the PDF obtained from DNS database, plotted on a logarithmic scale, for locally averaged velocity profiles carried out at each wall-normal position.

In figure 9, the streamwise rms velocity profiles as a function of the wall distance are shown in inner units.

Figures 8, 9 show that the periodic solution displays distinct phases in its dynamics. During the bursting ( $P_3$  event), its trajectory exhibits active turbulence on rare occasions where the friction velocity peaks. In this case, the mean flow profile exhibits a small intermediate region (considering the low value of the friction Reynolds number) where it

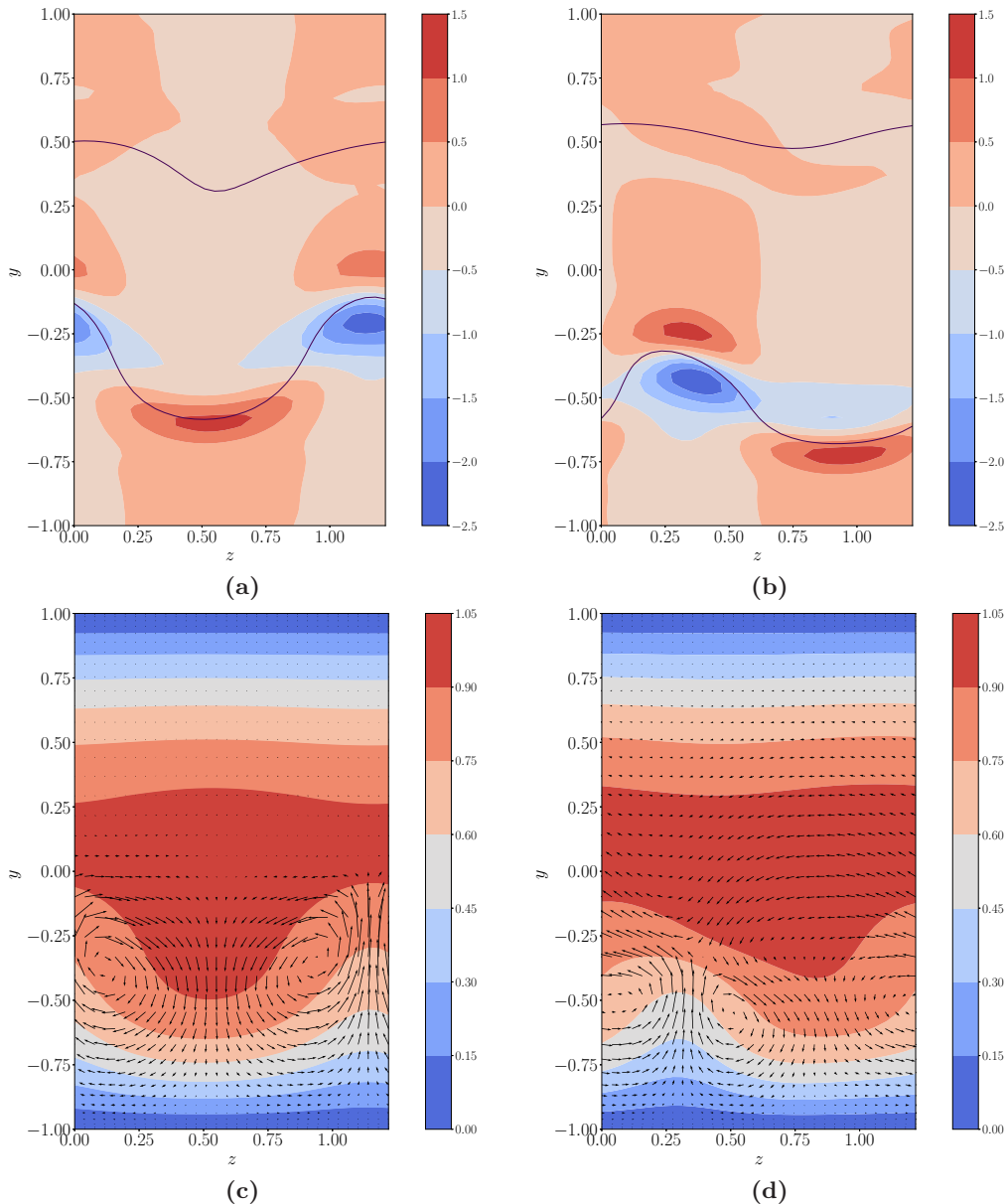


FIGURE 7. Spanwise component of the quasi-neutral mode for  $P_1$  (a) and  $P_2$  (b). The critical layer is reported in full line. Mean streamwise velocity contours with superimposed cross-stream vectors for  $P_1$  (c) and  $P_2$  (d) when assuming a self-sustained equilibrium state.

approaches a logarithmic law with a constant close to the von-Karman value. In addition, at this specific time the friction Reynolds number  $Re_\tau \approx 112$  which is close to the value provided by the full DNS ( $\approx 133$ ). During  $P_1$  events (*i.e.