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From KAM Tori to Isospectral Invariants and Spectral Rigidity of Billiard Tables.

G.Popov, P.Topalov

February 9, 2019

Abstract

This article is a part of a project investigating the relationship between the dynamics of completely integrable or “close” to completely integrable billiard tables, the integral geometry on them, and the spectrum of the corresponding Laplace-Beltrami operators. It is concerned with new isospectral invariants and with the spectral rigidity problem for the Laplace-Beltrami operators Δ_t , $t \in [0, 1]$, with Dirichlet, Neumann or Robin boundary conditions, associated with C^1 families of billiard tables (X, g_t) . We introduce a notion of weak isospectrality for such deformations.

The main dynamical assumption on (X, g_0) is that the corresponding billiard ball map B_0 or an iterate $P_0 = B_0^m$ of it posses a Kronecker invariant torus with a Diophantine frequency ω_0 and that the corresponding Birkhoff Normal Form is nondegenerate in Kolmogorov sense. Then we prove that there exists $\delta_0 > 0$ and a set Ξ of Diophantine frequencies containing ω_0 and of full Lebesgue measure around ω_0 such that for each $\omega \in \Xi$ and $0 < \delta < \delta_0$ there exists a C^1 family of Kronecker tori $\Lambda_t(\omega)$ of P_t for $t \in [0, \delta]$. If the family Δ_t , $t \in [0, 1]$, satisfies the weak isospectral condition we prove that the average action $\beta_t(\omega)$ on $\Lambda_t(\omega)$ and the Birkhoff Normal Form of P_t at $\Lambda_t(\omega)$ are independent of $t \in [0, \delta]$ for each $\omega \in \Xi$.

As an application we obtain infinitesimal spectral rigidity for Liouville billiard tables in dimensions 2 and 3. In particular infinitesimal spectral rigidity for the ellipse and the ellipsoid is obtained under the weak isospectral condition. Applications are obtained also for strictly convex billiard tables in \mathbb{R}^2 as well as in the case when (X, g_0) admits an elliptic periodic billiard trajectory with no resonances of order ≤ 4 .

In particular we obtain spectral rigidity (under the weak isospectral condition) of elliptical billiard tables in the class of analytic and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric billiard tables in \mathbb{R}^2 . We prove also that billiard tables with boundaries close to ellipses are spectrally rigid in this class.

The results are based on a construction of C^1 families of quasi-modes associated with the Kronecker tori $\Lambda_t(\omega)$ and on suitable KAM theorems for C^1 families of Hamiltonians. We propose a new iteration schema (a modified iterative lemma) in the proof of the KAM theorem with parameters, which avoids the Whitney extension theorem for C^∞ jets and allows one to obtain global estimates of the corresponding canonical transformations and Hamiltonians in the scale of all Hölder norms. The classical and quantum Birkhoff Normal Forms for C^1 or analytic families of symplectic mappings (Hamiltonians) obtained here can be used as well in order to investigate problems related to the quantum non-ergodicity of C^∞ -smooth KAM systems.

Contents

1	Introduction	3
I	Isospectral invariants and rigidity	17
2	Main Results	17
2.1	Billiard ball map	17
2.2	Main Results	18
2.3	Proof of Theorem 3	21
3	Birkhoff Normal Forms of C^k deformations.	22
4	Infinitesimal spectral rigidity of Liouville billiard tables	26
5	Isospectral deformations in the presence of elliptic geodesics.	30
6	Isospectral deformation of locally strictly geodesically convex billiard tables of dimension two.	39
7	Microlocal Birkhoff Normal Form of the monodromy operator	48
7.1	C^1 families of PDOs and FIOs with a large parameter λ	49
7.1.1	C^1 families of symbols and λ -PDOs	49
7.1.2	C^1 families of λ -FIOs.	50
7.1.3	Quantization of C^1 families of billiard ball maps.	53
7.2	Reduction to the boundary	61
7.3	Quantum Birkhoff Normal Form	64
8	C^1 families of quasi-modes and iso-spectral invariants	72
8.1	Quantization condition	72
8.2	Construction of C^1 families of quasi-modes	74
8.3	From quasi-modes to isospectral invariants	76
II	KAM theorems and Birkhoff Normal Forms	79
9	KAM theorems	79
9.1	KAM theorems for C^k families of Hamiltonians	80
9.2	KAM theorem with parameters for symplectic maps	87
9.3	KAM theorems for C^k families of symplectic maps	93
9.4	Birkhoff Normal Forms for C^k -families of symplectic maps	97
10	KAM theorem with parameters	105

11 Proof of Theorem 10.1	108
11.1 The KAM Step	108
11.1.1 <i>The KAM Lemma</i>	108
11.1.2 <i>Preparing next iteration.</i>	117
11.2 Iteration	118
11.2.1 <i>Choice of the small parameters.</i>	118
11.2.2 <i>Analytic smoothing of P_t.</i>	121
11.2.3 <i>Iterative Lemma.</i>	122
11.2.4 <i>Modified Iterative Lemma.</i>	126
11.2.5 <i>Choice of the sequence ν and the small parameters ϵ and $\hat{\epsilon}$.</i>	133
11.3 KAM theorem with parameters in Hölder classes.	138
A Appendix.	139
A.1 Approximation Lemma	139
A.2 Almost analytic Gevrey extensions	141
A.3 Borel's Theorem	146
A.4 Higher order Hölder estimates of a composition and of the inverse function.	146
A.4.1 Estimates of the composition and the inverse function.	146
A.4.2 Higher order Hölder estimates and Interpolation inequalities.	150
A.4.3 Weighted Hölder norms and interpolation inequalities.	153
A.4.4 Symplectic transformations and generating functions.	154
B Appendix.	157
B.1 Invariant characterization of Liouville billiards	157
B.2 Kolmogorov Nondegeneracy of the bouncing ball map for Liouville billiards	162

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1 Introduction

This article is a part of a project (cf. [60]-[63]) investigating the relationship between the dynamics of completely integrable or “close” to completely integrable billiard tables, the integral geometry on them, and the spectrum of the corresponding Laplace-Beltrami operators. It is concerned with new isospectral invariants and the spectral rigidity of the Laplace-Beltrami operator associated with C^1 deformations (X, g_t) , $0 \leq t \leq 1$, of a billiard table (X, g) , where X is a C^∞ smooth compact manifold with a connected boundary $\Gamma := \partial X$ of dimension $\dim X = n \geq 2$ and $t \rightarrow g_t$ is a C^1 family of smooth Riemannian metric on X .

Substantial progress in the inverse spectral geometry has been made by means of the wave-trace formula [18], [20], [33, 34], [46], [74]-[78], and by its semi-classical analogue - the Gutzwiller trace formula [47], [33, 34], [22, 23]. The wave-trace formula, known in physics as the Balian-Bloch formula and treated rigorously by Y. Colin de Verdière [7], J. Duistermaat and V. Guillemin [13], V. Guillemin and R. Melrose [21] and S. Zelditch [77] (see also [8], [46], [56], [65], [66]), as well as the Gutzwiller trace formula relate the spectrum of the operator with certain invariants of the corresponding closed geodesics such as their lengths and the spectrum of the linear Poincaré map.

It has been proved in [18], [74, 75], [33, 34], that for certain nondegenerate closed geodesics one can extract the Birkhoff Normal Form (BNF) from the singularity expansions of the wave-trace. S. Zelditch [76] - [79] and H. Hezari and S. Zelditch [27] have reconstructed the boundary for a large class of analytic domains on \mathbb{R}^n having certain symmetries. Hezari and Zelditch [28] have proven infinitesimal rigidity of isospectral deformations of the ellipse.

Spectral rigidity of closed Riemannian manifolds of negative sectional curvature has been obtained by V. Guillemin and D. Kazhdan [19] (in dimension two), C. Croke and V. Sharafutdinov [9] (in any dimension) and by G. Paternain, M. Salo, and G. Uhlmann [52] for closed oriented Anosov surfaces. In order to link the spectrum of the Laplace-Beltrami operator with the length spectrum of the manifold the wave-trace formula is used. The wave-trace formula is especially useful for C^1 -deformations (X, g_t) of a closed Riemannian manifold (X, g_0) with an Anosov geodesic flow since every closed geodesic of (X, g_0) is hyperbolic, hence, nondegenerate and it gives rise to a C^1 family of closed hyperbolic geodesics of (X, g_t) for $|t|$ small enough. This reduces the problem of the infinitesimal spectral rigidity of Anosov manifolds to the injectivity of a geodesic ray transform which has been proved for negatively curved closed manifolds of any dimension [9] and for closed oriented Anosov surfaces [52]. Moreover, infinitesimal rigidity implies spectral rigidity because of the structural stability. Non of these properties is valid for deformations of a billiard table “close” to an integrable billiard table which makes the spectral rigidity problem much more difficult in that case. The wave-trace method requires certain technical assumptions such as simplicity of the length spectrum (a non-coincidence condition) and non-degeneracy of the corresponding closed geodesic and its iterates which are not fulfilled in general.

The main dynamical assumption on (X, g_0) in the present work is that the corresponding billiard ball map or an iterate of it posses a Kronecker invariant torus (see Definition 1.1) with a Diophantine frequency vector and that the corresponding Birkhoff Normal Form (BNF) is nondegenerate in a Kolmogorov sense. Such Hamiltonian systems are said to be of Kolmogorov-Arnold-Moser (KAM) type. The dynamics of such systems is quite complex. In particular, the non-coincidence and the non-degeneracy conditions may not hold for the corresponding closed geodesics. This makes the wave-trace method useless for such systems in general. On the other hand, the Kronecker tori with Diophantine frequencies survive under small perturbations which makes them the right objects to look for. For this reason we propose another method which is based on the construction of C^1 -families of quasi-modes associated with these tori.

Du to the Kolmogorov-Arnold-Moser (KAM) theory, if the initial Hamiltonian system ($t = 0$) is completely integrable and if it satisfies the Kolmogorov nondegeneracy condition, then a large part of the invariant tori of the initial system having Diophantine frequencies ω survive under the perturbation for t in a small interval $[0, \delta_0)$ and give rise to cylinders of invariant tori $t \rightarrow \Lambda_t(\omega)$, $0 \leq t < \delta_0$ along the perturbation. The positive number δ_0 depends on the small constant κ and on the exponent τ in the Diophantine condition (1.2). The aim of this paper is to prove that the invariant tori form C^1 -families with respect to t and that the value at ω of the corresponding Mather’s β -function does not depend on t , or equivalently that the Birkhoff Normal Form (BNF) of the system at each torus $\Lambda_t(\omega)$ does not depend on t for any C^1 -smooth isospectral deformation. Applications will be obtain in the following three cases: for deformations of Liouville billiard tables, in the case of deformations of strictly convex domains and in the case when g_0 admits an elliptic (broken) geodesic which has no resonances of order ≤ 4 and has a nondegenerate BNF.

Let us formulate the main problems that we are going to investigate. Denote by Δ_t the

“geometric” Laplace-Beltrami operator corresponding to the Riemannian manifold (X, g_t) with Dirichlet, Neumann or Robin boundary conditions. This is a self-adjoint operator in $L^2(X)$ with discrete spectrum accumulating at $+\infty$. The corresponding eigenvalues λ solve the spectral problem

$$\begin{cases} \Delta_t u &= \lambda u & \text{in } X, \\ \mathcal{B}_t u &= 0, \end{cases} \quad (1.1)$$

where $\mathcal{B}_t u = u|_\Gamma$ in the case of Dirichlet boundary conditions, $\mathcal{B}_t u = \frac{\partial u}{\partial \nu_t}|_\Gamma$ in the case of Neumann boundary conditions, and $\mathcal{B}_t u = \frac{\partial u}{\partial \nu_t}|_\Gamma - f u|_\Gamma$ in the case of Robin boundary conditions, where $\nu_t(x)$, $x \in \Gamma_t$, is the outward unit normal to Γ with respect to the metric g_t and f is a smooth real valued function on Γ .

The method we use is based on the construction of C^1 smooth with respect to t quasi-modes. This method has been applied in [63] in order to investigate the spectral rigidity of the problem (1.1) with Robin boundary conditions in the case when the metric g is fixed and $t \rightarrow f_t$ is a continuous deformation of the function appearing in the Robin boundary condition. Let us formulate the isospectral condition.

Consider a union \mathcal{I} of infinitely many disjoint intervals $[a_k, b_k]$ going to infinity, of length $o(\sqrt{a_k})$, and which are polynomially separated.

More precisely, fix two positive constants $d \geq 0$ and $c > 0$, and suppose that

(H₁) $\mathcal{I} \subset (0, \infty)$ is a union of infinitely many disjoint intervals $[a_k, b_k]$, $k \in \mathbb{N}$, such that

- $\lim a_k = \lim b_k = +\infty$;
- $\lim \frac{b_k - a_k}{\sqrt{a_k}} = 0$;
- $a_{k+1} - b_k \geq c b_k^{-d}$ for any $k \in \mathbb{N}$.

Given a set \mathcal{I} satisfying (H₁), we impose the following “weak isospectral assumption”

(H₂) There is $a \geq 1$ such that $\text{Spec}(\Delta_t) \cap [a, +\infty) \subset \mathcal{I} \quad \forall t \in [0, 1]$.

Note that the length of the intervals $[a_k, b_k]$ can increase and even go to infinity as $k \rightarrow \infty$ but not faster than $o(\sqrt{a_k})$. Physically this means that we allow noise in the system. Using the asymptotic behavior of the eigenvalues λ_j as $j \rightarrow \infty$ one can show that conditions (H₁)-(H₂) are “natural” for any $d > n/2$ and $c > 0$. By “natural” we mean that for any $d > n/2$ and $c > 0$ the usual isospectral condition

$$\text{Spec}(\Delta_t) = \text{Spec}(\Delta_0) \quad \forall t \in [0, 1]$$

implies that there exists $a \geq 1$ and a family of infinitely many disjoint intervals $[a_k, b_k]$ such that (H₁)-(H₂) are satisfied – see [63], Lemma 2.2, for details. The exponent d depends on the level spacing of the spectrum of Δ_0 .

The elastic reflection of geodesics of (X, g_t) at Γ determines continuous curves on X called *billiard trajectories* as well as a discontinuous dynamical system on the corresponding cosphere bundle $S_t^* X$ – the “billiard flow” consisting of broken bicharacteristics of the Hamiltonian h_t associated to g_t via the Legendre transform. The latter induces a discrete dynamical system B_t defined on an open subset $\tilde{\mathbf{B}}_t^* \Gamma$ (depending on t) of the open coball bundle $\mathbf{B}_t^* \Gamma$ of Γ called

billiard ball map (see Section 2.1 for a definition). The map $B_t : \tilde{\mathbf{B}}_t^* \Gamma \rightarrow \mathbf{B}_t^* \Gamma$ is exact symplectic. Fix an integer $m \geq 1$ and consider the exact symplectic map

$$P_t = B_t^m : U_t \rightarrow \mathbf{B}_t^* \Gamma$$

where U_t is an open subset of $\mathbf{B}_t^* \Gamma_t$ such that $B_t^j(U_t) \subset \tilde{\mathbf{B}}_t^* \Gamma$ for any $0 \leq j < m$. Given an interval $J \subset [0, 1]$ we say that $P_t = B_t^m$, $t \in J$, is a C^1 family of exact symplectic maps if for every $t_0 \in J$ and $\rho^0 \in U_{t_0}$ there exist neighborhoods $J_0 \subset J$ of t_0 and $V \subset U_{t_0}$ of ρ^0 such that $V \subset U_t$ for every $t \in J_0$ and the map $J_0 \ni t \mapsto P_t|_V \in C^\infty(V, T^* \Gamma)$ is C^1 .

We are interested in Kronecker invariant tori of P_t of Diophantine frequencies, which are defined as follows.

We denote by \mathbb{T}^d the torus $\mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d$ of dimension $d \geq 1$ and by $\text{pr} : \mathbb{R}^d \rightarrow \mathbb{T}^d$ the canonical projection and we consider \mathbb{T}^d as a \mathbb{Z} -module. A “distance” from a given $\alpha \in \mathbb{T} := \mathbb{T}^1$ to 0 can be defined by

$$|\alpha|_{\mathbb{T}} := \inf\{|a| : a \in \text{pr}^{-1}(\alpha)\}.$$

Fix $\kappa \in (0, 1)$ and $\tau > n - 1$ and denote by $D(\kappa, \tau)$ the set all $\omega \in \mathbb{T}^{n-1} = \mathbb{R}^{n-1} / 2\pi\mathbb{Z}^{n-1}$ satisfying the “strong” (κ, τ) -Diophantine condition

$$|\langle k, \omega \rangle|_{\mathbb{T}} = |k_1\omega_1 + \cdots + k_{n-1}\omega_{n-1}|_{\mathbb{T}} \geq \frac{\kappa}{\left(\sum_{j=1}^{n-1} |k_j|\right)^\tau} \quad \forall k = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1} \setminus \{0\}. \quad (1.2)$$

The condition (1.2) is equivalent to the following one. There exists $\tilde{\omega}' \in \text{pr}^{-1}(\omega)$ such that

$$|\langle \tilde{\omega}', k \rangle + 2\pi k_n| \geq \frac{\kappa}{\left(\sum_{j=1}^{n-1} |k_j|\right)^\tau} \quad \forall (k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}, \quad k \neq 0.$$

Obviously, if this condition is satisfied for one $\tilde{\omega}' \in \text{pr}^{-1}(\omega)$ then it holds for each $\tilde{\omega}' \in \text{pr}^{-1}(\omega)$. We denote by $\tilde{D}(\kappa, \tau)$ the set all $\tilde{\omega} \in \mathbb{R}^n$ satisfying the following “weak” (κ, τ) -Diophantine condition:

$$|\langle \tilde{\omega}, k \rangle| \geq \frac{\kappa}{\left(\sum_{j=1}^n |k_j|\right)^\tau} \quad \forall k \in \mathbb{Z}^n, \quad k \neq 0. \quad (1.3)$$

Thus the relation $\omega \in D(\kappa, \tau)$ implies that $\tilde{\omega} := (\tilde{\omega}', 2\pi) \in \tilde{D}(\kappa, \tau)$ for at least one (and then for all) $\tilde{\omega}' \in \text{pr}^{-1}(\omega)$.

The set $D(\kappa, \tau)$ ($\tilde{D}(\kappa, \tau)$) is closed and nowhere dense in \mathbb{R}^{n-1} (\mathbb{R}^n), respectively. Moreover, the union $\cup_{0 < \kappa \leq 1} D(\kappa, \tau)$ of (κ, τ) -Diophantine vectors is of full Lebesgue measure in \mathbb{R}^{n-1} for $\tau > n - 1$ fixed, and $D(\kappa', \tau) \subset D(\kappa, \tau)$ for $0 < \kappa < \kappa'$. Denote by $D^0(\kappa, \tau)$ the set of *points of positive Lebesgue density* in $D(\kappa, \tau)$, i.e. $\omega \in D^0(\kappa, \tau)$ if the Lebesgue measure $\text{meas}(D(\kappa, \tau) \cap V) > 0$ for any neighborhood V of ω_0 in \mathbb{R}^{n-1} . By definition, the complement of $D^0(\kappa, \tau)$ in $D(\kappa, \tau)$ is of zero Lebesgue measure. In the same way we define the subset $\tilde{D}^0(\kappa, \tau)$ of points of positive Lebesgue density in $\tilde{D}(\kappa, \tau)$.

Definition 1.1. *A Kronecker torus of P_t of a frequency ω is an embedded submanifold $\Lambda_t(\omega)$ of $\mathbf{B}_t^* \Gamma$ diffeomorphic to \mathbb{T}^{n-1} such that*

- (i) $B_t^j(\Lambda_t(\omega))$ is a subset of $\tilde{\mathbf{B}}_t^* \Gamma$ for each $0 \leq j \leq m - 1$;
- (ii) $\Lambda_t(\omega)$ is invariant with respect to $P_t = B_t^m$;

(iii) The restriction of P_t to $\Lambda_t(\omega)$ is C^∞ conjugated to the translation $R_\omega : \mathbb{T}^{n-1} \rightarrow \mathbb{T}^{n-1}$ given by $R_\omega(\varphi) = \varphi + \omega$.

This means that there is a smooth embedding $f_{t,\omega} : \mathbb{T}^{n-1} \rightarrow \mathbf{B}_t^*\Gamma$ such that $\Lambda_t(\omega) = f_{t,\omega}(\mathbb{T}^{n-1})$ and the diagram

$$\begin{array}{ccc} \mathbb{T}^{n-1} & \xrightarrow{R_\omega} & \mathbb{T}^{n-1} \\ \downarrow f_{t,\omega} & & \downarrow f_{t,\omega} \\ \Lambda_t(\omega) & \xrightarrow{P_t} & \Lambda_t(\omega) \end{array} \quad (1.4)$$

is commutative.

Definition 1.2. By a C^1 -smooth family of Kronecker tori $\Lambda_t(\omega)$ of P_t , $t \in [0, \delta]$, with a frequency ω we mean a C^1 family of smooth embeddings $[0, \delta] \ni t \mapsto f_{t,\omega} \in C^\infty(\mathbb{T}^{n-1}, T^*\Gamma)$ satisfying (i)-(iii) of Definition 1.1.

For each Diophantine frequency $\omega \in D(\kappa, \tau)$ the embedding $f_{t,\omega} : \mathbb{T}^{n-1} \rightarrow \mathbf{B}_t^*\Gamma$ is a *Lagrange embedding* (see [26], Sect. I.3.2). We simply say that each Kronecker torus $\Lambda_t(\omega) \subset \mathbf{B}_t^*\Gamma$ is Lagrangian for such frequencies. Note that the map $P_t : \Lambda_t(\omega) \rightarrow \Lambda_t(\omega)$ is uniquely ergodic for $\omega \in D(\kappa, \tau)$, i.e. there is a unique probability measure μ_t on $\Lambda_t(\omega)$ which is P_t invariant. Evidently, its pull-back $f_{t,\omega}^*(d\mu_t)$ by the diffeomorphism $f_{t,\omega}$ coincides with the Lebesgue-Haar measure $d\theta$ of \mathbb{T}^{n-1} . The automorphism $x \rightarrow \frac{x}{2\pi}$ of \mathbb{R}^{n-1} induces an isomorphism of groups $j : \mathbb{R}^{n-1}/2\pi\mathbb{Z}^{n-1} \rightarrow \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ assigning to any frequency vector ω the corresponding rotation vector which will be denoted by $\omega/2\pi$. Hereafter we will deal mainly with frequency vectors which is motivated by the extensive use of the Fourier analysis.

To any Kronecker torus $\Lambda_t(\omega)$ with a Diophantine frequency $\omega \in D(\kappa, \tau)$ one can associate three dynamical invariants as follows.

The first one is the *average action* on the torus, which corresponds to the Mather's β -function in the case of twist maps. Given $\varrho \in \tilde{\mathbf{B}}_t^*\Gamma$ we denote by

$$A_t(\varrho) := \int_{\tilde{\gamma}_t(\varrho)} \xi dx$$

the action on the broken bicharacteristic $\tilde{\gamma}_t(\varrho)$ “issuing from” $\varrho_0 := \varrho$ and “having endpoint” at $\varrho_m := P_t(\varrho)$, where ξdx is the fundamental one-form on T^*X . Denote by X_{h_t} the Hamiltonian vector field where h_t is the Legendre transform of the metric tensor g_t . The broken bicharacteristic $\tilde{\gamma}_t(\varrho)$ is a disjoint union of integral curves $\gamma_t(\varrho_j)$ of the Hamiltonian vector field X_{h_t} “issuing” from $\varrho_j := B_t^j(\varrho)$ and “ending” at $\varrho_{j+1} = B_t^{j+1}(\varrho)$ and lying on the cosphere bundle $\Sigma_t := S_t^*X = \{h_t = 1\}$ (for a more precise definition see Section 2.1). The vertices of $\tilde{\gamma}_t(\varrho)$ can be identified with ϱ_j , $0 \leq j \leq m$, and we have

$$A_t(\varrho) = \sum_{j=0}^{m-1} \int_{\gamma_t(\varrho_j)} \xi dx.$$

Notice that $2A_t(\varrho)$ is just the length of the broken geodesic in (X, g_t) obtained by projecting the broken bicharacteristic $\tilde{\gamma}_t(\varrho)$ to X . In particular, $A_t(\varrho) > 0$. By Birkhoff's ergodic theorem

$$\beta_t(\omega) := -2 \lim_{N \rightarrow +\infty} \frac{1}{2N} \sum_{k=-N}^{N-1} A_t(P_t^k \varrho) = -2 \int_{\Lambda_t(\omega)} A_t d\mu_t < 0 \quad (1.5)$$

does not depend on the choice of $\varrho \in \Lambda_t(\omega)$. The function β_t can be extended as a convex function in the case when $n = 2$ and P_t is a monotone twist map. It can be related to the Mather's β -function β_t^M [70] (cf. also [69]) by the isomorphism $j : \mathbb{R}^{n-1}/2\pi\mathbb{Z}^{n-1} \rightarrow \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, i.e. $\beta_t = \beta_t^M \circ j$.

Another invariant of a Kronecker torus $\Lambda_t(\omega)$ with a Diophantine frequency is the *Liouville class* on it which is defined as the cohomology class $[f_{t,\omega}^*(\xi dx)] \in H^1(\mathbb{T}^{n-1}, \mathbb{R})$, where ξdx stands for the fundamental one-form of $T^*\Gamma$ (recall that $f_{t,\omega} : \mathbb{T}^{n-1} \rightarrow \mathbf{B}_t^*\Gamma$ is a Lagrange embedding). Let e_1, \dots, e_{n-1} be the canonical basis of \mathbb{R}^{n-1} and $s \rightarrow c_j(s) = \text{pr}(se_j)$, $j = 1, \dots, n-1$, be the corresponding loops on \mathbb{T}^{n-1} . Then $\gamma_{t,\omega}^j := f_{t,\omega} \circ c_j$, $j = 1, \dots, n-1$, provide a basis of loops of $H_1(\Lambda_t(\omega), \mathbb{Z})$. In the dual basis of $H^1(\mathbb{T}^{n-1}, \mathbb{R})$ we write $[f_{t,\omega}^*(\xi dx)]$ as

$$I_t(\omega) = \left(\int_{\gamma_{t,\omega}^1} \xi dx, \dots, \int_{\gamma_{t,\omega}^{n-1}} \xi dx \right). \quad (1.6)$$

The *Birkhoff Normal Form* (BNF) of P_t is another invariant related to a Kronecker torus. To each Kronecker torus $\Lambda_t(\omega)$ with a Diophantine frequency ω one can associate a BNF of P_t as follows. There exist an exact symplectic map χ_t from a neighborhood of $\mathbb{T}^{n-1} \times \{I_t(\omega)\}$ in $T^*\mathbb{T}^{n-1}$ to a neighborhood of $\Lambda_t(\omega)$ in $T^*\Gamma$ a smooth function L_t and a map R_t such that $\Lambda_t(\omega) = \chi_t(\mathbb{T}^{n-1} \times \{I_t(\omega)\})$ and

$$(\chi_t^{-1} \circ P_t \circ \chi_t)(\varphi, I) = (\varphi + \nabla L_t(I), I) + R_t(\varphi, I), \quad \partial_I^\alpha R_t(\varphi, I_t(\omega)) = 0 \quad \forall \alpha \in \mathbb{N}^{n-1}, \quad (1.7)$$

(see Sect. 3). The BNF of P_t at the torus $\Lambda_t(\omega)$ is said to be *nondegenerate* if the Hessian matrix of L_t at $I = I_t(\omega)$ is nondegenerate, i.e.

$$\det \partial_I^2 L_t(I_t(\omega)) \neq 0. \quad (1.8)$$

One can choose L_t so that

$$\beta_t(\omega) + L_t(I_t(\omega)) = \langle \omega, I_t(\omega) \rangle \quad \text{and} \quad \nabla L_t(I_t(\omega)) = \omega \quad (1.9)$$

(see Lemma 3.5).

Given an interval $J \subset [0, 1]$ and a C^1 family of Kronecker tori $J \ni t \mapsto \Lambda_t(\omega)$, we say that (1.7) provides a C^1 family of BNFs in J if $t \rightarrow \chi_t$, $t \rightarrow L_t$ and $t \rightarrow R_t$ are C^1 families with values in the corresponding C^∞ spaces (see Definition 3.3).

Let $[0, \bar{\delta}] \ni t \rightarrow P_t$ be a C^1 family of exact symplectic maps and $0 < \bar{\delta} \leq 1$. We are interested in the following problems.

Problem I. *Let $\Lambda_0(\omega_0)$ be a Kronecker torus of P_0 with a (κ_0, τ) -Diophantine frequency $\omega_0 \in D(\kappa_0, \tau)$, where $0 < \kappa_0 \leq 1$ and $\tau > n - 1$. Suppose that the BNF of P_0 at $\Lambda_0(\omega_0)$ is nondegenerate. Do there exist $\Xi \subset \mathbb{T}^{n-1}$ and $0 < \delta \leq \bar{\delta}$ such that*

1. $\omega_0 \in \Xi$ and Ξ is a set of Diophantine frequencies of full Lebesgue measure at ω_0 which means that $\text{meas}(\Xi \cap W) = \text{meas}(W) + o(\text{meas}(W))$ as $\text{meas}(W) \rightarrow 0$ for any open neighborhood W of ω_0 ;
2. For each $\omega \in \Xi$ there exists a C^1 family of Kronecker tori $[0, \delta] \ni t \mapsto \Lambda_t(\omega)$ of P_t .

A positive answer of this question is given by Theorem 1, item 2, and Theorem 3.2. The set Ξ is of the form (see Section 3)

$$\Xi = \bigcup_{0 < \kappa \leq \kappa_1} \Omega_\kappa^0 \quad (1.10)$$

where $\kappa_1 \leq \kappa_0$, $\omega_0 \in \Omega_{\kappa_0}^0$ and the set Ω_κ^0 consists only of points of positive Lebesgue density for any κ fixed. Moreover, Theorem 3.2 gives a C^1 family of simultaneous BNFs associated with the C^1 families of invariant tori

$$[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \quad \forall \omega \in \Omega_\kappa^0.$$

which means that the family of symplectic maps χ_t , $t \in [0, \delta]$, is C^1 and for any fixed t the map χ_t provides a BNF (1.7) of P_t at $\Lambda_t(\omega)$ for all $\omega \in \Omega_\kappa^0$ at once. These families are analytic in t if the map $t \rightarrow P_t$ is analytic. We apply that to the following three situations

1. (X, g_0) is a nondegenerate Classical Liouville Billiard table as defined in Section 4. Then the billiard ball map $P_0 = B_0$ is completely integrable and the Kolmogorov non-degeneracy condition is fulfilled. Hence Theorem 1, 1-2, and Theorem 3.2 hold for every invariant torus of Diophantine frequency in this case.
2. (X, g_0) has an elliptic closed broken geodesic with $m \geq 2$ vertices of no resonances of order ≤ 4 and with a nondegenerate BNF (see Section 5). The corresponding return map $P_0 = B_0^m$ has a large family of Kronecker tori with Diophantine frequencies, the BNF of each of them is nondegenerate and one can apply Theorem 1, 1-2, and Theorem 3.2.
3. (X, g_0) is a locally strictly geodesically convex billiard table of dimension two (see Section 6). Then there exists a large family of Kronecker tori with Diophantine frequencies of the billiard ball map $P_0 = B_0$. These invariant circles accumulate at the boundary $S_0^*\Gamma$ of the coball bundle $\mathbf{B}_0^*\Gamma$ and give rise of the so called Lazutkin caustics in the interior of X . Close to $S_0^*\Gamma$ the map $P_0 = B_0$ is twisted. This implies that the BNF of P_0 is nondegenerate at each Kronecker torus sufficiently close to $S_0^*\Gamma$ and we can apply Theorem 1, 1-2, as well as Theorem 3.2.

From now on we denote by Ξ a set of Diophantine frequencies of the form (1.10) such that the items 1. and 2. in Problem I are satisfied in Ξ .

Problem II. *Let (X, g_t) , $t \in [0, 1]$, be a C^1 family of billiard tables satisfying the weak isospectral condition (H_1) - (H_2) . Consider a C^1 family of Kronecker tori $[0, \delta] \ni t \mapsto \Lambda_t(\omega)$ of P_t for each $\omega \in \Xi$. Are the functions $t \rightarrow \beta_t(\omega)$, $t \rightarrow I_t(\omega)$ and $t \rightarrow L_t(I_t(\omega))$ independent of $t \in [0, \delta]$ for each $\omega \in \Xi$?*

Affirmative answer of this question is given in Theorem 1 and Theorem 2. This result can be applied in the cases 1. – 3. listed above. The proof is based on the construction of C^1 families of quasi-modes of the spectral problem (1.1) in Theorem 8.2. We present below the main idea of the proof.

Problem III. *Does the weak isospectral condition (H_1) - (H_2) imply the existence of a C^1 family of Kronecker tori $[0, 1] \ni t \mapsto \Lambda_t(\omega)$ of P_t all along the perturbation for each $\omega \in \Xi$?*

This problem is closely related with a mysterious phenomena in the Hamiltonian dynamics of close to integrable systems - the destruction of Kronecker tori with Diophantine frequencies along a perturbation. The C^1 family of Kronecker tori $t \mapsto \Lambda_t(\omega)$ exists in a certain interval $[0, \delta_0)$ but it may cease to exist at $t = \delta_0$. Does the “weak isospectral condition” prevent the tori from destroying? We give a positive answer of Problem III in the following two cases - in the case 2. mentioned above if the elliptic periodic broken geodesic has no resonances of order ≤ 12 (see Theorem 5.2 and Proposition 5.3) and for the Lazutkin caustics in the case of a C^1 deformation of a strictly convex billiard table in \mathbb{R}^2 (see Theorem 6). The proof of these two results is rather involved. It requires a KAM theorem and BNF theorem where the constant ϵ appearing in the smallness condition essentially depends only on the dimension n and the exponent $\tau > n - 1$ but not on the particular completely integrable Hamiltonian (see Theorem 9.8 and Theorem 9.11). We need as well suitable uniform with respect to t *global* estimates of the Hölder C^ℓ -norms ($\ell \geq 1$) of the functions L_t in the BNF (1.7). These estimates are obtained in Theorem 9.11.

Problem IV. *Spectral rigidity under the weak isospectral condition.*

We show in Proposition 2.2 that the variation $\dot{\beta}_t(\omega)$, $\omega \in \Xi$, can be written by means of a suitable Radon transform at the family of Kronecker tori $\Lambda_t(\omega)$, $\omega \in \Xi$, applied to the “vertical component” of the variation of the boundary Γ_t . In particular, the equality $\beta_t(\omega) = \beta_0(\omega)$, $t \in [0, \delta]$, $\omega \in \Xi$, obtained in Theorem 2 implies that the image of the Radon transform is zero for any weakly isospectral family (see Theorem 3). Hence, to prove infinitesimal rigidity one has to obtain injectivity of that Radon transform. In this way we obtain infinitesimal spectral rigidity under the weak isospectral conditions for classical Liouville Billiard Tables of dimension 2 and 3 in Theorem 4.3 and Theorem 4.5. We obtain in particular that the billiard tables inside the ellipse in \mathbb{R}^2 and inside the ellipsoid in \mathbb{R}^3 are infinitesimally spectrally rigid under the weak isospectral conditions (H₁) – (H₂). Infinitesimal spectral rigidity of the billiard table inside the ellipse has been obtained by Hezari and Zelditch [27] under the usual isospectral condition using the wave-trace method. Unfortunately infinitesimal spectral rigidity does not always apply spectral rigidity as in the case of negatively curved manifolds because of the phenomena of destruction of Kronecker tori with Diophantine frequencies.

As an application of Theorem 3 we prove in Theorem 4 spectral rigidity of analytic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetric billiard tables (X_t, g) of dimension two if one of the corresponding bouncing ball trajectories is elliptic, it has no resonances of order ≤ 4 and the Poincaré map is nondegenerate.

Problem V. *Are classical Liouville Billiard Tables spectrally rigid?*

It turns out (see Corollary B.6) that the map $P = B^2$ is always Kolmogorov nondegenerate (twisted) at the elliptic point for elliptical billiard tables (bounded by an ellipse). Moreover, except of five confocal families of ellipses given explicitly by (A.73), the geodesic γ_1 is 4-elementary. These two conditions are open in the C^5 topology, and applying Theorem 4 we obtain spectral rigidity not only of such elliptical billiard tables but also of analytic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetric billiard tables close to them.

Problem VI. *Estimates of the canonical transformation and the transformed Hamiltonian in the KAM theorem in the scale of C^α norms.*

In order to prove the main theorems in the first part of the article we need certain global estimates (in the whole domain of frequencies) of the canonical transformations and the transformed Hamiltonian in the KAM theorem in the whole scale of Hölder norms. Such estimates are obtained in Theorem 10.1, (iii), and in Theorem 11.22, using a new iteration schema, which allows us to avoid the Whitney extension theorem for C^∞ jets.

Before giving the structure of the paper we would like to compare different features of the spectral rigidity problem in the cases of negatively curved closed manifolds and of close to integrable Hamiltonian systems.

<u>negative curvature</u>	<u>close to integrable</u>
Anosov system	KAM system
C^1 families of hyperbolic closed geodesics	C^1 families of Kronecker tori
Labeled length spectrum	Average action on the Kronecker tori, Mather's β function
Wave trace formula	C^1 families of quasi-modes
Geodesic ray transform	Radon transform on Kronecker tori
Structural stability of Anosov dynamics	Phenomena of destruction of invariant tori with Diophantine frequencies
Infinitesimal rigidity easily implies spectral rigidity	Passing from infinitesimal rigidity to spectral rigidity is a hard problem

We are going to describe now the structure of the paper.

In Section 2 we recall first the definition of the billiard ball map and then we formulate some of the main results. We give as well a proof of Theorem 3 which reduces the spectral rigidity problem to the injectivity of a suitable Radon transform.

In Section 3 we obtain by Theorem 3.2 a C^1 family of BNFs of P_t associated with C^1 families of Kronecker tori $\Lambda_t(\omega)$, where $\omega \in \Omega_\kappa^0$, $t \in J$, and $J \subset [0, \delta_0]$ is an interval. This family is analytic in t if the map $t \rightarrow P_t$ is analytic. The theorem is based on the BNF obtained in Theorem 9.11.

Section 4 is devoted to Liouville billiard tables of dimension $n = 2$ or $n = 3$. Liouville billiard tables of dimension two were defined in [60, Sec. 2] by using a branched double covering map. We give here an invariant definition of Liouville billiard tables in dimension two and we prove the equivalence of the two definitions in Appendix B.1. Then we recall the definition of Liouville billiard tables of classical type in dimension two. Infinitesimal spectral rigidity of such billiard tables under the “weak isospectral condition” is obtained in Theorem 4.3. Infinitesimal spectral rigidity of nondegenerate Liouville billiard tables of classical type in dimension three is obtained in Theorem 4.5. Here we essentially use the injectivity of the corresponding Radon transform which has been proven in [62]. In particular we obtain infinitesimal spectral rigidity of the ellipse in \mathbb{R}^2 and the ellipsoid in \mathbb{R}^3 under the “weak isospectral conditions”.

In Section 5 we consider C^1 isospectral deformations $[0, 1] \ni t \rightarrow (X, g_t)$ of a given billiard table (X, g_0) admitting an elliptic closed broken geodesic γ with $m \geq 2$ vertices and we denote

by $P_0 := B_0^m$ the corresponding Poincaré map. We suppose that γ admits no resonances of order ≤ 4 and that the BNF of P_0 is nondegenerate. By the implicit function theorem there exists $\bar{\delta} > 0$ and a C^1 family of elliptic closed broken geodesic γ_t , $t \in [0, \bar{\delta})$, with $m \geq 2$ vertices in (X, g_t) having no resonances of order ≤ 4 and such that the BNF of the corresponding Poincaré maps P_t are nondegenerate. Theorem 4 gives spectral rigidity for a class of analytic $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetric billiard tables of dimension two. Then we address the following questions. Suppose that the C^1 family of billiard tables (X, g_t) , $0 \leq t \leq 1$, is weakly isospectral. Assume that (X, g_0) admits a periodic elliptic broken geodesic γ_0 and that the corresponding local Poincaré map is twisted. Does there exist a C^1 family of periodic elliptic broken geodesics $[0, 1] \ni t \rightarrow \gamma_t$ in (X, g_t) *along the whole perturbation*? Do the corresponding local Poincaré map remain twisted? Do the invariant tori $\Lambda_0(\omega)$ associated to γ_0 give rise to C^1 families of invariant tori $[0, 1] \ni t \rightarrow \Lambda_t(\omega)$ *along the whole perturbation*? We give an answer of these questions in Theorem 5.2 and Proposition 5.3.

Section 6 is devoted to isospectral deformations of locally strictly geodesically convex billiard tables of dimension two. Firstly we obtain in Proposition 6.1 a C^1 family of BNFs for the billiard ball maps B_t in a neighborhood of $S_t^*\Gamma$ in terms of the interpolating Hamiltonian ζ_t introduced by Marvizi and Melrose [46]. Then Theorem 6.2 gives an affirmative answer of Problems I-III in the case of C^1 families (X, g_t) , $t \in [0, \delta]$, of locally strictly geodesically convex billiard tables of dimension two satisfying the weak isospectral condition $(H_1) - (H_2)$. Moreover, if (X_t, g) , $t \in [0, 1]$, is a C^1 family of billiard tables in \mathbb{R}^2 equipped with the Euclidean metric and satisfying the weak isospectral condition and if X_0 is strictly convex then we prove in that X_t remains strictly convex for each $t \in [0, 1]$ and we get an affirmative answer of Problem III for $t \in [0, 1]$ (see Theorem 6).

In Section 7 we reduce the problem (1.1) microlocally at the boundary. The main idea is explained in the beginning of Section 7. Let $f_t = f_t(\cdot, \lambda)$ be a $\frac{1}{2}$ -density on Γ depending on a large parameter λ and with a frequency support contained in a small neighborhood of the union of the invariant tori $\Lambda_t(\omega)$. We consider the corresponding outgoing solution u_t of the reduced wave equation (the Helmholtz equation) in X_t with initial data f_t and we “reflect it at the boundary” $m - 1$ times if $m \geq 2$. To this end we use the outgoing parametrix of the reduced wave equation which is a Fourier Integral Operator with a large parameter λ (λ -FIO). Taking the pull-back to Γ of the last branch of the solution $u_t(\cdot, \lambda)$ we get a $\frac{1}{2}$ -density $M_t^0(\lambda)f_t$, where $M_t^0(\lambda)$ is a λ -FIO of order zero at Γ the canonical relation of which is just the graph of P_t . We call $M_t^0(\lambda)$ a monodromy operator. In this way obtain that

$$(-\Delta_t + \lambda^2)u_t = O_N(|\lambda|^{-N})f_t, \quad \mathcal{B}_t u_t = O_N(|\lambda|^{-N})f_t$$

if and only if

$$M_t^0(\lambda)f_t = f_t + O_N(|\lambda|^{-N})f.$$

We are looking for couples (λ, f_t) solving the last equation. To this end using the BNF of P_t we obtain a suitable microlocal (quantum) Birkhoff normal form of $M_t^0(\lambda)$ for $t \in J$ (see Proposition 7.11 and Proposition 7.12 in Sect. 7.3). This enables us to “separate the variables” microlocally near the whole family of invariant tori $\Lambda_t(\omega)$, $\omega \in \Omega_\kappa^0$, and to obtain a microlocal spectral decomposition of $M_t^0(\lambda)$ in Proposition 7.15. Then the problem of finding λ is reduced to an algebraic equation $\mu_t(\lambda) = 1 + O_N(|\lambda|^{-N})$ where $\mu_t(\lambda)$ are suitable eigenvalues of $M_t^0(\lambda)$ with eigenfunctions f_t . In this way we obtain that λ should satisfy (7.125) and (7.126) and we solve that system of equations recursively. This is done in Section 8, where we obtain C^1

families of quasi-modes. Using these quasi-modes we prove item 3 of Theorem 1 which claims that the function $t \rightarrow \beta_t(\omega)$ is independent of t for Diophantine frequencies ω provided that the weak isospectral condition (H_1) - (H_1) is fulfilled. We are going to give the idea of the proof.

Sketch of the Proof.

We fix $t \in [0, \delta)$, $\kappa > 0$ and $\omega \in \Omega_\kappa^0$ and we impose the following

Strong Quantization Condition on the torus $\Lambda_t(\omega)$. There exists an infinite sequence $\widetilde{\mathcal{M}}(\omega)$ of $(q, \lambda) \in \mathbb{Z}^n \times [1, \infty)$ such that $q = (k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ and $\lambda = \mu_q^0 \geq 1$ satisfy

$$\left\{ \begin{array}{l} c_0^{-1}|q| \leq \mu_q^0 \leq c_0|q| \\ \lim_{|q| \rightarrow \infty} \left| \mu_q^0 \left(I_t(\omega), L_t(I_t(\omega)) \right) - \left(k + \frac{\vartheta_0}{4}, 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right) \right| = 0. \end{array} \right. \quad (1.11)$$

Here $c_0 > 1$ is a constant, $(\vartheta, \vartheta_0) \in \mathbb{Z}^n$, ϑ is related with the Maslov class of $\Lambda_t(\omega)$ and ϑ_0 is a Maslov index. It turns out that condition (1.11) is satisfied for each ω in a subset $\Xi_\kappa^t \subset \Omega_\kappa^0$ of full Lebesgue measure in Ω_κ^0 (see Lemma 8.1).

Now we fix $\omega \in \Xi_\kappa^t$ and denote by $\mathcal{M} \subset \mathbb{Z}^n$ the image of the projection of $\widetilde{\mathcal{M}}(\omega) \subset \mathbb{Z}^n \times [1, \infty)$ on the first factor. The set \mathcal{M} will be the index set of the C^1 family of quasi-modes that we are going to construct. To obtain a quantization condition for the tori $\Lambda_s(\omega)$ for s close to t we introduce for any $q \in \mathcal{M}$ the interval

$$J_q := [t, t + 2|q|^{-1}].$$

Since the maps $s \mapsto L_s$ and $s \mapsto I_s$ are C^1 in a neighborhood of t with values in the corresponding C^∞ spaces (see Theorem 3.2 and Definition 3.3), the following quantization condition of the tori $\Lambda_s(\omega)$ is satisfied

Quantization Condition. There exists a constant $C = C(\omega) > 0$ independent of $q \in \mathcal{M}$ and $s \in J_q$ such that

$$\left| \mu_q^0 \left(I_s(\omega), L_s(I_s(\omega)) \right) - \left(k + \frac{\vartheta_0}{4}, 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right) \right| \leq C \quad \forall q \in \mathcal{M}, s \in J_q. \quad (1.12)$$

Using this condition we construct C^1 families quasi-modes $(\mu_q(s)^2, u_{s,q})$, $q \in \mathcal{M}$, $s \in J_q$, of order M for the problem (1.1) such that (see Theorem 8.2)

- (1) $u_{s,q} \in D(\Delta_s)$, $\|u_{s,q}\|_{L^2(X)} = 1$, and there exists a constant $C_M > 0$ such that

$$\left\{ \begin{array}{l} \|\Delta u_{s,q} - \mu_q^2(s) u_{s,q}\| \leq C_M \mu_q^{-M}(s) \quad \text{in } L^2(X), \\ \mathcal{B} u_{s,q}|_\Gamma = 0 \end{array} \right.$$

for every $q \in \mathcal{M}$ and $s \in J_q$;

(2) We have

$$\mu_q(s) = \mu_q^0 + c_{q,0}(s) + c_{q,1}(s) \frac{1}{\mu_q^0} + \cdots + c_{q,M}(s) \frac{1}{(\mu_q^0)^M}$$

where the functions $s \mapsto c_{q,j}(s)$ are real valued and C^1 on the interval J_q and there exists a constant $C'_M > 0$ such that $|c_{q,j}(s)| \leq C'_M$ for every $q \in \mathcal{M}$, $0 \leq j \leq M$, and any $s \in J_q$;

(3) There exists $C > 0$ such that

$$\left| \mu_q(s) L_s \left(\frac{k + \vartheta_0/4}{\mu_q(s)} \right) - 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right| \leq \frac{C}{\mu_q(s)}$$

for every $q \in \mathcal{M}$ and $s \in J_q$;

(4) We have

$$\frac{k + \vartheta_0/4}{\mu_q(t)} = I_t(\omega) + o\left(\frac{1}{|q|}\right) \quad \text{as } |q| \rightarrow \infty.$$

We point out that the strong quantization condition (1.11) is needed only in the proof of item (4). Items (1)-(3) follow from the weaker quantization condition (1.12). The estimate in (3) is related to the nullity of the subprincipal symbol of the Laplace-Beltrami operator. Note also that the quasi-eigenvalues $\mu_q(s)^2$ are defined only in the intervals J_q which shrink to t as $q \rightarrow \infty$.

Consider now the self-adjoint operator Δ_s with Dirichlet, Neumann or Robin boundary conditions and the corresponding spectral problem (1.1). The relation between the spectrum of Δ_s and the quasi-eigenvalues $\mu_q(s)^2$ is given by

$$\text{dist}\left(\text{Spec}(\Delta_s), \mu_q(s)^2\right) \leq C_M \mu_q^{-M}(s)$$

where C_M is the constant in (1) and M is the order of the quasi-mode. We fix $M > 2d + 2$ where $d \geq 0$ is the exponent in (H₁). It follows then from (H₂) that the quasi-eigenvalues $\mu_q(s)^2$, $|q| \geq q_0 \gg 1$, $s \in J_q$, belong to the union of intervals

$$A_k := \left[a_k - \frac{c}{2} a_k^{-d-1}, b_k + \frac{c}{2} a_k^{-d-1} \right] \quad k \geq k_0 \gg 1$$

where c is the positive constant in the third hypothesis of (H₁). These intervals do not intersect each other for $k \geq k_0 \gg 1$ in view of the third hypothesis of (H₁). The function $J_q \ni s \mapsto \mu_q(s)^2$ being continuous in J_q can not jump from one interval to another, hence, it is trapped in a certain interval A_k . Then using the first and second hypothesis of (H₁) we obtain

$$|\mu_q(s) - \mu_q(t)| \leq \frac{1}{\mu_q(t)} |\mu_q(s)^2 - \mu_q(t)^2| \leq 2a_k^{-\frac{1}{2}} (b_k - a_k) = o(a_k^{\frac{1}{2}}) = o(\mu_q(t)) = o\left(\frac{1}{|q|}\right)$$

for $|q| \geq q_0$ where $q_0 \gg 1$ does not depend on the choice of $s \in J_q$. This implies

$$\mu_q(s) = \mu_q(t) \left(1 + o\left(\frac{1}{|q|}\right) \right) \quad \text{as } q \rightarrow \infty$$

uniformly with respect to $s \in J_q$. A detailed proof of this statement is given in Lemma 8.3. Now using (4) we get

$$\zeta_q(s) := \frac{k + \vartheta_0/4}{\mu_q(s)} = \frac{k + \vartheta_0/4}{\mu_q(t)(1 + o(1/|q|))} = \frac{k + \vartheta_0/4}{\mu_q(t)} + o\left(\frac{1}{|q|}\right) = I_t(\omega) + o\left(\frac{1}{|q|}\right), \quad q \rightarrow \infty$$

uniformly with respect to $s \in J_q$. In the same way we get from (3)

$$L_s(\zeta_q(s)) = 2\pi \frac{k_n - \vartheta/4}{\mu_q(s)} + O(|q|^{-2}) = 2\pi \frac{k_n - \vartheta/4}{\mu_q(t)} + o\left(\frac{1}{|q|}\right) = L_t(\zeta_q(t)) + o\left(\frac{1}{|q|}\right), \quad q \rightarrow \infty$$

uniformly with respect to $s \in J_q$. Setting $\eta := 1/|q|$ we obtain from the above equalities that

$$\begin{aligned} L_{t+\eta}(I_t(\omega)) &= L_{t+\eta}(\zeta_q(t+\eta) + o(\eta)) = L_{t+\eta}(\zeta_q(t+\eta)) + o(\eta) \\ &= L_t(\zeta_q(t)) + o(\eta) = L_t(I_t(\omega)) + o(\eta), \quad \eta = 1/|q| \rightarrow 0. \end{aligned}$$

Recall that the map $[0, \delta] \rightarrow L_s(\cdot)$ is C^1 with values in the corresponding C^∞ space. Hence,

$$\dot{L}_t(I_t(\omega)) = \frac{d}{ds} L_s(I_t(\omega)) \Big|_{s=t} = 0 \quad \forall \omega \in \Xi_\kappa^t.$$

On the other hand, Ξ_κ^t is dense in Ω_κ^0 since any point of Ω_κ^0 is of positive Lebesgue density and $\Omega_\kappa^0 \setminus \Xi_\kappa^t$ has measure zero, and by continuity (the function $I \rightarrow \dot{L}_t(I)$ is smooth) we get this equality for each $\omega \in \Omega_\kappa^0$. The point t has been fixed arbitrary in $[0, \delta)$, hence, $\dot{L}_t(I_t(\omega)) = 0$ for every $t \in [0, \delta)$ and $\omega \in \Omega_\kappa^0$. Now differentiating the first equation of (1.9) with respect to t we obtain

$$\dot{\beta}_t(\omega) = \langle \omega, \dot{I}_t(\omega) \rangle - \dot{L}_t(I_t(\omega)) - \langle \nabla L_t(I_t(\omega)), \dot{I}_t(\omega) \rangle = 0 \quad \forall \omega \in \Omega_\kappa^0$$

since $\nabla L_t(I_t(\omega)) = \omega$. Hence, $\beta_t(\omega) = \beta_0(\omega)$ for every $t \in [0, \delta)$ and $\omega \in \Omega_\kappa^0$. By continuity we get the last equality for every $t \in [0, \delta]$ as well. \square

We point out that the classical and quantum BNFs are analytic in t if the perturbation is analytic in t which leads to analytic in t quasi-modes. This can be used as in [15, 16] to extend the results of S. Gomes and A. Hassell about the quantum non-ergodicity of C^∞ -smooth KAM systems. Moreover, using Theorem 11.22 and the pseudodifferential calculus of operators with symbols of finite smoothness [63], one may extend them to KAM systems of finite smoothness.

The second part of the manuscript is devoted to the KAM theorem and the BNF around families of invariant tori in both the continuous and discrete cases. The main novelty in it can be briefly summarized as follows

- the constant ϵ in the smallness condition essentially depends only on the dimension of the configuration space and on the exponent in the Diophantine condition;
- C^k smooth (analytic) families of invariant tori $t \rightarrow \Lambda_t(\omega)$ with Diophantine frequencies are obtained;
- C^k smooth (analytic) with respect to the parameter t BNF is obtained around the union of $\Lambda_t(\omega)$;
- global estimates in the whole scale of Hölder norms with universal constants are obtained. To this end a new iterative schema is proposed. The Modified Iterative Lemma proven in Sect. 11.9 provides in a limit smooth functions in the whole domain Ω (not only smooth Whitney jets on the Cantor set Ω_κ) with a good control of the Hölder norms.

We need all these properties un the first part of the manuscript. The KAM theorems here are based on Theorem 10.1 which is a KAM theorem for C^k ($k=0; 1$) or analytic families of C^∞ smooth Hamiltonians H_t in $\mathbb{T}^n \times D$ with parameters $\omega \in \Omega$ where H_t are small perturbations of the normal form $\mathcal{N}(I; \omega) := \langle \omega, I \rangle$. The proof of the theorem, especially of the so called KAM step follows that of J. Pöschel [54] in the case of analytic Hamiltonians but it requires additional work in order to adapt it to the case of C^k families of Hamiltonians $[0, 1] \ni t \rightarrow C^\infty(\mathbb{T}^n \times D; \Omega)$. For this reason we give a complete proof of the KAM step. Next we adapt the Iterative Lemma in [54] to the case of smooth Hamiltonians. In this way one obtains an iteration schema which gives in a limit C^∞ Whitney jets on Ω_κ . The Whitney extension theorem in the C^∞ case does not provide in general global estimates of the Hölder norms in Ω without loss of derivatives [14]. For this reason we provide another iteration scheme based on a Modified Iteration Lemma given by Proposition 11.13, which involves Gevrey almost analytic extensions of certain cut-off Gevrey functions. The $\bar{\delta}$ derivatives of such functions are exponentially small near the reals and one can use Cauchy (Green's) formula. This allows one to obtain a convergent iteration schema on the the whole space of frequencies and to obtain the desired global (in Ω) Hölder estimates of any order.

Using Theorem 10.1 we obtain a KAM theorem for C^1 families of Hamiltonians H_t which are perturbations of a given C^1 family of completely integrable nondegenerate in Kolmogorov sense Hamiltonians H_t^0 . The family H_t^0 is given as follows. We consider a C^1 family of nondegenerate real valued functions $K_t \in C^\infty(\Omega)$, $t \in [0, \delta]$, in a domain $\Omega \subset \mathbb{R}^n$, where by nondegenerate we mean that the gradient map

$$\Omega \ni \omega \mapsto \nabla K_t(\omega) \in D_t := \nabla K_t(D_t)$$

is a diffeomorphism for each $t \in [0, \delta]$. We denote by $H_t^0 \in C^\infty(D_t)$ the Legendre transform K_t^* of K_t given by

$$H_t^0(I) = K_t^*(I) := \text{Crit.val.}_{\omega \in \Omega} \{ \langle \omega, I \rangle - K_t(\omega) \}.$$

Then $H_t^0 \in C^\infty(D_t)$ is nondegenerate and $(H_t^0)^* = K_t$. Theorem 9.1 provides a result of KAM type for C^k families of Hamiltonians $[0, \delta] \ni t \mapsto H_t \in C^\infty(\mathbb{T}^n \times D_t)$ which is a small perturbation of the family H_t^0 . The constant ϵ in the smallness condition essentially depends only on the dimension n and on the exponent $\tau > n - 1$ in the Diophantine condition (1.3). To this end, given $\omega \in \Omega$ we linearize H_t^0 at $I = \nabla H_t^{0*}(\omega)$ applying Taylor's formula and sending the nonlinear part of it to the functions with perturbation. The smallness condition and the estimates in Theorem 9.1 are given in terms of suitable weighted Hölder norms. In order to obtain estimates in Hölder norms with universal constants of a composition of functions with H_t^{0*} , we suppose that Ω is a strictly convex bounded domain in \mathbb{R}^n and that $t \rightarrow H_t^{0*}$ is a C^k family with values in $C^\infty(\bar{\Omega}, \mathbb{R})$, $\bar{\Omega}$ being the closure of Ω . The idea is to use the interpolation inequalities for Hölder norms in \mathbb{R}^n or in $\mathbb{T}^n \times \mathbb{R}^n$ which simplify a lot the estimates of higher order Hölder norms for the inverse function and for the composition of functions. The problem about the composition of functions in Hölder spaces is quite delicate. It has been investigated recently by R. de la Llave and R. Obaya [42]. We can not use directly their results here since we need estimates with universal constants. These estimates are obtained in Appendix A.4.

Theorem 9.5 is a counterpart of Theorem 10.1 in the discrete case for C^1 families of exact symplectic maps P_t . Theorem 9.5 is obtained from Theorem 10.1 using an idea of R. Douady [11]. The BNF of C^k families of exact symplectic maps at C^k families of Kronecker tori is obtained in Section 9.4. We point out that the constant ϵ in the corresponding smallness conditions (9.182)

and (9.188) essentially depend only on the dimension and on the exponent τ . Moreover, the constants C_m in the corresponding Hölder estimates (9.183), (9.189), (9.190) and in Theorem 11.22 are universal. This makes these results especially useful in the case when the symplectic maps P_t have singularities. They can be applied for example for the billiard ball map B_t near the singular set $S_t^*\Gamma$ in the case when (X, g_t) is locally strictly geodesically convex and $\dim X = 2$ (see Theorem 6.2).

Part I

Isospectral invariants and rigidity

2 Main Results

Before formulating the main results we recall from Birkhoff [2] (see also [72]) the definition of the billiard ball map B associated to a billiard table (X, g) with a smooth boundary Γ .

2.1 Billiard ball map

Let (X, g) be a smooth billiard table with boundary Γ . The “broken geodesic flow” given by the elastic reflection of geodesics hitting transversely the boundary induces a discrete dynamical system at the boundary which can be described as follows.

Denote by h the Hamiltonian on T^*X corresponding to the Riemannian metric g on X via the Legendre transformation and by h^0 the Hamiltonian on $T^*\Gamma$ corresponding to the induced Riemannian metric on Γ . The billiard ball map B lives in an open subset of the open coball bundle $\mathbf{B}^*\Gamma = \{(x, \xi) \in T^*\Gamma : h^0(x, \xi) < 1\}$. It is defined as follows. Denote by $S^*X := \{(x, \xi) \in T^*X : h(x, \xi) = 1\}$ the cosphere bundle, and set

$$\Sigma = S^*X|_{\Gamma} := \{(x, \xi) \in S^*X : x \in \Gamma\} \text{ and } \Sigma^{\pm} := \{(x, \xi) \in \Sigma : \pm\langle \xi, \nu(x) \rangle > 0\}$$

where $\nu(x) \in T_xX$, $x \in \Gamma$, is the outward unit normal to Γ with respect to the metric g . Let $\overline{\mathbf{B}^*\Gamma}$ be closed coball bundle, i.e. the closure of $\mathbf{B}^*\Gamma$ in $T^*\Gamma$. Consider the natural projection $\pi_{\Sigma} : \Sigma \rightarrow \overline{\mathbf{B}^*\Gamma}$ assigning to each $(x, \eta) \in \Sigma$ the covector $(x, \eta|_{T_x\Gamma})$. Its restriction to $\Sigma^+ \cup \Sigma^-$ admits two smooth inverses

$$\pi_{\Sigma}^{\pm} : \mathbf{B}^*\Gamma \rightarrow \Sigma^{\pm}, \quad \pi_{\Sigma}^{\pm}(x, \xi) = (x, \xi^{\pm}). \quad (2.13)$$

The maps π_{Σ}^{\pm} can be extended continuously on the closed coball bundle $\overline{\mathbf{B}^*\Gamma}$. Given $\varrho^{\pm} \in \Sigma^{\pm}$ we consider the integral curve $\exp(sX_h)(\varrho)$ of the Hamiltonian vector field X_h starting at ϱ . If it intersects Σ transversely at a time $T = T(\varrho)$ and lies entirely in the interior of S^*X for t between 0 and T we set

$$J(\varrho^{\pm}) = \exp(TX_h)(\varrho^{\pm}) \in \Sigma^{\mp}.$$

Notice that J is a smooth involution defined in an open dense subset of Σ . In this way we obtain a smooth exact symplectic map $B : \widetilde{\mathbf{B}^*\Gamma} \rightarrow \mathbf{B}^*\Gamma$, given by $B = \pi_{\Sigma} \circ J \circ \pi_{\Sigma}^+$, where $\widetilde{\mathbf{B}^*\Gamma}$ is an open dense subset of $\mathbf{B}^*\Gamma$. The map J can be extended to a smooth involution of Σ in the case when X is a strictly convex billiard table in \mathbb{R}^n . In this case the billiard ball map is well defined and smooth in $\mathbf{B}^*\Gamma$ and can be extended by continuity as the identity map on its boundary $S^*\Gamma$. This case will be considered in more details in Sect. 6.

Suppose now that $t \rightarrow g_t$ is a C^1 family of Riemannian metrics on X . For any t we denote the corresponding cosphere bundle by $S_t^*X := \{h_t = 1\}$ and the corresponding open coball bundle of Γ by $\mathbf{B}_t^*\Gamma := \{h_t^0 < 1\}$. Let $\pi_t : \Sigma_t \rightarrow \overline{B_t^*\Gamma}$ be the natural projection and $\pi_t^\pm : \mathbf{B}_t^*\Gamma \rightarrow \Sigma_t^\pm$, $\pi_{\Sigma_t}^\pm(x, \xi) = (x, \xi_t^\pm)$ its inverses. Denote by J_t the corresponding involution in Σ_t and consider the billiard ball map $B_t : \tilde{\mathbf{B}}_t^*\Gamma \rightarrow \mathbf{B}_t^*\Gamma$. If $(x, \xi) \in \tilde{\mathbf{B}}_{t_0}^*\Gamma$, then $(x, \xi) \in \tilde{\mathbf{B}}_t^*\Gamma$ for any t in a neighborhood of t_0 because of the transversality and one can show that the map $t \rightarrow B_t \in C^\infty(\tilde{\mathbf{B}}_t^*\Gamma, \mathbf{B}_t^*\Gamma)$ is C^1 . In this way we obtain a C^1 family of symplectic mappings $t \rightarrow (B_t : \tilde{\mathbf{B}}_t^*\Gamma \rightarrow \mathbf{B}_t^*\Gamma)$.

2.2 Main Results

Recall that P_0 admits a C^∞ Birkhoff normal form at any Kronecker invariant torus $\Lambda_0(\omega)$ with a Diophantine frequency ω (cf [43], Proposition 9.13). The Birkhoff normal form of P_0 at $\Lambda_0(\omega)$ is said to be nondegenerate if the quadratic part of it is a nondegenerate quadratic form. The non-degeneracy of the Birkhoff normal form enables one to apply the KAM theorem. Recall that $D^0(\kappa, \tau)$ is the set of points of positive Lebesgue measure in $D(\kappa, \tau)$, defined in the Introduction.

Theorem 1. *Let (X, g_t) , $t \in [0, 1]$, be a C^1 family of billiard tables. Let $\Lambda_0(\omega_0) \subset \mathbf{B}_0^*\Gamma$ be a Kronecker invariant torus of $P_0 := B_0^m$ of a Diophantine frequency $\omega_0 \in D^0(\kappa_0, \tau)$. Suppose that the Birkhoff normal form of P_0 at $\Lambda_0(\omega_0)$ is nondegenerate. Then there exists $\delta_0 = \delta_0(\kappa_0) > 0$ such that the following holds.*

1. *There exists a C^1 -family of Kronecker invariant tori*

$$[0, \delta_0] \ni t \rightarrow \Lambda_t(\omega_0) \subset \mathbf{B}_t^*\Gamma$$

of $P_t := B_t^m$ of a frequency ω_0 .

2. *For any $0 < \delta < \delta_0$ there exists a set $\Xi \subset \mathbb{T}^{n-1}$ of Diophantine frequencies such that $\omega_0 \in \Xi$,*

$$\frac{\text{meas}(B(\omega_0, \varepsilon) \cap \Xi)}{\text{meas}(B(\omega_0, \varepsilon))} = 1 - O_\delta(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0,$$

and for any $\omega \in \Xi$ there exists a C^1 -family of Kronecker invariant tori

$$[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \subset \mathbf{B}_t^*\Gamma$$

of $P_t := B_t^m$ of a frequency ω .

3. *If the billiard tables satisfy the weak isospectral condition $(H_1) - (H_2)$ then $\beta_t(\omega)$, $I_t(\omega)$ and $L_t(I_t(\omega))$ are independent of $t \in [0, \delta]$ for any $\omega \in \Xi$.*

We are going to apply this result for C^1 deformations of the boundary keeping the Riemannian metric fixed. Let X be a smooth compact manifold of dimension $n \geq 2$ with non-empty boundary $\Gamma := \partial X$ which is smoothly embedded into a Riemannian manifold (\tilde{X}, g) of the same dimension and without boundary. We say that (X_t, g) , $t \in [0, \varepsilon]$, is a C^1 variation of (X, g) if (X_t, g) is a billiard table in (\tilde{X}, g) with boundary $\Gamma_t = \partial X_t$, $X_0 = X$, and if there exists a C^1 family of embeddings

$$[0, \varepsilon] \ni t \rightarrow \psi_t \in C^\infty(\Gamma, \tilde{X}) \tag{2.14}$$

such that $\psi_t(\Gamma) = \Gamma_t$. In this case we say that (X_t, g) is a C^1 family of billiard tables. Then there exists a C^1 family of diffeomorphisms onto their images $[0, \varepsilon] \ni t \rightarrow \Psi_t \in C^\infty(X, \tilde{X})$ such that $\Psi_t(X) = X_t$, $\Psi_t|_\Gamma = \psi_t$, and Ψ_t is identity outside an open neighborhood of Γ in X . The family Ψ_t can be constructed parameterizing a neighborhood of Γ in \tilde{X} by the exponential map $(x, s) \mapsto \exp_x(s\nu(x))$ corresponding to the Riemannian metric $g = g_0$ in \tilde{X} , where $\nu(x)$ is the outward unit normal to Γ . In particular, we get a family of billiard tables (X_t, g) , $t \in [0, \varepsilon]$, which are isometric to (X, g_t) , $g_t := \Psi_t^*g$. We say that a family of Kronecker invariant tori $[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \subset \mathbf{B}^*\Gamma_t$ is C^1 -smooth if the family $[0, \delta] \ni t \rightarrow \psi_t^*(\Lambda_t(\omega)) \subset \mathbf{B}^*\Gamma$ is C^1 . Consider the corresponding Laplace-Beltrami operator Δ_t in (X_t, g) with Dirichlet, Neumann or Robin boundary conditions on Γ_t . As a corollary of the main theorem we obtain

Theorem 2. *Let (X_t, g) , $t \in [0, 1]$, be a C^1 family of billiard tables. Let $\Lambda_0(\omega_0) \subset \mathbf{B}^*\Gamma$ be a Kronecker invariant torus of $P_0 := B_0^m$ of a Diophantine frequency $\omega_0 \in D^0(\kappa_0, \tau)$. Suppose that the Birkhoff normal form of P_0 at $\Lambda_0(\omega_0)$ is non-degenerate. Then there exists $\delta_0 = \delta_0(\kappa_0) > 0$ such that the following holds.*

1. *There exists a C^1 -family of Kronecker invariant tori*

$$[0, \delta_0] \ni t \rightarrow \Lambda_t(\omega_0) \subset \mathbf{B}^*\Gamma_t$$

of $P_t := B_t^m$ of a frequency ω_0 .

2. *For any $0 < \delta < \delta_0$ there exists a set $\Xi \subset \mathbb{T}^{n-1}$ of Diophantine frequencies such that $\omega_0 \in \Xi$,*

$$\frac{\text{meas}(B(\omega_0, \varepsilon) \cap \Xi)}{\text{meas}(B(\omega_0, \varepsilon))} = 1 - O_\delta(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0,$$

and for any $\omega \in \Xi$ there exists a C^1 -family of Kronecker invariant tori

$$[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \subset \mathbf{B}^*\Gamma_t$$

of $P_t := B_t^m$ of a frequency ω .

3. *If the billiard tables satisfy the weak isospectral condition $(H_1) - (H_2)$ then $\beta_t(\omega)$, $I_t(\omega)$ and $L_t(I_t(\omega))$ are independent of $t \in [0, \delta]$ for any $\omega \in \Xi$.*

We shall denote by $\delta_\nu \Gamma_t : \Gamma_t \rightarrow \mathbb{R}$ the vertical component of the variation Γ_s of Γ_t which is defined by

$$\forall x \in \Gamma_t, \quad \delta_\nu \Gamma_t(x) := \left\langle \dot{\psi}_t(\psi_t^{-1}(x)), \nu_t(x) \right\rangle = \left\langle \frac{d\psi_s}{ds} \Big|_{s=t}(\psi_t^{-1}(x)), \nu_t(x) \right\rangle, \quad (2.15)$$

where $\nu_t(x)$ is the outward unit normal to Γ_t at x with respect to the metric g .

Let $\pi_t : T^*\Gamma_t \rightarrow \Gamma_t$ be the natural projection. Given $\zeta = (x, \xi) \in B^*\Gamma_t$, we denote by $\xi_t^+(\zeta) \in T_x^*X$ the corresponding outgoing unit covector which means that the restriction of the covector ξ_t^+ to $T_x\Gamma_t$ equals ξ , $h_t(\xi_t^+) = 1$, and $\langle \xi_t^+(\zeta), \nu_t(\pi_t(\zeta)) \rangle_x \geq 0$, where $\langle \cdot, \cdot \rangle_x$ stands for the pairing between covectors in T_x^*X with vectors in T_xX (see Sect. 2.1). In this case the second part of Corollary 2 can be stated as follows.

Theorem 3. Let (X_t, g) , $t \in [0, \varepsilon]$, be a C^1 -family of billiard tables satisfying the isospectral condition $(H_1) - (H_2)$. Let $[0, \delta) \ni t \rightarrow \Lambda_t(\omega)$, $0 < \delta \leq \varepsilon$, be a C^1 family of invariant tori of $P_t = B_t^m$ with a Diophantine vector of rotation ω . Then

$$\sum_{j=0}^{m-1} \int_{\Lambda_t(\omega)} \left(\left\langle \xi_t^+(\zeta), \nu_t(\pi_t(\zeta)) \right\rangle (\delta_\nu \Gamma_t)(\pi_t(\zeta)) \right) \Big|_{\zeta=B_t^j(\rho)} d\mu_t(\rho) = 0 \quad (2.16)$$

for any $t \in [0, \delta)$.

There are three particular cases we will focus our attention at, namely, C^1 deformations of nondegenerate Liouville billiard tables, deformations of a manifold having non-degenerate elliptic periodic geodesics and deformations of strictly convex planar domains.

We shall prove a spectral rigidity result for analytic billiard tables of dimension two having the symmetries of the ellipse provided that one of the bouncing ball rays of the initial billiard table is elliptic, 4-elementary, and has a nondegenerate BNF. Define a class of billiard tables as follows. Let (\tilde{X}, g) , $\dim \tilde{X} = 2$ be a Riemannian manifold of dimension two. Suppose that it admits two commuting involutions \mathcal{J}_k , $k = 1, 2$, acting as isometries. Consider the family \mathcal{B} of billiard tables (X, g) in (\tilde{X}, g) such that the boundary $\Gamma = \partial X$ of X is connected and invariant with respect to \mathcal{J}_k , $k = 1, 2$. Then the set of fixed points of \mathcal{J}_k , $k = 1, 2$, in X defines a bouncing ball geodesic γ_k of any $(X, g) \in \mathcal{B}$. Denote by \mathcal{B}_{an} the set of analytic billiard tables which belong to \mathcal{B} .

Theorem 4. Let $(X, g) \in \mathcal{B}$. Assume that the broken geodesic γ_1 given by the set of fixed points of \mathcal{J}_1 in X is elliptic 4-elementary and that the corresponding Poincaré map admits a non-degenerate BNF. Suppose that $(X_t, g) \in \mathcal{B}$, $t \in [0, 1]$, is a C^1 deformation of (X, g) satisfying the weak isospectral condition $(H_1) - (H_2)$. Then γ_1 is a bouncing ball geodesic of (X_t, g) for each $t \in [0, 1]$ and Γ_t has a contact of infinite order to Γ_0 at the vertexes of γ_1 . In particular, $X_1 = X_0$ if the boundaries $\Gamma_1 = \partial X_1$ and $\Gamma_2 = \partial X_2$ are both analytic.

It turns out (see Corollary B.6) that the Poincaré map associated with the elliptic bouncing ball geodesic γ_1 is always non-degenerate (twisted) for Liouville Billiard Tables in surfaces of constant curvature. Let us fix the foci $F_1 \neq F_2$ and consider the corresponding confocal ellipses. Then, except of five confocal families of ellipses given explicitly by (A.73), the geodesic γ_1 is 4-elementary. Denote by \mathcal{E} the set of ellipses which do not belong to these families. A billiard table in \mathbb{R}^2 is said to be elliptical if its boundary is an ellipse.

Theorem 5. We have the following.

1. Each elliptical billiard table with boundary in \mathcal{E} is spectrally rigid in the class \mathcal{B}_{an} under the weak isospectral condition $(H_1) - (H_2)$;
2. Each billiard table X in \mathcal{B}_{an} with boundary sufficiently close to an ellipse $\Gamma \in \mathcal{E}$ in the C^5 topology is spectrally rigid in the class \mathcal{B}_{an} under the weak isospectral condition $(H_1) - (H_2)$.

Theorems 4 and 5 are proved in Section 5.

Remark 2.1. To obtain the preceding results we use only the leading term of the quasi-mode. Replacing the second condition of (H_1) with a stronger one

$$\lim a^{s/2}(b_k - a_k) = 0 \quad \text{as } k \rightarrow +\infty,$$

where $s \in \mathbb{N}$ is fixed, and using an analogue of Lemma 2.5 [63], one can obtain further isospectral invariants, which could be used to remove at least one of the symmetries.

Consider a strictly convex billiard table X in \mathbb{R}^2 . Lazutkin has proved that any fixed $\tau > 1$ and any $0 < \kappa < \kappa_0 \ll 1$ there is a subset $\Omega_\kappa \subset D(\kappa, \tau)$ of positive Lebesgue measure consisting (κ, τ) -Diophantine frequencies such that for any $\omega \in \Omega_\kappa$ there is a Kronecker invariant circle $\Lambda(\omega)$ of the billiard ball map B of a frequency ω (see [43] and the references there). Moreover, the corresponding caustic $\mathcal{C}(\omega)$ - the envelope of the rays issuing from $\Lambda(\omega)$ - is a closed smooth convex curve lying in the interior of X . As κ tends to 0 the invariant curves accumulate at the boundary $S^*(\Gamma)$ of $\mathbf{B}^*(\Gamma)$.

Theorem 6. *Let $X_t \subset \mathbb{R}^2$, $t \in [0, 1]$, be a C^1 family of compact billiard tables satisfying the weak isospectral condition $(H_1) - (H_2)$. Suppose that X_0 is strictly convex. Then*

1. X_t is strictly convex for each $t \in [0, 1]$
2. There is a Cantor set $\Xi \subset (0, 1]$ consisting of Diophantine numbers such that

$$\text{meas}(\Xi \cap (0, \varepsilon)) = \varepsilon(1 - O(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0^+$$

and such that $\forall \omega \in \Xi$ there is a C^1 family of Kronecker invariant circles $[0, 1] \ni t \rightarrow \Lambda_t(\omega)$ of B_t of frequency ω ,

3. $\forall \omega \in \Xi$ and $t \in [0, 1]$, $\beta_t(\omega) = \beta_0(\omega)$, $L_t(I_0(\omega)) = L_0(I_0(\omega))$, and $I_t(\omega) = I_0(\omega)$.

2.3 Proof of Theorem 3

Theorem 3 follows from Theorem 2 and the following statement.

Proposition 2.2. *Let $[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \subset \mathbf{B}^*\Gamma_t$ be a C^1 family of invariant tori of $P_t = B_t^m$ with a frequency $\omega \in \Omega_\kappa^0$. Then*

$$\frac{d}{dt}\beta_t(\omega) = -\frac{2}{\mu_t(\Lambda_t(\omega))} \sum_{j=0}^{m-1} \int_{\Lambda_t(\omega)} (\delta_\nu \Gamma_t)(\pi_t(\zeta)) \left\langle \xi_t^+(\zeta), \nu_t(\pi_t(\zeta)) \right\rangle \Big|_{\zeta=B_t^j(\rho)} d\mu_t(\rho) \quad (2.17)$$

for each $t \in [0, \delta]$ and $\omega \in \Omega_\kappa^0$.

We are going to use the following

Lemma 2.3.. *Consider a C^1 family of curves $c : (-\epsilon, \epsilon) \times [0, l] \rightarrow X$, $c_s(\cdot) \equiv c(s, \cdot) : [0, l] \rightarrow X$, such that $c(\theta) := c_0(\theta)$ is a geodesic of the metric g and denote by $l(s) := l_g(c_s)$ the length of the curve $c_s(\cdot) : [0, l] \rightarrow X$ with respect to the metric g . Then*

$$\frac{dl}{ds}(0) = -g \left(\frac{\dot{c}_0(0)}{\|\dot{c}_0(0)\|_g}, \frac{\partial c}{\partial s}(0, 0) \right) + g \left(\frac{\dot{c}_0(l)}{\|\dot{c}_0(l)\|_g}, \frac{\partial c}{\partial s}(0, l) \right),$$

where $\dot{c}_0(\theta) := \frac{dc}{d\theta}(0, \theta)$.

Proof. The Lemma follows from a straightforward differentiation of the length function $l(s)$ and the Euler-Lagrange equation. \square

Proof of Proposition 2.3. Using the notations introduced just before Theorem 2 we set $p(\theta, t) := f_t(\theta)$, $\theta \in \mathbb{T}^{n-1}$, where $f_t \equiv f_{t,\omega} : \mathbb{T}^{n-1} \rightarrow \mathbf{B}^*\Gamma_t$ is the embedding of the Kronecker torus $\Lambda_t(\omega) \subset \mathbf{B}^*\Gamma_t$. Consider the function $x : \mathbb{T}^{n-1} \times [0, \delta] \rightarrow \Gamma_t$ given by

$$x(\theta, t) := \pi_t(p(\theta, t)) = \psi(\pi_0(\tilde{f}_t(\theta)), t). \quad (2.18)$$

where $\tilde{f}_t(\theta) := \psi_t^*(f_t(\theta))$. Clearly, $x \in C^1(\mathbb{T}^{n-1} \times [0, \delta], \tilde{X})$ and $x(\theta, 0) = \pi_0(f_0(\theta))$ for any $\theta \in \mathbb{T}^{n-1}$. Suppose first that $m = 1$, that means that $P_t = B_t$. By (1.5) we have

$$\beta_s(\omega) = -\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} l(x(\theta, s), x(R_{2\pi\omega}(\theta), s)) d\theta$$

where $l(x, x')$ is the corresponding length function which is well defined and smooth in a neighborhood of the projection of $\Lambda_t(\omega) \times \Lambda_t(\omega)$ in $\tilde{X} \times \tilde{X}$.