* where friction velocity reaches a minimum), the mean flow profile seems to tend to the Virk log-law profile. Nonetheless, although it seems clear from figure 1 that turbulent trajectories experience low drag events, as shown in figure 8 it appears difficult to compare these states as close excursions through the vicinity of the  $P_1$  (low-drag) solution.

While  $u_{rms}$  exhibits one peaks for  $P_1$  (figure 9(a)), a second peak near the wall appears



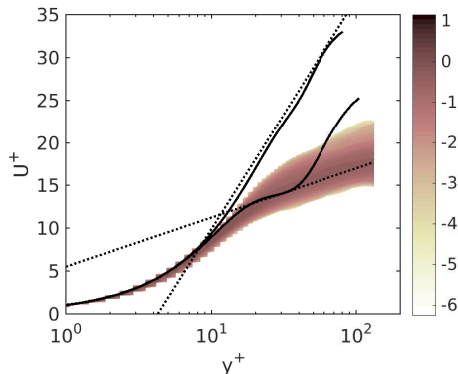


FIGURE 8. Full line: streamwise and spanwise averaged flow:  $\bar{U}$  as a function of the distance from the lower wall in inner units  $y^+$  for solutions  $P_1$  (low-drag) and  $P_3$  (high drag). (see figure 5). The Virk log-law and the standard wall turbulence laws are represented in black dashed-lines. Probability density functions from the DNS are presented in log-scale. To be consistent with RNL model, the instantaneous DNS profiles used for the PDF are averaged over an area of streamwise to spanwise lengths:  $\pi \times 0.4\pi$ .

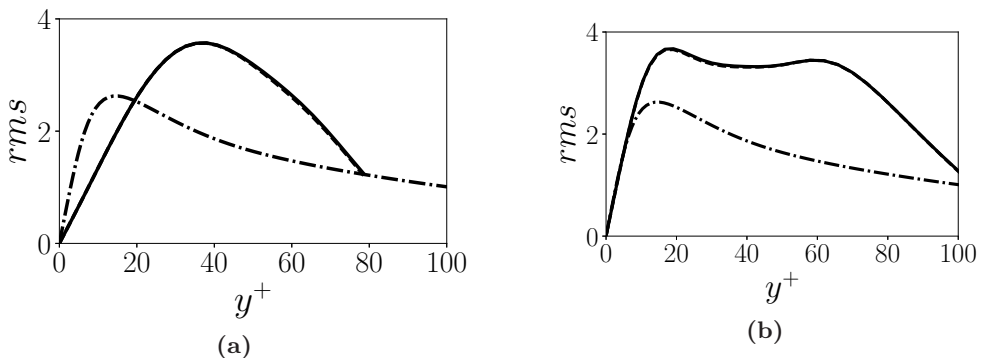


FIGURE 9. Streamwise rms velocities normalized by  $u_\tau$  as a function of the distance from the lower wall in inner units  $y^+$  for solutions (a)  $P_1$  and (b)  $P_3$ . (see figure 5). The total contribution for the RNL system (*i.e.*  $U + u'$ ) is shown in full line. The contribution of  $U$  is depicted in dashed lines. Statistics for the turbulent regime obtained by DNS are represented in dotted lines.