First we prove (2.17) for $t = 0$. Setting $\tilde{\theta} := R_{2\pi\omega}(\theta)$ and using Lemma 2.3 we get

$$\begin{aligned} -\frac{d}{ds}\Big|_{s=0} \beta_s(\omega) &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \left[\frac{d}{ds} l(x(\theta, s), x(R_{2\pi\omega}(\theta), s)) \right] \Big|_{s=0} d\theta \\ &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \left[\left\langle \pi_{\Sigma_s}^+(p(\tilde{\theta}, s)), \frac{\partial x}{\partial s}(\tilde{\theta}, s) \right\rangle - \left\langle \pi_{\Sigma_s}^-(p(\theta, s)), \frac{\partial x}{\partial s}(\theta, s) \right\rangle \right] \Big|_{s=0} d\theta \\ &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \left\langle (\pi_{\Sigma}^+ - \pi_{\Sigma}^-)(p(\theta, 0)), \frac{\partial x}{\partial t}(\theta, 0) \right\rangle d\theta \\ &= \frac{2}{(2\pi)^{n-1}} \int_{\mathbb{T}^{n-1}} \langle \pi_{\Sigma}^+(f_0(\theta)), \nu(x(\theta, 0)) \rangle g \left(\frac{\partial \psi}{\partial t}(\pi_0(f_0(\theta)), 0), \nu(\pi_0(f_0(\theta))) \right) d\theta. \end{aligned}$$

where $\pi_{\Sigma_s}^{\pm} : \mathbf{B}^*\Gamma_s \rightarrow \Sigma_s^{\pm}$ is the map defined in (2.13). In the variables $(x, \xi) = f_0(\theta)$, we get

$$\frac{d}{ds}\Big|_{s=0} \beta_s(\omega) = -\frac{2}{\mu_0(\Lambda_0(\omega))} \int_{\Lambda_0(\omega)} \langle \xi^+(\xi), \nu(\pi_0(\xi)) \rangle (\delta_{\nu}\Gamma)(\pi_0(\xi)) d\mu_0,$$

where $\delta_{\nu}\Gamma(x) = \langle \frac{\partial \psi}{\partial t}(x, 0), \nu(x) \rangle$. The same argument holds for $m \geq 2$. Finally, to prove (2.17) for any $t \in [0, \delta]$ we replace Γ_0 by Γ_t and apply the same arguments. \square

3 Birkhoff Normal Forms of C^k deformations.

The aim of this section is to prove items 1 and 2 of Theorem 1. We shall obtain a C^k , $k \in \{0, 1\}$, family of Birkhoff Normal Forms for the C^k family of exact symplectic maps $t \rightarrow P_t$ around the corresponding invariant tori. This BNF will be used to construct a C^k family of Quantum Birkhoff Normal Forms. In order to obtain the BNF we will apply Theorem 9.11 to a suitable C^k family of exact symplectic mappings \tilde{P}_t which will be constructed below.

Let $[0, 1] \ni t \mapsto P_t \in C^{\infty}(U, U)$ be a C^k family of exact symplectic maps where $U \subset T^*\Gamma$ is an open set. Suppose that P_0 has a Kronecker torus $\Lambda_0(\omega_0)$ with frequency $\omega_0 \in D(\kappa_0, \tau)$ where $0 < \kappa_0 < 1$ and $\tau > n - 1$. Then P_0 admits a Birkhoff Normal Form (BNF) at $\Lambda_0(\omega_0)$

(cf [43], Proposition 9.13 and [63], Proposition 3.3) which means the following. There exists an exact symplectic transformation $\tilde{\chi} : \mathbb{A} \rightarrow T^*\Gamma$, where $\mathbb{A} := \mathbb{T}^{n-1} \times D \subset T^*\mathbb{T}^{n-1}$ and D is a neighborhood of $I_0(\omega_0)$ given by (1.6), $\tilde{\chi}(\mathbb{A})$ is a neighborhood of $\Lambda_0(\omega_0)$, $\tilde{\chi}(\mathbb{T}^{n-1} \times \{I_0(\omega_0)\}) = \Lambda_0(\omega_0)$ and the exact symplectic map $\tilde{P}_0 := \tilde{\chi}^{-1} \circ P_0 \circ \tilde{\chi}$ has the form

$$\begin{cases} \tilde{P}_0(\theta, r) = (\theta + \nabla K(r), r) + R(\theta, r), \\ \partial_r^\alpha R(\theta, I_0(\omega_0)) = 0, \quad \forall \theta \in \mathbb{T}^{n-1}, \quad \forall \alpha \in \mathbb{N}^{n-1}. \end{cases} \quad (3.19)$$

We are going to use the following definition of a generating function of a symplectic map. Denote by $\text{pr} : \mathbb{R}^{n-1} \rightarrow \mathbb{T}^{n-1}$ the canonical projection.

Definition 3.1. *Let $D \subset \mathbb{R}^{n-1}$ be an open set and $F \in C^\infty(\mathbb{T}^{n-1} \times D)$. The function $S \in C^\infty(\mathbb{R}^{n-1} \times D)$ given by $S(x, r) = \langle x, r \rangle - F(\text{pr}(x), r)$ is said to be a generating function of a symplectic map P in $\mathbb{T}^{n-1} \times D$ if*

- *the map $x \rightarrow \nabla_r S(x, r) = x - \nabla_r F(\text{pr}(x), r)$ projects to a diffeomorphism of \mathbb{T}^{n-1} homothope to the identity for any fixed $r \in D$*
- *for any $(\theta, r) \in \mathbb{T}^{n-1} \times D$*

$$P(\theta - \nabla_r F(\theta, r), r) = (\theta, r - \nabla_\theta F(\theta, r))$$

Hereafter, we slightly abuse the notations identifying $y = \nabla_r F(\theta, r) \in \mathbb{R}^{n-1}$ with its image $\text{pr}(y) \in \mathbb{T}^{n-1}$.

Shrinking U if necessary we set $U = \tilde{\chi}(\mathbb{A})$. We suppose that the BNF is nondegenerate, which means that the Hessian matrix $\partial^2 K(r_0)$ is nondegenerate at $r_0 = I_0(\omega_0)$. Shrinking D if necessary we suppose that the map $\nabla K : D \rightarrow \nabla K(D) \subset \mathbb{R}^{n-1}$ is a diffeomorphism. Then there exists $\bar{\delta} > 0$ such that the exact symplectic maps $\tilde{P}_t = \tilde{\chi}^{-1} \circ P_t \circ \tilde{\chi}$ admit for $0 \leq t \leq \bar{\delta}$ a C^1 family of generating functions $[0, \bar{\delta}] \ni t \rightarrow \tilde{G}_t \in C^\infty(\mathbb{R}^{n-1} \times D)$ such that

$$\tilde{G}_t(x, r) = \langle x, r \rangle - K(r) - G_t(\text{pr}(x), r) \quad (3.20)$$

where the map $t \rightarrow G_t \in C^\infty(\mathbb{T}^{n-1} \times D)$ is C^1 , $\text{pr} : \mathbb{R}^{n-1} \rightarrow \mathbb{T}^{n-1}$ is the canonical projection and

$$\partial_r^\alpha G_0(\theta, I_0(\omega_0)) = 0 \quad \forall \theta \in \mathbb{T}^{n-1}, \quad \alpha \in \mathbb{N}^{n-1}. \quad (3.21)$$

On the other hand,

$$|\partial_\theta^\alpha \partial_r^\beta (G(\theta, r) - G_0(\theta, r))| \leq C_{\alpha, \beta} t \quad \forall (\theta, r) \in \mathbb{T}^{n-1} \times D(\kappa), \quad t \in [0, \bar{\delta}].$$

These inequalities allow us to apply Theorem 9.8. Consider the Legendre transform K^* of K in a ball $B(\omega_0, \varepsilon)$ centered at ω_0 and with sufficiently small radius $0 < \varepsilon \ll 1$. Then $\nabla K^* : B(\omega_0, \varepsilon) \rightarrow \nabla K^*(B(\omega_0, \varepsilon))$ becomes a diffeomorphism. Using Theorem 9.8 with $\kappa = \varrho = \kappa_1$, where $\kappa_1 \leq \kappa_0$ is sufficiently small, we obtain a C^k family of Kronecker tori $[0, \delta_1] \ni t \mapsto \Lambda_t(\omega_0)$ of P_t with a frequency ω_0 , where $\delta_1 = \delta_1(\kappa_1) > 0$. Following Lazutkin (cf [43], Proposition 9.13 and [63], Proposition 3.3) we obtain a C^k family of Birkhoff Normal Forms of \tilde{P}_t around the tori $\Lambda_t(\omega_0)$ up to any order, which means the following. Fix $N \geq 4$. There exist C^k families of exact symplectic mappings $t \rightarrow \chi_t^0 \in C^\infty(\mathbb{T}^{n-1} \times D, \mathbb{T}^{n-1} \times D)$ and vector valued functions $t \rightarrow I_t(\omega_0) \in D$, $t \in [0, \delta_1]$ (analytic in t if the map $t \rightarrow P_t$ is analytic) with the properties

- $\chi_t^0(\mathbb{T}^{n-1} \times \{I_t(\omega_0)\}) = \Lambda_t(\omega_0)$ for each $t \in [0, \delta_1]$;
- the exact symplectic map $\tilde{P}_t^0 = (\chi_t^0)^{-1} \circ \tilde{P}_t \circ \chi_t^0$ admits for each $0 \leq t \leq \delta_1$ a generating function

$$\tilde{G}_t^0(x, r) = \langle x, r \rangle - K_t(r) - G_t^0(\text{pr}(x), r), \quad (3.22)$$

such that the maps $t \rightarrow G_t^0 \in C^\infty(\mathbb{T}^{n-1} \times D)$ and $t \rightarrow K_t \in C^\infty(D)$ are C^k (analytic in t if the map $t \rightarrow P_t$ is analytic) and

$$\partial_\theta^\alpha \partial_r^\beta G_t^0(\theta, I_t(\omega_0)) = 0 \quad \forall \theta \in \mathbb{T}^{n-1}, \quad \forall \alpha, \beta \in \mathbb{N}^{n-1}, \quad \text{with } |\beta| \leq 2N, \quad (3.23)$$

and for each $0 \leq t \leq \delta_1$. Moreover, K_t is a polynomial of degree $2N$ for each t fixed.

Let ω_0 be a point of positive Lebesgue density in $D(\kappa_0, \tau)$ ($\omega_0 \in D^0(\kappa_0, \tau)$), which means that the Lebesgue measure $\text{meas}(D(\kappa_0, \tau) \cap V) > 0$ for any neighborhood V of ω_0 in \mathbb{R}^{n-1} . Then $\omega_0 \in D^0(\kappa, \tau)$ for each $0 < \kappa \leq \kappa_0$. We suppose that $\kappa_1 \leq \varepsilon^2$ and for every $0 < \kappa \leq \kappa_1 < 1$ we set

$$\Omega(\kappa) := B(\omega_0, \sqrt{\kappa}), \quad \Omega_\kappa := D(\kappa, \tau) \cap B(\omega_0, \sqrt{\kappa} - \kappa), \quad \Omega_\kappa^0 := D^0(\kappa, \tau) \cap B(\omega_0, \sqrt{\kappa} - \kappa) \quad (3.24)$$

It follows from [43], Proposition 9.9, that

$$\frac{\text{meas}(\Omega(\kappa) \setminus \Omega_\kappa)}{\text{meas}(\Omega(\kappa))} \leq C \kappa. \quad (3.25)$$

Moreover, $\text{meas}(\Omega_\kappa^0) = \text{meas}(\Omega_\kappa)$. Set $D_t(\kappa) := \nabla K_t^*(\Omega(\kappa))$ and $\mathbb{A}_t = \mathbb{T}^{n-1} \times D_t(\kappa)$. Notice that there exists $c > 0$ such that $D_t(\kappa) \subset B(I_t(\omega_0), c\sqrt{\kappa})$ and (3.23) implies that for every $\alpha, \beta \in \mathbb{N}^{n-1}$ there exists $C_{\alpha, \beta, N}$ such that

$$|\partial_\theta^\alpha (\kappa \partial_r)^\beta G_t(\theta, r)| \leq C_{\alpha, \beta, N} \kappa^N \quad \forall (\theta, r) \in \mathbb{T}^{n-1} \times D_t(\kappa), \quad t \in [0, \delta_1(\kappa_1)].$$

This inequality allows one to apply Theorem 9.11 taking $\varrho = \varepsilon \kappa$ and $0 < \kappa \leq \kappa_1$, where ε and κ_1 are sufficiently small in order. In this way we obtain the following

Theorem 3.2. (*Birkhoff Normal Form*) *Let $[0, \delta] \ni t \rightarrow P_t \in C^\infty(U, U)$, $U \subset T^*\Gamma$, be a C^1 family of symplectic maps. Let P_0 have a Kronecker torus $\Lambda_0(\omega_0)$ with a frequency $\omega_0 \in D^0(\kappa_0, \tau)$, where $\tau > n - 1$. Suppose that the BNF of P_0 at $\Lambda_0(\omega_0)$ is nondegenerate. Then there exists $0 < \kappa_1 \leq \kappa_0$ and $\delta_1 = \delta_1(\kappa_1) > 0$ such that for each $M > 0$ fixed and $0 < \kappa \leq \kappa_1$ the following holds*

- (i) *For each $\omega \in \Omega_\kappa^0$ there exists a C^1 family of Kronecker invariant tori $[0, \delta_1] \ni t \rightarrow \Lambda_t(\omega)$ of P_t with a frequency ω ;*
- (ii) *There exists a C^1 -smooth with respect to $t \in [0, \delta_1]$ family of exact symplectic maps $\chi_t : \mathbb{A}_t \rightarrow U$ and of real valued functions $L_t \in C^\infty(D_t(\kappa))$ and $R_t^0 \in C^\infty(\mathbb{A}_t)$ (analytic in t if the map $t \rightarrow P_t$ is analytic) such that*

1. $\Lambda_t(\omega) = \chi_t(\mathbb{T}^{n-1} \times \{I_t(\omega)\})$ for each $t \in [0, \delta_1]$ and $\omega \in \Omega_\kappa^0$, where $I_t(\omega)$ is given by (1.6);

2. The function $\mathbb{R}^{n-1} \times D \ni (x, I) \mapsto \phi_t(x, I) := \langle x, I \rangle - L_t(I) - R_t^0(x, I)$ is a generating function in the sense of Definition 3.1 of the exact symplectic map

$$P_t^0 := \chi_t^{-1} \circ P_t \circ \chi_t : \mathbb{A} \rightarrow \mathbb{A};$$

3. $\nabla L_t : D_t \rightarrow \Omega$ is a diffeomorphism, $L_t = K_t$ outside $D_t^1 := \nabla K_t^*(B(\omega_0, \sqrt{\kappa} - \frac{1}{2}\kappa))$ and $\nabla L_t^*(\omega) = I_t(\omega)$ is given by (1.6) for each $\omega \in \Omega_\kappa^0$;
4. R_t^0 is flat at $\mathbb{T}^{n-1} \times E_t^\kappa$, where $E_t^\kappa := \nabla L_t^*(\Omega_\kappa^0)$.
5. $\|\nabla L_t - \nabla K_t\|_{m, D_t; \kappa} + \|\nabla R_t^0\|_{m, D_t; \kappa} \leq C_{m, M} \kappa^M$ for each $\alpha, \beta \in \mathbb{N}^{n-1}$ and $m \in \mathbb{N}$.

Moreover, if the map $t \rightarrow P_t$ is analytic in a disc $B(0, \delta)$ in \mathbb{C} , then the maps $t \rightarrow \chi_t$, $t \rightarrow L_t$, $t \rightarrow R_t^0$, are analytic in a disc $B(0, \delta_1)$ and the estimate in 5. holds for $t \in B(0, \delta_1)$.

To prove the Theorem we apply Theorem 9.11 taking $N \gg M$, $\varrho = \varepsilon\kappa$ and $0 < \kappa \leq \kappa_1$, where ε and κ_1 are sufficiently small in order to satisfy (9.188).

Observe that each $I \in E_t^\kappa$ is an element of positive Lebesgue density of E_t^κ since the map $\nabla L_t^* : \Omega(\kappa) \rightarrow D_t(\kappa)$ is a diffeomorphism. For any $0 < \kappa \leq \kappa_1$ fixed, we extend I_t as a C^1 family of smooth functions setting

$$I_t(\omega) := \nabla L_t^*(\omega) \quad \forall \omega \in \Omega.$$

Then we have

$$P_t^0(\varphi, I) = (\varphi + \nabla L_t(I), I) + O_N(|I - I_t(\omega)|^N)$$

for each $\omega \in \Omega_\kappa^0$ and $N \in \mathbb{N}$. This formula can be differentiated with respect to (φ, I) as many times as we want. To summarize we give the following

Definition 3.3. We say that the C^1 family of exact symplectic maps P_t , $t \in [0, \delta]$, admits a C^1 -smooth family of nondegenerate Birkhoff Normal Forms associated with a C^1 family of invariant tori $\Lambda_t(\omega)$ with frequencies $\omega \in \Omega_\kappa^0$ if item (ii) of Theorem 9.11 holds true.

Recall as well that the complement of Ω_κ^0 in Ω_κ is of Lebesgue measure zero.

Setting

$$\Xi := \bigcup_{0 < \kappa \leq \kappa_1} \Omega_\kappa^0 \tag{3.26}$$

we prove items 1 and 2 of Theorem 1.

The advantage of working with Ω_κ^0 instead of Ω_κ is given by the following

Lemma 3.4. Let Ω be an open subset of \mathbb{R}^d , $d \geq 1$. Let $E \subset \Omega$ be a measurable set of positive Lebesgue measure and let $E^0 \subset E$ be the set of points of positive Lebesgue density in E . Then any smooth function f on Ω which is zero on E^0 is flat at E^0 , i.e. the equality $f|_{E^0} = 0$ implies $\partial^k f|_{E^0} = 0$ for any $k \in \mathbb{N}^d$. Moreover, the Lebesgue measure of $E \setminus E^0$ is zero by construction.

Proof. The result is evident when $d = 1$. Suppose that $d \geq 2$. Let $\omega = (\omega_1, \omega') \in E^0$. By Fubini's theorem, for any neighborhood $U_1 \subset \mathbb{R}$ of ω_1 and $U' \subset \mathbb{R}^{d-1}$ of ω' there is $z' \in U'$ and a set of positive Lebesgue measure $V_1 \subset U_1$ such that $(z_1, z') \in E^0$ for any $z_1 \in V_1$. Then $f(z_1, z') = 0$ for any $z_1 \in V_1$ and there is $y_1 \in U_1$ such that $\partial_1 f(y_1, z') = 0$. By continuity we obtain $\partial_1 f(\omega) = 0$. In the same way we prove by induction that the restriction of $\partial^\alpha f$ to Ω_E^0 is zero for any $\alpha \in \mathbb{N}^d$. \square

We are going to give a relation between the function β_t defined by (1.5) and the restrictions on Ω_κ^0 of the functions I_t and L_t given by Theorem 3.2. As $\chi_t : \mathbb{A} \rightarrow U \subset T^*\Gamma$ is exact symplectic for each $t \in J := [0, \delta(\kappa)]$ there exists a C^1 family of functions $\Psi_t \in C^\infty(\mathbb{A})$ such that

$$\chi_t^*(\xi dx) = Id\varphi + d\Psi_t.$$

Notice that the generating functions ϕ_t of P_t^0 are uniquely defined up to additive constants C_t such that the function $J \ni t \rightarrow C_t$ is C^1 .

Lemma 3.5. *Choosing appropriately the C^1 function $J \ni t \rightarrow C_t$ we obtain the following*

(i) *We have*

$$\langle I, \nabla L_t(I) \rangle - L_t(I) = A_t(\chi_t(\varphi, I)) + \Psi_t(\varphi, I) - \Psi_t(P_t^0(\varphi, I)) + R_t^1(\varphi, I)$$

where the function R_t^1 is flat at $\mathbb{T}^{n-1} \times E_t^\kappa$ for each $t \in J$;

(ii) $\beta_t(\omega) + L_t(I_t(\omega)) = \langle \omega, I_t(\omega) \rangle \quad \forall \omega \in \Omega_\kappa^0, t \in J.$

Proof. The Poincaré identity implies

$$P_t^*(\xi dx) = \xi dx + dA_t, \tag{3.27}$$

where ξdx is the fundamental one-form on $T^*\Gamma$ and $A_t(\rho)$, $\rho = \chi(\varphi, I) \in U$ is the action

$$A_t(\rho) = \int_{\tilde{\gamma}_t(\rho)} \xi dx$$

along the broken bicharacteristic $\tilde{\gamma}_t(\rho)$ starting at $\pi_t^+(\rho)$ and ending at $\pi_t^-(P_t(\rho))$. Then we obtain $(P^0)^*(Id\varphi) - Id\varphi = d((A \circ \chi) + \Psi - \Psi \circ P^0)$. On the other hand, $(P^0)^*(Id\varphi) = d(L_t(I) - \langle I, \nabla L_t(I) \rangle + R_t^1(\varphi, I))$ where R_t^1 is a flat function at $\mathbb{T}^{n-1} \times E_t^\kappa$ and we obtain (i). To prove (ii) we use (1.5). \square

4 Infinitesimal spectral rigidity of Liouville billiard tables

A Liouville billiard table of dimension $n \geq 2$, is a completely integrable billiard table (X, g) admitting n functionally independent and Poisson commuting integrals of the billiard flow on T^*X which are quadratic forms in the momentum. It can be viewed as a 2^{n-1} -folded branched covering of a disk-like domain in \mathbb{R}^n by the cylinder $\mathbb{T}^{n-1} \times [-N, N]$, $N > 0$.

Liouville billiard tables of dimension two were defined in [60, Sec. 2] by using a branched double covering map. Here we give an invariant definition of Liouville billiard tables in dimension two. The equivalence of the two definitions is proven in Appendix B.1.

Definition 4.1. *A Liouville billiard table is a smooth oriented compact and connected Riemannian manifold of dimension two (X, g) with connected boundary $\Gamma \equiv \partial X$ such that the following two conditions are satisfied:*

- (a) *There exists a smooth quadratic in velocities integral of the geodesic flow $I : TX \rightarrow \mathbb{R}$ that is invariant with respect to the reflection at the boundary $TM|_\Gamma \rightarrow TM|_\Gamma$, $\xi \mapsto \xi - 2g(\nu, \xi)$, where ν is the outward unit normal to Γ . In addition, we assume that the metric g does not allow global Killing symmetries;*

- (b) *There is no point $x_0 \in \Gamma$ and a constant $c \in \mathbb{R}$ such that $g_{x_0}(\xi, \xi) = cI_{x_0}(\xi, \xi)$ for any $\xi \in T_{x_0}X$.*

In view of Theorem 7 in Appendix B.1 there exists a double covering map with two branched points,

$$\tau : C \rightarrow X, \quad (4.28)$$

where C denotes the cylinder $(\mathbb{R}/\mathbb{Z}) \times [-N, N]$, $N > 0$, coordinatized by the variables x and y respectively, so that the metric $\tau^*(g)$ and the integral $\tau^*(I)$ have the following form on C ,

$$\begin{aligned} dg^2 &= (f(x) - q(y))(dx^2 + dy^2) \\ dI^2 &= \alpha dF^2 + \beta dg^2 \end{aligned} \quad (4.29)$$

where $\alpha \neq 0$ and β are real constants and

$$dF^2 := (f(x) - q(y))(q(y) dx^2 + f(x) dy^2). \quad (4.30)$$

In other words, the integral dI^2 belongs to the pencil of dg^2 and dF^2 . Here $f \in C^\infty(\mathbb{R})$ is 1-periodic, $q \in C^\infty([-N, N])$, and

- (i) f is even, $f > 0$ if $x \notin \frac{1}{2}\mathbb{Z}$, and $f(0) = f(1/2) = 0$;
- (ii) q is even, $q < 0$ if $y \neq 0$, $q(0) = 0$ and $q''(0) < 0$;
- (iii) $f^{(2k)}(1/2) = (-1)^k q^{(2k)}(0)$, $l = 0, 1$, for every natural $k \in \mathbb{N}$.

In particular, if $f \sim \sum_{k=1}^{\infty} f_k x^{2k}$ is the Taylor expansion of f at 0, then, by (iii), the Taylor expansion of q at 0 is $q \sim \sum_{k=1}^{\infty} (-1)^k f_k x^{2k}$. A Liouville billiard table is said to be of *classical type* if it satisfies the following additional conditions,

- (iv) the boundary Γ of X is locally geodesically convex which amounts to $q'(N) < 0$;
- (v) $f(x) = f(1/2 - x)$ for any x and f is strictly increasing on the interval $[0, 1/4]$;

Remark 4.2. *Note that in contrast to [60, Sec. 2] we do not assume that the functions f and g are analytic Morse functions.*

The points $F_1 := \tau(0, 0)$ and $F_2 := \tau(1/2, 0)$ on X are the two branched points of the covering map $\tau : C \rightarrow X$. All other points on X are regular values of τ . The preimage of any regular value consists of two points. Note also that $\tau : C \rightarrow X$ commutes with the involution on the cylinder C induced by the map $\sigma : (x, y) \mapsto (-x, -y)$. The fixed points of this involution are precisely the singular points $(0, 0)$ and $(1/2, 0)$ of the covering map τ . One can see that any Liouville billiard table possesses the *string property* which means that any broken geodesic starting from the singular point F_1 (F_2) passes through F_2 (F_1) after the first reflection at the boundary and the sum of distances from any point of Γ to F_1 and F_2 is constant [60]. In particular, the only Liouville billiard table in \mathbb{R}^2 equipped with the Euclidean metric is the interior of the ellipse. Thus Liouville billiard tables can be regarded as a natural generalization of elliptic billiards to curved space. In view of condition (v) in the definition of the Liouville billiard tables of classical type, there is a group $I(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ acting on (X, g) by isometries. This group is generated by the involutions $\sigma_1(x, y) = (x, -y)$ and $\sigma_2(x, y) = (\pi - x, y)$ of the cylinder.

The construction of Liouville billiards of dimension two involving the covering map τ was generalized to any dimension in [61, §5.3] (cf. also [62, §3]). As in the two dimensional case, one defines the subclass of Liouville billiards of classical type in a similar way. An important example of a Liouville billiard table of classical type is the interior of the n -axial ellipsoid in \mathbb{R}^n equipped with the Euclidean metric. More generally, there is a non-trivial two-parameter family of analytic Liouville billiard tables of classical type of constant scalar curvature K having the same broken geodesics (considered as non-parameterized curves) as the ellipsoid [61, Theorem 3]. This family includes the ellipsoid ($K = 0$), and Liouville billiard tables on the sphere ($K = 1$) and in the hyperbolic space ($K = -1$).

In what follows we will apply Theorem 3 to Liouville billiard tables of classical type in dimensions two and three for obtaining several new isospectral results. The main idea is to interpret the integrals in (2.16) as values of a suitable Radon transform which is one-to-one.

Let (X, g) be a Liouville billiard table of classical type. The group of isometries of (X, g) has a subgroup $I(X, g)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. One can extend (X, g) to an open Riemannian manifold (M, g) so that any isometry in $I(X, g)$ can be extended to an isometry of (M, g) . In this way, the group of isometries of (M, g) contains a subgroup $I(M, g)$ isomorphic to $I(X, g)$. Denote by $\text{Symm}(M, g)$ the class of C^∞ -smooth billiard tables (Y, g) , Y isometrically embedded in M , so that any isometry in $I(M, g)$ is an isometry of (Y, g) . Recall from Sec. 2 that $\delta_\nu\Gamma : \Gamma \rightarrow \mathbb{R}$ is the vertical component of the variation $\psi_0 : \Gamma \rightarrow TM|_\Gamma$, where $\psi_0 = id$ and $\Gamma_t = \psi_t(\Gamma_0)$ is a C^1 -deformation of $\Gamma = \Gamma_0$ and (X_t, g) is the billiard table with boundary $\Gamma_t = \partial X_t$.

Theorem 4.3. *Let (X, g) be a Liouville billiard table of classical type dimension 2 and let (X_t, g) , $t \in (-\varepsilon, \varepsilon)$, be a C^1 -family of billiard tables in $\text{Symm}(M, g)$ satisfying the weak isospectral condition $(H_1) - (H_2)$ and such that $X_0 = X$. Then $\delta_\nu\Gamma \equiv 0$.*

This means that any Liouville billiard table of classical type (X, g) is *infinitesimally spectrally rigid* in $\text{Symm}(M, g)$ under the weak-isospectral condition $(H_1) - (H_2)$.

Proof of Theorem 4.3. The theorem follows from Theorem 2 as in the proof of Corollary 1.4 in [63]. A first integral of B in $B^*\Gamma$ is the function $\mathcal{I}(x, \xi) = f(x) - \xi^2$ the regular values h of which belong to $(q(N), 0) \cup (0, f(1/4))$ (see [60], Lemma 4.1 and Proposition 4.2). Moreover, for each regular value $h \in (q(N), 0)$ the corresponding level set $L_h : \{\mathcal{I} = h\}$ consists of two connected circles which are invariant with respect to B , having rotation numbers $\omega = \pm\rho(h)$. By Proposition 4.4 [60], the rotation function ρ is smooth and strictly increasing in an interval $(q(N), q(N) + \varepsilon)$, and we obtain a diffeomorphism $\rho : (q(N), q(N) + \varepsilon) \rightarrow (0, \omega_0)$. Then the Kolmogorov non-degeneracy condition is satisfied in that interval. Hence, one can apply Corollary 2 to any Kronecker invariant circle $\Lambda_0(\omega)$, with a Diophantine vector of rotation $\omega = \rho(h) \in (0, \omega_0)$.

We are going to interpret (2.16) as a value of a suitable Radon transform. The Leray form on the invariant circle $\Lambda_0(\omega) \subset L_h$ is

$$\lambda_h = \begin{cases} \frac{dx}{\sqrt{f(x)-h}}, & \xi > 0, \\ -\frac{dx}{\sqrt{f(x)-h}}, & \xi < 0. \end{cases}$$

Since $\omega \in (0, \omega_0)$ is Diophantine and the Leray form is invariant with respect to B , there exists a constant $c(h) \neq 0$ such that $\lambda_h = c(h)d\mu_0$, where $d\mu_0$ is the unique probability measure on $\Lambda_0(\omega)$

which is invariant with respect to B . Setting $K := \pi_{\Gamma_t}^* (\delta_\nu \Gamma_t)$ we consider the corresponding Radon transform which assigns to each circle $\Lambda_0(\omega) := \{(x, y(h)) : x \in \mathbb{T}\}$, $h = \rho^{-1}(\omega)$, the integral

$$R_K(\Lambda_0(\omega)) = \int_{\Lambda_0(\omega)} \langle \xi^\pm, \nu \rangle K \circ \pi_\Gamma \lambda_h.$$

We have

$$\langle \xi^\pm(x, y(h)), \nu(x) \rangle = \sqrt{\frac{h - q(N)}{f(x) - q(N)}},$$

hence,

$$R_K(\Lambda_0(\omega)) = \pm c(h) \sqrt{h - q(N)} \int_0^1 \frac{K_1(x)}{\sqrt{f(x) - h}} dx, \quad h \in (q(N), 0),$$

where $K_1(x) = K(x)/\sqrt{f(x) - q(N)}$. Since K is invariant with respect to the group of isometries $I(X)$ then so is K_1 and applying (2.16) for $t = 0$ we get

$$0 = \int_0^{1/4} \frac{K_1(x)}{\sqrt{f(x) - h}} dx = \int_0^{f(1/4)} \frac{K_2(s)}{\sqrt{s - h}} ds \quad (4.31)$$

for any $h \in (q(N), q(N) + \varepsilon)$ such that $\rho(h) \in D(\kappa, \tau)$, where $K_2 = (K_1/f') \circ f^{-1} \in L^1(0, f(1/4))$. On the other hand, the right hand side of (4.31) is analytic in $h \in (q(N), 0)$ and the set of $h = \rho^{-1}(\omega)$, $\omega \in D(\kappa, \tau) \cap (0, \omega_0)$ is of positive measure, and we obtain (4.31) for any $h \in (q(N), 0)$. Differentiating (4.31) with respect to h at $h = q(N)$ we get

$$\int_0^{f(1/4)} \frac{K_2(s)}{\sqrt{s - q(N)}} (s - q(N))^{-k} ds = 0$$

for any $k \in \mathbb{N}$, which implies $K_2 = 0$ since the set $\{(s - q(N))^{-k} : k \in \mathbb{N}\}$ is dense in $L^1(0, f(1/4))$ and K_2 is continuous in $(0, f(1/4))$. This completes the proof of the theorem. \square

In order to apply Theorem 2 to Liouville billiard tables of dimension three we need to ensure that the following Kolmogorov nondegeneracy condition is satisfied. Consider a Liouville-Arnold chart which consists of an open set U of the phase space of the billiard ball map B and a symplectic map $(\varphi, I) : U \rightarrow \mathbb{T}^n \times D$, D being an open subset of \mathbb{R}^n , giving ‘‘action-angle’’ coordinates on U , i.e. B is given by the map $(\varphi, I) \mapsto (\varphi + \nabla K(I), I)$ in these coordinates. The Kolmogorov condition means that the map $\nabla K : D \rightarrow D^* := \nabla K(D)$ is a diffeomorphism. We are interested in maximal charts with this property. It turns out that Liouville billiard tables of classical type of dimension two are always non-degenerate in a Kolmogorov sense close to the boundary [60]. The non-degeneracy property of Liouville billiard tables of classical type of dimension three has been investigated in [62].

It is proved in [62] that any Liouville Billiard Table of classical type of dimension 3 admits four not necessarily connected maximal Liouville-Arnold charts U_j , $1 \leq j \leq 4$, of action-angle variables in $B^*\Gamma$. Two of them, say U_1 and U_2 , have the property that any unparameterized geodesic in $S^*\Gamma$ can be obtained as a limit of orbits of B lying either in U_1 or in U_2 (then

the corresponding broken geodesics approximate geodesics of the boundary). Moreover, in any connected component of U_1 and U_2 there is such a sequence of orbits of B , while any orbit of B in U_3 and U_4 is far away from $S^*\Gamma$. In other words, the charts U_1 and U_2 can be characterized by the property that there is a family of “*whispering gallery rays*” issuing from any of their connected components. For this reason the two cases $j = 1, 2$ are referred as to boundary cases. Denote by \mathcal{F}_b the set of all regular invariant tori $\Lambda \in \mathcal{F}$ lying either in U_1 or in U_2 . We say that a Liouville Billiard Table is Kolmogorov nondegenerate if B satisfies the Kolmogorov nondegeneracy condition in U_1 and U_2 . It is shown in [62], Theorem 5.1, that *any analytic 3-dimensional Liouville billiard table of classical type having at least one non-periodic geodesic on the boundary is Kolmogorov nondegenerate*. An example of such billiard tables is the ellipsoid.

It is proved in [62], Theorem 4.4, that the Radon transform is one-to-one for Liouville billiard tables of classical type of dimension 3. More precisely, we have

Theorem 4.4. [62] *Let (X, g) , $\dim X = 3$, be a Liouville billiard table of classical type. If $K \in C(\Gamma)$ is invariant under the group of symmetries G of Γ and the Radon transform $\mathcal{R}_K(\Lambda) = 0$ for any $\Lambda \in \mathcal{F}_b$, then $K \equiv 0$.*

We point out that Liouville billiard tables of classical type are smooth by construction but they are not supposed to be analytic.

Using Corollary 2 and Theorem 4.4, we obtain as above the following

Theorem 4.5. *Any nondegenerate Liouville billiard table of dimension 3 of classical type (X, g) is infinitesimally spectrally rigid in $\text{Symm}(M, g)$ under the weak-isospectral condition $(H_1) - (H_2)$.*

This means that if (X_t, g) , $t \in (-\varepsilon, \varepsilon)$ is a C^1 -family of billiard tables in $\text{Symm}(M, g)$ satisfying the weak isospectral condition $(H_1) - (H_2)$ and such that $X_0 = X$, then $\delta_\nu \Gamma \equiv 0$.

A smooth deformation (X_t, g) , $t \in (-\varepsilon, \varepsilon)$, is said to be flat at $t = 0$ if (2.14) is C^∞ smooth with respect to t in an interval $(-\varepsilon, \varepsilon)$ and the vertical component of k -th variation $\delta_\nu^k \Gamma$ is zero for any integer $k \geq 1$, where

$$\delta_\nu^k \Gamma(x) := \left\langle \left(\frac{d}{dt} \right)^k \psi_t|_{t=0}(x), \nu(x) \right\rangle, \quad x \in \Gamma.$$

Corollary 4.6. *Let (X, g) be a classical Liouville billiard table of dimension 2 or a classical non-degenerate Liouville billiard table of dimension 3 and let (X_t, g) , $t \in (-\varepsilon, \varepsilon)$, be a C^∞ -family of billiard tables in $\text{Symm}(M, g)$ satisfying the weak isospectral condition $(H_1) - (H_2)$ and such that $X_0 = X$. Then the deformation is flat at $t = 0$. In particular, $\Gamma_t = \Gamma_0$ for $t \in (-\varepsilon, \varepsilon)$ if the family is analytic with respect to t .*

In the case of the ellipse similar results have been obtained by Hezari and Zelditch [27] under the usual isospectral condition using the wave-trace method.

5 Isospectral deformations in the presence of elliptic geodesics.

Let (X, g) be a smooth billiard table with boundary $\Gamma := \partial X$. Consider a C^1 -smooth family

$$[0, 1] \ni t \rightarrow (X, g_t) \tag{5.32}$$

of Riemannian metrics on X with $g_0 = g$. Suppose that (X, g) admits an elliptic closed broken geodesic γ with $m \geq 2$ vertices. Denote by $\{B^j(\rho) : 0 \leq j \leq m-1\}$ the corresponding periodic trajectory of the billiard ball map B . Then $\rho = (x, \xi) \in \mathbf{B}^*\Gamma$ is a fixed point of the local Poincaré map $P = B^m$ which is symplectic. Recall that γ is *elliptic* if ρ is an *elliptic* fixed point of P which means that the eigenvalues of the linear Poincaré map $dP(\rho) : T_\rho\Gamma \rightarrow T_\rho\Gamma$ are all distinct, different from one and of modulus one, hence,

$$\text{Spec}(dP(\rho)) = \{e^{\pm i\phi_j} : 1 \leq j \leq n-1\},$$

where $0 < \phi_1 < \dots < \phi_{n-1} \leq \pi$. Set $\phi = (\phi_1, \dots, \phi_{n-1})$ and fix a positive integer N . The linear Poincaré map $dP(\rho)$ is said to be *N-elementary* if the scalar product

$$\langle \phi, k \rangle \notin 2\pi\mathbb{Z}$$

for any integer vector $k = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}$ such that $0 < |k_1| + \dots + |k_{n-1}| \leq N$. We say as well that γ admits no resonances of order less or equal to N .

From now on we fix $N \geq 4$ and suppose that $\gamma_0 = \gamma$ is elliptic in (X, g_0) and that it admits no resonances of order less or equal to N . By the implicit function theorem there exists $\bar{\delta} > 0$ such that the following holds. There is a unique C^1 curve $[0, \bar{\delta}] \ni t \mapsto \rho_t \in T^*\Gamma$ starting from $\rho_0 = \rho$ such that $\rho_t \in \mathbf{B}_t^*\Gamma$ is an elliptic fixed point of $P_t = B_t^m$ for any $t \in [0, \bar{\delta}]$. Moreover, the linear Poincaré map $dP_t(\rho_t)$ is *N-elementary*. The eigenvalues of P_t have the form $\exp(i\phi_j(t))$, where $0 < \phi_1(t) < \dots < \phi_{n-1}(t) < \pi$ and the map $t \rightarrow \phi(t) := (\phi_1(t), \dots, \phi_{n-1}(t))$ is C^1 in $[0, \bar{\delta}]$. Moreover, P_t admits a Birkhoff normal form of order $[N/2] \geq 2$ ($[a]$ denotes the integer part of $a \in \mathbb{R}$) in suitable polar symplectic coordinates which will be described below.

In order to avoid eventual singularities at $r_j = 0$, $j = 1, \dots, n-1$, we fix $0 < c_0 \ll 1$ and $r_0 > 0$, and set

$$\mathbb{D} = \mathbb{D}(c_0) := \{r = (r_1, \dots, r_{n-1}) \in \mathbb{R}^{n-1} : 0 < c_0|r| < |r_j| < r_0, 1 \leq j \leq n-1\}. \quad (5.33)$$

and $\mathbb{A} := \mathbb{T}^{n-1} \times \mathbb{D}$. Denote by $\text{pr} : \mathbb{R}^{n-1} \rightarrow \mathbb{T}^{n-1}$ the canonical projection.

Proposition 5.1. (*Birkhoff Normal Form*). *For any $0 < \delta < \bar{\delta}$ there exists*

- a C^1 -family of exact symplectic transformation $[0, \delta] \ni t \rightarrow (\tilde{\chi}_t : \mathbb{A} \rightarrow U_t := \chi_t(\mathbb{A}))$, where $U_t \subset \mathbf{B}_t^*\Gamma$ is an open set
- a C^1 -family of polynomials $K_t \in \mathbb{R}_{[N/2]}[\xi_1, \dots, \xi_{n-1}]$ with real coefficients of $n-1$ variables ξ_1, \dots, ξ_{n-1} and of degree $[N/2]$
- a C^1 -family of real valued functions $G_t \in C^\infty(\mathbb{A})$

such that the following holds

1. the function $\tilde{G}_t \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{D})$ defined by $\tilde{G}_t(x, r) := \langle x, r \rangle - K_t(r) - G_t(\text{pr}(x), r)$ is a generating function of the symplectic map $\tilde{P}_t := \tilde{\chi}_t^{-1} \circ P_t \circ \tilde{\chi}_t$ in $\mathbb{T}^{n-1} \times \mathbb{D}$,
2. for any $\alpha, \beta \in \mathbb{N}^{n-1}$ there exists $C_{\alpha, \beta} > 0$ such that

$$|\partial_\theta^\alpha \partial_r^\beta G_t(\theta, r)| \leq C_{\alpha, \beta} |r|^{\frac{N+1}{2} - |\beta|}$$

for any $t \in [0, \delta]$, $(\theta, r) \in \mathbb{A}$, and

3. $\lim_{r \rightarrow 0} \tilde{\chi}_t(\theta, r) = \rho_t$.

The first condition is satisfied for $N \geq 2$ and $|r| \leq r_0 \ll 1$ in view of the estimate in 2. and the inverse function theorem. Multiplying G_t by a smooth cut-off function of the form $f_0(r) = f(r/r_0)$ where f is compactly supported in the unit ball and $f(r) = 1$ for $|r| \leq 1/2$ we obtain a smooth function with support contained in the ball of radius r_0 . Notice that the estimates of 2 still hold for the function $G_t f_0$ with constants $C_{\alpha, \beta}$ depending on f but not on r_0 .

The polynomial K_t is the so called Birkhoff polynomial,

$$\nabla K_t(0) = \phi(t)$$

and (r, θ) are local polar symplectic coordinates. The construction of the Birkhoff normal form follows from that of Moser [51] (see also [37], Lemma 3.3.2).

The Birkhoff normal form of P_t is said to be *nondegenerate* if the Hessian of K_t at 0 does not vanish, i.e. $\det \partial^2 K_t(0) \neq 0$. We say as well that P_t is a *twisted map* at ρ_t in this case. Suppose now that P_0 is twisted. Choosing $\delta > 0$ sufficiently small we obtain by continuity that P_t is twisted for any $t \in [0, \delta]$, i.e.

$$\det \partial^2 K_t(0) \neq 0 \quad \forall t \in [0, \delta]. \quad (5.34)$$

Then $\nabla K_t : \mathbb{D} \rightarrow \mathbb{D}_t^* := \nabla K_t(\mathbb{D})$ is a diffeomorphism for any $t \in [0, \delta]$ provided that $r_0 \ll 1$. Denote by $Q_t : \mathbb{A} \rightarrow \mathbb{A}$ the corresponding non-degenerate completely integrable map defined by $Q_t(\theta, r) := (\theta + \nabla K_t(r), r)$. The set of frequency vectors of Q_t is $\Omega_t := \mathbb{D}_t^*$. This is an open cone-like set in \mathbb{R}^{n-1} with vertex at $\phi(t)$.

The remaining of this Section is devoted to the spectral rigidity of the Kronecker tori in a vicinity of an elliptic geodesic. We address the following questions. Suppose that the C^1 family of billiard tables (X, g_t) , $0 \leq t \leq 1$, is weakly isospectral. Assume that (X, g_0) admits a periodic elliptic broken geodesic γ_0 and that the corresponding local Poincaré map is twisted. Does there exist a C^1 family of periodic elliptic broken geodesics $[0, 1] \ni t \rightarrow \gamma_t$ in (X, g_t) along the whole perturbation? Do the corresponding local Poincaré map remain twisted? Do the invariant tori $\Lambda_0(\omega)$ associated to γ_0 give rise to C^1 families of invariant tori $[0, 1] \ni t \rightarrow \Lambda_t(\omega)$ along the whole perturbation? We give an answer of these questions in the following Theorem.

Denote by $\mathbb{B}(\alpha, \epsilon) = \mathbb{B}^{n-1}(\alpha, \epsilon)$ the ball of radius ϵ and center α in \mathbb{R}^{n-1} . Recall that the functions $\beta_t(\omega)$, $I_t(\omega)$ and $L_t(I)$ are defined by (1.5), (1.6) and (1.7) respectively.

Theorem 5.2. *Let (X, g_t) , $t \in [0, 1]$, be a C^1 family of billiard tables of dimension $n \geq 2$ satisfying the weak isospectral condition $(H_1) - (H_2)$. Suppose that (X, g_0) admits a closed elliptic broken billiard trajectory γ_0 with $m \geq 2$ vertices. Suppose as well that the corresponding linear Poincaré map $dP(\rho_0)$ is $N \geq 8$ elementary and that $P = B^m$ is twisted at ρ_0 . Then there exists $\delta_0 > 0$ such that for any interval $I = [0, \delta]$, $0 < \delta < \delta_0$ the following holds.*

- (i) *There exist a C^1 -family of elliptic fixed points $I \ni t \rightarrow \rho_t \in \mathbf{B}_t^* \Gamma$ of $P_t = B_t^m$, the corresponding linear Poincaré map $dP_t(\rho_t)$ is N -elementary and P_t is twisted at ρ_t . Moreover for any $t \in I$ and $|\alpha| \leq \frac{N}{4} - 1$*

$$\partial^\alpha \nabla K_t(0) = \partial^\alpha \nabla K_0(0). \quad (5.35)$$

(ii) There is a set Ξ of positive Lebesgue measure consisting of Diophantine frequencies such that

$$\lim_{\epsilon \searrow 0} \frac{\text{meas}(\Xi \cap \mathbb{B}(\phi(0), \epsilon))}{\text{meas}(\mathbb{B}(\phi(0), \epsilon))} = 0$$

and for any $\omega \in \Xi$ there is a C^1 family of Kronecker invariant tori $I \ni t \rightarrow \Lambda_t(\omega) \subset \mathbf{B}_t^* \Gamma$ of P_t of a frequency ω .

(iii) $\beta_t(\omega) = \beta_0(\omega)$, $L_t(I_0(\omega)) = L_0(I_0(\omega))$, and $I_t(\omega) = I_0(\omega)$ for any $t \in I$ and $\omega \in \Xi$.

It follows from (5.35) that the function $[0, \delta_0) \ni t \rightarrow \phi(t) = \nabla K_t(0)$ is constant, i.e.

$$\forall t \in [0, \delta_0), \quad \text{Spec}(dP_t(\rho_t)) = \text{Spec}(dP_0(\rho_0)). \quad (5.36)$$

A natural question is to describe the largest interval I , if it exists, for which Theorem 5.2 holds. The answer is given by

Proposition 5.3. *Let $N \geq 12$. Then there are only two possibilities that may occur.*

(i) the conclusion (i) – (iii) of Theorem 5.2 holds with $I = [0, 1]$

(ii) there is $0 < \delta_0 \leq 1$ such that Theorem 5.2 holds in any interval $I = [0, \delta]$ with $0 < \delta < \delta_0$ and the limit set

$$\Sigma_{\delta_0} := \lim_{t \rightarrow \delta_0} \{\rho_t, B(\rho_t), \dots, B^{m-1}(\rho_t)\}$$

intersects the boundary of $B_{\delta_0}^* \Gamma$.

The case (ii) means that there is a generalized (glancing to the boundary at certain point) geodesic issuing from the limit set Σ_{δ_0} . If the billiard tables (X, g_t) , $t \in [0, 1]$, are locally strictly geodesically convex then each generalized geodesics of (X, g_t) lies entirely on Γ and the second case can not occur.

We are going to prove Theorem 4, Theorem 5.2 and Proposition 5.3. Firstly using Theorem 9.11 we will obtain a KAM theorem for the C^1 family of symplectic maps \tilde{P}_t given by Proposition 5.1. To this end we will determine the convex set Ω , fix the parameters κ and ϱ , and then estimate the corresponding quantities \mathcal{B}_ℓ which appear in Theorem 9.11.

Consider the C^1 family of exact symplectic mappings \tilde{P}_t in $\mathbb{A} = \mathbb{T}^{n-1} \times \mathbb{D}$ with generating functions

$$\tilde{G}_t(x, r) := \langle x, r \rangle - K_t(r) - G_t(\text{pr}(x), r), \quad (x, r) \in \mathbb{R}^{n-1} \times \mathbb{D}, \quad (5.37)$$

given by Proposition 5.1. Recall that $t \rightarrow K_t$ is a C^1 -family of polynomials with real coefficients of $n - 1$ variables and of degree $\lceil \frac{N}{2} \rceil$, i.e. $K_t \in \mathbb{R}_{\lceil N/2 \rceil}(\xi_1, \dots, \xi_{n-1})$, while $t \rightarrow G_t \in C^\infty(\mathbb{A})$ is a C^1 -family of real valued functions with support in $\mathbb{B}(0, r_0)$ such that

$$|\partial_\theta^\alpha \partial_r^\beta G_t(\theta, r)| \leq C_{\alpha, \beta} |r|^{\frac{N+1}{2} - |\beta|} \quad (5.38)$$

for any $t \in [0, \delta]$, $(\theta, r) \in \mathbb{A}$, and $\alpha, \beta \in \mathbb{N}^{n-1}$.

There exists a constant $A \geq 1$ such that

$$\forall t, s \in [0, \delta], \quad \|K_t - K_s\|_{C^{\lceil N/2 \rceil}} \leq A|t - s|, \quad (5.39)$$

where the norm is taken in $C^{[N/2]}(\mathbb{B}(0, 1))$. The map P_0 is twisted, then by continuity P_t remains twisted for any $t \in [0, \delta]$ provided that $\delta > 0$ is sufficiently small. Choosing $\delta > 0$ small enough, there exists $\varepsilon > 0$ such that the Legendre transform K_t^* of K_t given by (9.146) is well defined in $\mathbb{B}(\phi(0), \varepsilon)$ and

$$\nabla K_t^* : \mathbb{B}(\phi(0), \varepsilon) \rightarrow V_t := \nabla K_t^*(\mathbb{B}(\phi(0), \varepsilon))$$

is a C^1 family of diffeomorphisms with respect to $t \in [0, \delta]$, where V_t is a neighborhood of 0. Moreover, the corresponding inverse maps are $\nabla K_t : V_t \rightarrow \mathbb{B}(\phi(0), \varepsilon)$, hence, $\nabla K_t \circ \nabla K_t^* = \text{id}$ on $\mathbb{B}(\phi(0), \varepsilon)$ for any $t \in [0, \delta]$. In particular the inverse map of $d\nabla K_t(0) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ is $d\nabla K_t^*(\phi(t))$.

We are ready to define suitable convex sets of frequencies Ω . Choose $e = (e_1, \dots, e_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$2c_0 < |e_j| < \frac{1}{2n}, \quad j = 1, \dots, n-1, \quad (5.40)$$

where $0 < c_0 < 1/4n$ is fixed in (5.33) and set $e^* := d\nabla K_t(0)e$. Given $0 < a_0 < \varepsilon$ and $0 < \eta_0 < 1$ we consider for any $0 < a \leq a_0$ the cube of center $\phi(t) + ae^*$ with sides of length $2\eta_0 a$ defined by

$$\Omega = \Omega(t, a) := \left\{ \omega \in \mathbb{R}^{n-1} : |\omega_j - \phi_j(t) - ae_j^*| < \eta_0 a, 1 \leq j \leq n-1 \right\},$$

Obviously, $\overline{\Omega}(t, a) \subset \mathbb{B}(\phi(0), \varepsilon)$ for $a_0 \ll \varepsilon$, hence, ∇K_t^* is well defined and smooth on the convex set $\overline{\Omega}(t, a)$. Denote by $I_{t,a}$ the set of all $s \in [0, \delta]$ such that $|t - s| \leq \eta_0 a$. Denote by \mathbb{D}_a the connected component of the set

$$\left\{ r = (r_1, \dots, r_{n-1}) \in \mathbb{R}^{n-1} : c_0 < |r_j| < a/n, 1 \leq j \leq n-1 \right\}$$

containing ae . Then \mathbb{D}_a is a *convex* open set and $ae \in \mathbb{D}_a$. We claim that there exist $0 < a_0 < 1$ and $0 < \eta_0 < 1$ such that for any $0 < a \leq a_0$, $t \in [0, \delta]$ and $s \in I_{t,a}$ the following relation holds

$$\mathbb{D}^s(t, a) := \nabla K_s^*(\Omega(t, a)) \subset \mathbb{D}_a \subset \mathbb{D}, \quad (5.41)$$

where \mathbb{D} is defined by (5.33). Indeed, for any $\omega \in \Omega(t, a)$, using Taylor's formula up to order three for the function $\omega \rightarrow \nabla K_t^*(\omega)$ at $\omega = \phi(t)$ and the identity $\nabla K_t^*(\phi(t)) = \nabla K_t^*(\nabla K_t(0)) = 0$ we obtain

$$\begin{aligned} |\nabla K_s^*(\omega) - ae| &\leq A\eta_0 a + |\nabla K_t^*(\omega) - a(d\nabla K_t^*)(\phi(t))e^*| \\ &\leq aC_n((A+B)\eta_0 + Ba) \end{aligned}$$

where C_n depends only on n , $A > 0$ is the constant in (5.39) and

$$B := 1 + \sup_{0 \leq t \leq \delta} \|\nabla K_t^*\|_{C^2(\mathbb{B}(\phi(t), \varepsilon))}.$$

Then the inclusion $\mathbb{D}^s(t, a) \subset \mathbb{D}_a$ follows from (5.40) choosing η_0 and a_0 so that

$$C_n((A+B)\eta_0 + Ba_0) < c_0 < 1/2n.$$

On the other hand, the inequalities $c_0 a < |r_j| < a/n$, $j = 1, \dots, n-1$, imply $c_0 |r| < c_0 a < |r_j|$ and we obtain the second inclusion in (11.273), which proves the claim.

Set $\mathbb{A}^s(t, a) := \mathbb{T}^{n-1} \times \mathbb{D}^s(t, a)$ for $s \in I_{t,a}$. The relation (11.273) allows one to apply Proposition 5.1 in $\mathbb{D}^s(t, a)$ for any $t \in [0, \delta]$ fixed, where the parameter of the deformation s

varies in $I_{t,a}$. We point out that both $\Omega(t,a)$ and \mathbb{D}_a are *convex* open sets which allows us to apply Theorem 9.11 and to obtain the corresponding Hölder estimates.

Fix $\tau > n - 1$ and choose $\kappa = \eta a$ in the Diophantine condition (1.2), where $0 < \eta < \eta_0$. Denote by $\Omega_{t,\kappa}$ the set of all $\omega \in \Omega(t,a) \cap D(\tau,\kappa)$ such that $\text{dist}(\omega, \mathbb{R}^{n-1} \setminus \Omega) \geq \kappa$. There exists $0 < \eta_1 = c(n,\tau)\eta_0$, where $0 < c(n,\tau) < 1$ depends only on n and τ such that the Lebesgue measure of $\Omega_{t,\kappa}$ is positive for any $t \in [0,\delta]$, $0 < \eta < \eta_1$ and $a \in (0,a_0]$. Indeed, it follows from [43], Proposition 9.9 that

$$\text{meas}(\Omega(t,a) \setminus \Omega_{t,\eta a}) \leq C \frac{\eta}{\eta_0} \text{meas}(\Omega(t,a)), \quad (5.42)$$

where the positive constant C depends only on n and τ , and we take $c = 1/C$. Let us fix $0 < \eta < \eta_1$ and denote by $\Omega_{t,\kappa}^0$ the set of points of positive Lebesgue density in $\Omega_{t,\kappa}$ (see Sect. 9.4).

Theorem 5.4. *Let $[0,\delta] \ni t \rightarrow \tilde{P}_t$ be a C^1 family of symplectic mappings with generating functions \tilde{G}_t given by (5.37), where $K_t \in \mathbb{R}_{[N/2]}[\xi_1, \dots, \xi_{n-1}]$ while $G_t \in C^\infty(\mathbb{A})$ satisfies (5.38) with $N \geq 4$. Then for any $t \in [0,\delta]$ there exists a C^1 -family of exact symplectic maps*

$$I_{t,a} \ni s \rightarrow (\chi_s : \mathbb{A}^s(t,a) \rightarrow \mathbb{A}^s(t,a))$$

and of real valued functions $L_s \in C^\infty(\mathbb{D}^s(t,a))$ and $R_t \in C^\infty(\mathbb{A}^s(t,a))$ such that for any $s \in I_{t,a}$ the following holds

1. $\tilde{G}_s^0(x, I) = \langle x, I \rangle - L_s(I) - R_s(\text{pr}(x), I)$ is a generating function of $P_s^0 := \chi_s^{-1} \circ \tilde{P}_s \circ \chi_s$
2. $\nabla L_s : \mathbb{D}^s(t,a) \rightarrow \Omega(t,a)$ is a diffeomorphism
3. R_s is flat at $\mathbb{T}^{n-1} \times \nabla L_s^*(\Omega_{t,\kappa}^0)$
4. for any $m \in \mathbb{N}$ there exists a constant $C_m > 0$ independent of $a \in (0,a_0]$ and $t \in [0,\delta]$ such that the following estimates hold

$$|\partial_\varphi^\alpha (a\partial_\omega)^\beta \sigma_a^{-1}(\chi_s - \text{id})| + |\partial_\varphi^\alpha (a\partial_\omega)^\beta \sigma_a^{-1}(\chi_s^{-1} - \text{id})| \leq C_m a^{\frac{N-3}{4}}$$

on $\mathbb{A}^s(t,a)$, and

$$|(a\partial_I)^\beta (\nabla L_s(I) - \nabla K_s(I))| \leq C_m a^{\frac{N+1}{4}} \quad (5.43)$$

on $\mathbb{D}^s(t,a)$ for any $s \in I_{t,a}$ and $|\alpha| + |\beta| \leq m$.

Moreover, if $\nabla K_t(0) = \nabla K_0(0)$ for any $t \in [0,\delta]$, then for any $0 < a \leq a_0$ there exists a C^1 -family of exact symplectic maps $\chi_s : \mathbb{A}^0(0,a) \rightarrow \mathbb{A}^0(0,a)$ in $I = [0,\delta]$ and of real valued functions $L_s \in C^\infty(\mathbb{D}^0(0,a))$ and $R_t \in C^\infty(\mathbb{A}^0(0,a))$ such that 1.) - 4.) hold for any $s \in I$ and $t = 0$.

Proof. To prove the first part of the Theorem we apply Theorem 9.11 to the C^1 family of symplectic mappings $I_{t,a} \ni s \rightarrow \tilde{P}_s \in C^\infty(\mathbb{A}^s(t,a), \mathbb{A}^s(t,a))$.

Let us estimate the corresponding quantities \mathcal{B}_l for $l \geq 1$ defined by (9.179)-(9.181). First of all the constant λ in (9.187) can be fixed by

$$\lambda = \sup_{t \in [0,\delta]} \|\partial^2 K_t\|_{C^{[N/2]}(\mathbb{B}(0,1))}.$$

Given $\ell = m + \mu$ with $m \in \mathbb{N}_*$ and $0 \leq \mu < 1$ we get by (11.273) that for any $s \in I_{t,a}$

$$\|G_s\|_{\ell, \mathbb{A}^s(t,a); \kappa} \leq \|G_s\|_{\ell, \mathbb{T}^{n-1} \times \mathbb{D}_a; \kappa} \leq \|G_s\|_{m+1, \mathbb{T}^{n-1} \times \mathbb{D}_a; \kappa}.$$

The second inequality follows from the fact that \mathbb{D}_a is convex. On the other hand,

$$\|G_s\|_{m+1, \mathbb{T}^{n-1} \times \mathbb{D}_a; \kappa} = \sup_{|\alpha|+|\beta| \leq m+1} \|\partial_\theta^\alpha (\kappa \partial_r)^\beta G_t\|_{C^0(\mathbb{T}^{n-1} \times \mathbb{D}_a)}$$

and using Proposition 5.1, 2, we obtain $\|G_s\|_{\ell, \mathbb{A}^s(t,a); \kappa} \leq C_m a^{\frac{N+1}{2}}$ for any $s \in I_{t,a}$. We choose $\varrho = \kappa^{\frac{N+1}{4}} < \kappa$, where $N \geq 4$. Then

$$\|G_s\|_{\ell, \mathbb{A}^s(t,a); \kappa} \leq \kappa \varrho C'_m a^{\frac{N-3}{4}}$$

for any $s \in I_{t,a}$, where $C'_m = C_m \eta^{-\frac{N+5}{4}}$. Moreover,

$$\|\partial^2 K_s\|_{\ell, \mathbb{A}^s(t,a); \kappa} \leq \|\partial^2 K_s\|_{\ell, \mathbb{T}^{n-1} \times \mathbb{B}(0,1); \kappa} \leq C \|\partial^2 K_t\|_{C^{[N/2]}(\mathbb{B}(0,1))}$$

and

$$S_\ell(\nabla K^*) \leq \sup_{0 \leq t \leq \delta} (1 + \|\nabla K_t^*\|_{C^1(\mathbb{B}(\phi(0), \varepsilon))})^{\ell-1} (1 + \|\nabla K_t^*\|_{C^\ell(\mathbb{B}(\phi(0), \varepsilon))}).$$

Thus for any $\ell \geq 1$ we obtain

$$\mathcal{B}_\ell \leq \kappa \varrho C_\ell a^{\frac{N-3}{4}}$$

where $C_\ell > 0$ depends neither on $a \in (0, a_0]$ nor on $t \in [0, \delta]$. Choosing $a_0 \ll 1$ we get $\mathcal{B}_2 \leq \epsilon \kappa \varrho \lambda^{-4}$, which gives (9.188). Applying Theorem 9.11 we obtain 1-4.

The equality $\nabla K_t(0) = \nabla K_0(0)$ means that $\phi(t) = \phi(0)$ which implies $\Omega(t, a) = \Omega(0, a)$. Then one can take $I_{t,a} = [0, \delta]$ in (11.273) which implies 1-4 in $[0, \delta]$. \square

Proof of Theorem 4. The set of fixed points of \mathcal{J}_1 in $X_t \in \mathcal{B}$ defines a bouncing ball geodesic γ_t in (X_t, g) which is preserved by \mathcal{J}_2 . We are going to apply Theorem 5.4 to the Birkhoff Normal Forms of the local Poincaré maps P_t associated to γ_t .

Let $t \rightarrow \rho_t$ be a C^1 family of fixed points of $P_t = B_t^2$. Denote by $\rho_{t,j} = (x_{t,j}, 0) = B_t^j(\rho_t)$, $j = 1, 2$, the corresponding periodic orbit of B_t . Fix $t \in [0, \delta]$. Denote by $U \subset \tilde{X}$ a neighborhood of the vertices $x_{t,0}$ and $x_{t,1}$ of $\gamma_{t,1}$ such that $\mathcal{J}_k(U) = U$, $k = 1, 2$. Denote the restrictions of the two involutions to $\Gamma_t \cap U$ by J_1 and J_2 and by $\tilde{J}_j : T^*(\Gamma_t \cap U) \rightarrow T^*(\Gamma_t \cap U)$ the corresponding lifts. The set $\Gamma_t \cap U$ has two connected components Γ_t^j , $j = 1, 2$, and $J_1(\Gamma_t^1) = \Gamma_t^1$ while $J_2(\Gamma_t^1) = \Gamma_t^2$. Since J_1 and J_2 act as isometries and commute with each other, using the definition of B_t in Sect. 2.1, we obtain that the involutions \tilde{J}_j , $j = 1, 2$, commute with each other and also with B_t .

Denote by Ξ the union of $\Omega_{t, \eta a}$, $0 < a \leq a_0$. For any $\omega \in \Xi$ we set $\Lambda_t^1(\omega) = \Lambda_t(\omega)$ and $\Lambda_t^2(\omega) = B_t(\Lambda_t(\omega))$. Then $\tilde{J}_1(\Lambda_t^j(\omega))$, $j = 1, 2$, are also invariant circles of $P_t = B_t^2$ of frequency $\omega \in \Xi$ and $\Lambda_t^j(\omega) = \tilde{J}_1(\Lambda_t^j(\omega))$ for $j = 1, 2$, while $\Lambda_\omega^2 = \tilde{J}_2(\Lambda_t^1(\omega))$. To prove it we use the following argument. Since $\dim T^*\Gamma_{t,j} = 2$ the KAM circle $\Lambda_t^j(\omega)$ divides $T^*\Gamma_{t,j}$ into two connected components, and it contains the elliptic fixed point $\rho_{t,j} = (x_{t,j}, 0)$ of P_t in its interior D_j . Moreover, $\tilde{J}_1(\rho_j) = \rho_j$, hence, $\tilde{J}_1(\Lambda_t^j(\omega))$ contains $\rho_{t,j}$ in its interior $\tilde{J}_1(D_j)$ as well. On the

other hand, \tilde{J}_1 preserves the volume form of $T^*\Gamma_{t,1}$, hence, $\Lambda_t^j(\omega)$ intersects $\tilde{J}_1(\Lambda_t^j(\omega))$. This implies $\Lambda_t^1(\omega) = \tilde{J}_1(\Lambda_t^1(\omega))$, since P_t acts transitively on both of them. In the same way we prove that $\Lambda_t^2(\omega) = \tilde{J}_2(\Lambda_t^1(\omega))$.

Recall that the family Γ_t , $t \in [0, 1]$, is given by a C^1 family of embeddings $\psi_t \in C^\infty(\Gamma, \tilde{X})$, where $\psi_t(\Gamma) = \Gamma_t$. Without loss of generality we suppose that $\gamma_0 = \text{id}_\Gamma$ is the identity at Γ . Notice that the vectors $\frac{\partial \psi_t}{\partial x}(x)$ and $\nu_t(\psi_t(x))$ provide a base of $T_{\psi_t(x)}\tilde{X}$ for any $x \in \Gamma$, hence,

$$\forall x \in \Gamma_t, \quad \dot{\psi}_t(x) = \lambda(t, x) \frac{\partial \psi_t}{\partial x}(x) + \delta_\nu \Gamma_t(\psi_t(x)) \nu_t(\psi_t(x)) \quad (5.44)$$

where $t \rightarrow \lambda(t, \cdot) \in C^\infty(\Gamma)$ is continuous on $[0, \delta]$ and the function $\delta_\nu \Gamma_t$ is defined by (2.15) and it belongs to $C^\infty(\Gamma_t)$. We are going to show that the function $\delta_\nu \Gamma_t$ is flat at $x_{t,1}$.

Using Corollary 3 and the symmetry with respect to \tilde{J}_1 given above we obtain

$$\int_{\Lambda_t(\omega)} \langle \xi_t^+(\rho), \nu_t(\pi_t(\rho)) \rangle \delta_\nu \Gamma_t(\pi_t(\rho)) d\mu_t(\rho) = 0 \quad (5.45)$$

for any $\omega \in \Xi$, with $0 < a \leq a_0$. Moreover, the functions

$$\rho \rightarrow f_t(\rho) := \langle \xi_t^+(\rho), \nu_t(\pi_t(\rho)) \rangle, \quad \rho \rightarrow h_t(\rho) := \delta_\nu \Gamma_t(\pi_t(\rho))$$

are invariant with respect to the involution \tilde{J}_1 . Let us parametrize Γ_t^1 by its arclength $y \in [-c, c]$ so that $y(x_{t,1}) = 0$ and denote by (y, η) the corresponding local coordinates in $T^*\Gamma_{t,1}$. Then $J_1(y) = -y$ for any y . For any invariant circle $\Lambda_t^1(\omega)$, $\omega \in \Xi$, there is $y(\omega) > 0$ such that $\pi_t(\Lambda_t^1(\omega)) = [-y(\omega), y(\omega)]$. Notice that $f_t(y, \eta) = \langle \xi_t^+(y, \eta), \nu_t(y) \rangle > 0$ for $(y, \eta) \in \Lambda_t^1(\omega)$ since $\Lambda_t^1(\omega)$ is contained in $\mathbf{B}_t^*(\Gamma_t)$. On the other hand, $h_t(y, \eta) = h_t(y)$ depends only on y . We are going to show that there exists an infinite sequence $(y_j)_{j \in \mathbb{N}} \subset (0, c)$ such that $\lim y_j = 0$ and $h_t(y_j) = 0$. Indeed, suppose that $h_t(y) \neq 0$ in $(0, b)$ for some $b > 0$. Take $\omega \in \Xi$ such that $0 < y(\omega) < b$. The function $h_t(y)$ is *even* because it is invariant with respect to \tilde{J}_1 , hence it will not change its sign in the interval $[-y(\omega), y(\omega)]$. Then $h_t(y, \eta) f_t(y)$ will not change its sign on $\Lambda_t^1(\omega)$ and it is not identically null, which contradicts (5.45). This proves the existence of an infinite sequence $\{y_j\}_{j \in \mathbb{N}}$ such that $h_t(y_j) = 0$, $y_j \neq 0$ for any $j \in \mathbb{N}$ and $\lim y_j = 0$. Now there exists an infinite sequence $(y'_j)_{j \in \mathbb{N}} \subset (0, b)$ such that $y_j \leq y'_j \leq y_{j+1}$ and $\frac{dh_t}{dy}(y'_j) = 0$, and so on. This implies that the Taylor series of $h_t(y)$ vanishes at $y = 0$. In particular we obtain that $x_{t,j} = x_{0,j}$ since $\lambda(t, x_{t,j}) = 0$. Hence, the function $\Gamma_t \ni x \rightarrow \delta_\nu \Gamma_t(x)$ is flat at $x = x_{0,1}$.

Take local coordinates $x : \Gamma_0^1 \rightarrow \mathbb{R}$ in the neighborhood Γ_0^1 of $x_{0,1}$ in $\Gamma = \Gamma_0$ such that $x(x_{0,1}) = 0$ and consider the equation

$$\frac{d}{dt} u_t(x) = -\lambda(t, u_t(x)) \quad (5.46)$$

with initial data $u_0(x) = x$. This problem has a unique solution $u_t(x)$ for t in a neighborhood of 0 and x in an open interval $V \subset \mathbb{R}$ containing $x = 0$. Moreover, $u_t : V \rightarrow \mathbb{R}$ is a C^1 family of local diffeomorphisms. Consider the C^1 family of embeddings $v_t = \psi_t \circ u_t : V \rightarrow \tilde{X}$. The set $v_t(V)$ is an open neighborhood of $x_{0,1}$ in Γ_t , v_t gives a local parametrization of Γ_t in $v_t(V)$ and $v_0(x) = x$. Using (5.44) and (5.46) one obtains that the map

$$V \ni x \rightarrow \dot{v}_t(x) = \delta_\nu \Gamma_t(v_t(x)) \nu_t(v_t(x))$$

is flat at $x = 0$ for any $s \in [0, \delta]$. Then for any $\varphi \in C^\infty(\tilde{X})$, the function

$$V \ni x \rightarrow \varphi(v_t(x)) - \varphi(x) = \int_0^t d\varphi(v_s(x)) \dot{v}_s(x) ds$$

is flat at $x = 0$ which means that Γ_t is tangent to infinite order to Γ_0 at $x_{0,1}$ for $t > 0$ sufficiently small. Replacing Γ by Γ_t , $t \in [0, 1]$, we complete the proof of the Theorem. \square

Proof of Theorem 5. Corollary B.6 implies that the Poincaré map associated with the elliptic bouncing ball geodesic γ_1 is always non-degenerate (twisted) for elliptical billiard tables. Fix the foci $F_1 \neq F_2$. Except of five confocal families of ellipses given explicitly by (A.73), the geodesic γ_1 is 4-elementary. The two conditions are open in the C^5 topology and the Theorem follows from Theorem 4. \square

Proof of Theorem 5.2. It follows from Theorem 5.4 that for any $0 < a \leq a_0 \ll 1$ and $\omega \in \Omega_{t,\kappa}$ with $\kappa = \eta a$ there is a C^1 family of Kronecker invariant tori $I_{t,a} \ni s \rightarrow \Lambda_s(\omega)$ of P_s . Moreover, Corollary 2 implies that $I_s(\omega) = I_t(\omega)$ and $\nabla L_s(I_t(\omega)) = \nabla L_t(I_t(\omega))$, hence, $\partial_I^\alpha \nabla L_s(I_t(\omega)) = \partial_I^\alpha \nabla L_t(I_t(\omega))$ for any $\alpha \in \mathbb{N}^{n-1}$ in view of Lemma 3.4. Then for any $\omega \in \Omega_{t,\kappa}$, $s \in I_{t,a}$ and $\alpha \in \mathbb{N}^{n-1}$ of length $|\alpha| \leq N/4 - 1$ using (5.43) we obtain

$$\begin{aligned} |\partial_I^\alpha \nabla K_s(I_t(\omega)) - \partial_I^\alpha \nabla K_t(I_t(\omega))| &\leq |\partial_I^\alpha \nabla K_s(I_t(\omega)) - \partial_I^\alpha \nabla L_s(I_t(\omega))| \\ &\quad + |\partial_I^\alpha \nabla K_t(I_t(\omega)) - \partial_I^\alpha \nabla L_t(I_t(\omega))| \\ &\leq C a^{\frac{N+1}{4} - |\alpha|} \leq C a^{\frac{5}{4}}. \end{aligned}$$

Taking $s = t + a$ there is $t(a) \in [t, t + a]$ such that

$$\left| \frac{d}{ds} \Big|_{s=t(a)} \partial_I^\alpha \nabla K_s(I_t(\omega)) \right| \leq C a^{\frac{1}{4}},$$

where the positive constant C is independent of $a \in (0, a_0)$ and $\omega \in \Omega_{t,\kappa}^0$. Let $a \rightarrow 0$. Then $\Omega_{t,\kappa}^0 \subset \Omega(t, a)$ shrinks to $\phi(t)$, hence,

$$\lim_{a \rightarrow 0} \sup \{ |I_t(\omega)| : \omega \in \Omega_{t,\kappa}^0 \} = 0$$

and we get

$$\frac{d}{dt} \partial_I^\alpha \nabla K_t(0) = 0$$

for any $t \in [0, \delta]$. This implies (5.35) in that interval. By assumption $N \geq 8$, hence (5.36) holds for $t \in [0, \delta]$.

We are going to define the set of frequencies Ξ as a union of $\Omega_{0,\kappa}^0$. Recall that $\kappa = \eta a$, where $0 < \eta \leq \eta_1 = c(n, \tau) \eta_0$. Moreover, $\Omega(0, a)$ depends on the choice of $e^* := d\nabla K_t(0)e$, where $d\nabla K_t(0)$ is an isomorphisme of \mathbb{R}^{n-1} and e satisfies (5.40), hence, $\Omega(0, a)$ depends as well as on the parameter $0 < c_0 < 1/4n$ defined in (5.33). Now varying the parameters $0 < c_0 < 1/2n$, e satisfying (5.40), $0 < a \leq a_0$ and $0 < \eta \leq \eta_1 = c(n, \tau) \eta_0$ we denote by Ξ the union of the corresponding sets $\Omega_{0,\kappa}^0$. The set of frequencies Ξ satisfies (ii) by construction in view of (5.42).

Using the second part of Theorem 5.4 and Corollary 2 in $I = [0, \delta]$ we complete the proof of the Theorem. \square

Proof of Proposition 5.3. Denote by $0 < \delta_0 \leq 1$ the supremum of all $\delta > 0$ such that (i), Theorem 5.2 holds in $[0, \delta]$. Suppose that $\delta_0 < 1$ and that $\Sigma_{\delta_0} \subset \mathbf{B}_{\delta_0}^* \Gamma$. Fix ρ in the limit set $\lim_{t \rightarrow \delta_0} \rho_t$. Then $P_{\delta_0} = B_{\delta_0}^m$ is well-defined and smooth in a neighborhood of ρ . By continuity, ρ is a fixed point of P_{δ_0} and (5.36) holds true for $t \in [0, \delta_0]$. Hence, ρ is an elliptic fixed point of P_{δ_0} and there are no resonances of order less or equal to N . Now Proposition 5.1 provides a C^1 family of Birkhoff normal forms of P_t in an interval $t \in [0, \delta_0 + \varepsilon]$, where $\varepsilon > 0$. On the other hand, (5.35) implies that $\partial^2 K_t(0) = \partial^2 K_0(0)$ for $t \in [0, \delta_0[$ since $N \geq 12$. By continuity, this equality is true for $t \in [0, \delta_0]$. Then P_t is twisted for any $t \in [0, \delta_0 + \varepsilon]$, provided that $\varepsilon > 0$ is sufficiently small. Applying Theorem 5.4 we show as above that (i) and (ii) in Theorem 5.2 hold in $[0, \delta_0 + \varepsilon]$. This contradicts the choice of δ_0 . If $\delta_0 = 1$ and $\Sigma_1 \subset \mathbf{B}_{\delta_0}^* \Gamma$, then (i) holds in $[0, 1]$ and Theorem 5.4 holds in $I = [0, 1]$. \square

6 Isospectral deformation of locally strictly geodesically convex billiard tables of dimension two.

The aim of this Section is to prove Theorem 6. More generally we consider isospectral deformations of a billiard table (X, g) in an ambient Riemannian manifold (\tilde{X}, g) with a locally strictly geodesically convex (with respect to the outward normal) boundary $\Gamma = \partial X$. This means that if a geodesic $s \rightarrow \gamma(s)$ of (\tilde{X}, g) is tangent to Γ at $s = 0$ then the order of the tangency is exactly two and $\gamma(s) \notin X$ for $0 < |s| \ll 1$. The behavior of the billiard ball map near $S^* \Gamma$ is investigated by Melrose [48] and Marvizi and Melrose [46] in the more general context of pairs of glancing surfaces.

Consider the hypersurfaces $\Sigma_1 := S^* \tilde{X}$ and $\Sigma_2 := T^* \tilde{X}|_{\Gamma}$ in $T^* \tilde{X}$. Set $f_1 = h - 1$, where the Hamiltonian h is just the Legendre transform $h(x, \xi) = g^{ij}(x) \xi_i \xi_j$ of the Riemannian metric defined locally by $g(x, v) = g_{ij}(x) v^i v^j$ and denote by $f_2 \in C^\infty(T^* \tilde{X})$ a smooth function which is constant on the fibers ($f_2(x, \xi) = f_2(x)$) and such that $f_2(x) > 0$ for x in the interior of X , $f_2(x) < 0$ in the exterior of X and $f_2(x) = 0$, $df_2(x) \neq 0$ for $x \in \Gamma$. Then the hypersurfaces Σ_j , $j = 1, 2$, are just the zero level sets of the non-degenerate Hamiltonians f_j ($df_j \neq 0$ on $\Sigma_j = \{f_j = 0\}$). One can show that Γ is locally strictly geodesically convex with respect to the outward normal to Γ if and only if the following relation holds

$$f_1(\varrho) = f_2(\varrho) = \{f_1, f_2\}(\varrho) = 0 \implies \{f_1, \{f_1, f_2\}\}(\varrho) < 0 \text{ and } \{f_2, \{f_2, f_1\}\}(\varrho) > 0, \quad (6.47)$$

where $\{, \}$ is the Poisson bracket related to the canonical symplectic two-form $\tilde{\omega}$ of $T^* \tilde{X}$. In particular

$$\mathcal{K} := \{\varrho \in T^* \tilde{X} : f_1(\varrho) = f_2(\varrho) = \{f_1, f_2\}(\varrho) = 0\}$$

is a smooth submanifold of $T^* \tilde{X}$ of co-dimension two. The characteristic foliations of the two-form $\tilde{\omega}|_{\Sigma_j}$ given by the non-parametrized integral curves of the hamiltonian vector fields of f_j define two involutions \mathcal{J}_j in a neighborhood U of the glancing manifold \mathcal{K} in $\Sigma := \Sigma_1 \cap \Sigma_2$. For any $\varrho \in U \setminus \mathcal{K}$, the point $\mathcal{J}_j(\varrho) \in U$ is just the second point of intersection of the characteristic of

$\tilde{\omega}|_{\Sigma_j}$ passing through ρ with U . The set of fixed points of \mathcal{J}_j is just \mathcal{K} . Moreover, (6.47) implies that the differentials of \mathcal{J}_j are linearly independent at any point of \mathcal{K} . The billiard ball map is given by the composition $\mathcal{J} := \mathcal{J}_2 \circ \mathcal{J}_1 : U \rightarrow U$. Moreover, $\mathcal{J}_j^* \omega_\Sigma = \omega_\Sigma$, where $\omega_\Sigma := \tilde{\omega}|_\Sigma$. Then the billiard ball map \mathcal{J} preserves $\omega|_\Sigma$ as well. Notice that the map \mathcal{J} is smooth in U but the two-form $\omega|_\Sigma$ is degenerate at \mathcal{K} . To make the later symplectic one considers the quotient space U/\mathcal{J}_2 of U by the action of \mathcal{J}_2 . In our case it is given by the closed co-ball bundle $\overline{B^*}\Gamma \subset T^*\Gamma$ equipped with the canonical symplectic two-form. Let $\pi : U \rightarrow U/\mathcal{J}_2 \cong \overline{B^*}\Gamma$ be the canonical projection. Then the billiard ball map is represented by the boundary map $B = \pi \circ \mathcal{J}_1 \circ \pi^+$, where π^+ is defined by (2.13). We call B a billiard ball map as well.

A local normal form of the pair of involutions \mathcal{J}_j , $j = 1, 2$, and of the two form ω has been obtained by Melrose [48] in a neighborhood of any point of the glancing manifold \mathcal{K} . This normal form leads to a local symplectic normal form of the billiard ball map B at any point of the projection $\pi(\mathcal{K}) = S^*\Gamma$ (see also [31], Theorem 21.4.8).

Consider a C^1 family of Riemannian metrics $[0, \delta_0] \ni t \rightarrow g_t$ in \tilde{X} and suppose that Γ is locally strictly geodesically convex in (\tilde{X}, g_0) with respect to the outward normal field at Γ . Choosing $0 < \delta \leq \delta_0$ sufficiently small we obtain by (6.47) that Γ remains locally strictly geodesically convex in (\tilde{X}, g_t) for any $t \in [0, \delta]$. We denote by $\Sigma_{j,t} = \{f_{j,t} = 0\}$, $j = 1, 2$, the corresponding pairs of glancing hypersurfaces. Here $f_{1,t} + 1$ is the Hamiltonian corresponding to the Riemannian metric g_t via the Legendre transform and $f_{2,t} = f_2$, hence, both families of Hamiltonians are C^1 smooth with respect to t . Moreover, $f_{j,t}$ satisfy (6.47) for $t \in [0, \delta]$ and we denote by \mathcal{K}_t the corresponding glancing manifolds. Consider the corresponding C^1 family of billiard ball maps $B_t : U_t \rightarrow \mathbf{B}_t^*\Gamma$, $t \in [0, \delta]$, where U_t are suitable open subsets of $\mathbf{B}_t^*\Gamma$. The map B_t is exact symplectic and smooth in U_t , and it is extended by continuity as the identity map on $S^*\Gamma$. Using the construction of the local symplectic normal form of B_t at $S^*\Gamma$ in [48] and the interpolating Hamiltonian introduced by Marvizi and Melrose [46] we obtain below a C^1 family of Birkhoff Normal Forms of B_t , $t \in [0, \delta]$.