for  $P_3$  events at a wall normal position  $y^+ \approx 15$  (figure 9(b)). One may notice that the latter peak closely matches the one corresponding to the inner most layer distribution found in DNS. In addition, near the wall, a close match is observed between minimal RNL system and full DNS. It is also interesting to remark that the contribution of the fluctuating part is negligible for  $P_1$  and  $P_3$  for rms. By recalling that the contribution of the streamwise component due to  $U$  is associated with the streak, it further indicates the strong correlation between the near-wall peak of the streamwise velocity fluctuations observed in experiments and DNS with streaks. Hence, the periodic edge found by Toh & Itano (2003) and computed here using a minimal RNL model exhibits also strong similarities with travelling wave solutions given recently by Park & Graham (2015). The latter authors found ECSs where travelling waves arising from a saddle-node bifurcation induce either a mean velocity profile that resemble Virk log-law (the lower branch solution) or turbulent mean flow profile (the upper branch solution). However, a clear connection between this bifurcation and the periodic edge is still missing. Finally, we may remark that the streak spacing during the burst is  $\approx 70$  in inner units which is lower

than the near-wall streak spacing observed usually in DNS ( $\approx 80 - 120$ ). It may explain why the roll-streak inner layer dynamics observed here appears only during a short time.

## 4. Discussion

Previously, the Reynolds number, streamwise and spanwise lengths match values given by Toh & Itano (2003). In the present section, the structural sensitivity of the solution with respect to these three parameters is discussed.

### 4.1. Effect of the Reynolds number

The increase of the Reynolds number will increase the distance, in the phase space, between the turbulent attractor  $P_3$  and the travelling wave solutions  $P_1$  and  $P_2$ . As a consequence, the period of the trajectory on the edge is expected to strongly increase with the Reynolds number. Moreover the solution  $P_3$ , which captures the key statistics of turbulent shear, is inconsistent with the RNL model for large Reynolds numbers. On the other hand, the two travelling wave solutions ( $P_1$  and  $P_2$ ) are lower branch coherent states. As shown by Wang *et al.* (2007), for large Reynolds number, their asymptotic structure consists of  $\mathcal{O}(1)$  streaks,  $\mathcal{O}(Re^{-1})$  streamwise rolls and a weak streamwise wave that develops on the critical layer and higher harmonics become negligible. Thus, for these equilibrium states, the RNL model is expected to be consistent with higher values of the Reynolds number.

For large Reynolds numbers, a feasible solution is to extend the RNL equations with turbulent model. Recently, Rawat *et al.* (2014, 2016) and Hwang *et al.* (2016) have computed invariant solutions using an over-damped large-eddy simulation at  $Re_\tau > 10^3$  that closely match with very-large and large-scale coherent structures observed in turbulent channel flows. In particular, by increasing artificially the Smagorinsky constant to account for the dissipation of surrounding small-scale motions, they were able to isolate a self-sustained exact coherent state in the outer region. Bretheim *et al.* (2018) extended the RNL model to high Reynolds numbers by using a RNL large eddy simulation (LES) framework where appropriate streamwise wavenumbers are identified: the “band-limited RNL-LES”. The authors showed that such a simulation reproduces successfully some of the most important statistical properties (*i.e.* the mean velocity and second-order moment profiles). Hence, the use of a RNL system where an additional term modelling dissipative effects of surrounding small-scales is included constitutes a promising route to push forward the present analysis to high Reynolds number wall-bounded turbulent flows. In this context, invariant states at various scales (*i.e.* different distances from the wall) could be investigated using this RNL model to identify the dynamical trajectory embedded in the chaotic turbulent attractor.

### 4.2. Effect of the spanwise length

While edge states (*i.e.* structures that guide the transition towards a turbulent state) have been widely studied in minimal periodic domains (*i.e.* MFU), the resulting constrained dynamics may prevent a direct connection with experimental observations of incipient turbulent spot generated by localized initial perturbation. In this context, some authors (Khapko *et al.* 2013; Schneider *et al.* 2010; Eckhardt 2014) have recently computed spanwise and/or spanwise/streamwise localized edge states in sufficiently extended computational domains for Couette and asymptotic suction boundary layer flows (ASBL).