From now on we suppose that $\dim \tilde{X} = 2$ and we denote by $2\pi l_t$ the length of Γ with respect to the Riemannian metric g_t . Set $\mathbb{A}_t := \mathbb{T} \times (l_t - \varepsilon, l_t + \varepsilon)$, where $\varepsilon > 0$ will be chosen below small enough. Denote by $\text{pr} : \mathbb{R} \rightarrow \mathbb{T}$ the canonical projection.

Proposition 6.1. *Let (X, g_t) , $t \in [0, \delta]$, be a C^1 family of connected locally strictly geodesically convex billiard tables in \tilde{X} . Then there exists*

- (1) a C^1 -family of exact symplectic transformation $[0, \delta] \ni t \rightarrow \tilde{\chi}_t \in C^\infty(\mathbb{A}_t, V_t)$, where $V_t := \tilde{\chi}_t(\mathbb{A}_t) \subset T^*\Gamma$ is a neighborhood of $S_t^*\Gamma$, $\tilde{\chi}_t(\mathbb{T} \times \{l_t\}) = S_t^*\Gamma$ and $\tilde{\chi}_t(\mathbb{T} \times (l_t - \varepsilon, l_t)) \subset \mathbf{B}_t^*\Gamma$,
- (2) a C^1 -family of real valued functions $\zeta_t \in C^\infty(\mathbb{R})$ and $G_t \in C_0^\infty(\mathbb{A}_t)$, $t \in [0, \delta]$, with $\zeta_t(l_t) = 0$ and $\zeta'_t(l_t) < 0$

such that the following holds

- (i) the function $\tilde{G}_t \in C^\infty(\mathbb{R} \times \mathbb{R})$ defined by

$$\tilde{G}_t(x, r) := xr - \frac{2}{3}\zeta_t(r)^{\frac{3}{2}} - G_t(\text{pr}(x), r)$$

is a generating function of the symplectic map $\tilde{P}_t := \tilde{\chi}_t^{-1} \circ B_t \circ \tilde{\chi}_t$ in $\mathbb{T} \times (l_t - \varepsilon, l_t)$,

- (ii) G_t is flat at $r = l_t$, which means that $\partial_r^\alpha G_t(\theta, l_t) = 0$ for any $\theta \in \mathbb{T}$ and $\alpha \in \mathbb{N}$.

The function $\tilde{\zeta}_t := \zeta_t \circ \chi_t^{-1}$ is an interpolating Hamiltonian of B_t in the sense of Marvizi and Melrose [46], which means that for any $\varphi \in C^\infty(T^*\Gamma)$, the function

$$\varphi \circ B_t - \varphi \circ \exp\left(\tilde{\zeta}_t^{\frac{1}{2}} X_{\tilde{\zeta}_t}\right)$$

is *flat* at $S_t^*\Gamma$, where $t \rightarrow \exp(tX_{\tilde{\zeta}_t})$ is the flow of the Hamiltonian vector field $X_{\tilde{\zeta}_t}$ of $\tilde{\zeta}_t$ on $T^*\Gamma$.

Proof. The proof of the Proposition is based on a local normal form of pairs of glancing hypersurfaces near the glancing manifold obtained by Melrose in [48]. Consider the two involutions $\mathcal{J}_{j,t}$ associated to the characteristic foliations of $\Sigma_{j,t} = \{f_{j,t} = 0\}$ in a neighborhood U_t of the glancing manifold \mathcal{K}_t in $\Sigma_t := \Sigma_{1,t} \cap \Sigma_{2,t}$ for $t \in [0, \delta]$. In this way one obtains a C^1 family of billiard ball maps given by the compositions $\mathcal{J}_t = \mathcal{J}_{2,t} \circ \mathcal{J}_{1,t} : U_t \rightarrow U_t$ and $B_t = \pi_t \circ \mathcal{J}_{1,t} \circ \pi_t^+$. Arguing as in the proof of [31], Theorem C.4.8, we first obtain a C^1 family of normal forms of the two involutions $\mathcal{J}_{j,t}$. More precisely, following the first part of the proof of that theorem we get a C^1 family of diffeomorphisms $\tilde{\Psi}_t : V \rightarrow U_t$, where V is a neighborhood of a point $z^0 = (z_1^0, 0, z^{0'}) \in \mathbb{R}^{2n-2}$ such that

$$\tilde{\Psi}_t^{-1} \circ \mathcal{J}_{1,t} \circ \tilde{\Psi}_t(z_1, z_2, z') = (z_1 + z_2, -z_2, z') + O_N(z_2^N),$$

$$\tilde{\Psi}_t^{-1} \circ \mathcal{J}_{2,t} \circ \tilde{\Psi}_t(z_1, z_2, z') = (z_1, -z_2, z')$$

for any $N \in \mathbb{N}$. In order to do this we consider an asymptotic expansion of $\tilde{\Psi}_t$ in formal power series $\tilde{\Psi}_t(z_1, z_2, z') \approx \sum \tilde{\Psi}_{t,k}(z_1, z') z_2^k$. The functions $\tilde{\Psi}_{t,k}(z_1, z')$ are obtained by solving linear systems of ordinary differential equations (see the proof of [31], Theorem C.4.8). In this way we obtain that the maps $t \rightarrow \tilde{\Psi}_{t,k} \in C^\infty(V)$ are C^1 smooth, and then using Borel's extension theorem we get a C^1 smooth family of maps $\psi_t : V \rightarrow U_t$ such that

$$|\tilde{\Psi}_t(z_1, z_2, z') - \sum_{k=0}^n \tilde{\Psi}_{t,k}(z_1, z') z_2^k| \leq C_N |z_2|^N.$$

Then following the proof of Theorem 21.4.4 in [31] (see also [17]) one finds a C^1 family of diffeomorphisms Ψ_t defined by $\Psi_t^{-1} : W \rightarrow U_t$, where W is a neighborhood of a point $\varrho^0 = (x_1^0, x^{0'}, 0, \xi^{0'}) \in T^*\mathbb{R}^{n-1}$, $U_t := \Psi_t^{-1}(W)$ is an open neighborhood of the glancing manifold \mathcal{K}_t in $\Sigma_{1,t} \cap \Sigma_{2,t}$ and such that

$$\left\{ \begin{array}{l} (\Psi_t^{-1})^*(\omega|_{\Sigma_t}) = dx_1 \wedge d(\xi_1^2) + \sum_{k=2}^{n-1} dx_k \wedge d\xi_k \\ \Psi_t \circ \mathcal{J}_{1,t} \circ \Psi_t^{-1}(x_1, x', \xi_1, \xi') = (x_1 + \xi_1, x', -\xi_1, \xi') + \tilde{R}_t(x, \xi), \\ \Psi_t \circ \mathcal{J}_{2,t} \circ \Psi_t^{-1}(x_1, x', \xi_1, \xi') = (x_1, x', -\xi_1, \xi') \\ \forall j \in \mathbb{N}, \partial_{\xi_1}^j \tilde{R}_t(x, 0, \xi') = 0. \end{array} \right. \quad (6.48)$$

We mention just for information that the formal power series are not convergent in general even when the hypersurfaces are analytic. It has been proved in [17] that for any t fixed the corresponding functions in (6.48) belong to the Gevrey class G^2 of index two if the glancing hypersurfaces are analytic.

Suppose now that $\Sigma_{1,t} = S_t^* \tilde{X}$ and $\Sigma_{2,t} = T^* \tilde{X}|_{\Gamma}$. Then U_t is an open subset of $S_t^* \tilde{X}|_{\Gamma}$. Choosing normal to Γ coordinates with respect to the metric g_t one can assume that locally $f_{1,t}(y, y_n, \eta, \eta_n) = y_n$ and $f_{2,t}(y, y_n, \eta, \eta_n) = \eta_n^2 + q_t(y, y_n, \eta)$, where $t \rightarrow q_t(y, y_n, \eta)$ is a C^1 family of quadratic forms with respect to $\eta = (\eta_1, \dots, \eta_{n-1})$ and $q_t(y, 0, \eta)$ is the Hamiltonian corresponding to the induced metric on Γ via the Legendre transform. In these coordinates U_t can be identified with the set of (y, y_n, η, η_n) , where $y_n = 0$, (y, η) are local coordinates in $T^*\Gamma$ near a point $\varrho_t^0 \in S_t^*\Gamma$ and $\eta_n^2 + q_t(y, 0, \eta, \eta_n) = 1$, while $\mathcal{K}_t \subset U_t$ is given by $y_n = \eta_n = 0$. Moreover, $\mathcal{J}_{2,t}(y, 0, \eta, -\eta_n) = -\mathcal{J}_{2,t}(y, 0, \eta, \eta_n)$ and $\pi_t^\pm(y, \eta) = \pm\sqrt{1 - q_t(y, 0, \eta)}$ for $(y, \eta) \in \mathbf{B}_t^*\Gamma$. Setting

$$\Psi_t(y, \eta, \eta_n) = (x_t(y, \eta, \eta_n), \xi_{t1}(y, \eta, \eta_n), \xi'_t(y, \eta, \eta_n))$$

where $\eta_n^2 + q_t(y, 0, \eta, \eta_n) = 1$ one obtains from the third relation of (6.48) that x_t and ξ'_t are even functions of η_n while ξ_{t1} is odd. Then there exists a C^1 family of functions $t \rightarrow (\tilde{x}_t, \tilde{\xi}_t) \in C^\infty(T^*\mathbb{R}^{n-1})$ such that

$$\Psi_t(y, 0, \eta, \eta_n) = (\tilde{x}_t(y, \eta, \eta_n^2), \eta_n \tilde{\xi}_{t1}(y, \eta, \eta_n^2), \tilde{\xi}'_t(y, \eta, \eta_n^2)),$$

where $\eta_n^2 + q_t(y, 0, \eta) = 1$.

We define a C^1 family of diffeomorphisms $\tilde{\chi}_t$ by

$$\tilde{\chi}_t^{-1}(y, \eta) = (\tilde{x}_t(y, \eta, \tilde{\eta}_n), \tilde{\eta}_n \tilde{\xi}_{t1}(y, \eta, \tilde{\eta}_n^2), \tilde{\xi}'_t(y, \eta, \tilde{\eta}_n)),$$

where $\tilde{\eta}_n := 1 - q_t(y, 0, \eta)$. Then $\tilde{\chi}_t : V \rightarrow V_t := \tilde{\chi}_t(V) \subset T^*\Gamma$ is a C^1 family of symplectic mappings, i.e.

$$\tilde{\chi}_t^* \left(\sum_{j=1}^{n-1} dy_j \wedge d\eta_j \right) = \sum_{j=1}^{n-1} dx_j \wedge d\xi_j,$$

where $V \subset T^*\mathbb{R}^{n-1}$ is an open neighborhood of a given point $(x^0, 0, \xi^{0'})$ and we get the following symplectic normal form of the billiard ball maps

$$\tilde{\chi}_t^{-1} \circ B_t \circ \tilde{\chi}_t(x_1, x', \xi_1, \xi') = (x_1 + \sqrt{\xi_1}, x', \xi_1, \xi') + R_t(x, \xi),$$

where $t \rightarrow R_t \in C^\infty(T^*\mathbb{R}^{n-1}, T^*\mathbb{R}^{n-1})$ is a C^1 family of maps such that $\partial_{\xi_1}^j R_t(x, 0, \xi') = 0$ for any $j \in \mathbb{N}$. The interpolating Hamiltonian $\tilde{\zeta}_t$ is defined by the ξ_1 component of $\tilde{\chi}_t^{-1}$, i.e.

$$\tilde{\zeta}_t(y, \eta) = (1 - q_t(y, 0, \eta)) \tilde{\xi}_{t1}(y, \eta, 1 - q_t(y, 0, \eta))^2.$$

As in [46] and [58] one obtains that $\tilde{\zeta}_t$ is uniquely defined modulo a flat function on $S_t^*\Gamma$.

We suppose now that $\dim \Gamma = 2$. To obtain the Hamiltonian ζ_t we find action-angle coordinates of $\tilde{\zeta}_t$ as in [58]. To simplify the notations we drop the index t . Denote by M_u the closed curve $\{\varrho \in T^*\Gamma : \tilde{\zeta}(\varrho) = u\}$ in $T^*\Gamma$ where u varies in a small neighborhood of the origin. For any $\varrho \in M_u$ consider the map $\mathbb{R} \ni t \rightarrow \exp(tX_{\tilde{\zeta}})(\varrho) \in M_u$ and denote by $2\pi\Pi(u)$ its period. Let S be a section transversal to M_0 in $T^*\Gamma$. It is equipped with local coordinates $S \ni \varrho \rightarrow u = \tilde{\zeta}(\varrho)$. Denote by \mathcal{O} the discrete group in $\mathbb{R} \times S$ generated by

$$\mathbb{R} \times S \ni (t, u) \rightarrow (t + 2\pi\Pi(u), u), \quad u = \tilde{\zeta}(\varrho).$$

Let $(\mathbb{R} \times S)/\mathcal{O}$ be the corresponding factor space. It is a symplectic manifold equipped with the symplectic two-form $dt \wedge du$ and the mapping

$$\mathbb{R} \times S \ni (t, \varrho) \rightarrow \exp(tX_{\tilde{\zeta}})(\varrho) \in T^*\Gamma$$

lifts to a symplectic diffeomorphism from $(\mathbb{R} \times S)/\mathcal{O}$ to a neighborhood of M_0 . Making suitable symplectic change of the variables

$$\theta = t/\Pi(u), \quad r = g(u),$$

in $\mathbb{R} \times S$ we can suppose that \mathcal{O} is generated by $(\theta, r) \rightarrow (\theta + 2\pi, r)$ while the symplectic two-form becomes $d\theta \wedge dr$. Then $g'(u) = -\Pi(u)$ which yields

$$r(u) = l - \int_0^u \Pi(t) dt, \quad (6.49)$$

where $l = \text{length}(\Gamma)/2\pi$. Denote by $\zeta(r)$ the function inverse to $r(u)$.

We have obtained symplectic coordinates $(\theta_t(x, \xi), r_t(x, \xi))$, $t \in [0, \delta]$, in a neighborhood of the boundary $S_t^* \Gamma$ in the co-ball bundle of Γ with values in $\mathbb{T} \times \mathbb{R}$ such that $S_t^* \Gamma = \{r_t = l_t\}$ and $\mathbf{B}_t^* \subset \{r_t < l_t\}$. The map $t \rightarrow (\theta_t, r_t) \in C^\infty(T^* \Gamma)$ is C^1 by construction. The exact symplectic map B_t is generated in this coordinates by the function G_t . \square

Recall that the functions $\beta_t(\omega)$, $I_t(\omega)$ and $L_t(I)$ are defined by (1.5), (1.6) and (1.7) respectively.

Theorem 6.2. *Let (X, g_t) , $t \in [0, \delta]$, be a C^1 family of compact locally strictly geodesically convex billiard tables of dimension two satisfying the weak isospectral condition $(H_1) - (H_2)$. Then*

(i) *There is a Cantor set $\Xi \subset (0, 1]$ consisting of Diophantine numbers such that*

$$\frac{\text{meas}(\Xi \cap (0, \varepsilon))}{\varepsilon} = 1 - O(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0^+$$

and for any $\omega \in \Xi$ there exists a C^1 family of Kronecker invariant circles $[0, \delta] \ni t \rightarrow \Lambda_t(\omega)$ of B_t of frequency ω ,

(ii) *$\forall \omega \in \Xi$ and $t \in [0, \delta]$, $\beta_t(\omega) = \beta_0(\omega)$, $I_t(\omega) = I_0(\omega)$ and $L_t(I_0(\omega)) = L_0(I_0(\omega))$,*

(iii) *$l_t = r_0$ and the function $\zeta_t - \zeta_0$ is flat at r_0 for any $t \in [0, \delta]$.*

We are going to prove Theorem 6.2. Firstly, using Theorem 9.11 we will obtain a suitable KAM theorem and a BNF at the corresponding family of invariant circles for the C^1 family of symplectic maps \tilde{P}_t given by Proposition 6.1. To this end we will determine the convex set Ω , fix the parameters κ and ϱ , and then estimate the corresponding quantities \mathcal{B}_t and λ which appear in Theorem 9.11.

Consider the function $K_t := -\frac{2}{3}\zeta_t^{\frac{3}{2}}$ in $[l_t - \varepsilon, l_t]$. Fix $\varepsilon > 0$ so that $\zeta_t'(r) < 0$ for $(r, t) \in [l_t - \varepsilon, l_t] \times [0, \delta]$, and denote by K_t^* the Legendre transform of K_t in an interval $[0, a_0]$, $0 < a_0 \ll 1$. One can easily show that the family $t \rightarrow K_t^*$ can be extended as a C^1 family of smooth odd functions $[0, \delta] \ni t \mapsto K_t^* \in C^\infty([-a_0, a_0])$. Indeed, the function K_t admits an asymptotic expansion of the form

$$K_t(r) \approx - \sum_{k=1}^{\infty} \frac{1}{\frac{1}{2} + k} (l_t - r)^{\frac{1}{2} + k} u_k(t) \quad \text{as } r \nearrow l_t,$$

where $u_k \in C^1([0, \delta])$ and $u_1(t) = (-\zeta'_t(l_t))^{\frac{3}{2}} > 0$. Moreover, this asymptotic expansion can be differentiated infinitely many times with respect to r . Recall that for any $t \in [0, \delta]$ fixed the derivative $K_t^{*'}$ of K_t^* satisfies the identity $K_t'(K_t^{*'}(\omega)) = \omega$ for any $\omega \in (0, a_0]$. Moreover, $K_t(K_t^{*'}(\omega)) + K_t^*(\omega) = \omega K_t^{*'}(\omega)$, $K_t^{*'}(0) = l_t$, and we easily obtain the asymptotic expansion

$$K_t^*(r) \approx \sum_{k=0}^{\infty} \frac{1}{2k+1} \omega^{2k+1} v_k(t) \quad \text{as } \omega \searrow 0,$$

where $v_k \in C^1([0, \delta])$,

$$v_0(t) = l_t, \quad v_1(t) = -u_1(t)^{-2} = -(-\zeta'_t(l_t))^{-3} < 0,$$

and so on.

Fix $\tau > 1$, set $\Omega(a) := [a/2, 2a]$, choose $\kappa_a = a^2$ and denote by $\Omega_\kappa(a)$ the set of Diophantine frequencies $[a/2 + a^2, 2a - a^2] \cap D(\kappa, \tau)$. It follows from [43], Proposition 9.9, that

$$\frac{\text{meas}(\Omega(a) \setminus \Omega_\kappa(a))}{\text{meas}(\Omega(a))} \leq C\kappa = Ca^2. \quad (6.50)$$

Choose $a_0 > 0$ so that the Lebesgue measure of $\Omega_\kappa(a_0)$ is positive and denote by $\Omega_\kappa^0(a)$ the set of points of $\Omega_\kappa(a)$ of positive Lebesgue density. We have

$$\mathbb{D}(t, a) := K_t^{*'}(\Omega(a)) = [l_t + v_1(t)a^2/4 + O(a^4), l_t + 4v_1(t)a^2 + O(a^4)] \subset [l_t - \varepsilon, l_t] \quad (6.51)$$

for $0 < a \leq a_0 \ll 1$. Set $\mathbb{A}(t, a) = \mathbb{T} \times \mathbb{D}(t, a)$. We are ready to announce the corresponding KAM theorem for the C^1 family of symplectic mappings $[0, \delta] \ni t \mapsto \tilde{P}_t$ with generating functions \tilde{G}_t satisfying (i) and (ii) in Proposition 6.1.

Theorem 6.3. *For any $a \in (0, a_0]$ there exists a C^1 -family of exact symplectic maps*

$$[0, \delta] \ni t \mapsto (\chi_t : \mathbb{A}(t, a) \rightarrow \mathbb{A}(t, a))$$

and of real valued functions $[0, \delta] \ni t \mapsto L_t \in C^\infty(\mathbb{D}(t, a))$ and $[0, \delta] \ni t \mapsto R_t \in C^\infty(\mathbb{A}(t, a))$ such that for any $t \in [0, \delta]$ the following holds

1. $\tilde{G}_t^0(x, I) = xI - L_t(I) - R_t(\text{pr}(x), I)$ is a generating function of $P_t^0 := \chi_t^{-1} \circ \tilde{P}_t \circ \chi_t$
2. $L'_t : \mathbb{D}(t, a) \rightarrow \Omega(a)$ is a diffeomorphism with inverse $L_t^{*'} : \Omega(a) \rightarrow \mathbb{D}(t, a)$, where L_t^* is the Legendre transform of L_t
3. R_t is flat at $\mathbb{T} \times L_t^{*'}(\Omega_\kappa^0(a))$
4. for any integer $N \geq 1$ and $m \in \mathbb{N}$ there exists a constant $C = C_{m, N} > 0$ independent of $a \in (0, a_0]$ and $t \in [0, \delta]$ such that the following estimates hold

$$|\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta \sigma_\kappa^{-1}(\chi_s - \text{id})| + |\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta \sigma_\kappa^{-1}(\chi_s^{-1} - \text{id})| \leq C\kappa^{2N-m-\frac{3}{4}}$$

on $\mathbb{A}(t, a)$, and

$$\left| \left(\kappa \frac{d}{dI} \right)^m (L'_t(I) - K'_t(I)) \right| \leq C\kappa^{2N-m+\frac{1}{4}} \quad (6.52)$$

on $\mathbb{D}(t, a)$ for any $t \in [0, \delta]$ and $m \in \mathbb{N}$.

Proof. We are going to apply Theorem 9.11 to the C^1 family of symplectic mappings $[0, \delta] \ni t \mapsto \tilde{P}_t \in C^\infty(\mathbb{A}(t, a), \mathbb{A}(t, a))$ given by Proposition 6.1.

Let us estimate the corresponding quantities \mathcal{B}_m for $m \in \mathbb{N}$ and λ defined by (9.179) - (9.181) and (9.187). The constant λ can be fixed by

$$\lambda = \lambda_a = \sup_{t \in [0, \delta]} \|K_t''\|_{\mathbb{D}(t, a); \kappa} = C_0 a^{-1} = C_0 \kappa^{-1/2}, \quad (6.53)$$

where C_0 is a positive constant independent of a . Given $\ell = m + \mu$ with $m \in \mathbb{N}_*$ we get by (11.273) that for any $t \in [0, \delta)$ the following inequality holds

$$\|G_t\|_{\ell, \mathbb{A}(t, a); \kappa} \leq \|G_t\|_{m+1, \mathbb{T} \times \mathbb{D}(t, a); \kappa}.$$

since $\mathbb{D}(t, a)$ is an interval. Moreover,

$$\|G_t\|_{m+1, \mathbb{T} \times \mathbb{D}_a; \kappa} = \sup_{|\alpha| + |\beta| \leq m+1} \|\partial_\theta^\alpha (\kappa \partial_r)^\beta G_t\|_{C^0(\mathbb{T} \times \mathbb{D}(t, a))}.$$

Fix $N \geq 2$. It follows from Proposition 6.1, (ii), and the definition of $\mathbb{D}(t, a)$ that

$$\|G_t\|_{\ell, \mathbb{A}(t, a); \kappa} \leq C_{m, N} a^{8N+4} = C_{m, N} \kappa^{4N+2}$$

for any $t \in [0, \delta]$, where $C_{m, N}$ is a positive constant. Choosing $\varrho = \kappa^{2N+5/4} < \kappa$ we obtain

$$\|G_t\|_{\ell, \mathbb{A}(t, a); \kappa} \leq C'_m \kappa \varrho \kappa^{2N-1/4}$$

for any $t \in [0, \delta]$. Moreover,

$$\|K_t''\|_{\ell, \mathbb{D}(t, a); \kappa} \leq \|K_t''\|_{m+1, \mathbb{D}(t, a); \kappa} \leq C_m a^{-1} = C_m \kappa^{-1/2} \quad (6.54)$$

and

$$S_\ell(\nabla K^*) \leq \sup_{0 \leq t \leq \delta} (1 + \|\nabla K_t^*\|_{C^1([-a_0, a_0])})^{\ell-1} (1 + \|\nabla K_t^*\|_{C^\ell([-a_0, a_0])}) \leq C_m$$

where C_m is a positive constant. Thus for any $m \in \mathbb{N}$ we obtain from (9.179) - (9.181) that

$$\mathcal{B}_m \leq C'_m \mathcal{B}_m^0 \leq C_\ell \varrho \kappa^{2N+3/4} = C_\ell \kappa \varrho \kappa^{2N-1/4} \quad (6.55)$$

where $C_m, C'_m > 0$ depends neither on $a \in (0, a_0]$ nor on $t \in [0, \delta]$. Choosing $a_0 \ll 1$ we get $\mathcal{B}_2 \leq \epsilon \kappa \varrho \lambda^{-4}$ for any $a \in (0, a_0]$ since $N \geq 2$ and $\lambda = C_0 \kappa^{-1/2}$, which gives (9.188). Applying Theorem 9.11 we obtain 1-4. In particular, taking into account (6.53) - (6.55) we obtain from (9.190) the estimate

$$\sup_{t \in [0, \delta]} \sup_{I \in \mathbb{D}(t, a)} \left| \left(\kappa \frac{d}{dI} \right)^m (L_t'(I) - K_t'(I)) \right| \leq C'_m \kappa^{2N+\frac{3}{4}} \lambda^{2m} (\lambda + \kappa^{-\frac{1}{2}}) \leq C_m \kappa^{2N-m+\frac{1}{4}}$$

for any $m \in \mathbb{N}$, where C'_m and C_m are positive constants. \square

Proof of Theorem 6.2. It follows from Theorem 6.3 that for any $0 < a \leq a_0 \ll 1$ and $\omega \in \Omega_\kappa^0(a)$ with $\kappa = a^2$ there exists a C^1 family of Kronecker invariant tori $[0, \delta] \ni t \rightarrow \Lambda_t(\omega)$ of B_t . Corollary 2 implies that $I_t(\omega) = I_0(\omega)$ and $L'_t(I_0(\omega)) = L'_0(I_0(\omega))$. Notice that

$$\limsup_{a \rightarrow 0} \{I_t(\omega) - l_t : \omega \in [a/2, 2a]\} = 0.$$

Then

$$|l_t - l_0| \leq \limsup_{a \rightarrow 0} \{I_t(\omega) - l_t : \omega \in [a/2, 2a]\} + \limsup_{a \rightarrow 0} \{I_0(\omega) - r_0 : \omega \in [a/2, 2a]\} = 0$$

hence, $l_t = l_0$ for any $t \in [0, \delta]$. Moreover, the function $\omega \mapsto L'_t(I_0(\omega)) - L'_0(I_0(\omega))$ is flat at the set $\Omega_\kappa^0(a)$ in view of Lemma 3.4. Then for any $\omega \in \Omega_\kappa^0(a)$, $t \in [0, \delta]$, and any $m \in \mathbb{N}$ using the equality $I_t(\omega) = I_0(\omega)$ and the estimate (6.52) with $N = m$ we obtain

$$\begin{aligned} |(d/dI)^m (K'_t - K'_0)(I_0(\omega))| &\leq |(d/dI)^m (K'_t - L'_t)(I_t(\omega))| \\ &\quad + |(d/dI)^m (K'_0 - L'_0)(I_0(\omega))| \\ &\leq C\kappa^{1/4} = Ca^{1/2}. \end{aligned}$$

Taking the limit as $a \searrow 0$ we obtain that the function $K_t - K_0$ is smooth in $[r_0, r_0 + \varepsilon]$ and flat at r_0 . Then $\zeta_t - \zeta_0 = (3K_t/2)^{2/3} - (3K_0/2)^{2/3}$ is also flat at r_0 . The set of frequencies Ξ is defined as the union of $\Omega_\kappa^0(a)$. \square

Proof of Theorem 6. It remains to show that Γ_t is strictly convex for any $t \in [0, 1]$. To do this we are going to use an argument from [58]. To simplify the notations we will omit the index $t \in [0, 1]$.

Consider the interpolating Hamiltonian $\tilde{\zeta}(x, \xi) = \zeta(\tilde{\chi}^{-1}(x, \xi))$ of B , where the function ζ and the symplectic transformation $\tilde{\chi}$ are given by Proposition 6.1. For any r with $|r|$ small enough the level set

$$M(r) = \{(x, \xi) \in T^*\Gamma : \tilde{\zeta}(x, \xi) = r\}$$

is an ‘‘circle’’ and we set

$$\nu(r) = \int_{M(r)} d(z|_{M(r)}) \tag{6.56}$$

where $r \rightarrow d(z|_{M(r)})$ is a smooth family of 1-forms on $M(r)$ such that $d(z|_{M(r)})(X_{\tilde{\zeta}}) = 1$. One can consider $z|_{M(r)}$ as a multivalued function on the circle $M(r)$ which is well defined on the corresponding covering space $\mathbb{R} \rightarrow M(r)$ so that

$$\{\tilde{\zeta}, z\} = dz(X_{\tilde{\zeta}}) = 1. \tag{6.57}$$

It is easy to show that the set of Taylor coefficients of $\nu(r)$ at $r = 0$ is algebraically equivalent to the set of Taylor’s coefficients of $\zeta(I)$ at $I = l$. Indeed, performing the symplectic change of the variables $(x, \xi) = \tilde{\chi}(\varphi, I)$, $(\varphi, I) \in \mathbb{A}$, and using (6.57) we easily get

$$\zeta'(I)\nu(\zeta(I)) = 2\pi.$$

Denote by \mathcal{R}_r the function inverse to $I \rightarrow \zeta(I)$. Then (6.56) implies

$$\nu(r) = 2\pi\mathcal{R}'(r) \quad (6.58)$$

and we obtain that the Taylor coefficients of $\nu(r)$ at $r = 0$ determine those of ζ at $I = \ell$ and vice versa.

The Taylor coefficients of $\nu(r)$ at $r = 0$, also called integral invariants, have been investigated by Sh. Marvizi and R. Melrose [46]. They are given by integrals on Γ of certain polynomials of the curvature $\kappa(x)$ of Γ and its derivatives. In particular, (4.6) in [46] and (6.58) yield together

$$\mathcal{R}'(0) = -\frac{1}{\pi} \int_0^\ell \kappa(x)^{2/3} dx \quad (6.59)$$

$$\mathcal{R}''(0) = \frac{1}{2160\pi} \int_0^\ell (9\kappa(x)^{4/3} + 8\kappa(x)^{-8/3}\kappa'(x)^2) dx \quad (6.60)$$

(see also [70]).

Suppose now that X_t is strictly convex for $0 \leq t < \delta$ but only convex for $t = \delta$. Consider the function $\mathcal{R}_t(r)$ inverse to $r = \zeta_t(I)$. Then Theorem 6.2, (iii), yields

$$\mathcal{R}_t(r) = \mathcal{R}_0(r) + O_N(r^N) \quad \text{as } r \rightarrow 0$$

for any $N \in \mathbb{N}$ and we obtain

$$\mathcal{R}'_t(0) = \mathcal{R}'_0(0), \quad \mathcal{R}''_t(0) = \mathcal{R}''_0(0), \quad s \in [0, b_0). \quad (6.61)$$

Denote by $\kappa_t(x) > 0$, $x \in \Gamma_t$ the curvature of Γ_t and define f_δ by $f_t(x) = \kappa_t(x)^{-1/3}$ for $t < \delta$ and $f_\delta(x) = \kappa_\delta(x)^{-1/3}$ if $\kappa_\delta(x) \neq 0$ and $f_\delta(x) = 0$ if $\kappa_\delta(x) = 0$. The second equality of (6.61) and (6.60) yield together

$$\int_{\Gamma_t} |f'_t(x)|^2 dx \leq C, \quad s \in [0, \delta). \quad (6.62)$$

where C is a positive constant. On the other hand, the first equality of (6.61) and (6.59) imply that for any $t \in [0, \delta)$ there exists $x_t \in \Gamma_t$ such that

$$\kappa_t(x_t) \geq C_1 := \left(-\frac{\pi}{l_0} \mathcal{R}'_0(0) \right)^{3/2} > 0.$$

Then $f_t(x_t) \leq C_1^{-1/3}$ for $t \in [0, \delta)$, and using Taylor's formula and (6.62) we obtain the estimate

$$\int_{\Gamma_t} (|f_t(x)|^2 + |f'_t(x)|^2) dx \leq C_2, \quad s \in [0, \delta),$$

where C_2 is a positive constant. Let $[0, \delta] \ni t \rightarrow \psi_t : \Gamma \rightarrow \mathbb{R}^2$ be a C^1 family of embeddings such that $\Gamma_0 = \Gamma$ and $\psi_t(\Gamma) = \Gamma_t$. Then $\{f_t \circ \psi_t : t \in [0, \delta)\}$ is a compact subset of $L^2(\Gamma)$ and we obtain that $f_\delta \circ \psi_\delta \in L^2(\Gamma)$ as well. On the other hand, Γ_δ is convex but not strictly convex, hence the curvature its curvature k_δ is a non-negative function and it has a zero of at least second order at a point $x_0 \in \Gamma$. Then

$$|f_\delta(x)| \geq C |x - x_0|^{-2/3}$$

in any local coordinates in a neighborhood of x_0 in Γ_δ . Hence $f_\delta \notin L^2(\Gamma)$ which leads to a contradiction. This implies that Γ_t is strictly convex for any $t \in [0, 1]$. \square

7 Microlocal Birkhoff Normal Form of the monodromy operator

Starting from the BNF in Theorem 3.2 we are going to find a microlocal (quantum) Birkhoff normal form (shortly QBNF) at the union of the invariant tori $\Lambda_t(\omega)$, $\omega \in \Omega_\kappa^0$, of the corresponding microlocal monodromy operator for the family of Laplace-Beltrami operators Δ_t in X with Dirichlet boundary conditions. A similar QBNF has been obtained in [63] for perturbations of the function in the Robin boundary conditions around a single Kronecker torus. In contrast to [63] the BNF of the tori here is nondegenerate which simplifies the construction.

Let us present the main steps in the construction. At first we reduce the problem to the boundary and introduce the corresponding microlocal monodromy operator $M_t^0(\lambda)(\lambda)$, $t \in J$. The reduction to the boundary is obtained by a variant of the reflection method for the wave equation which consists in the following. Given a suitable function $f(\cdot; \lambda)$ on Γ depending on a large parameter λ the frequency support of which is contained in a small neighborhood of the union of the invariant tori $\Lambda_t(\omega)$, we consider the corresponding outgoing solution of the reduced wave equation (the Helmholtz equation) in X and we reflect it at the boundary $m - 1$ times if $m \geq 2$. After each reflection at the boundary we consider the corresponding branch of the solution u_t of the Helmholtz equation given by the outgoing parametrix. We denote by $M_t(\lambda)f$ the restriction at Γ of the last branch of the solution u_t . We call $M_t(\lambda)$ a monodromy operator. By construction, the function $f(\cdot, \lambda)$ on Γ gives rise to an asymptotic solution $u_t(\cdot, \lambda)$ of the Dirichlet problem of the Helmholtz equation

$$(-\Delta_t + \lambda^2)u_t = O_N(|\lambda|^{-N})f_t, \quad u_t|_\Gamma = O_N(|\lambda|^{-N})f_t,$$

or a quasi-mode (λ, u_t) of the Laplace-Beltrami operator with Dirichlet boundary conditions when $\|u_t\|_{L^2} = 1$ if and only if

$$M_t^0(\lambda)(\lambda)f = f + O_N(|\lambda|^{-N})f.$$

The family of operators $M_t^0(\lambda)(\lambda)$, $t \in J$, is a C^1 family of Fourier Integral Operator with a large parameter λ (λ -FIO) the canonical relation of each of them being the graph of P_t . For this reason we recall in Sect. 7.1 some properties of the λ -FIOs associated with a C^1 family of Lagrange immersions. The reduction to the boundary and the construction of the microlocal monodromy operator is done in Sect. 7.2.

Our next goal is to "separate the variables" microlocally near the whole family of invariant tori $\Lambda_t(\omega)$, $\omega \in \Omega_\kappa^0$. This is done in Sect. 7.3. To this end we use the Birkhoff normal form of P_t given by Theorem 3.2. First we conjugate $M_t^0(\lambda)$ with a microlocally unitary λ -FIO $T_t(\lambda)$ the canonical relation of which is the graph of the symplectic transformation χ_t given by Theorem 3.2. In this way we obtain a λ -FIO $W_t(\lambda)$ the canonical relation of which is just the graph of P_t^0 (see Proposition 7.11). Then we obtain a microlocal Birkhoff normal form $W_t^0(\lambda)$ of $W_t(\lambda)$ by conjugating it with a suitable λ -PDO and solving at any step the corresponding homological equation. In this way we separate microlocally the variables near the whole family of invariant tori. This means that the amplitude of $W_t^0(\lambda)$ does not depend on the angular variables but only on the action variables at the family of invariant tori, which allows us to obtain a microlocal "spectral decomposition" of $W_t(\lambda)$ near the family $\Lambda_t(\omega)$, $\omega \in \Omega_\kappa^0$. At any step the corresponding phase functions and amplitudes are C^1 with respect to the parameter t .

7.1 C^1 families of PDOs and FIOs with a large parameter λ .

7.1.1 C^1 families of symbols and λ -PDOs .

Let M^d be a smooth paracompact manifold of dimension d . We are going to define a class of C^1 families of pseudo-differential operators depending on a large parameter λ (shortly λ -PDOs) acting on the half-density bundle $\Omega^{\frac{1}{2}}(M^d)$ of M^d . The large parameter λ will belong to the set

$$\mathcal{D} := \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \geq C_0, |\operatorname{Im} \lambda| \leq C_1\}, \quad \sup_{\lambda \in \mathcal{D}} |\lambda| = +\infty, \quad (7.63)$$

where $C_0, C_1 > 0$. One can switch to the semi-classical setting by introducing $\hbar := 1/\lambda$.

Let us first define the symbols we are going to deal with. Given an interval $J \subset \mathbb{R}$ we define a C^1 family of symbols $J \ni t \rightarrow a_t$ of order 0 in $T^*\mathbb{R}^d$ as a map

$$J \times \mathcal{D} \longrightarrow C_0^\infty(T^*\mathbb{R}^d), \quad (t, \lambda) \longmapsto a_t(\cdot, \lambda),$$

such that

- The map $J \ni t \rightarrow a_t(\cdot, \lambda) \in C^\infty(T^*\mathbb{R}^d)$ is C^1 for any $\lambda \in \mathcal{D}$ fixed;
- The support $\operatorname{supp} a_t(\cdot, \lambda)$ is contained in a fixed compact subset of $T^*\mathbb{R}^d$ independent of $(t, \lambda) \in I \times \mathcal{D}$;
- For any $\alpha, \beta \in \mathbb{N}^d$ there exists a positive constant $C_{\alpha, \beta}$ such that

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a_t(x, \xi, \lambda)| \leq C_{\alpha, \beta}$$

for every $(t, \lambda) \in I \times \mathcal{D}$, $(x, \xi) \in T^*\mathbb{R}^d$ and $k \in \{0, 1\}$.

In this case we say that a_t is a C^1 family of symbols in $S^0(T^*\mathbb{R}^d \times \mathcal{D})$ with respect to the parameter $t \in J$. We set $S^p(T^*\mathbb{R}^d \times \mathcal{D}) = \lambda^p S^0(T^*\mathbb{R}^d \times \mathcal{D})$ for $p \in \mathbb{R}$ and denote by $S^{-\infty}(T^*\mathbb{R}^d \times \mathcal{D})$ the residual set $\bigcap_{p \geq 0} S^{-p}(T^*\mathbb{R}^d \times \mathcal{D})$. We say that

$$J \ni t \longmapsto \sum_{j \in \mathbb{N}} a_{t,j} \lambda^{-j} \quad (7.64)$$

is a C^1 family of formal symbols of order 0 if for any $j \in \mathbb{N}$ the map $J \ni t \mapsto a_{t,j} \in C^\infty(T^*\mathbb{R}^d)$ is C^1 smooth and the support $\operatorname{supp} a_{t,j}$ is contained in a fixed compact subset of $T^*\mathbb{R}^d$ independent of $(t, j) \in I \times \mathbb{N}$. A C^1 family of symbols $J \ni t \mapsto a_t \in S^0(T^*\mathbb{R}^d \times \mathcal{D})$ is said to be a realizations of the C^1 family of formal symbols (7.64) if for any $N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^d$ there exists a positive constant $C_{N, \alpha, \beta}$ such that

$$\sup_{(t, x, \xi, \lambda) \in J \times T^*\mathbb{R}^d \times \mathcal{D}} \left| \partial_t^k \partial_x^\alpha \partial_\xi^\beta \left(a_t(x, \xi, \lambda) - \sum_{j=0}^{N-1} a_{t,j}(x, \xi) \lambda^{-j} \right) \right| \leq C_{N, \alpha, \beta} |\lambda|^{-N} \quad (7.65)$$

for $k \in \{0, 1\}$. Symbols admitting an asymptotic expansion of the form (7.65) for any N are said to be classical. We denote by $S_{\text{cl}}^0(T^*\mathbb{R}^d \times \mathcal{D})$ the class of the classical symbols. Any C^1 family of formal symbols of order zero admits a C^1 family of realizations by Borel's theorem.