In this section, we investigate the evolution of the periodic solution obtained at MFU

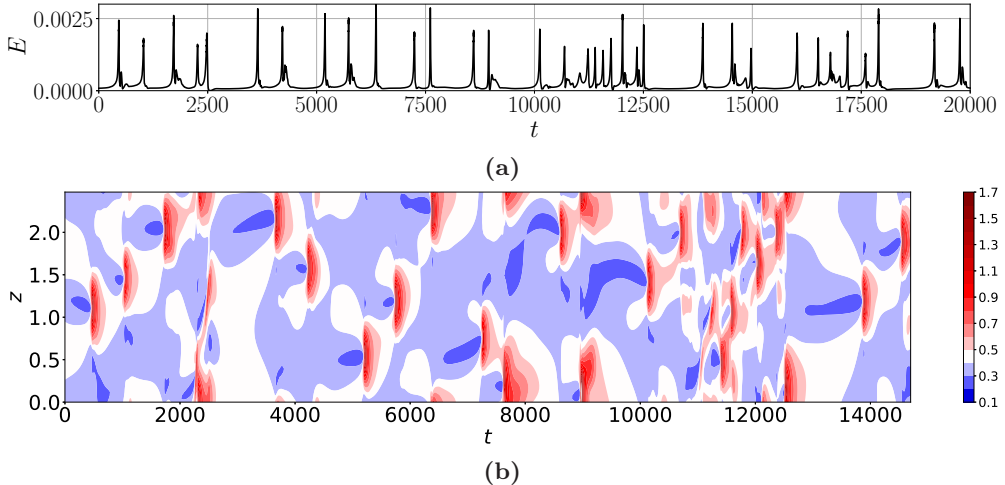


FIGURE 10. RNL simulation. Edge solution for  $Re = 3000$ ,  $L_z = 0.8\pi$  and  $L_x = \pi$ ; (a) Time evolution of kinetic energy of the fluctuation; (b) Spanwise and time evolution of the wall-shear stress, normalized with the turbulent value.

when doubling both the spanwise extend and number of spanwise Fourier modes. For that purpose, we construct a spanwise localized solution by windowing periodic solution obtained at MFU (Gibson & Brand 2014). In figures 10(a) and (b), we report RNL results given by the edge-tracking algorithm for  $L_z = 0.8\pi$ . The figure 10(b) shows that solution is mostly localized in the spanwise direction in contrast with the previous case ( $L_z = 0.4\pi$ ) for which the domain is almost filled entirely. Similar behaviour is recently observed by Xi & Bai (2016) using DNS. In particular, we observe the alternation of calm phases and bursting events before being shifted in the spanwise direction. However, the solution is not periodic anymore, and low-speed streaks are either shifted towards the right or the left direction, similarly as what it is observed for the ASBL case (Khapko *et al.* 2013). In addition, the skin friction reaches a higher peak than its minimal box counterpart (see figure 3). Nevertheless, we can not draw definite conclusions onto a chaotic behaviour or if a periodic state exists because transient regime appears to be extremely long for such a case.

#### 4.3. Effect of the streamwise length

The streamwise length  $L_x = \pi$ , or  $L_x^+ = 410$ , is close to a minimal streamwise unit. Under this restriction large-scale motions are exactly two-dimensional, infinitely long structures, prevented from evolving spatially in the flow direction, but can evolve in the cross-stream plane, hence, dynamics is thus basically temporal rather than spatio-temporal. Direct numerical simulations of channel flow with a minimal streamwise flow unit were performed by Toh & Itano (2005) for relatively low Reynolds numbers ( $Re_\tau = 137$  and 349), and extended by Abe *et al.* (2018) up to  $Re_\tau = 1020$ . The large-scale structures observed in large computational domains are shown to exist, even in a minimal streamwise flow unit. Moreover, the authors observed a co-supporting cycle where large-scale structures are generated by the collective behaviour of near-wall structures and the generation of the latter is in turn enhanced by the large-scale structures.

The extension of the present results, based on a RNL model, to very large streamwise domains is not just a numerical challenge, it mainly raises the issue of the decomposition between a (streamwise averaged) base flow and fluctuations (travelling wave).

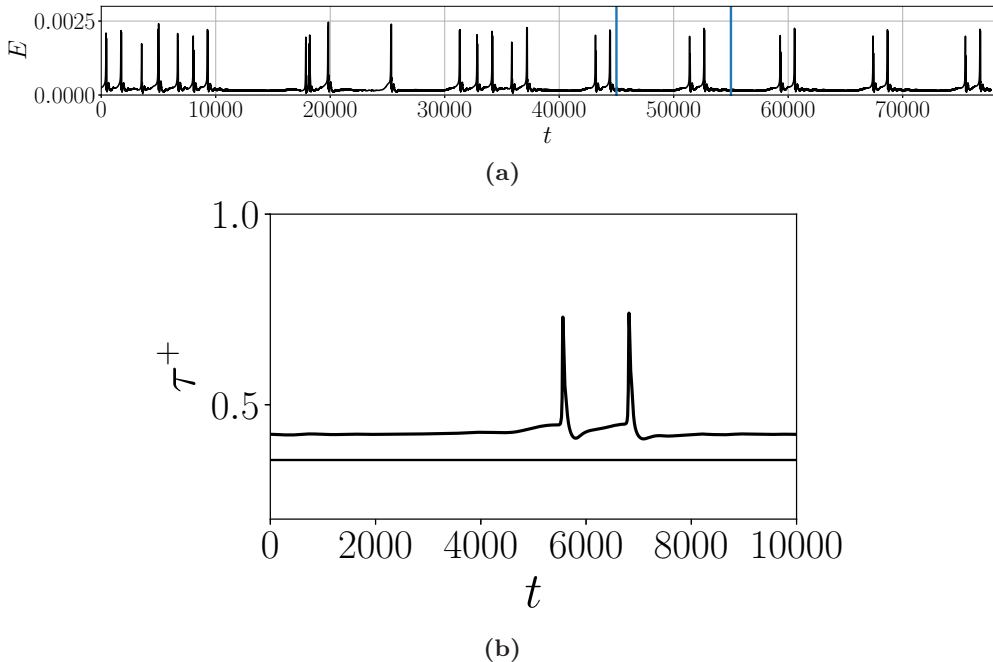


FIGURE 11. RNL simulation. Edge solution for  $Re = 3000$ ,  $L_z = 0.4\pi$  and  $L_x = 0.9\pi$ ; (a) Time evolution of kinetic energy of the fluctuation; (b) time evolution of the streamwise wall-shear stress normalized with the turbulent value for a time interval represented by vertical lines in panel (a). The laminar value is indicated with the horizontal line.

Nonetheless, the effect of small variation of streamwise length have been investigated. In figure 11, we report a minimal RNL simulation for the same Reynolds number and spanwise extend but with a slightly smaller length in streamwise direction:  $L_x = 0.9\pi$ . After a long transient regime, the solution reaches a periodic state where a pair of bursting events are separated by very long quiescent intervals ( $\approx 6000$ ). The figure shows that the solution exhibits similar behaviour that the one observed previously, for  $L_x = \pi$ . In particular, for bursting events, the flow pattern also exhibits a competition between the fundamental and its first harmonic spanwise Fourier mode, *i.e.*  $E_1$  and  $E_2$  (not shown here for the sake of conciseness). However such event appear to append within a periodic process that has strongly extended in time. The latter remark is also consistent with the minimal box for self-sustained turbulence which was close to the one used by Toh & Itano (2003).