Proposition 7.1. *Any C^1 family of formal symbols (7.64) of order 0 admits a realization as a C^1 family of symbols $J \ni t \mapsto a_t \in S^0(T^*\mathbb{R}^d \times \mathcal{D})$. Moreover, if a_t and a'_t are two C^1 family of realizations of (7.64) then $J \ni t \mapsto a_t - a'_t \in S^{-p}(T^*\mathbb{R}^d \times \mathcal{D})$ is a C^1 family for every $p \geq 0$.*

We give a prove of Borel's theorem in Appendix A.3.

We say that the family of operators $J \ni t \mapsto \text{Op}(a_t)$ with Schwartz kernels

$$K_{\text{Op}(a_t)}(x, y, \lambda) := \left(\frac{\lambda}{2\pi} \right)^d \left(\int_{\mathbb{R}^d} e^{i\lambda\langle x-y, \xi \rangle} a_t(x, \xi, \lambda) d\xi \right) |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}} \quad (7.66)$$

is a C^1 family of λ -PDOs of order zero acting on $\frac{1}{2}$ -densities if $J \ni t \mapsto a_t \in S^0(T^*\mathbb{R}^d \times \mathcal{D})$ is a C^1 family of symbols. We say that a family of operators $J \ni t \mapsto A_t$ acting on the smooth sections of the half-density bundle $\Omega^{\frac{1}{2}}(M^d)$ of the manifold M^d is a C^1 family of λ -PDOs if it is given by a C^1 family of λ -PDOs with Schwartz kernels of the form (7.66) in any local coordinates.

7.1.2 C^1 families of λ -FIOs.

Consider a C^1 family of exact Lagrange immersions

$$\iota_t : \Lambda \rightarrow T^*M^d, \quad t \in [0, \delta], \quad (7.67)$$

which means that the map $[0, \delta] \ni t \mapsto \iota_t \in C^\infty(\Lambda, T^*M^d)$ is C^1 , ι_t is an immersion and the pull-back $\iota_t^*(\xi dx)$ of the canonical one-form ξdx of T^*M^d is exact for each $t \in [0, \delta]$. Then there exists a C^1 mapping $[0, \delta] \ni t \mapsto f_t \in C^\infty(\Lambda)$ such that

$$\iota_t^*(\xi dx) = df_t \quad t \in [0, \delta]. \quad (7.68)$$

Fix $t \in [0, \delta]$. Recall from [12] and [31] that a real valued phase function $\Phi_t(x, \theta)$ defined in a neighborhood of a point $(x^0, \theta^0) \in \mathbb{R}^d \times \mathbb{R}^N$ with $d_\theta \Phi_t(x^0, \theta^0) = 0$ is nondegenerate at (x^0, θ^0) if

$$\text{rank } d_{(x, \theta)} d_\theta \Phi_t(x^0, \theta^0) = N. \quad (7.69)$$

Then there exists a neighborhood $V \subset \mathbb{R}^d \times \mathbb{R}^N$ of (x^0, θ^0) such that (7.69) holds for any $(x, \theta) \in V$ and

$$C_{\Phi_t} := \{(x, \theta) \in V : d_\theta \Phi_t = 0\}$$

is a smooth manifold of dimension d . Moreover, the differential of the map

$$\iota_{\Phi_t} : C_{\Phi_t} \ni (x, \theta) \longrightarrow (x, d_x \Phi(x, \theta)) \in \Lambda_{\Phi_t} := \iota_{\Phi_t}(C_{\Phi_t}) \quad (7.70)$$

is of rank d and shrinking V if necessary we obtain that Λ_{Φ_t} is an embedded Lagrangian submanifold of T^*M^d . We say that the nondegenerate function $\Phi_t(x, \theta)$, $(x, \theta) \in V$, defines locally the Lagrange immersion $\iota_t : \Lambda \rightarrow T^*M^d$ if there is an open subset $W_t \subset \Lambda$ such that

$$\iota_t : W_t \rightarrow \Lambda_{\Phi_t} \quad \text{is a diffeomorphism.} \quad (7.71)$$

We can take $\Phi_t = \Phi_t(x)$ depending only on the coordinates h ($N = 0$) if the corresponding Lagrangian manifold is "horizontal" which means that the projection to the base is a local diffeomorphism. The collection $(\iota_t^{-1}(\Lambda_{\Phi_t}), \iota_{\Phi_t}^{-1} \circ \iota_t)$ provides the Lagrangian immersion $\iota_t : \Lambda \rightarrow T^*M^d$ with an atlas of local charts. Given an interval $J \subset [0, \delta]$ we say that a C^1 map $J \ni t \rightarrow$

$\Phi_t \in C^\infty(V, \mathbb{R})$ is a C^1 family of nondegenerate phase functions generating locally the C^1 family of Lagrange immersions (7.67) in J if for any $t \in J$ the phase function Φ_t is nondegenerate in V and (7.71) holds. Such C^1 families of phase functions can always be constructed locally. Consider the function $\Phi_t^\Lambda := (k_t \circ \iota_{\Phi_t}^{-1} \circ \iota_t)^*(\Phi_t)$ on $\iota_t^{-1}(\Lambda_{\Phi_t})$ where $k_t : C_{\Phi_t} \rightarrow \mathbb{R}^d \times \mathbb{R}^N$ is the inclusion map. Observe that

$$\begin{aligned} d\Phi_t^\Lambda &= (\iota_{\Phi_t}^{-1} \circ \iota_t)^* \left(k_t^* \left(\frac{\partial \Phi_t}{\partial x} dx + \frac{\partial \Phi_t}{\partial \theta} d\theta \right) \right) \\ &= \iota_t^* \left((k_t \circ \iota_{\Phi_t}^{-1})^* \left(\frac{\partial \Phi_t}{\partial x} dx \right) \right) = \iota_t^* (\xi dx) = df_t \end{aligned}$$

and we choose Φ_t so that $\Phi_t^\Lambda = f_t$ on $\iota_t^{-1}(\Lambda_{\Phi_t})$, where f_t is defined in (7.68).

Given a C^1 family of classical amplitudes $[0, \delta] \ni t \rightarrow a_t \in S_{\text{cl}}^0(V \times \mathcal{D})$ such that $a_t = 0$ for $t \notin J$ we consider the C^1 family of oscillatory $\frac{1}{2}$ -densities

$$I_{\Phi_t, a_t}(x, \lambda) |dx|^{\frac{1}{2}} = \left(\frac{\lambda}{2\pi} \right)^{m + \frac{d+2N}{4}} \left(\int_{\mathbb{R}^N} e^{i\lambda \Phi_t(x, \theta)} a_t(x, \theta, \lambda) d\theta \right) |dx|^{\frac{1}{2}} \quad (7.72)$$

with the convention that there is no integration when $N = 0$. Notice that the function $[0, \delta] \ni t \mapsto I_{\Phi_t, a_t} \in C^\infty(M^d)$ is C^1 for each $\lambda \in \mathcal{D}$ fixed. Its oscillation is detected as $\lambda \rightarrow \infty$ by the corresponding semi-classical wave front set. Integrating by parts one obtains $\text{WF}_\lambda(I_{\Phi_t, a_t}(\cdot, \lambda)) \subset \Lambda_{\Phi_t}$, where WF_λ is the frequency set (or semi-classical \hbar -wave-front with $\hbar = 1/\lambda$) (cf. [1], [10], [80], [24]). A (global) C^1 family of oscillatory $\frac{1}{2}$ -densities is given by

$$u_t(x, \lambda) = \sum_j I_{\Phi_t^j, a_t^j}(x, \lambda) |dx|^{\frac{1}{2}} \quad (7.73)$$

where Φ_t^j are nondegenerate phase functions in $V_j \subset \mathbb{R}^d \times \mathbb{R}^{N_j}$ such that $\iota_t^{-1}(\Lambda_{\Phi_t^j})$, $j = 1, 2, \dots$, is a locally finite covering of Λ with open sets for each t fixed.

We denote the class of these oscillatory $\frac{1}{2}$ -densities by $I^m(M^d, \Lambda_t; \Omega^{\frac{1}{2}}(M^d))$ or simply by $I^m(M^d, \Lambda_t)$. In order to simplify the notations we denote the immersion $\iota_t : \Lambda \rightarrow T^*M^d$ by Λ_t .

To any oscillatory integral $u_t(x, \lambda)$ of the form (7.72) one can associate a principal symbol

$$e^{i\lambda f_t} \sigma_t \quad \text{where} \quad \sigma_t = \left(\frac{\lambda}{2\pi} \right)^m \sigma_{1,t} \otimes \sigma_{2,t} \quad (7.74)$$

$t \rightarrow \sigma_{1,t}$ is a C^1 family of sections of the half-density bundle $\Omega^{\frac{1}{2}}(\Lambda)$ and $\sigma_{2,t}$ is a section of the Keller-Maslov bundle $M(\Lambda_t)$ for each t fixed (cf. [12, 47, 24]).

In any local chart the half-density part $\sigma_{1,t}$ can be written in terms of the nondegenerate phase functions Φ_t and the leading part $a_{0,t}$ of the amplitude t in (7.72) as follows

$$(\iota_t^{-1} \circ \iota_{\Phi_t})^*(\sigma_{1,t}) = a_{t,0} |d_{C_{\Phi_t}}|^{\frac{1}{2}} \quad (7.75)$$

(cf. [31], Sect. 25.3), where $d_{C_{\Phi_t}}$ is a Leray form on C_{Φ_t} , i.e. $d_{C_{\Phi_t}} = k_t^*(\tilde{d}_{C_{\Phi_t}})$ is the pull-back via the inclusion map $k_t : C_{\Phi_t} \rightarrow \mathbb{R}^d \times \mathbb{R}^N$ of a form $\tilde{d}_{C_{\Phi_t}}$ such that

$$\tilde{d}_{C_{\Phi_t}} \wedge d \frac{\partial \Phi_t}{\partial \theta_1} \wedge \dots \wedge d \frac{\partial \Phi_t}{\partial \theta_N} = dx_1 \wedge \dots \wedge dx_d \wedge d\theta_1 \wedge \dots \wedge d\theta_N.$$

Given for any t a suitable system of coordinates $\mu = (\mu_1, \dots, \mu_d)$ on C_{Φ_t} extended to a neighborhood of C_{Φ_t} one obtains

$$\begin{cases} d_{C_{\Phi_t}} = b_t d\mu_1 \wedge \dots \wedge d\mu_d & \text{with} \\ b_t = \frac{dx_1 \wedge \dots \wedge dx_d \wedge d\theta_1 \wedge \dots \wedge d\theta_N}{d\mu_1 \wedge \dots \wedge d\mu_d \wedge d\frac{\partial\Phi_t}{\partial\theta_1} \wedge \dots \wedge d\frac{\partial\Phi_t}{\partial\theta_N}} = \left| \frac{D(\mu, (\Phi_t)'_{\theta})}{D(x, \theta)} \right|^{-1}. \end{cases} \quad (7.76)$$

More generally, given a vector bundle E over M^d , we denote by $I^m(M^d, \Lambda_t; \Omega^{\frac{1}{2}}(M^d) \otimes E)$ the corresponding class of oscillatory $\frac{1}{2}$ -densities of order m with values in the space of sections $\Gamma(E)$, and by $S^m(\Lambda_t, \Omega^{\frac{1}{2}}(\Lambda) \otimes M(\Lambda_t) \otimes E_t)$ the corresponding class of symbols, where E_t is the lifting of E to Λ_t .

Given two manifolds M_j , $j = 1, 2$, we denote by ω_j the corresponding canonical symplectic forms on $T^*(M_j)$ and consider the symplectic manifold $T^*(M_2) \times T^*(M_1)$ equipped with the exact symplectic form $\omega_2 - \omega_1$. A C^1 family of (exact) canonical relations \mathcal{C}_t , $t \in [0, \delta]$, “from $T^*(M_1)$ to $T^*(M_2)$ ” is given by a C^1 family of (exact) Lagrange immersions $\iota_t : \mathcal{C} \rightarrow T^*(M_2) \times T^*(M_1)$. To any C^1 family of (exact) canonical relations \mathcal{C}_t one associates a C^1 family of (exact) Lagrangian submanifolds \mathcal{C}'_t of $T^*(M_2 \times M_1)$ defined by the exact Lagrange immersions $\iota'_t : \mathcal{C} \rightarrow T^*(M_2 \times M_1)$ where $\iota'_t = j \circ \iota_t$ and

$$j : T^*(M_2 \times M_1) \rightarrow T^*(M_2) \times T^*(M_1), \quad j(x_2, x_1, \xi_2, \xi_1) = (x_2, \xi_2, x_1, -\xi_1). \quad (7.77)$$

We use the same notations as in [31], Sect. 25, for the corresponding classes of λ -FIOs. Given vector bundles E_j on M_j and a C^1 family of exact canonical relations \mathcal{C}_t from $T^*(M_1)$ to $T^*(M_2)$ we say that

$$A_t : C_0^\infty(M_1, \Omega^{\frac{1}{2}}(M_1) \otimes E_1) \rightarrow C^\infty(M_2, \Omega^{\frac{1}{2}}(M_2) \otimes E_2)$$

is a C^1 family of λ -FIOs of order m if the family of the corresponding Schwartz kernels K_{A_t} is a C^1 family of oscillatory $\frac{1}{2}$ -densities belonging to $I^m(M_2 \times M_1, \mathcal{C}_t; \Omega^{\frac{1}{2}}(M_2 \times M_1) \otimes \text{Hom}(E_1, E_2))$. The composition of λ -FIOs with exact canonical relations having transversal and more generally a clean composition can be defined in the same way as in the case of classical FIOs [24, 47]. The microlocal calculus is even simpler since the amplitudes are uniformly compactly supported with respect to θ . In particular we have the following analogue of Theorem 25.2.4 [31] (see [12], [24])

Theorem 7.2. *Let P_t be a C^1 family of classical λ -PDOs of order 0 acting on $\frac{1}{2}$ -densities in M_2 with principal symbol p_t and subprincipal symbol c_t . Let \mathcal{C}_t be a C^1 family of exact canonical relations from $T^*(M_1)$ to $T^*(M_2)$ with Schwartz kernels $K_{A_t} \in I^k(M_2 \times M_1, \mathcal{C}'_t; \Omega^{\frac{1}{2}}(M_2 \times M_1))$ with principal symbols $e^{i\lambda f_t} \sigma_t$. Suppose that p_t vanishes on the projection of \mathcal{C}'_t to X_2 . Then $P_t A_t$ is a C^1 family of λ -FIOs of order $k - 1$ with kernels $K_{P_t A_t}$ in $I^{k-1}(M_2 \times M_1, \mathcal{C}'_t; \Omega^{\frac{1}{2}}(M_2 \times M_1))$ and principal symbols*

$$e^{i\lambda f_t} \left(i^{-1} \mathcal{L}_{X_{p_t}} \sigma_t + c_t \sigma_t \right)$$

where X_{p_t} is the Hamiltonian vector field of p_t lifted to functions in $T^*(M_2 \times M_1)$ and $\mathcal{L}_{X_{p_t}}$ is the Lie derivative.

7.1.3 Quantization of C^1 families of billiard ball maps.

The aim of this section is to construct a family of monodromy operators quantizing billiard ball maps of a C^1 family of billiard tables. The monodromy operators will arise as boundary values of the microlocal outgoing parametrizes $H_t(\lambda) : L^2(\Gamma) \rightarrow C^\infty(\tilde{X})$, $t \in [0, \delta]$, of the Dirichlet problem for the Helmholtz equation. We will construct $H_t(\lambda)$ for $t \in [0, \delta]$ as a C^1 family of λ -FIOs satisfying asymptotically the Helmholtz equation at high frequencies ($|\lambda| \rightarrow \infty$), i.e.

$$\forall N \in \mathbb{N}, \quad (\Delta_t - \lambda^2)H_t(\lambda)u = O_N(|\lambda|^{-N})u \quad (7.78)$$

in a neighborhood of X in a smooth extension (\tilde{X}, g_t) of the Riemannian manifold of (X, g_t) . Hereafter,

$$O_N(|\lambda|^{-N}) : L^2(\Gamma) \rightarrow L^2(\tilde{X})$$

stands for a C^1 family with respect to t of operators $A_t(\lambda) : L^2(\Gamma) \rightarrow L^2(\tilde{X})$ depending on $\lambda \in \mathcal{D}$ such that

$$\|A_t(\lambda)\|_{L^2} \leq C_N(1 + |\lambda|)^{-N}$$

for each t and $\lambda \in \mathcal{D}$ where $C_N > 0$ is a constant independent of t and of λ . Moreover, u are suitable “initial data” on Γ . Set

$$\Lambda := \{(s, \rho) \in \mathbb{R} \times T^*\Gamma : \rho \in U, -\varepsilon < s < T_0(\rho) + 2\varepsilon\}, \quad (7.79)$$

where U is a compact subset of the domain of definition $\tilde{B}_0^*\Gamma$ of the billiard ball map B_0 , $0 < \varepsilon \ll 1$, and $T_t : U \rightarrow (0, +\infty)$ is the “return time function” which assigns to each $\rho \in U$ the time of the first impact at the boundary, i.e. the smallest positive $s = T_t(\rho)$ such that $\exp(sX_{h_t})(\pi_t^+(\rho)) \in \Sigma_t^-$. Recall from Sect. 2.1 that h_t is the Hamiltonian corresponding to the Riemannian metric g_t via the Legendre transform, X_{h_t} is the corresponding Hamiltonian vector field, and the map $\pi_t^+ : B^*\Gamma \rightarrow \Sigma_t^+$ is defined by (2.13). In particular, $\exp(sX_{h_t})(\pi_t^+(\rho))$ lies on the cosphere bundle

$$\tilde{\Sigma}_t := S_t^*\tilde{X} = \{(x, \xi) \in T^*\tilde{X} : h_t(x, \xi) = 1\}. \quad (7.80)$$

The FIOs $H_t(\lambda)$, $t \in [0, \delta]$, will be associated to the C^1 family of canonical relations \mathcal{C}_t in $T^*\tilde{X} \times T^*\Gamma$ given by the C^1 family of immersions

$$\iota_t : \Lambda \rightarrow T^*\tilde{X} \times T^*\Gamma, \quad \iota_t(s, \rho) = (\exp(sX_{h_t})(\pi_t^+(\rho)), \rho). \quad (7.81)$$

Choosing $\delta > 0$ sufficiently small we suppose that the set U in (7.79) is a connected open subset of $T^*\Gamma$ such that

- U is contained in $\tilde{B}_t^*\Gamma$ for any $t \in [0, \delta]$;
- $T_t(\rho) < T_0(\rho) + \varepsilon$ for any $t \in [0, \delta]$ and $\rho \in \bar{U}$.

Then T_t is a smooth function on \bar{U} , its image is a compact interval and there exist $0 < a < b$ such that $T_t(U) \subset [a, b]$ for any t . Moreover, Lemma A.1. in [63] implies that

$$\iota'_t = j \circ \iota_t : \Lambda \rightarrow T^*(\tilde{X} \times \Gamma).$$

is a C^1 family of exact Lagrangian immersions which will be denoted by \mathcal{C}'_t , $t \in [0, \delta]$. We choose the corresponding function f_t in (7.68) to be just the action $f_t(s, \rho) = 2s$ on the bicharacteristic arc associated with $\iota_t(s, \rho) \in \mathcal{C}_t$, where $(s, \rho) \in \Lambda \subset \mathbb{R} \times T^*\Gamma$.

Our aim now is to define the immersed Lagrangian manifold \mathcal{C}'_t locally by a nondegenerate phase function. Fix $t^0 \in [0, \delta]$ and take $\varrho^0 = (x^0, y^0, \xi^0, -\eta^0) \in \mathcal{C}'_{t^0}$. Choose a smooth submanifold M^0 of \tilde{X} of dimension $n-1$ passing through x^0 and transversal at x^0 to the geodesic of g_{t^0} starting from y^0 with codirection $(\eta^0)^+$. Consider the symplectic map $\chi_t : U^0 \rightarrow T^*M^0$ defined in a neighborhood U^0 of $\rho^0 = (y^0, \eta^0)$ in $\tilde{B}_t^*\Gamma$ by

$$\chi_t(\rho) = (\pi \circ \exp(s(\rho)X_{h_t}) \circ \pi_t^+)(\rho), \quad \rho \in U^0 \subset \tilde{B}_t^*\Gamma,$$

where $s(\rho) > 0$ is the arrival time at $T^*X|_{M^0}$ and $\pi(x', x_n, \xi', \xi_n) = (x', \xi')$. If $M^0 = \Gamma$, this is just the billiard ball map B_t defined in Sect. 2.1. Denote by $\mathcal{C}'_{\chi_t} \subset T^*(M \times \Gamma)$ the Lagrangian manifold corresponding to the canonical relation $\mathcal{C}_{\chi_t} := \{(\chi_t(\rho), \rho) : \rho \in U^0\}$.

Let $J \ni t \rightarrow x_t = (x'_t, x_{t,n}) \in C^\infty(\mathcal{O}, \mathbb{R}^n)$ be a C^1 family of normal coordinates to M^0 with respect to the metrics g_t , where $J \subset [0, \delta]$ is an interval containing t^0 and \mathcal{O} is a sufficiently small neighborhood of x^0 . For any fixed $t \in J$ we have $M^0 \cap \mathcal{O} = \{x_n = 0\} \cap \mathcal{O}$ and the normal vector field to $M^0 \cap \mathcal{O}$ associated to g_t and determined by $\xi^0(\nu_t(x^0)) > 0$ becomes $\nu_t = (0, \dots, 0, 1)$ in these coordinates. Then the Hamiltonian h_t is of the form

$$h_t(x, \xi) = \xi_n^2 + r_t(x, \xi') \quad (7.82)$$

in these coordinates, where $J \ni t \rightarrow r_t$ is a C^1 family of smooth functions in a neighborhood of (x^0, ξ^0) . If $x^0 \in \Gamma$, we take M^0 to be a neighborhood of x^0 in Γ , then $r_t(x', 0, \xi') = h_t^0(x', \xi')$ is the Hamiltonian corresponding to the induced Riemannian metric g_t^0 on Γ .

Following the proof of Hörmander [31], Proposition 25.3.3, we can find local coordinates $y \in \mathbb{R}^{n-1}$ in a neighborhood of y^0 in Γ such that projection $\mathcal{C}'_{\chi_{t^0}} \ni (x', \xi', y, \eta) \rightarrow (x', \eta) \in T^*\mathbb{R}^{n-1}$ is a local diffeomorphism in a neighborhood of $(x^0, \xi^0, y^0, \eta^0)$. Shrinking J if necessary we obtain that the map $\mathcal{C}'_{\chi_t} \ni (x', \xi', y, \eta) \rightarrow (x', \eta) \in T^*\mathbb{R}^{n-1}$ is a local diffeomorphism as well for any $t \in J$. Then there exists a C^1 family of smooth functions ϕ_t^0 defined in a neighborhood V^0 of (x^0, η^0) in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ such that

$$\det \frac{\partial^2 \phi_t^0}{\partial x' \partial \eta}(x', \eta) \neq 0 \quad \text{for } (x', \eta) \in V^0. \quad (7.83)$$

and

$$\text{graph}(\chi_t) = \{(x', (\phi_t^0)'_{x'}(x', \eta)); (\phi_t^0)'_{\eta}(x', \eta), \eta); (x', \eta) \in V^0\} \quad (7.84)$$

(see [31], Theorem 22.2.18). Then solving a suitable Hamilton-Jacobi equation we obtain a C^1 family of nondegenerate phase functions

$$\Phi_t(x, y, \eta) = \phi_t(x, \eta) - \langle y, \eta \rangle \quad (7.85)$$

in a neighborhood of (x^0, y^0, η^0) in $\mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ generating locally \mathcal{C}'_t in a neighborhood \mathcal{C}'_{Φ_t} of ϱ_0 where $\phi_t(x^0, \eta) = \phi_t^0(x', \eta)$. In particular, we have

$$\det \frac{\partial^2 \phi_t}{\partial x' \partial \eta}(x, \eta) \neq 0 \quad \text{in a neighborhood } V \text{ of } (x^0, \eta^0). \quad (7.86)$$

We summarize this construction as follows.

Lemma 7.3. *There exists an open interval $J \subset [0, \delta]$ containing t^0 , local coordinates $y \in \mathbb{R}^{n-1}$ in a neighborhood of y^0 in Γ and independent of t , a neighborhood $V^0 \subset \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ of (x^0, η^0) and a C^1 family of function $\phi_t^0 \in C^\infty(V^0)$ satisfying (7.83) and such that the following holds*

1. *the function $\Phi_t^0(x', y, \eta) := \phi_t^0(x', \eta) - \langle y, \eta \rangle$ is a local generating function of the Lagrangian manifold $\mathcal{C}'_{\chi_t} \subset T^*(M^0 \times \Gamma)$ for every $t \in J$;*
2. *the Lagrangian manifold \mathcal{C}'_t is defined in a neighborhood of ϱ_0 by a phase function*

$$\Phi_t(x, y, \eta) = \phi_t(x, \eta) - \langle y, \eta \rangle,$$

where $\phi_t(x, \eta)$ is a local solution of the Hamilton-Jacobi equation

$$\partial_{x_n} \phi_t(x, \eta) = \sqrt{1 - r_t(x, \partial_{x'} \phi_t(x, \eta))}, \quad \phi_t(x', 0, \eta) = \phi_t^0(x', \eta), \quad (7.87)$$

and r_t is given by (7.82).

3. *there exists a neighborhood $V \subset \mathbb{R}^n \times \mathbb{R}^{n-1}$ of (x^0, η^0) such that the family of functions $J \ni t \rightarrow \phi_t \in C^\infty(V, \mathbb{R})$ is C^1 and (7.86) holds for any $t \in J$.*

In particular,

$$\Phi_t(x, y, \eta) = \phi_t^0(x', \eta) - \langle y, \eta \rangle + x_n \phi_t^1(x, \eta), \quad (7.88)$$

where $J \ni t \rightarrow \phi_t^1 \in C^\infty(V, \mathbb{R})$ is a C^1 map. We take $\phi_t^0(x', \eta) = \langle x', \eta \rangle$ if the image of ϱ_0 by the involution in (7.77) belongs to $\Sigma_{t^0}^+ \times U$, which means that $x^0 = (y^0, 0)$ and $\xi^0 = (\eta^0)^+$.

In order to obtain the Maslov part of the principal symbol picked up by the phase functions constructed by the Lemma we need the following. Fix $\tilde{t}^0 \in [0, \delta]$, take $\tilde{\varrho}^0 = (\tilde{x}^0, \tilde{y}^0, \tilde{\xi}^0, -\tilde{\eta}^0) \in \mathcal{C}'_{\tilde{t}^0}$ and denote by \tilde{M}^0 the corresponding submanifold transversal at \tilde{x}^0 to the geodesic of $g_{\tilde{t}^0}$ starting from \tilde{y}^0 with codirection $(\tilde{\eta}^0)^+$ and by $\tilde{\chi}_t : \tilde{U}^0 \rightarrow T^*\tilde{M}^0$ the corresponding symplectic map. Let \tilde{J} be the corresponding interval about \tilde{t}^0 and $\tilde{\Phi}_t(\tilde{x}, \tilde{y}, \tilde{\eta}) = \phi_t(\tilde{x}, \tilde{\eta}) - \langle \tilde{y}, \tilde{\eta} \rangle$, $t \in \tilde{J}$, the corresponding C^1 family of phase functions given by Lemma 7.3. Suppose that $\tau \in J \cap \tilde{J} \neq \emptyset$ and that there exists $\zeta \in \Lambda$ such that

$$i'_\tau(\zeta) \in \mathcal{C}'_{\Phi_t} \cap \mathcal{C}'_{\tilde{\Phi}_t}.$$

Lemma 7.4. *There exists a neighborhood $I \subset J \cap \tilde{J}$ of τ and a neighborhood V of ζ in Λ such that the function $\mu : V \times I \rightarrow 2\mathbb{Z}$ defined by*

$$\mu(\varrho, t) := \operatorname{sgn}(\phi_t)''_{\eta\eta}(x, \eta) - \operatorname{sgn}(\tilde{\phi}_t)''_{\tilde{\eta}\tilde{\eta}}(\tilde{x}, \tilde{\eta}),$$

$$i_t(\rho) = i_{\Phi_t}(x, (\phi_t)'_\eta(x, \eta), \eta) = i_{\tilde{\Phi}_t}(\tilde{x}, (\tilde{\phi}_t)'_{\tilde{\eta}}(\tilde{x}, \tilde{\eta}), \tilde{\eta}).$$

is constant on $V \times I$.

Proof. The assertion follows from an argument in [13] using Hörmander's index $\sigma(M_1, M_2; L_1, L_2) \in \mathbb{Z}$ of four Lagrangian spaces M_1, M_2, L_1, L_2 in the Lagrangian Grassmannian $\Lambda(n-1)$, where L_1 and L_2 are transversal to both M_1 and M_2 in $T^*\mathbb{R}^{n-1}$ (see [12], Definition 3.4.2). It is known that σ is locally constant and continuous with respect to all the variables (M_1, M_2, L_1, L_2) . Set $i_\tau(\zeta) = (u, (\phi_\tau)'_u(u, v), (\phi_\tau)'_v(u, v), v) \in T^*\tilde{X} \times T^*\Gamma$ where ϕ_t , $t \in J$, is the phase function in Lemma 7.3 corresponding to the coordinates $x_t : \mathcal{O} \rightarrow \mathbb{R}^n$. Take a section M in \tilde{X} passing through

the point u and transversal to the geodesic starting from $\partial_v \phi_\tau(u, v) \in \Gamma$ and having a codirection v^+ . We can suppose that $M = \{x_n = g(x')\}$ in these coordinates with some smooth function g . Let us change the x coordinates in a neighborhood of M by $z' = x'$ and $z_n = x_n - g(x')$ and set $\psi_t(z, \eta) = \phi_t(z', z_n + g(z'), \eta)$. Consider the (local) symplectic transformation $\chi_t^0 : T^*\Gamma \rightarrow T^*M$ defined by

$$\chi_t^0(\rho) = (\pi^0 \circ \exp(s_t^0(\rho)X_{h_t}) \circ \pi_t^+)(\rho), \quad \rho \in U_0 \subset \tilde{B}_t^*\Gamma,$$

for t sufficiently close to τ , where $s_t^0(\rho) > 0$ is the arrival time at $T^*X|_M$ and $\pi^0(z', z_n, \xi', \xi_n) = (z', \xi')$. Then $(z', \eta) \rightarrow \psi_t(z', 0, \eta)$ is a generating function of χ_t^0 in the sense of (7.84). Given $(x, y, \eta) \in C_{\Phi_t}$ in a neighborhood of $(u, \partial_u \phi_\tau(v), v)$ with $x \in M$, we obtain as in [13] p. 69

$$\operatorname{sgn}(\Phi_t)''_{\eta\eta}(x', g(x'), y, \eta) = \operatorname{sgn}(\psi_t)''_{\eta\eta}(z', 0, \eta) = \operatorname{sgn}(V, H_t; (d\chi_t^0)^{-1}(V))$$

where $\operatorname{sgn}(M_1, M_2; L)$ is defined in [12], Definition 3.4.3, $V = \{(\delta z', \delta \xi') : \delta z' = 0\}$ is the vertical space (the tangent space to the fiber) and H_t is the horizontal space $\{(\delta z', \delta \xi') : \delta \xi' = 0\}$ for the local coordinates x_t in \mathcal{O} used in the construction of Φ_t . Repeating this procedure in $\tilde{\mathcal{O}}$ for the phase function $\tilde{\Phi}_t$ obtained by Lemma 7.3 corresponding to the coordinates $\tilde{x}_t : \mathcal{O} \rightarrow \mathbb{R}^n$, we obtain

$$\begin{aligned} & \operatorname{sgn}(\Phi_t)''_{\eta\eta}(x, y, \eta) - \operatorname{sgn}(\tilde{\Phi}_t)''_{\tilde{\eta}\tilde{\eta}}(\tilde{x}, \tilde{y}, \tilde{\eta}) \\ &= \operatorname{sgn}(V, H_t; (d\chi_t^0)^{-1}(V)) - \operatorname{sgn}(V, \tilde{H}_t; (d\chi_t^0)^{-1}(V)) = 2\sigma(H_t, \tilde{H}_t; (d\chi_t^0)^{-1}(V), V) \end{aligned}$$

where \tilde{H}_t is the horizontal space $\{(\delta z', \delta \xi') : \delta \xi' = 0\}$ for the corresponding local coordinates $\tilde{x}_t : \tilde{\mathcal{O}} \rightarrow \mathbb{R}^n$ used in the construction of $\tilde{\Phi}_t$. This shows that μ is independent of t and of ρ in a small neighborhood of (τ, ζ) . \square

Using the phase functions obtained in Lemma 7.3 one can define the space of λ -FIOs corresponding to the C^1 family of canonical relations \mathcal{C}_t . We are looking for solutions $H_t(\lambda)$ of (7.78) with Schwartz kernels in $I^{-1/4}(\tilde{X} \times \Gamma, \mathcal{C}'_t)$. To any C^1 family of nondegenerate phase function $\Phi_t(x, y, \eta)$ of the form (7.85) generating \mathcal{C}'_t in a neighborhood of a point $\rho_0 = (x^0, y^0, \xi^0, -\eta^0) \in \mathcal{C}'$ (Φ_t is given by Lemma 7.3) there is a C^1 family of classical amplitudes

$$b_t(x, \eta, \lambda) \sim b_{0,t}(x, \eta) + b_{1,t}(x, \eta)\lambda^{-1} + \dots$$

such that the Schwartz kernel of $H_t(\lambda)$ can be written microlocally near ρ_0 as a C^1 family of oscillatory $\frac{1}{2}$ -densities

$$I_{\Phi_t}(x, y, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \left(\int_{\mathbb{R}^{n-1}} e^{i\lambda\Phi_t(x, y, \eta)} b_t(x, \eta, \lambda) d\eta \right) |dx|^{1/2} |dy|^{1/2}. \quad (7.89)$$

(see (7.72)).

Notice that the Hamiltonian p_t in $T^*(X \times \Gamma)$ obtained by lifting of the principal symbol $h_t - 1$ of the operator $\lambda^{-2}\Delta_t - 1$ vanishes on \mathcal{C}'_t . Thus to compute the principal symbol of $(\Delta_t - \lambda^2)H_t(\lambda)$ we can use Theorem 7.2. Note also that the corresponding subprincipal symbol is $c_t = 0$.

We are going to define suitable sections $\sigma_{1,t}$ and $\sigma_{2,t}$ of the half density bundle and of the Keller-Maslov bundle of \mathcal{C}'_t . The lifting of the Hamiltonian vector field X_{h_t} to $T^*X \times T^*\Gamma$ is $Y_t = (X_{h_t}, 0)$ and its flow S_t^τ restricted to \mathcal{C}_t is given by

$$S_t^\tau(x, \xi, y, \eta) = S_t^\tau(\exp(sX_h)(\pi_\Sigma^+(y, \eta)), y, \eta) = (\exp((s + \tau)X_{h_t})(\pi_t^+(y, \eta)), y, \eta) \quad (7.90)$$

for any $(x, \xi, y, \eta) \in \mathcal{C}_t$. The volume form β_t on \mathcal{C}_t given by the pull-back by ι_t of

$$ds \wedge (dy_1 \wedge d\eta_1) \wedge \cdots \wedge (dy_{n-1} \wedge d\eta_{n-1}) \in \Omega(\Lambda)$$

is invariant with respect to the flow S_t^r or equivalently, the Lie derivative $\mathcal{L}_{Y_t}\beta_0$ vanishes. Then the Lie derivative of the $\frac{1}{2}$ -density $\sigma_{0,t} := |j^*(\beta_0)|^{\frac{1}{2}} \in |\Omega|^{\frac{1}{2}}(\mathcal{C}'_t)$ with respect to X_{p_t} is zero. Recall that j is given by (7.77) and that X_{p_t} is the Hamiltonian vector field of $h_t - 1$ lifted to functions in $T^*(\tilde{X} \times \Gamma)$. We set

$$\sigma_{1,t} = \tilde{b}_{0,t}\sigma_{0,t}, \quad \tilde{b}_{0,t} \in C_0^\infty(\mathcal{C}'_t). \quad (7.91)$$

The relation between $\tilde{b}_{0,t}$ and the principal part $b_{0,t}$ of the amplitude b_t in (7.89) is obtained in [63], (A.23). More precisely, let us denote by $\eta_t = \eta_t(x, \xi')$ the local solution of $\xi' = (\phi_t)'_{x'}(x, \eta)$ obtained by the implicit function theorem and set

$$\tilde{b}'_{0,t}(x, \xi') = \tilde{b}_{0,t}(\pi_1^{-1}(x, \xi', \xi_n)),$$

where $\pi_1 : \mathcal{C}'_t \rightarrow \tilde{\Sigma}_t \subset T^*\tilde{X}$ is the projection $\pi_1(x, y, \xi, -\eta) = (x, \xi)$. Then [63], (A.23), yields

$$b_{0,t}(x, \eta_t(x, \xi')) = \frac{\tilde{b}'_{0,t}(x, \xi')}{\sqrt{2|\xi_n|}} \left| \det(\phi_t)''_{x'\eta}(x, \eta(x, \xi')) \right|^{\frac{1}{2}} \quad (7.92)$$

in a neighborhood of (x^0, ξ^0) , where $\xi_n = \sqrt{1 - r_t(x, \xi')}$.

The Keller-Maslov bundle $M(\mathcal{C}'_t)$ of \mathcal{C}'_t admits a natural trivialization by locally constant sections. Recall from Hörmander [30], p. 148, that a section of the line bundle $M(\mathcal{C}'_t)$ is given by a family of functions $f_{\Phi_t} : \mathcal{C}'_{\Phi_t} \rightarrow \mathbb{C}$, where Φ_t is a nondegenerate phase function generating locally \mathcal{C}'_t at $\mathcal{C}'_{\Phi_t} = \iota_{\Phi_t}(C_{\Phi_t})$ (see (7.70)) such that $f_{\tilde{\Phi}_t} = i^{\mu_t} f_{\Phi_t}$ on $\mathcal{C}'_{\tilde{\Phi}_t} \cap \mathcal{C}'_{\Phi_t}$. The function $\mu_t = \mu_{\tilde{\Phi}_t\Phi_t}$ is defined by

$$\mu_{\tilde{\Phi}_t\Phi_t}(\varrho) := \frac{1}{2} \left((\text{sgn}(\Phi_t)''_{\theta\theta}(x, y, \theta) - N) - (\text{sgn}(\tilde{\Phi}_t)''_{\tilde{\theta}\tilde{\theta}}(x, y, \tilde{\theta}) - \tilde{N}) \right), \quad (7.93)$$

where $\theta \in \mathbb{R}^N$, $\tilde{\theta} \in \mathbb{R}^{\tilde{N}}$, $(\Phi_t)'_{\theta}(x, y, \theta) = 0$, $(\tilde{\Phi}_t)'_{\tilde{\theta}}(x, y, \tilde{\theta}) = 0$ and $\varrho = \iota_{\Phi_t}(x, y, \theta) = \iota_{\tilde{\Phi}_t}(x, y, \tilde{\theta}) \in \mathcal{C}'_{\tilde{\Phi}_t} \cap \mathcal{C}'_{\Phi_t}$. Moreover, $\mu_{\tilde{\Phi}_t\Phi_t} \in \mathbb{Z}$ and it is constant on each connected component of $\mathcal{C}'_{\tilde{\Phi}_t} \cap \mathcal{C}'_{\Phi_t}$. The section will be called “natural” if f_{Φ_t} are constant functions taking values in $\{i^k : k \in \mathbb{Z}\}$. In our case $M(\mathcal{C}'_t)$ can be trivialized in a band $|s| < \epsilon$ using the phase functions Φ_t given by Lemma 7.3, where $\phi_t^0(x', \eta) = \langle x', \eta \rangle$. Then $(\Phi_t)''_{\eta\eta}(y, 0, y, \eta) = 0$ in view (7.88) and we get $\text{sgn}(\Phi_t)''_{\eta\eta}(y, 0, y, \eta) = 0$. This yields a natural trivialization of the Keller-Maslov bundle in a band $\mathcal{C}' \cap \{|s| < \epsilon\}$ for some $\epsilon > 0$, choosing a locally constant section which equals 1 in that band. In particular, the Lie derivative $\mathcal{L}_{Y_t}\sigma_{2,t}$ vanishes for each t . This argument holds whenever $\mathcal{C}'|_{s=0}$ is a conormal bundle of a smooth submanifold (see [30] Sect. 3.3 and [13], p. 65).