## 5. Conclusion

In this paper, we show that the nonlinear periodic solution computed by Toh & Itano (2003), Zammert & Eckhardt (2014) and more recently by Rinaldi *et al.* (2018) using DNS can be tracked with a restricted non linear (RNL) system where only a single streamwise Fourier mode is retained for the fluctuation. In particular, similar characteristics are found for both simulations such as long quiet states, shifted states along the spanwise direction and bursting events where flow patterns exhibit either a single streak or two streaks that compete each others. Interestingly, mean flow and root mean square profiles provided by the RNL system exhibit strong similarities with wall-bounded turbulent statistics obtained both in experiments and DNS for data extracted when the mean

friction velocity (*i.e.* averaged in streamwise and spanwise direction) peaks. The friction Reynolds number computed by collecting data at the burst almost matches the one derived from DNS database. The present results are then consistent with those of Farrell *et al.* (2016) and Brethiem *et al.* (2015) who have found that a realistic turbulence state can be maintained with a RNL system with low streamwise Fourier components for the fluctuation. Furthermore, the contributions of the streamwise averaged quantities and streamwise-varying fluctuations about that mean onto rms extracted at the maximum drag indicate that its peak is mainly due to the mean flow variables. It further supports that near-wall streaks are “active” scales in the sense they are mainly responsible for the turbulent energy production and momentum transfer in comparison with the fluctuating part. Interestingly, the present analysis also shows that the trajectory of the periodic solution initially found by Toh & Itano (2003) exhibits close excursions near an asymptotic state with maximum drag reduction. Hence, the solution seems to be closely tied to intermittent behaviour found in near-wall turbulence for low Reynolds number flows where chaotic movement of turbulence is seen to spend some times around high and low drag events. In particular, our study shows that the periodic solution seems to be intimately connected to lower and upper branch states found by Park & Graham (2015) in plane channel flows. However, the link between this periodic solution and upper/lower branch solutions of Park & Graham (2015) remains unclear. It needs to be pointed out that in comparison to the work of Park & Graham (2015), no symmetry conditions are imposed here, neither in time nor space, which allows the solution to travel around different fixed points. Nevertheless, results obtained using the RNL system agree with the edge tracked solution recently found by Xi & Bai (2016) using the full Navier-Stokes equations for a similar flow case (*i.e.* trajectories closely approaching a MDR state are also found).

In addition, while it seems clear from statistics provided by DNS database that some turbulent trajectories pass through low drag events, present results seem also to indicate a connection in the phase space between the turbulent attractor, associated with large drag events, and travelling wave equilibrium states, associated with low drag. A projection of the dynamical trajectory onto a more appropriate 2D plane seems to be necessary to draw definite conclusion.

Our analysis also gives strong indication that more complex nonlinear finite amplitude solutions than travelling waves solution may be well approximated by minimal RNL system. In that respect, it extends the recent study carried out by Pausch *et al.* (2018), focusing on travelling wave states associated with upper branch solutions in both Couette and plane channel flow. From this discussion, it appears that investigations of invariant solutions using RNL system can provide an efficient tool to select appropriate streamwise wave numbers for RNL simulation aiming to reproduce turbulence in wall-bounded flows. It also further confirms the important role of the mean flow in shaping invariant states for wall-bounded flows. It is consistent with other models relying on mean flow analysis as the resolvent operator technique (Sharma & McKeon 2013). In particular, Sharma *et al.* (2016) have recently shown that exact invariant solutions lying on the edge for channel geometries are well approximated using a projection onto a limited number of resolvent modes.

In the present study, it is observed that a long transient time is necessary to reach an invariant state. Hence, the use of a RNL system may be useful to provide initial condition for tracking periodic edge having long quiescent phases with DNS.

Finally, to better characterize similarities between observations at MFU and states lying on extended domains, a spanwise localized state has been investigated by doubling width of the computational box and windowing the periodic solution obtained at MFU.

We show that the solution bears a strong resemblance to the one obtained in a minimal box (*i.e.* alternation of calm phases and bursting events, spanwise shifted pattern) with the exception of periodicity. In this case, the edge dynamics is erratic. In a future work, the spanwise extend should be further increased to prevent any influence of periodicity. The RNL system may thus provide an efficient model to characterize complex spanwise localized states that may constitute nuclei for the transition of turbulence.

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