Using Lemma 7.4 one can obtain a natural section $\sigma_{2,t}$ of $M(\mathcal{C}'_t)$ which is independent of t in a small neighborhood of t^0 for any $t^0 \in [0, \delta]$. The section $\sigma_{2,t}$ of $M(\mathcal{C}'_t)$ can be described as in [13], [30] and [47] as a Maslov index of a suitable path. Let $\varrho_1 = (x^0, y^0, \xi^0, -\eta^0) \in \mathcal{C}'_{t^0}$ and $(x^0, \xi^0) = \exp(TX_{h_{t^0}})(y^0, \eta^0)$. Let M be a submanifold transversal to the corresponding geodesic in x^0 and let Φ_t be a C^1 family of generating function of \mathcal{C}'_t in a neighborhood of ϱ_1 given by Lemma 7.3. Consider the path $\tilde{\gamma}_t$ on \mathcal{C}'_t defined by

$$\pi_1(\tilde{\gamma}_t(s)) = \exp(sX_{h_t})(\pi_t^+(y^0, \eta^0)), \quad s \in [0, T].$$

We have

$$\tilde{\gamma}_{t^0}(0) := \varrho_0 = (y^0, y^0, (\eta^0)^+, -\eta^0) \in \mathcal{C}'_{t^0}|_{s=0} \quad \text{and} \quad \tilde{\gamma}(T) = \varrho_1 \in \mathcal{C}'_{t^0}.$$

Choose a partition $0 = s_0 < s_1 < \dots < s_k = T$ and phase functions $\Phi_{t,j}$, $j = 1, \dots, k$, as in Lemma 7.3 generating locally \mathcal{C}'_t in a neighborhood of $\tilde{\gamma}_t(s_j)$ for t in a small neighborhood of t^0 and such that

$$\tilde{\gamma}_t([s_{j-1}, s_j]) \subset \mathcal{C}'_{\Phi_{t,j}}, \quad \Phi_{t,k} = \Phi_t \quad \text{and} \quad \Phi_{t,1}(x', 0, y, \eta) = \langle x' - y, \eta \rangle.$$

Then trivializing $M(\mathcal{C}'_t)$ in a neighborhood of ϱ_1 by the phase function Φ_t we get

$$(\sigma_{2t})_{\Phi_t} = i^{\mu(\tilde{\gamma}_t)}, \quad \text{where}$$

$$\mu(\tilde{\gamma}_t) := \frac{1}{2} \sum_{j=1}^{k-1} \left(\text{sgn}(\Phi_{t,j})''_{\eta\eta}(\tilde{\gamma}_t(s_j)) - \text{sgn}(\Phi_{t,j+1})''_{\eta\eta}(\tilde{\gamma}_t(s_j)) \right) \in \mathbb{Z}. \quad (7.94)$$

Now Lemma 7.4 implies that $\mu(\tilde{\gamma}_t)$ is independent of $t \in I$ where I is a sufficiently small neighborhood of t^0 in $[0, \delta]$. In order to construct $\sigma_{2,t}$ one can use finitely many paths $\tilde{\gamma}_t$ since \mathcal{C}'_{t^0} is compact, hence I can be chosen to be common for all the paths. We set

$$\sigma_t = \sigma_{1,t} \times \sigma_{2,t} = \tilde{b}_{0,t} \sigma_{0,t} \times \sigma_{2,t}.$$

According to Theorem 7.2 the oscillatory integral $(\Delta_t - \lambda^2)K_{H_t}(x, y, \lambda)$ belongs to $I^{3/4}(\tilde{X} \times \Gamma, \mathcal{C}'_t)$ and its principal symbol is just the Lie derivative $\mathcal{L}_{Y_t}\sigma_t$ multiplied by $(\lambda/2\pi)^{3/4}$ since the subprincipal symbol of the Laplace-Beltrami operator is 0. Moreover, the Lie derivative with respect to Y of the sections $\sigma_{0,t}$ and $\sigma_{2,t}$ vanishes, hence, the transport equation $\mathcal{L}_{Y_t}\sigma_t = 0$ becomes

$$(S^t)^* \tilde{b}_{0,t} = \tilde{b}_{0,t}. \quad (7.95)$$

Multiplying \tilde{b}_0 with a suitable cut-off function, which equals 1 in a neighborhood of $\mathcal{C}'_t \cap T^*(X \times \Gamma)$, we can suppose that \tilde{b}_0 has a compact support with respect to $(s, y, \eta) \in \Lambda$. In this way we obtain a C^1 family of λ -FIOs $H_{0,t}(\lambda)$ with Schwartz kernels in $I^{-1/4}(\tilde{X} \times \Gamma, \mathcal{C}'_t)$ such that the Schwartz kernel of $(\Delta_t - \lambda^2)H_{0,t}(\lambda)$ belongs to $I^{-1/4}(X \times \Gamma, \mathcal{C}'_t)$. Repeating this procedure we get an operator $H_{1,t}(\lambda)$ such that $H_0(\lambda) + H_1(\lambda)$ solves (7.78) modulo a λ -FIO of order $-5/4$ and so on. The initial data $\tilde{b}_0|_{s=0}$ will be determined by Lemma 7.5 below.

Denote by $\iota_\Gamma^* : C^\infty(\tilde{X}) \rightarrow C^\infty(\Gamma)$ the operator of restriction $\iota_\Gamma^*(u) = u|_\Gamma$. We would like to represent ι_Γ^* microlocally as a λ -FIO. To this end, denote by \mathcal{N} the conormal bundle of the graph of the inclusion map $\iota_\Gamma : \Gamma \rightarrow \tilde{X}$ and by $\mathcal{R} = \mathcal{N}^{-1}$ the corresponding inverse canonical relation. In other words,

$$\mathcal{R} := \{(x, \xi; x, \tilde{\xi}) \in T^*\Gamma \times T^*\tilde{X} : x \in \Gamma, \xi = \tilde{\xi}|_{T_x\Gamma}\}.$$

The operator ι_Γ^* can be considered microlocally as a λ -FIO with Schwartz kernel of the class $I^{1/4}(\Gamma \times \tilde{X}, \mathcal{R}; \Omega^{\frac{1}{2}})$ which means that the composition $\iota_\Gamma^* \circ A(\lambda)$ belongs to that class for any classical λ -PDO $A(\lambda)$ of order 0. Moreover, its principal symbol can be identified with $(\lambda/2\pi)^{1/4}$ modulo the corresponding $\frac{1}{2}$ -density (see [63], Sect. A.1.4). In what follows, we shall investigate the composition $\iota_\Gamma^* H_t(\lambda)$ of λ -FIOs. Firstly, notice that the composition $\mathcal{R} \circ \mathcal{C}'_t$ of the corresponding canonical relations is transversal (see [63], Sect. A.1.4). Recall that $\pi_1 : \mathcal{C}'_t \rightarrow T^*X$

and $\pi_2 : \mathcal{C}'_t \rightarrow T^*\Gamma$ are given by $\pi_1(x, y, \xi, -\eta) = (x, \xi)$ and $\pi_2(x, y, \xi, -\eta) = (y, \eta)$. Denote by $dv(\rho) := dy \wedge d\eta$ the symplectic volume form on $T^*\Gamma$. Recall that $\nu_t(x) \in T_x X|_\Gamma$ is the unit inward normal to Γ related to the metric g_t and that $\pi_t^\pm(x, \xi) = (x, \xi_t^\pm) \in \Sigma_t^\pm$ for $(x, \xi) \in B_t^*\Gamma$. Moreover,

$$\langle \xi_t^\pm, \nu_t \rangle(x, \xi) := \langle \xi_t^\pm(x, \xi), \nu_t(x) \rangle = \pm \sqrt{1 - r_t(x, \xi')}$$

in the normal coordinates used in Lemma 7.3. Using (7.91) and (7.92) and the theorem about the composition of h -FIOs one obtains

Lemma 7.5. *The composition of canonical relations $\mathcal{R} \circ \mathcal{C}_t$ is transversal and it is a disjoint union $\Delta^0 \sqcup \mathcal{C}_t^0$ of the diagonal Δ^0 in $U \times U$ (for $s = 0$) and the graph \mathcal{C}_t^0 of the billiard ball map $B_t : U \rightarrow B_t(U)$ (for $s = T$). Moreover,*

$$i_\Gamma^* H_t(\lambda) = P_t(\lambda) + G_t(\lambda) + O_M(|\lambda|^{-M}), \quad (7.96)$$

where $P_t(\lambda)$ is a C^1 family of classical λ -PDOs on Γ of order 0 and $G_t(\lambda)$ is a C^1 family of λ -FIOs with Schwartz kernels in $I^0(\Gamma, \Gamma, \mathcal{C}_t^0)$. The principal symbol of the operator $P_t(\lambda)$ can be identified with

$$\tilde{b}_0(\pi_2^{-1}(\rho)) (2|\langle \xi_t^+, \nu_t \rangle(\rho)|)^{-1/2} |dv(\rho)|^{1/2}, \quad \rho \in U. \quad (7.97)$$

The principal symbol of $G_t(\lambda)$ can be identified with

$$\tilde{b}_{0,t}(\pi_1^{-1}(\pi_t^-(\rho))) |2|\langle \xi_t^-, \nu_t \rangle(\rho)|^{-1/2} e^{i\lambda A_{\gamma_t}(\rho)} |dv(\rho)|^{1/2} \otimes \sigma'_t, \quad \rho \in B_t(U), \quad (7.98)$$

where $A_{\gamma_t} = \int_{\gamma_t} \xi dx$ is the action along the integral curve γ_t of the Hamiltonian vector field X_{h_t} starting from $\pi_t^+(B_t^{-1}(\rho))$ and with endpoint $\pi_t^-(\rho)$ and σ'_t is a natural section of the Maslov bundle $M(\mathcal{C}_t^0)$. Moreover, for each $t^0 \in [0, \delta]$ one can choose σ'_t to be independent of t in a neighborhood of t^0 .

The Lemma is proved in [63], Sect. A.1.4.

Let $\Psi(\lambda)$ be a classical λ -PDO of order 0 with frequency set in U and principal symbol $\Psi_0(\rho)$, $\rho \in U$. We take $\Psi(\lambda)$ as initial data of $H_0(\lambda)$ as $s = 0$ setting $P_t(\lambda) = \Psi(\lambda)$ in Lemma 7.5. Recall that \tilde{b}_0 satisfies (7.95). On the other hand

$$(x', \xi') = B_t(y, \eta) \quad \text{if and only if} \quad \pi_1^{-1}(x', 0, \xi^-) = S^{T_t(y, \eta)}(\pi_2^{-1}(y, \eta))$$

where T_t is the return time function. Then (7.95) and (7.97) imply

$$\tilde{b}_0(\pi_1^{-1}(x', 0, \xi^-)) = \tilde{b}_0((\pi_2^{-1}(y, \eta))) = \Psi_0(y, \eta) (2|\langle \eta^+, \nu \rangle(y, \eta)|)^{1/2}.$$

Then parameterizing \mathcal{C}_t^0 by the variables $(y, \eta) \in U$ and using (7.98) we write the principal symbol of $G_t(\lambda)$ as follows

$$\sigma(G_t(\lambda)) = \Psi_0(y, \eta) \frac{|\langle \eta_t^+, \nu_t \rangle(y, \eta)|^{1/2}}{|\langle \xi_t^-, \nu_t \rangle(B_t(y, \eta))|^{1/2}} e^{i\lambda A_t(y, \eta)} |dy \wedge d\eta|^{1/2} \otimes \sigma'_t, \quad (7.99)$$

where $A_t(y, \eta) = \int_{\gamma_t} \xi dx$ is the action along the integral curve γ_t of the Hamiltonian vector field X_{h_t} starting from $\pi_t^+(y, \eta)$ and with endpoint $\pi_t^-(B_t(y, \eta))$.

In the same way, using Lemma 7.5 we determine the initial conditions of $H_{1,t}(\lambda)$ and so on. In this way we obtain a C^1 family of λ -FIOs

$$H_t(\lambda) = H_{0,t}(\lambda) + H_{1,t}(\lambda) + \dots \quad (7.100)$$

with Schwartz kernels in $I^{-1/4}(\tilde{X}, \Gamma, \mathcal{C}'_t)$ satisfying (7.78) and such that $P_t(\lambda) = \Psi(\lambda)$. From now on, to simplify the notations we drop the corresponding $\frac{1}{2}$ -density. Denote by $E_t(\lambda)$ a C^1 family of classical λ -PDOs of order 0 on Γ with principal symbols $E_{0,t} \in C_0^\infty(\tilde{B}_t^* \Gamma)$ such that

$$E_{0,t}(\rho) = |\langle \xi_t^+, \nu_t \rangle(\rho)|^{\frac{1}{2}} = |\langle \xi_t^-, \nu_t \rangle(\rho)|^{\frac{1}{2}} \quad (7.101)$$

in a compact neighborhood of \bar{U} in $\tilde{B}_t^* \Gamma$. Then using Egorov's theorem, (7.99) and (7.101) we obtain $\iota_\Gamma^* H_t(\lambda)$. We summarize this construction in the following

Proposition 7.6. *The C^1 family of λ -FIOs operators $t \rightarrow H_t(\lambda)$ gives for any half density in f in $L^2(\Gamma)$ a family of solution $u_t = H_t(\lambda)f$ of*

$$(\Delta_t - \lambda^2)u_t = O_N(|\lambda|^{-N})f.$$

Moreover,

$$\iota_\Gamma^* H_t(\lambda) = \Psi(\lambda) + G_t(\lambda) + O_M(|\lambda|^{-M}), \quad G_t(\lambda) = E_t(\lambda)^{-1} G_t^0(\lambda) E_t(\lambda), \quad (7.102)$$

where $E_t(\lambda)$ is a family λ -PDOs of order 0 which are of elliptic microlocally in a neighborhood of $WF'_\lambda(\Psi)$, the principal symbol of $G_t^0(\lambda)$ can be identified with

$$\Psi_0(\rho) e^{i\lambda A_t(\rho)} |dv(\rho)|^{\frac{1}{2}} \otimes \sigma'_t, \quad \rho \in U, \quad (7.103)$$

and σ'_t could be chosen to be independent of t in a small neighborhood of any t_0 .

In particular, the frequency set WF' of $G_t(\lambda)$ is contained in $B_t(U) \times U$.

We are going to estimate the L^2 -norm of $u_t = H_t(\lambda)f$, where f is a $\frac{1}{2}$ -density on Γ . Consider the L^2 -adjoint operator $H_t(\lambda)^*$ of $H_t(\lambda)$ which is well-defined for any $\lambda \in \mathcal{D}$ fixed as an operator from $L^2(\tilde{X})$ to $L^2(\Gamma)$. Moreover, it can be considered as a λ -FIO associated with the canonical relation \mathcal{C}_t^{-1} the Schwartz kernel of which belongs to $I^{-1/4}(\Gamma, \tilde{X}, (\mathcal{C}_t^{-1})')$. As in [63], Sect. A.1.4, we obtain

Proposition 7.7. *The family $t \rightarrow C_t(\lambda) := H_t(\lambda)^* H_t(\lambda) : L^2(\Gamma) \rightarrow L^2(\Gamma)$ is a C^1 family of classical λ -PDOs of order 0. The principal symbol of $C_t(\lambda)$ can be identified with by*

$$C_{0,t}(y, \eta) := \int_{\mathbb{R}} |\tilde{b}_0(s, y, \eta)|^2 ds, \quad (y, \eta), (x, \eta) \in U.$$

Moreover,

$$C_{0,t}(y, \eta) \geq T_t(y, \eta) |\langle \xi_t^+, \nu_t \rangle(y, \eta)| |\Psi_0(y, \eta)|^2, \quad (y, \eta) \in U,$$

where T_t is the return time function. In particular, there exists $C > 1$ such that

$$C^{-1} \|\Psi(\lambda)f\|_{L^2(\Gamma)} \leq \|H_t(\lambda)f\|_{L^2(X)} \leq C \|\Psi(\lambda)f\|_{L^2(\Gamma)}$$

for each $(t, \lambda) \in [0, \delta] \times \mathcal{D}$.

7.2 Reduction to the boundary

The reduction to the boundary is a variant of the reflection method for the wave equation. We shall describe it in the case of Dirichlet boundary conditions. In the case of Neumann and more generally of Robin boundary conditions it is done in [63].

Denote by (\tilde{X}, g_t) a C^∞ extension of (X, g_t) and by h_t the Hamiltonian corresponding to g_t via the Legendre transform. Consider a C^1 family of Kronecker invariant tori $[0, \delta] \ni t \rightarrow \Lambda_t(\omega) \subset \tilde{\mathbf{B}}_t^* \Gamma$ of B_t having frequencies in the set Ω_κ^0 of points of positive Lebesgue density in $\Omega_\kappa = (\Omega - \kappa) \cap D(\kappa, \tau)$, where $\Omega = B(\omega_0, \varepsilon)$, $0 < \kappa < \varepsilon/2 \ll 1$ and $0 \leq \delta \leq 1$. For $0 < \delta \ll 1$ such families of Kronecker invariant tori of B_t are provided by Theorem 3.2. Denote by

$$\mathcal{T}_t^j := \cup \{B_t^j(\Lambda_t(\omega)) : \omega \in \Omega_\kappa^0\} \subset \tilde{B}_t^* \Gamma$$

the cooresponding union of the invariant tori of $P_t \circ B_t^j$ for $0 \leq j < m$ and set

$$\mathcal{T}^j := \cup \{\mathcal{T}_t^j, 0 \leq j < m.$$

Fix $t_0 \in [0, \delta]$ and choose open sets $U_j \subset V_j \subset T^* \Gamma$ for $0 \leq j \leq m$ and a sufficiently small interval $J \subset [0, \delta]$ around t_0 such that

$$\mathcal{T}^j \subset U_j \subset\subset V_j \subset\subset \tilde{\mathbf{B}}_t \Gamma \quad \text{and} \quad B_t(V_j) \subset U_{j+1}$$

for each $t \in J$ and $0 \leq j \leq m - 1$ and

$$\bar{U}_0 \cup \bar{U}_m \subset V_m \subset\subset \tilde{\mathbf{B}}_t \Gamma$$

for each $t \in J$. The relation $U \subset\subset V$ means here that $\bar{U} \subset V$ where \bar{U} is the closure of U . We suppose that the C^1 family of exact symplectic mappings

$$J \ni t \rightarrow P_t$$

admits a C^1 family of BNFs in a neighborhood U of \bar{V}_0 in the sense of Definition 3.3 (see also Theorem 3.2). In other words, we suppose that there exist C^1 -smooth with respect to $t \in J$ families of exact symplectic diffeomorphisms $\chi_t : \mathbb{A} \rightarrow \chi_t(\mathbb{A}) \subset U$ and of real valued functions $L_t \in C^\infty(D)$ and $R_t^0 \in C^\infty(\mathbb{A})$ where $\mathbb{A} = \mathbb{T}^{n-1} \times D$ and $D = \nabla L_{t_0}^*(\Omega)$ such that for each $t \in J$ the following holds

1. $\bar{V}_0 \subset \chi_t(\mathbb{A})$;
2. $\Lambda_t(\omega) = \chi_t(\mathbb{T}^{n-1} \times \{I_t(\omega)\}) \subset U_0$ for $\omega \in \Omega_\kappa^0$, where $I_t(\omega)$ is given by (1.6);
3. The function

$$\mathbb{R}^{n-1} \times D \ni (x, I) \mapsto \phi_t(x, I) := \langle x, I \rangle - L_t(I) - R_t^0(x, I)$$

is a generating function of the exact symplectic map

$$P_t^0 := \chi_t^{-1} \circ P_t \circ \chi_t : \mathbb{A} \rightarrow \mathbb{A}$$

in the sense of Definition 3.1;

4. $\nabla L_t : D \rightarrow \Omega$ is a diffeomorphism and $L_t = L_{t_0}$ outside $D^1 := \nabla L_{t_0}^*(\Omega - \kappa/2)$;
5. R_t^0 is flat at $\mathbb{T}^{n-1} \times E_t^\kappa$, where $E_t^\kappa = \nabla L_t^*(\Omega_\kappa^0)$.

Chose the λ -PDO $\Psi(\lambda)$ giving the “initial data” of the operators $H_t(\lambda)$ in (7.102) such that

$$\text{WF}'_\lambda(\Psi - \text{Id}) \cap \bar{V}_j = \emptyset \quad \forall 0 \leq j \leq m. \quad (7.104)$$

Recall from Proposition 7.6 that

$$i_\Gamma^* H_t(\lambda) = \Psi(\lambda) + G_t(\lambda) + O_M(|\lambda|^{-M})$$

where $G_t(\lambda)$ is described in (7.102) and (7.102). Take now a classical λ -pseudodifferential operator λ -PDO $\Psi_0(\lambda)$ such that

$$\text{WF}'_\lambda(\Psi_0) \subset V_0 \quad \text{and} \quad \text{WF}'_\lambda(\Psi_0 - \text{Id}) \cap \bar{U}_0 = \emptyset.$$

Consider the “outgoing” solution of the Helmlotz equation

$$(\Delta_t - \lambda^2)u_t = O_N(|\lambda|^{-N})f \quad (7.105)$$

for $\lambda \in \mathcal{D}$ and $t \in J$ with “initial data” $\Psi_0(\lambda)f$ which is given by $u_t := H_t^0(\lambda)f$, where

$$H_t^0(\lambda) = H_t(\lambda)\Psi_0(\lambda).$$

Recall that $O_N(|\lambda|^{-N}) : L^2(\Gamma) \rightarrow L^2(\tilde{X})$ stands here for a family of operators

$$A_t(\lambda) : L^2(\Gamma) \rightarrow L^2(\tilde{X})$$

such that

$$\|A_t(\lambda)\|_{L^2} \leq C_N(1 + |\lambda|)^{-N}$$

for each $t \in J$ and $\lambda \in \mathcal{D}$ where $C_N > 0$ is constant independent of t and of λ . Then

$$i_\Gamma^* H_t^0(\lambda) = \Psi_0(\lambda) + G_t(\lambda)\Psi_0(\lambda) + O_M(|\lambda|^{-M})$$

since $\text{WF}'_\lambda((\Psi - \text{Id})\Psi_0) = \emptyset$ in view of (7.104).

To satisfy the “boundary conditions” on U^1 in the case when $m \geq 2$ we use the reflexion method. Let $\Psi_1(\lambda)$ be a classical λ -PDO such that

$$\text{WF}'_\lambda(\Psi_1) \subset V_1 \quad \text{and} \quad \text{WF}'_\lambda(\Psi_1 - \text{Id}) \cap \bar{U}_1 = \emptyset.$$

Set

$$H_t^1(\lambda) = H_t(\lambda)\Psi_1(\lambda)G_t(\lambda)\Psi_0(\lambda)$$

and consider

$$u_t(\lambda) = \tilde{H}_t(\lambda)f := H_t^0(\lambda)f - H_t^1(\lambda)f.$$

Then u_t satisfies (7.105) and it satisfies microlocally the Dirichlet boundary conditions on U_1 . Notice that $\text{WF}'_\lambda(i_\Gamma^*(u_t)) \subset U_0 \cap U_2$. Similarly if $m > 2$ one can treat the boundary conditions in U_j for any $0 < j < m$ which leads to a solution $u_t = \tilde{H}_t(\lambda)f$ satisfying the boundary conditions

microlocally in U_j for each $0 < j < m$. Let $\Psi_j(\lambda)$, $0 \leq j \leq m-1$ be a classical λ -PDO such that

$$\mathrm{WF}'_\lambda(\Psi_j) \subset V_j \quad \text{and} \quad \mathrm{WF}'_\lambda(\Psi_j - \mathrm{Id}) \cap \bar{U}_j = \emptyset.$$

We set $H_t^0(\lambda) = H_t(\lambda)\Psi_0(\lambda)$ if $m = 1$ and

$$\begin{cases} \tilde{H}_t(\lambda)f &= \sum_{j=0}^{m-1} (-1)^j H_t^j(\lambda)f \quad \text{where} \\ H_t^j(\lambda) &= H_t(\lambda)\Psi_{m-1}(\lambda)G_t(\lambda)\Psi_{m-2}(\lambda)\cdots G_t(\lambda)\Psi_0(\lambda) \end{cases} \quad (7.106)$$

if $m \geq 2$. Then u_t satisfies (7.105) and

$$\mathrm{WF}'_\lambda(i_\Gamma^*(u_t)) \subset U_0 \cap B_t(U_{m-1}) \subset\subset V_m.$$

More precisely,

$$i_\Gamma^*(u^t) = \Psi_0(\lambda)f - M_t(\lambda)f + O_N(\lambda^{-N})f$$

where $M_t(\lambda) := -G_t(\lambda)\Psi_0(\lambda)$ for $m = 1$ and

$$M_t(\lambda) := (-1)^{m-1}G_t(\lambda)\Psi_{m-1}(\lambda)\cdots G_t(\lambda)\Psi_0(\lambda)$$

if $m \geq 2$. Taking into account (7.102) we obtain $M_t(\lambda) = E(\lambda)^{-1}M_t^0(\lambda)E(\lambda)$, where

$$M_t^0(\lambda) := (-1)^{m-1}Q_t(\lambda)\Psi_{m-1}(\lambda)\cdots Q_t(\lambda)\Psi_0(\lambda). \quad (7.107)$$

Moreover, using Proposition 7.6 and the theorem about the composition of λ -FIOs (here we use it in the simple case of canonical transformations) and parameterizing $\mathrm{graph}(P_t) \subset V_m \times V_0$ by its projection on V_0 we obtain that for each $t \in J$ the principal symbol of $M_t^0(\lambda)$ is given by

$$(-1)^{m-1} \exp(i\lambda A_t(x, \xi)) |dv(\rho)|^{\frac{1}{2}} \otimes \sigma'_m$$

over U_0 , where

$$A_t(x, \xi) = \sum_{j=0}^{m-1} A_t(x_t^j, \xi_t^j), \quad (x_t^j, \xi_t^j) = B_t^j(x, \xi),$$

is the action along the corresponding broken geodesic and σ'_m is a ‘‘natural’’ section of the corresponding Keller-Maslov bundle which can be chosen to be independent of $t \in J$.

Let $\psi_0(\lambda)$ be a classical λ -PDO of order zero such that

$$\mathrm{WF}'(\psi_0(\lambda)) \subset U_0 \quad \text{and} \quad \mathrm{WF}'(\psi_0(\lambda) - \mathrm{Id}) \cap \mathcal{T}^0 = \emptyset.$$

We summarize the above construction by the following

Proposition 7.8. *Let $v_{t,\lambda} \in L^2(\Gamma)$ and $u_{t,\lambda} = \tilde{H}_t(\lambda)\psi_0(\lambda)v_{t,\lambda}$ where $(t, \lambda) \in J \times \mathcal{D}$. Then*

$$\begin{cases} (\Delta_t - \lambda^2)u_{t,\lambda} &= O_N(\lambda_q^{-N})u_{t,\lambda}, \\ i_\Gamma^* u_{t,\lambda} &= O_N(\lambda_q^{-N})u_{t,\lambda} \end{cases}$$

if and only if

$$(M_t^0(\lambda) - \mathrm{Id})\psi_0(\lambda)v_{t,\lambda} = O_N(\lambda_q^{-N})v_{t,\lambda}. \quad (7.108)$$

The structure of the monodromy operator $M_t^0(\lambda)$ is given by

Proposition 7.9. *The canonical relation of $M_t^0(\lambda) := E_t(\lambda)M_t(\lambda)E_t(\lambda)^{-1}$ is given by the graph $\text{graph}(P_t) \subset V_m \times V_0$ of the symplectic map $P_t = B_t^m : V_0 \rightarrow V_m$, which is C^1 with respect to t . The family $J \ni t \rightarrow M_t^0(\lambda)$ of classical λ -FIO of order 0 with a large parameter $\lambda \in \mathcal{D}$ is C^1 smooth with respect to $t \in J$. Parameterizing graph P_t by its projection on V_0 for $t \in J$, the principal symbol of $M_t^0(\lambda)$ becomes*

$$\sigma(M_t^0) = (-1)^{m-1} \exp(i\lambda A_t(x, \xi)) |dv(\rho)|^{\frac{1}{2}} \otimes \sigma'_m$$

over U_0 , where σ'_m is a “natural” section of the corresponding Keller-Maslov bundle which does not depend on $t \in J$.

7.3 Quantum Birkhoff Normal Form

Using the C^1 family of exact symplectic transformations χ_t given by Theorem 3.2 we identify the first cohomology groups $H^1(\Lambda_t(\omega), \mathbb{Z}) = H^1(\mathbb{T}^{n-1}, \mathbb{Z}) = \mathbb{Z}^{n-1}$ for $\omega \in \Omega_\kappa^0$ and $t \in J$, and we denote by $\vartheta_0 \in \mathbb{Z}^{n-1}$ the Maslov class of the invariant tori $\Lambda_t(\omega)$. Notice that $\vartheta_0 \in \mathbb{Z}^{n-1}$ does not depend on $t \in J$ and $\omega \in \Omega_\kappa^0$. Consider as in [6] the flat Hermitian line bundle \mathbb{L} over \mathbb{T}^{n-1} associated to the representation $\varrho : \mathbb{Z}^{n-1} \rightarrow SU(1)$ of the fundamental one group $\pi_1(\mathbb{T}^{n-1}) = \mathbb{Z}^{n-1}$ defined by $\varrho(k) = \exp(i\frac{\pi}{2}\langle \vartheta_0, k \rangle)$, $k \in \mathbb{Z}^{n-1}$ (see [38], Sect. 1.2). More precisely, \mathbb{L} is the quotient of $\mathbb{R}^{n-1} \times \mathbb{C}$ by the action of \mathbb{Z}^{n-1} given by $k.(x, z) = (x + 2\pi k, \varrho(k)z)$. Then sections s of \mathbb{L} can be identified canonically with smooth functions $\tilde{s} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}$ such that

$$\tilde{s}(x + 2\pi k) = e^{i\frac{\pi}{2}\langle \vartheta_0, k \rangle} \tilde{s}(x) \quad \forall x \in \mathbb{R}^{n-1}, k \in \mathbb{Z}^{n-1}. \quad (7.109)$$

An orthonormal basis of $L^2(\mathbb{T}^{n-1}, \mathbb{L})$ is given by e_k , $k \in \mathbb{Z}^{n-1}$, where

$$\tilde{e}_k(x) = \exp(i\langle k + \vartheta_0/4, x \rangle).$$

We quantize the family of exact symplectic transformations $\chi_t : \mathbb{A} = \mathbb{T}^{n-1} \times D \rightarrow T^*\Gamma$ as in [6], Sect. 5 and [63], Sect. 3.3. Denote by \mathcal{C}_{χ_t} the graph of χ_t in $T^*\Gamma \times T^*\mathbb{T}^{n-1}$ and by $\mathcal{C}'_{\chi_t} = j(\mathcal{C}_{\chi_t})$ the corresponding Lagrangian submanifold of $T^*(\Gamma \times \mathbb{T}^{n-1})$, where j is defined in (7.77). Consider the class of λ -FIOs $T_t(\lambda) : C^\infty(\mathbb{T}^{n-1}, \mathbb{L}) \rightarrow C^\infty(\Gamma, \mathbb{C})$ of order 0 associated with the canonical relation \mathcal{C}_{χ_t} . The Schwartz kernel $K_{T_t(\lambda)}$ of $T_t(\lambda)$ belongs to the class $I^0(\Gamma \times \mathbb{T}^{n-1}, \mathcal{C}'_{\chi_t}; p_2^*(\mathbb{L}'))$, where $p_2 : \Gamma \times \mathbb{T}^{n-1} \rightarrow \mathbb{T}^{n-1}$ is the projection on the second factor. Recall from [6], Sect. 5, that the principal symbol $\sigma(K_{T_t(\lambda)})$ of $K_{T_t(\lambda)}$ can be canonically identified with a smooth function in $T^*\mathbb{T}^{n-1}$. Indeed, $\sigma(K_t T)(\lambda)$ belongs to the symbol class $S^0(\mathcal{C}'_{\chi_t}, M(\mathcal{C}'_{\chi_t}) \otimes \pi_2^*(\mathbb{L}'))$, where $\pi_2 : \mathcal{C}'_{\chi_t} \rightarrow \mathbb{T}^{n-1} \times D$ and $\pi_2 \circ j : \mathcal{C}_{\chi_t} \rightarrow \mathbb{T}^{n-1} \times D$ is the restriction at \mathcal{C}_{χ_t} of the projection $T^*\Gamma \times T^*\mathbb{T}^{n-1} \rightarrow T^*\mathbb{T}^{n-1}$ on the second factor while \mathbb{L}' is the dual bundle to \mathbb{L} (the base manifold of \mathbb{L} and \mathbb{L}' here is $\mathbb{T}^{n-1} \times D$ instead of \mathbb{T}^{n-1}). On the other hand, $M(\mathcal{C}'_{\chi_t}) = \pi_2^*(\mathbb{L})$ and using the parametrization of \mathcal{C}'_{χ_t} given by π_2 we identify the above class of symbols with $S^0(\mathbb{T}^{n-1} \times D, \mathbb{L} \otimes \mathbb{L}')$ which can be canonically identified with $C_0^\infty(\mathbb{T}^{n-1} \times D)$ since $\mathbb{L} \otimes \mathbb{L}'$ is trivial (cf. [29], Chapter I, 3.7). This allows us to obtain a λ -FIO $T_t(\lambda)$ of order 0 associated to the canonical relation \mathcal{C}_{χ_t} , which is microlocally unitary over $\mathbb{A}^0 := \mathbb{T}^{n-1} \times D^0$, where D^0 is a neighborhood of $\cup_{t \in J} E_t^\kappa$ in D and

$$E_t^\kappa = \nabla L_t^*(\Omega_\kappa^0) = I_t(\Omega_\kappa^0)$$

has been defined in 4, Theorem 3.2. This means that

$$\text{WF}'(T_t(\lambda)^*T_t(\lambda) - \text{Id}) \cap \mathbb{A}^0 = \emptyset.$$

Trivializing the $\frac{1}{2}$ -density bundle of \mathcal{C}'_{χ_t} by $\pi_2^*|dv|^{\frac{1}{2}}$, where dv is the symplectic volume form on $T^*\mathbb{T}^{n-1}$, we take the principal symbol of $T_t(\lambda)$ to be equal to one in $\mathbb{T}^{n-1} \times D^0$ modulo a Liouville factor $\exp(i\lambda\Psi_t(\varphi, I))$, where the function Ψ_t is real valued. Consider the C^1 family of λ -FIOs of order zero

$$M_t^1(\lambda) := T_t(\lambda)^*M_t^0(\lambda)T_t(\lambda) : C^\infty(\mathbb{T}^{n-1}, \mathbb{L}) \rightarrow C^\infty(\mathbb{T}^{n-1}, \mathbb{L}).$$

The corresponding canonical relation \mathcal{C}_t is just the graph of $P_t^0 = \chi_t^{-1} \circ P_t \circ \chi_t$ i.e.

$$\mathcal{C}_t := \{(P_t^0(\rho), \rho) : \rho \in \mathbb{A}\}. \quad (7.110)$$

Denote by \mathcal{C}'_t the corresponding Lagrangian submanifold of $T^*(\mathbb{T}^{n-1} \times \mathbb{T}^{n-1})$. Using the theorem about the composition of λ -FIOs in the special case of canonical transformations we obtain that the Schwartz kernel of $M_t^1(\lambda)$ belongs to $I^0(\mathbb{T}^{n-1} \times \mathbb{T}^{n-1}, \mathcal{C}'_t; M(\mathcal{C}'_t) \otimes \text{End}(\mathbb{L}))$. Let us find its principal symbol, parameterizing \mathcal{C}'_t by the variables $\rho = (\varphi, I) \in \mathbb{A}$.

Lemma 7.10. *The principal symbol of $M_t^1(\lambda)$ is given by*

$$\sigma(M_t^1)(\lambda) = (-1)^m \exp(i\lambda f_t) s_{t,0} \otimes \sigma_0 \otimes |dv(\rho)|^{1/2}$$

where $s_{t,0}$ is a C^1 family of smooth function in $\mathbb{T}^{n-1} \times D$ such that $s_{t,0}(\varphi, I) = 1$ in $\mathbb{T}^{n-1} \times D^0$, $dv(\rho)$ the symplectic volume form on $T^*\mathbb{T}^{n-1}$, σ_0 is a natural section of the Keller-Maslov bundle $M(\mathcal{C}'_t)$ independent of t and

$$f_t(\varphi, I) = A_t(\chi_t(\varphi, I)) + \Psi_t(\varphi, I) - \Psi_t(P^0(\varphi, I)), \quad (\varphi, I) \in \mathbb{T}^{n-1} \times D. \quad (7.111)$$

Proof. Notice that $\text{End}(\mathbb{L}) \cong \mathbb{L} \otimes \mathbb{L}'$ is trivial as a bundle over $\mathbb{T}^{n-1} \times D$, hence, smooth sections can be canonically identified with smooth functions in $\mathbb{T}^{n-1} \times D$. Then parameterizing \mathcal{C}'_t by the variables $\rho = (\varphi, I) \in \mathbb{A}$ and using the λ -FIO calculus and Proposition 7.9 we obtain the principal symbol of $M_t^1(\lambda)$. To prove (7.111) we write microlocally the Schwartz kernels of the corresponding λ -FIOs in as oscillatory integrals of the form (7.89) with suitable phase functions and then we evaluate the phase function of the composition at the stationary points. The claim that σ_0 is natural and independent of t follows from the fact that the section σ'_m in Proposition 7.9 is natural and from the composition law of FIOs. \square

Recall that the Lagrangian manifolds \mathcal{C}'_t are generated by the C^1 family of functions $\Phi_t(x, y, I) = \phi_t(x, I) - \langle y, I \rangle$, where

$$\phi_t(x, I) = \langle x, I \rangle - L_t(I) - R_t^0(x, I)$$

satisfies 3-5 in Sect. 7.2 (see also Definition 3.3 and Theorem 3.2).

Proposition 7.11. *We have*

$$T_t(\lambda)^*M_t^0(\lambda)T_t(\lambda) = e^{i\pi\vartheta/2}W_t(\lambda)$$

where $\vartheta \in \mathbb{Z}$ is a Maslov's index independent of $t \in J$ and

$$J \ni t \rightarrow W_t(\lambda) : C^\infty(\mathbb{T}^{n-1}, \mathbb{L}) \rightarrow C^\infty(\mathbb{T}^{n-1}, \mathbb{L})$$

is a C^1 family of λ -FIOs of order zero with canonical relations given by the graph of P_t^0 over \mathbb{A} . Moreover, the Schwartz kernel of $W_t(\lambda)$ is of the form

$$\widetilde{W}_t(x, y, \lambda) |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}} = \left(\frac{\lambda}{2\pi} \right)^{n-1} \left(\int_{\mathbb{R}^{n-1}} e^{i\lambda(\phi_t(x, I) - \langle y, I \rangle)} w_t(x, I, \lambda) dI \right) |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}}, \quad (7.112)$$

where $t \rightarrow w_t = \sum_{j=0}^{\infty} w_{t,j}$ is a C^1 family of classical amplitudes 2π -periodic with respect to x and $w_{t,0}(x, I) = 1$ for $(x, I) \in \mathbb{R}^{n-1} \times D^0$.

Proof. The Schwartz kernel of $M_t^1(\lambda)$ can be written in the form (7.112) with a phase function $C + \Phi_t(x, y, I)$, where C is a constant since Φ_t is a globally defined generating function of \mathcal{C}'_t . We are going to show that $C = 0$. Indeed, the exponent on the Liouville factor picked up by these phase functions is

$$C + \langle I, \nabla L_t(I) \rangle - L_t(I) + \langle I, \nabla_I R_t^0(\varphi, I) \rangle - R_t^0(\varphi, I) = f_t^0(\varphi, I),$$

then taking $(\varphi, I) \in E_t^\kappa$ and using Lemma 3.5, (7.111) and the equality $R_t^0|_{E_t^\kappa} = 0$ we get $C = 0$. Trivializing the Maslov bundles $M(\mathcal{C}'_t)$ by the C^1 family of phase functions Φ_t we get $(\sigma_0)_{\Phi_t} = \exp(i\frac{\pi}{2}\vartheta_1)$ for some $\vartheta_1 \in \mathbb{Z}$ independent of t since σ_0 does not depend on t . We set $\vartheta = \vartheta_1 + m\pi$. Moreover, by (7.75) and (7.76) we obtain that $|d_{C_\Phi}| = dx dI$. Hence, $w_{t,0}(x, I) = s_{t,0}(\text{pr}(x), I) = 1$ for each $(x, I) \in \mathbb{R}^{n-1} \times D^0$. \square

In the case of Neumann and Robin boundary conditions we have $\vartheta = \vartheta_1$.

Our aim is to make $w_{t,j}(x, I)$ independent of the angle variable x for $I \in E_t^\kappa$ conjugating $W_t(\lambda)$ by a suitable C^1 in t family of λ -PDOs which are elliptic on $\mathbb{T}^{n-1} \times D^0$.

Proposition 7.12. *There exists a C^1 family of λ -PDOs $J \ni t \rightarrow A_t(\lambda)$ of order 0 acting on $C^\infty(\mathbb{T}^{n-1}, \mathbb{L})$ and a C^1 family of λ -FIO $J \ni t \rightarrow W_t^0(\lambda)$ of the form (7.112) such that*

$$W_t(\lambda)A_t(\lambda) = A_t(\lambda)W_t^0(\lambda) + Z_t(\lambda), \quad (7.113)$$

where

(1) the full symbols of $A_t(\lambda)$ and of $W_t^0(\lambda)$ are

$$\begin{aligned} \sigma(A_t)(\varphi, I, \lambda) &:= a_t(\varphi, I, \lambda) \sim \sum_{j=0}^{\infty} \lambda^{-j} a_t^j(\varphi, I, \lambda) \text{ and} \\ \sigma(W_t^0)(\varphi, I, \lambda) &:= p_t(I, \lambda) \sim \sum_{j=0}^{\infty} \lambda^{-j} p_t^j(I), \end{aligned}$$

where $J \ni t \rightarrow a_t(\varphi, I, \lambda)$ and $J \ni t \rightarrow p_t(I, \lambda)$ are C^1 families of classical symbols and $a_t^0(\varphi, I) = 1$ and $p_t^0(I) = 1$ for $I \in D^0$,

(2) $J \ni t \rightarrow Z_t(\lambda)$ is a C^1 family of λ -FIOs of order 0 of the form (7.112) with symbols

$$S_t(\varphi, I, \lambda) \sim \sum_{j=0}^{\infty} \lambda^{-j} S_t^j(\varphi, I)$$

such that the functions S_t^j , $j \geq 0$, are flat on $\mathbb{T}^{n-1} \times E_t^\kappa$.

Proof. The proof of the proposition is similar to that in [5] and [63]. First, comparing the symbols of order $-j$ of the left and the right hand side of (7.113) we shall derive the corresponding homological equation. Set

$$\phi_t^0(x, I) = L_t(I) + R_t(x, I).$$

We write the Schwartz kernel of the operator $W_t(\lambda)A_t(\lambda)$ of the form (7.112) with amplitude

$$u_t(x, I, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-z, \xi-I \rangle - (\phi_t^0(x, \xi) - \phi_t^0(x, I)))} w_t(x, \xi, \lambda) a_t(z, I, \lambda) d\xi dz ,$$

which belongs to $C^\infty(\mathbb{T}^{n-1} \times D)$ for each λ fixed. Changing the variables we write $u_t(x, I, \lambda)$ of the form

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v, \eta \rangle} w_t(x, I + \eta, \lambda) a_t(v + x + K_t(I, \eta) + H_t(x, I, \eta), I, \lambda) d\eta dv ,$$

where $K_t(I, \eta) := \int_0^1 \nabla_I L_t(I + s\eta) ds$ and $H_t(x, I, \eta) := \int_0^1 \nabla_I R_t(x, I + s\eta) ds$. Note that $K_t(I, 0) = \nabla L_t(I)$. Moreover, $\partial_I^\alpha \partial_\eta^\beta H_t(x, I, 0) = 0$ for each $I \in E_t^\kappa$ and any $\alpha, \beta \in \mathbb{Z}^{n-1}$ since the function $I \rightarrow R_t(x, I)$ is flat at E_t^κ for every $x \in \mathbb{R}^{n-1}$ in view of 5, Sect. 7.2. Using the Taylor formula for the amplitude at $v = 0$ and integrating by parts we get

$$u_t(x, I, \lambda) \sim \sum_{j=1}^{\infty} u_t^j(x, I) \lambda^{-j}$$

where

$$u_t^j(x, I) := \sum_{r+s+|\gamma|=j} \frac{1}{\gamma!} [D_\eta^\gamma (a_t^r(x, I + \eta) \partial_x^\gamma a_t^s(x + K_t(I, \eta) + H_t(x, I, \eta), I))]_{|\eta|=0} .$$

In the same way we write the Schwartz kernel of $A_t(\lambda)W_t^0(\lambda)$ in the form (7.112) with amplitude $q_t(x, I, \lambda)$ given by the oscillatory integral

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} p^0(I, \lambda) \int_{\mathbb{R}^{2n-2}} e^{i\lambda(\langle x-z, \xi-I \rangle - (\phi_t^0(z, I) - \phi_t^0(x, I)))} a_t(x, \xi, \lambda) d\xi dz .$$

Changing the variables we obtain $q_t = q_t^0 + q_t^1$, where $q_t^0(x, I, \lambda) = a_t(x, I, \lambda)p_t(I, \lambda)$ and $q_t^1(x, I, \lambda)$ is given by

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} p_t(I, \lambda) \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v, \eta \rangle} [a_t(x, \eta + I + H_t^1(x, v, I), \lambda) - a_t(x, \eta + I, \lambda)] d\eta dv ,$$

where $H_t^1(x, v, I) = \int_0^1 \nabla_x R_t(x + \tau v, I) d\tau$. Moreover, all the derivatives of $H_t^1(x, v, I)$ vanish for $I \in E_t^\kappa$ since the function $I \rightarrow R_t(x, I)$ is flat at E_t^κ for every $x \in \mathbb{R}^{n-1}$. In this way we obtain for any $j \geq 1$ that

$$S_t^j(\varphi, I) = a_t^j(\varphi + \nabla L_t(I), I) - a_t^j(\varphi, I) - p_t^j(I) - F^j(\varphi, I, t), \quad (7.114)$$

where F^j is a polynomial of $\partial_\varphi^\alpha \partial_I^\beta a_t^l$ and $\partial_I^\beta p_t^l$ for $l < j$ and $|\alpha| + |\beta| \leq 2j$ and of $\partial_I^\beta L_t$ for $|\beta| \leq 2j + 1$.

We are looking for functions a_t^j and $p_t^j(I)$ such that $S_t^j(\varphi, I) = 0$ on $\mathbb{T}^{n-1} \times E_t^\kappa$. We shall solve this equation recursively with respect to j changing the variables by $I = I_t(\omega)$, $\omega \in \Omega$, and we consider Ω as a subset of \mathbb{R}^{n-1} . Set $f(\varphi, \omega, t) := a_t^j(\varphi, I_t(\omega))$, $c(\omega, t) := p_t^j(I_t(\omega))$ and $F(\varphi, \omega, t) := F^j(\varphi, I_t(\omega), t)$. Then we get the homological equation

$$f(\varphi + \omega, \omega, t) - f(\varphi, I, t) = c(\omega, t) + F(\varphi, \omega, t), \quad \omega \in \Omega_\kappa^0. \quad (7.115)$$

We are looking for smooth functions f and c on $\mathbb{T}^{n-1} \times \Omega$ and Ω respectively, which solve (7.115) for every $\omega \in \Omega_\kappa^0$. We have the following

Lemma 7.13. *Let $J \ni t \rightarrow F(\cdot, \cdot, t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega)$ be a C^1 family of functions such that $F(\varphi, \omega, t) = 0$ for each $\omega \in \Omega_\kappa^0$. Then there exist C^1 families functions $J \ni t \rightarrow f(\cdot, \cdot, t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega)$ and $J \ni t \rightarrow c(\cdot, t) \in C^\infty(\Omega)$ such that the function*

$$(\varphi, \omega) \rightarrow S(\varphi, \omega, t) := f(\varphi + \omega, \omega, t) - f(\varphi, I, t) - c(\omega, t) - F(\varphi, \omega, t)$$

is flat at $\mathbb{T}^{n-1} \times \Omega_\kappa^0$ for each t fixed.

Proof. Given $g \in L^1(\mathbb{T}^{n-1})$ we denote by \hat{g}_k , $k \in \mathbb{Z}^{n-1}$, its Fourier coefficients. For any $k \in \mathbb{Z}^{n-1}$ we have

$$\hat{S}_k(\omega, t) = \hat{f}_k(\omega, t) \left(e^{i\langle \omega, k \rangle} - 1 \right) - c(\omega, t) - \hat{F}_k(\omega, t).$$

We set $c(\omega, t) = -\hat{F}_0(\omega, t)$, which gives $\hat{S}_0(\omega, t) = 0$. We are going to find $S_k(\omega, t)$ for $k \neq 0$. To this end we choose $\phi \in C_0^\infty(\mathbb{R})$ such that

$$0 \leq \phi \leq 1, \quad \phi(x) = 1 \text{ for } |x| \leq \pi/5 \text{ and } \phi(x) = 0 \text{ for } |x| \geq \pi/4.$$

For any $0 \neq k \in \mathbb{Z}^{n-1}$ set

$$\phi_k(x) := \sum_{j \in \mathbb{Z}} \phi((x - 2\pi j)|k|^\tau \kappa^{-1}).$$

We have $|k|^\tau \kappa^{-1} \geq \kappa^{-1} > 1$, hence, $\phi((x - 2\pi j)|k|^\tau \kappa^{-1}) = 0$ for $j \neq [x]_\pi$, where $[x]_\pi/2\pi \in \mathbb{Z}$ is the unique integer such that $-\pi \leq x - [x]_\pi < \pi$. Then $\phi_k(x) = \phi(\{x\}|k|^\tau \kappa^{-1})$, where $\{x\} = x - [x]_\pi$. Fix $k \neq 0$ in \mathbb{Z}^{n-1} and consider the smooth function

$$\omega \rightarrow z_k(\omega) = 1 - e^{i\langle \omega, k \rangle} + \frac{1}{3} \kappa (1 + |k|)^{-\tau} \phi_k(\langle \omega, k \rangle).$$

Lemma 7.14. *We have*

$$|z_k(\omega)| \geq \frac{1}{3} \kappa (1 + |k|)^{-\tau} \quad \forall \omega \in \Omega. \quad (7.116)$$

Moreover,

$$z_k(\omega) = 1 - e^{i\langle \omega, k \rangle} \quad \forall \omega \in \Omega_\kappa. \quad (7.117)$$

Proof. Let Ω^1 be the set of all $\omega \in \Omega$ such that

$$\pi/6 \leq |\{\langle \omega, k \rangle\}| |k|^\tau \kappa^{-1} \leq \pi$$

and Ω^2 the set of all $\omega \in \Omega$ such that

$$|\{\langle \omega, k \rangle\}| |k|^\tau \kappa^{-1} \leq \pi/6.$$

For every $\omega \in \Omega^1$ we have

$$|1 - \exp(i\langle k, \omega \rangle)| = 2 \left| \sin\left(\frac{1}{2}\{\langle \omega, k \rangle\}\right) \right| \geq \frac{4}{\pi} |\{\langle \omega, k \rangle\}| \geq \frac{2}{3} \kappa (1 + |k|)^{-\tau}.$$

This implies

$$|z_k(\omega)| \geq \frac{1}{3} \kappa (1 + |k|)^{-\tau} \quad \forall \omega \in \Omega^1.$$

If $\omega \in \Omega^2$, then $\phi_k(\langle \omega, k \rangle) = \phi(\{\langle \omega, k \rangle\} |k|^\tau \kappa^{-1}) = 1$, hence,

$$\operatorname{Re}(z_k(\omega)) \geq \frac{1}{3} \kappa (1 + |k|)^{-\tau}$$

which proves (7.116). Moreover, for any $\omega \in \Omega_\kappa$ we have $\phi_k(\langle \omega, k \rangle) = \phi(\{\langle \omega, k \rangle\} |k|^\tau \kappa^{-1}) = 0$ in view of (1.2), which implies (7.117). \square

Let us go back to the homological equation (7.115). For every $k \neq 0$ we set

$$\hat{f}_k(\omega, t) := -\frac{\hat{F}_k(\omega, t)}{z_k(\omega)}.$$

Using Lemma 7.14 we obtain that the function

$$(\varphi, I) \rightarrow f(\varphi, \omega, t) := \sum_{k \in \mathbb{Z}^{n-1}} \hat{f}_k(\omega, t) e^{i\langle \varphi, k \rangle}$$

belongs to $C^\infty(\mathbb{T}^{n-1} \times \Omega)$ for any t fixed and the map $J \ni t \rightarrow f(\cdot, \cdot, t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega)$ is C^1 . Hence, the map $J \ni t \rightarrow S(\cdot, \cdot, t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega)$ is C^1 as well. Moreover, $F_k(\varphi, \omega, t) = 0$ for each $\omega \in \Omega_\kappa^0$ and $k \in \mathbb{Z}^{n-1}$ and using (7.117) we obtain that $S(\varphi, \omega, t) = 0$ for $\omega \in \Omega_\kappa^0$. Now Lemma 3.4 implies that the function $\omega \rightarrow S(\varphi, \omega, t)$ is flat at Ω_κ^0 for each φ and t fixed. Now using This completes the proof of Lemma 7.13. \square

Using Lemma 7.13 we find a_t^j and $p_t^j(I)$ such that $\partial_I^\alpha S_t^j(\varphi, I, t) = 0$ for every $I \in E_t^\kappa$ and $\alpha \in \mathbb{Z}^{n-1}$. Using Lemma 7.1 we find C^1 of realisations $S_t(\varphi, I, \lambda)$ and $p_t(I, \lambda)$ of the formal symbols and such that which completes the proof of Proposition 7.12. \square

We are looking for C^1 families of solutions $t \rightarrow (\lambda(t), v(t))$ of the equation (7.108) of the form

$$v(t) = E_t(\lambda)^{-1} T_t(\lambda) A_t(\lambda) e(t)$$

for $t \in J$. In view of Proposition 7.12, $e(t)$ should satisfy the equation

$$e^{i\pi\theta/2} W_t^0(\lambda) e(t) + e^{i\pi\theta/2} S_t(\lambda) e(t) = e(t) + O_N(|\lambda|^{-N}) e(t). \quad (7.118)$$

Natural candidates for $e(t)$ are the sections e_k , $k \in \mathbb{Z}^{n-1}$. Since $\lambda \in \mathcal{D}$ may be complex, we consider almost analytic extensions of order $M \geq 2N + n + 2$ of the functions ϕ_t^0 , p_t and S_t^j in $\zeta = \xi + i\eta$, where $\xi \in D$ and $|\eta| \leq C$. The almost analytic extension of ϕ_t^0 is given by

$$\phi_t^0(x, \xi + i\eta) = L_t(\xi + i\eta) + R_t(x, \xi + i\eta)$$

where

$$L_t(\xi + i\eta) = \sum_{|\alpha| \leq M} \partial_\xi^\alpha L_t(\xi)(i\eta)^\alpha (\alpha!)^{-1} \text{ and } R_t(x, \xi + i\eta) = \sum_{|\alpha| \leq M} \partial_\xi^\alpha R_t(x, \xi)(i\eta)^\alpha (\alpha!)^{-1}.$$

It is easy to see that

$$\partial_\zeta \phi_t^0(x, \xi + i\eta) = O(|\eta|^M).$$

Moreover,

$$\partial_\zeta^\alpha \bar{\partial}_\zeta^\beta R_t(x, \xi + i\eta) = O_M(|\xi - E_t^\kappa|^M), \quad |\eta| \leq C, \quad (7.119)$$

for $\alpha, \beta \in \mathbb{N}^{n-1}$ since R_t is flat at $\mathbb{R}^{n-1} \times E_t^\kappa$. In the same way we obtain almost analytic extensions $S_t(\varphi, \zeta, \lambda)$ of S_t^j and $p_t^j(\zeta)$ of p_t^j , $\zeta := \xi + i\eta$, such that

$$\bar{\partial}_\zeta p_t^j(\xi + i\eta) = O(|\eta|^M), \quad \partial_\xi^\alpha \partial_\eta^\beta \bar{\partial}_\zeta S_t^j(x, \xi + i\eta) = O_{\alpha, \beta}(|\eta|^{M-|\beta|}), \quad (7.120)$$

for $\alpha, \beta \in \mathbb{N}^{n-1}$, $|\beta| \leq M$. Moreover,

$$\partial_\xi^\alpha \partial_\eta^\beta S_t^j(\varphi, \xi + i\eta) = O(|\xi - E_t^\kappa|^N), \quad (7.121)$$

for $|\eta| \leq C$, and $\text{supp}_\zeta S_t^j \subset K$, $\text{supp} p_t^j \subset K$ for $j \in \mathbb{N}$, where K is a fixed compact subset of \mathbb{R}^{n-1} . We have as well $p_t^0(\xi + i\eta) = 1$ whenever $\xi \in D^0$.

Proposition 7.15. *For each $t \in J$ we have*

$$\begin{aligned} W_t^0(\lambda) e_k(\varphi) &= \exp\left(-i\lambda \phi_t^0(\varphi, (k + \vartheta_0/4)\lambda^{-1})\right) \\ &\times \left(\sum_{j=0}^N p_t^j((k + \vartheta_0/4)\lambda^{-1}) \lambda^{-j} \right) e_k(\varphi) + O_N(|\lambda|^{-N-1}) e_k(\varphi) \end{aligned} \quad (7.122)$$

and

$$S_t(\lambda) e_k(\varphi) = O_N(|\lambda|^{-N-1} + |E_t^\kappa - (k + \vartheta_0/4)\lambda^{-1}|^{N+1}) e_k(\varphi) \quad (7.123)$$

where $\lambda \in \mathcal{D}$ and $k \in \mathbb{Z}^{n-1}$.

Proof. The proof of the proposition is close to that of Proposition 3.11, [63] but we give it for the sake of completeness. We have

$$\begin{aligned} \widetilde{W_t^0(\lambda) e_k(x)} &= \widetilde{e_k(x)} e^{-i\lambda \phi_t^0(x, \xi_k)} \\ &\times \lambda^{-j} \left(\frac{\lambda}{2\pi} \right)^{n-1} \sum_{j=0}^N \int_{\mathbb{R}^{2n-2}} e^{i\lambda \langle x-y+w_t(x, \xi_k, \eta_k), \eta_k \rangle} p_t^j(I) dI dy + O_N(|\lambda|^{-N-1}) \widetilde{e_k(x)}, \end{aligned}$$

where $\lambda \in \mathcal{D}$ and

$$w_t(x, \xi, \eta) = \int_0^1 \nabla_\xi \phi_t^0(x, \xi + \tau\eta) d\tau, \quad \xi_k = (k + \vartheta_0/4)/\lambda, \quad \eta_k = I - (k + \vartheta_0/4)/\lambda.$$

If $|k| \geq C_0|\lambda|$ and $C_0 \gg 1$ (C_0 depends only on the compact set $K \subset \mathbb{R}^{n-1}$ such that $\text{supp} p_t^j \subset K$ for every $j \in \mathbb{N}$) then $|\eta_k| \geq 1$ and we can integrate by parts with respect to y gaining $O_N(|\lambda|^{-N-1})$. Suppose now that $|k| \leq C_0|\lambda|$. We have

$$|\text{Im}((k + \vartheta_0/4)/\lambda)| \leq \frac{C}{|\lambda|} \quad \text{for } |k| \leq C_0|\lambda| \text{ and } \lambda \in \mathcal{D}. \quad (7.124)$$

Then deforming the contour of integration we obtain

$$\begin{aligned} W_t^0(\lambda)e_k(\varphi) &= e_k(\varphi) e^{-i\lambda\phi_t^0(\varphi, (k+\vartheta_0/4)/\lambda)} \\ &\times \sum_{j=0}^N \lambda^{-j} \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle u, v \rangle} p_t^j(v + (k + \vartheta_0/4)/\lambda) du dv + O_N(|\lambda|^{-N-1})e_k(\varphi), \end{aligned}$$

which implies (7.122).

To prove (7.123) we write $S_t(\lambda)\widetilde{e}_k(x)$ as an oscillatory integral as above, and then for $|k| \leq C_0|\lambda|$ we change the contour of integration with respect to y by

$$y \rightarrow v = y - x - w_t(x, (k + \vartheta_0/4)/\lambda, I - (k + \vartheta_0/4)/\lambda)$$

while for $|k| \geq C_0|\lambda|$ we integrate by parts to gain $O_N(|\lambda|^{-N-1})$. This implies, using (7.124), that

$$\begin{aligned} S_t(\lambda)e_k(\varphi) &= e_k(\varphi) e^{-i\lambda\phi_t^0(\varphi, (k+\vartheta_0/4)/\lambda)} \\ &\times \sum_{j=0}^N \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbb{R}^{2n-2}} e^{-i\lambda\langle v, I - (k+\vartheta_0/4)/\lambda \rangle} S_t^j(\varphi, I) \lambda^{-j} dI dv + O_N(|\lambda|^{-N-1})e_k(\varphi). \end{aligned}$$

Since $M > 2N + n + 2$, taking the Taylor expansion of order N of the function

$$[0, 1] \ni s \rightarrow \psi(s) := S_t^j(\varphi, (k + \vartheta_0/4)/\lambda) + s(I - (k + \vartheta_0/4)/\lambda)$$

at $s = 0$ with an integral remainder and using (7.120) and (7.124) we get

$$S_t^j(\varphi, I) = \sum_{|\alpha| \leq N} \partial_\zeta^\alpha S_t^j(\varphi, (k + \vartheta_0/4)/\lambda) (I - (k + \vartheta_0/4)/\lambda)^\alpha / \alpha! + T_N(\varphi, I) + O(|\lambda|^{-N-n-1})$$

where the reminder term is

$$T_N(\varphi, I) := (N + 1) \sum_{|\alpha|=N+1} \int_0^1 (1-s)^N \partial_I^\alpha S_t^j(\varphi, I + s(k + \vartheta_0/4)/\lambda) (I - (k + \vartheta_0/4)/\lambda)^\alpha / \alpha! ds.$$

We have

$$\partial_\zeta^\alpha S_t^j(\varphi, (k + \vartheta_0/4)/\lambda) = O_{N, \alpha, \beta} (|E_t^\kappa - (k + \vartheta_0/4)\lambda^{-1}|^{N+1}), \quad \lambda \in \mathcal{D},$$

for every $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$ in view of (7.121).

To estimate the reminder we integrate $N + 1$ times by parts with respect to v in the corresponding oscillatory integral with amplitude

$$(N + 1) \sum_{|\alpha|=N+1} \int_0^1 (1-s)^N \partial_I^\alpha S_t^j(\varphi, I + s(k + \vartheta_0/4)/\lambda) (I - (k + \vartheta_0/4)/\lambda)^\alpha / \alpha! ds$$

and we estimate it by $C_N |\lambda|^{-N-1}$. This implies (7.123). \square

Proposition 7.15 suggests that we should look for pairs $(\lambda, k) \in \mathcal{D} \times \mathbb{Z}^{n-1}$ such that $|\lambda| \gg 1$ and

$$|E_t^\kappa - (k + \vartheta_0/4)\lambda^{-1}| \leq \frac{C}{|\lambda|} \tag{7.125}$$

where $C > 0$ is a constant. Then (7.122) and (7.123) imply

$$\begin{aligned} W_t^0(\lambda)e_k(\varphi) &= \exp\left(-i\lambda L_t((k + \vartheta_0/4)\lambda^{-1})\right) \\ &\times \left(\sum_{j=0}^N p_t^j((k + \vartheta_0/4)\lambda^{-1})\lambda^{-j}\right)e_k(\varphi) + O_N(|\lambda|^{-N-1})e_k(\varphi) \end{aligned}$$

and

$$S_t(\lambda)e_k(\varphi) = O_N(|\lambda|^{-N-1})e_k(\varphi)$$

Thus taking $e = e_k$ in (7.118) we obtain

$$\begin{aligned} &\exp\left(-i\lambda L_t((k + \vartheta_0/4)\lambda^{-1}) + i\pi\vartheta/2\right) \left(\sum_{j=0}^N p_t^j((k + \vartheta_0/4)\lambda^{-1})\lambda^{-j}\right)e_k(\varphi) \\ &= O_N(|\lambda|^{-N-1})e_k(\varphi) \end{aligned}$$

for every $N \in \mathbb{N}$. Recall that $p_t^0(\xi + i\eta) = 1$ if $\xi \in D^0$. Then for $|\lambda| \gg 1$ and $t \in J$ we can write the above equation as follows

$$\begin{aligned} \lambda L_t\left(\frac{k + \vartheta_0/4}{\lambda}\right) &= 2\pi k_n \pi + \pi\vartheta/2 \\ &+ \frac{1}{i} \text{Log} \left(1 + \sum_{j=1}^N p_t^j\left(\frac{k + \vartheta_0/4}{\lambda}\right)\lambda^{-j}\right) + O_N(|\lambda|^{-N-1}) \end{aligned} \quad (7.126)$$

where $k_n \in \mathbb{Z}$ and $\text{Log } z = \ln |z| + i \arg z$, $-\pi < \arg z < \pi$.

Hence, to construct quasi-modes we have to find pairs (λ, k) satisfying both (7.125) and (7.126).

8 C^1 families of quasi-modes and iso-spectral invariants

Given $t \in J$ and $\omega \in \Omega_\kappa^0$ the formulas (7.125) and (7.126) suggest that the quantization condition of the Lagrangian torus $\Lambda_t(\omega)$ should be of the form

$$\left|\lambda I_t(\omega) - (k + \vartheta_0/4)\right| + \left|\lambda L_t\left(\frac{k + \vartheta_0/4}{\lambda}\right) - 2\pi k_n \pi - \pi\vartheta/2\right| \leq C$$

for some $C > 0$, where $I_t(\omega) \in E_t^\kappa$ is the corresponding action on the torus $\Lambda_t(\omega)$, $(k, k_n) \in \mathbb{Z}^n$ and $\lambda \in \mathcal{D}$. To obtain iso-spectral invariants from C^1 -families of quasi-modes we need a stronger quantization condition which will be formulated below.

8.1 Quantization condition

Fix $t \in J$. The quantization condition corresponding to a Lagrangian torus $\Lambda_t(\omega)$ with a frequency $\omega \in \Omega_\kappa^0$ will be given by means the following Lemma.

Lemma 8.1. *Given $t \in J$ there is a set $\Xi_\kappa^t \subset \Omega_\kappa^0$ of full Lebesgue measure in Ω_κ^0 such that the following holds.*

For any $\omega \in \Xi_\kappa^t$ there is an infinite sequence $\widetilde{\mathcal{M}}(\omega)$ of $(q, \lambda) \in \mathbb{Z}^n \times [1, \infty)$ such that

$$q = (k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}, \quad \lambda = \mu_q^0 \geq 1 \quad \text{satisfies} \quad c_0^{-1}|q| \leq \mu_q^0 \leq c_0|q| \quad \text{with } c_0 > 0, \quad (8.127)$$

and

$$\lim_{|q| \rightarrow \infty} \left| \mu_q^0 \left(I_t(\omega), L_t(I_t(\omega)) \right) - \left(k + \frac{\vartheta_0}{4}, 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right) \right| = 0. \quad (8.128)$$

Proof. Denote by Ξ_κ^t the set of all $\omega \in \Omega_\kappa^0$ such that

$$2\pi k_n I_t(\omega) \neq L_t(I_t(\omega))k \quad \text{for each } 0 \neq (k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}. \quad (8.129)$$

We claim that the complement $\Omega_\kappa^0 \setminus \Xi_\kappa^t$ of Ξ_κ^t in Ω_κ^0 is of Lebesgue measure zero. Suppose the contrary. Then there is $0 \neq (k, k_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ and a set of positive Lebesgue measure $R_t \subset \Omega_\kappa^0$ such that

$$2\pi k_n I_t(\omega) = L_t(I_t(\omega))k \quad \forall \omega \in R_t.$$

On the other hand, the map $\Omega \ni \omega \rightarrow I_t(\omega) = \nabla L_t^*(\omega) \in D$ is a local diffeomorphism with inverse $I \rightarrow \nabla L_t(I)$ by 4, Theorem 3.2, hence, the set $R_t^0 := \{I_t(\omega) : \omega \in R_t\}$ is of positive Lebesgue measure in \mathbb{R}^{n-1} . Moreover,

$$2\pi k_n I = L_t(I)k \quad \forall I \in R_t^0 \quad (8.130)$$

and $R_t := \{\nabla L_t(I) : I \in R_t^0\}$ by definition. Let $I^0 \in R_t^0$ be a point of positive Lebesgue density in R_t^0 . Set $\omega^0 = (\omega_1^0, \dots, \omega_{n-1}^0) := \nabla L_t(I^0) \in \Omega_\kappa^0$. Differentiating (8.130) with respect to I at I^0 and using Lemma 3.4 we get $2\pi k_n = k_j \omega_j^0$, for $j = 1, \dots, n-1$, which contradicts (1.2). Hence, the Lebesgue measure of $\Omega_\kappa^0 \setminus \Xi_\kappa^t$ is zero. On the other hand, (8.129) implies that for any $\omega \in \Omega_\kappa^0$ the trajectory

$$\{\lambda(I_t(\omega), L_t(I_t(\omega))) \pmod{\mathbb{Z}^n} : \lambda \geq 1\} \subset \mathbb{R}^n / \mathbb{Z}^n$$

is not periodic, hence, it is dense on the torus $\mathbb{R}^n / \mathbb{Z}^n$ which implies that there exists an infinite sequence $(q_j, \lambda_j)_{j \in \mathbb{N}}$ satisfying (8.128). The inequality in (8.127) follows from (8.128) since the continuous function

$$\omega \mapsto \|(I_t(\omega), L_t(I_t(\omega)))\| = \|(\nabla L_t^*(\omega), L_t(\nabla L_t^*(\omega)))\|$$

does not vanish on the compact set Ω_κ^0 in view of (1.5) and (1.9). \square

We point out that the set Ξ_κ^t and the sequence $\widetilde{\mathcal{M}}(\omega)$ may depend on t .

From now on we fix ω in the set Ξ_κ^t given by Lemma 8.1 and denote by $\mathcal{M} \subset \mathbb{Z}^n$ the image of the projection of $\widetilde{\mathcal{M}}(\omega) \subset \mathbb{Z}^n \times [1, \infty)$ on the first factor. \mathcal{M} will be the index set of the C^1 family of quasi-modes that we are going to construct and (8.128) - the quantization condition for $s = t$. To obtain a quantization condition for the tori $\Lambda_s(\omega)$ for s close to t we consider for any $q \in \mathcal{M}$ the interval

$$J_q := [t, t + 2|q|^{-1}].$$

Getting rid of finitely many elements $q \in \mathcal{M}$ we suppose that J_q is contained in $J \subset [0, \delta]$ for every $q \in \mathcal{M}$. Recall from Theorem 3.2 that the maps $s \mapsto L_s \in C^\infty(D)$ and $s \mapsto I_s \in C^\infty(\Omega; \mathbb{R}^{n-1})$ are C^1 on the interval J . Then using (8.127) and (8.128) we obtain that there exists a constant $C = C(\omega) > 0$ independent of $q \in \mathcal{M}$ and $s \in J_q$ such that

$$\left| \mu_q^0(I_s(\omega), L_s(I_s(\omega))) - \left(k + \frac{\vartheta_0}{4}, 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right) \right| \leq C \quad \forall q \in \mathcal{M}, s \in J_q. \quad (8.131)$$

The quantization condition (8.131) will be used below to construct a C^1 quasi-mode with an index set \mathcal{M} for $s \in J_q, q \in \mathcal{M}$. Condition (8.128) is not needed for the the construction of the quasi-mode, but it is essential for the proof of Lemma 8.4 below.

8.2 Construction of C^1 families of quasi-modes

Fix a positive integer $M \geq 0$. For any $q \in \mathcal{M}$ with $|q| \geq q_0 \gg 1$ we are going to construct a family of quasi-modes of order M depending on $s \in J_q$ such that the corresponding family of quasi-eigenvalues $s \mapsto \mu_q(s)^2$ belongs to $C^1(J_q)$.

Theorem 8.2. *For every $q = (k, k_n) \in \mathcal{M}$ and $s \in J_q$ there exists a quasi-mode $(\mu_q(s)^2, u_{s,q})$ of Δ_s of order M such that*

- (i) $u_{s,q} \in D(\Delta_s)$ and $\|u_{s,q}\|_{L^2(X)} = 1$;
- (ii) There exists a constant $C_M > 0$ such that

$$\begin{cases} \|\Delta u_{s,q} - \mu_q^2(s) u_{s,q}\| \leq C_M \mu_q^{-M}(s) & \text{in } L^2(X), \\ \mathcal{B} u_{s,q}|_\Gamma = 0 \end{cases} \quad (8.132)$$

for every $q \in \mathcal{M}$ and $s \in J_q$;

- (iii) We have

$$\mu_q(s) = \mu_q^0 + c_{q,0}(s) + c_{q,1}(s) \frac{1}{\mu_q^0} + \cdots + c_{q,M}(s) \frac{1}{(\mu_q^0)^M}, \quad \text{where}$$

- (iv) The functions $s \mapsto c_{q,j}(s)$ are real valued and C^1 on the interval J_q ;
- (v) There exists a constant $C'_M > 0$ such that $|c_{q,j}(s)| \leq C'_M$ for every $q \in \mathcal{M}, 0 \leq j \leq M$, and any $s \in J_q$;
- (vi) There exists $C > 0$ such that

$$\left| \mu_q(s) L_s \left(\frac{k + \vartheta_0/4}{\mu_q(s)} \right) - 2\pi \left(k_n + \frac{\vartheta}{4} \right) \right| \leq \frac{C}{\mu_q(s)}$$

for every $q \in \mathcal{M}$ and $s \in J_q$;

- (vii) We have

$$\frac{k + \vartheta_0/4}{\mu_q(t)} = I_t(\omega) + o\left(\frac{1}{|q|}\right) \quad \text{as } |q| \rightarrow \infty.$$

Proof. We are looking for a perturbation $\lambda = \mu_q(s)$ of μ_q^0 satisfying (7.126) which means that

$$\begin{aligned} \mu_q(s)L_s\left(\frac{k + \vartheta_0/4}{\mu_q(s)}\right) - \frac{1}{i}\text{Log}\left(1 + \sum_{j=1}^M p_s\left(\frac{k + \vartheta_0/4}{\mu_q(s)}\right)\mu_q(s)^{-j}\right) \\ = 2\pi\left(k_n + \frac{\vartheta}{4}\right) + O_M\left(\frac{1}{(\mu_q^0)^{M+1}}\right) \end{aligned}$$

uniformly with respect to $q \in \mathcal{M}$ and $s \in J_q$. Introducing a small parameter $\varepsilon_q = (\mu_q^0)^{-1}$ we are looking for

$$\begin{cases} \mu_q(s) &= \mu_q^0 + c_{q,0}(s) + c_{q,1}(s)\varepsilon_q + \cdots c_{q,M}(s)\varepsilon_q^M, \\ \zeta_q(s) &= I_s^0(\omega) + b_{q,0}(s)\varepsilon_q + \cdots b_{q,M}(s)\varepsilon_q^{M+1}(s) + b_{q,M+1}(s)\varepsilon_q^{M+2} \end{cases} \quad (8.133)$$

such that

$$\begin{cases} \mu_q(s)\zeta_q(s) &= k + \frac{\vartheta_0}{4} \\ \mu_q(s)L_s(\zeta_q(s)) &= 2\pi\left(k_n + \frac{\vartheta}{4}\right) + \frac{1}{i}\text{Log}\left(1 + \sum_{j=1}^M p_s^j(\zeta_q(s))\mu_q(s)^{-j}\right) + O_M(\varepsilon_q^{M+1}). \end{cases} \quad (8.134)$$

We are going to find $\mu_q(s)$. Using (8.133) we write

$$\begin{aligned} \mu_q(s)\zeta_q(s) - k - \vartheta_0/4 &= \sum_{j=0}^M \varepsilon_q^j [b_{q,j}(s) + c_{q,j}(s)I_s(\omega) - W_{q,j}(s)] \\ &+ \varepsilon_q^{M+1} [(\varepsilon_q\mu_q)b_{q,M+1}(s) - W_{q,M+1}(s)], \end{aligned}$$

where

$$\begin{cases} W_{q,0}(s) = k + \vartheta_0/4 - \mu_q^0 I_s(\omega), \\ W_{q,j}(s) = - \sum_{r+s=j-1} c_{q,r}(s)b_{q,s}(s) \text{ for } 1 \leq j \leq M, \text{ and} \\ W_{q,M+1}(s) = - \sum_{l=0}^M \sum_{r=M-l}^M c_{q,r}(s)b_{q,l}(s). \end{cases} \quad (8.135)$$

Expanding $L_s(\zeta_q(s))$ and $p_s^j(\zeta_q(s))$, $1 \leq j \leq M$, in Taylor series at $\zeta = I_s(\omega)$ up to order M we obtain from (8.134) the following linear systems

$$\begin{cases} b_{q,j}(s) + c_{q,j}(s)I_s(\omega) &= W_{q,j}(s) \\ L_s(I_s(\omega))c_{q,j}(s) + \langle \omega, b_{q,j}(s) \rangle &= V_{q,j}(s), \end{cases}$$

for $0 \leq j \leq M$, and we put $b_{q,M+1}(s) = (\varepsilon_q\mu_q)^{-1}W_{q,M+1}(s)$, where $W_{q,j}(s)$ is given by (8.135), and $V_{q,j}(s)$ is a polynomial of $c_{q,r}(s)$ and $b_{q,r'}(s)$ with $0 \leq r, r' \leq j-1$ and with C^1 with respect to s coefficients. By (1.5) and (1.9) the corresponding determinant is

$$D(I_s(\omega)) := L_s(I_s(\omega)) - \langle I_s(\omega), \omega \rangle = -\beta_s(\omega) = 2 \int_{\Lambda_s(\omega)} A_s(\rho) d\mu_s > 0$$

and we obtain a unique solution $(c_{q,j}(s), b_{q,j}(s))$, $0 \leq j \leq M-1$. More precisely,

$$\begin{cases} c_{q,j}(s) = D(I_s(\omega))^{-1} [V_{q,j}(s) - 2\pi \langle \omega, W_{q,j}(s) \rangle] \\ b_{q,j}(s) = W_{q,j}(s) - c_{q,j}(s) I_s(\omega). \end{cases} \quad (8.136)$$

We have

$$\begin{cases} W_{q,0}(s) = k + \vartheta_0/4 - \mu_q^0 I_s(\omega) = O(1), \\ V_{q,0}(s) = 2\pi k_n - \pi\vartheta/2 - \mu_q^0 L_s(I_s(\omega)) = O(1), \quad q \in \mathcal{M}, \end{cases} \quad (8.137)$$

uniformly with respect to $q \in \mathcal{M}$ and $s \in J_q$, in view of (8.131). Hence, $b_{q,0}(s)$ and $c_{q,0}(s)$, $q \in \mathcal{M}$, are C^1 in J_q and uniformly bounded. By recurrence we prove that $b_{q,j}(s)$ and $c_{q,j}(s)$, $q \in \mathcal{M}$, are C^1 in J_q and uniformly bounded with respect to $q \in \mathcal{M}$ and $s \in J_q$. To evaluate $b_{q,M+1}(s)$ observe that $\varepsilon_q \mu_q = 1 + O(\varepsilon_q)$.

For such $\mu_q(s)$ the quantization condition (8.131) gives the estimate

$$\frac{k + \vartheta_0/4}{\mu_q(s)} = \zeta_q(s) = I_s(\omega) + O(\mu_q(s)^{-1})$$

uniformly with respect to $s \in J_q$. Then Proposition 7.15 for $N = M$ implies that $(\lambda = \mu_q(s), e_k)$ satisfy (7.118) and we obtain that

$$\begin{cases} \|\Delta u_{s,q} - \mu_q^2(s) u_{s,q}\| \leq C_M \mu_q^{-M}(s) & \text{in } L^2(X), \\ \|\mathcal{B} u_{s,q}\| \leq C_M \mu_q^{-M}(s) & \text{in } L^2(\Gamma). \end{cases}$$

In order to prove the property (i) and to satisfy the boundary conditions in (ii) exactly we follow the proof given in [63], Sect. 3.6.3, using Proposition 7.7. The property (vi) follows from the second equation of (8.134). To prove (vii) observe that

$$\begin{cases} W_{q,0}(t) = k + \vartheta_0/4 - \mu_q^0 I_t(\omega) = o(1) \\ V_{q,0}(t) = 2\pi k_n - \pi\vartheta/2 - \mu_q^0 L_t(I_t(\omega)) = o(1) \end{cases}$$

as $|q| \rightarrow \infty$ in view of (8.128). Then (8.136) implies that $c_{q,0}(t) = o(1)$ and $b_{q,0}(t) = o(1)$ and we obtain

$$\frac{k + \vartheta_0/4}{\mu_q(t)} = \zeta_t(\omega) = I_t(\omega) + b_{q,0}(t)\varepsilon_q + O(\varepsilon_q^2) = I_t(\omega) + o\left(\frac{1}{|q|}\right) \quad \text{as } |q| \rightarrow \infty.$$

This completes the proof of the Theorem. \square

8.3 From quasi-modes to isospectral invariants

We are going to complete the proof of Theorem 1. The items (i) and (ii) have been proven in Sect. 3. We are going to prove item (iii) which states that the functions $\beta_t(\omega)$, $I_t(\omega)$ and $\alpha_t(I_t(\omega)) = L_t(I_t(\omega))$ are independent of $t \in [0, \delta]$ for any $\omega \in \Xi$ provided that the billiard tables

satisfy the weak isospectral condition (H₁) – (H₂). Recall that the set Ξ is of the form (3.26), hence, it suffices to prove the statement for each ω in $\Omega_\kappa^0 \subset \Xi$.

Given $\alpha \in \mathbb{R}$ we say that a family of functions $f_q : J_q \rightarrow \mathbb{C}$, $q \in \mathcal{M}$, is $o(|q|^\alpha)$ as $q \rightarrow \infty$ uniformly with respect to s in J_q if

$$\lim_{q \rightarrow \infty} \left(|q|^{-\alpha} \sup_{s \in J_q} |f_q(s)| \right) = 0.$$

We say that “ $f_q = O(|q|^\alpha)$ uniformly with respect to s in J_q ” if there is $C > 0$ such that $|q|^{-\alpha} |f_q(s)| \leq C$ for any $q \in \mathcal{M}$ and $s \in J_q$. The isospectral condition implies

Lemma 8.3. *Suppose that (H₁) – (H₂) holds. Fix an integer $M > 2d \geq 0$. Then*

$$\mu_q(s) - \mu_q(t) = o(1) \quad \text{as } q \rightarrow \infty$$

and

$$\mu_q(s) = \mu_q(t) \left(1 + o\left(\frac{1}{|q|}\right) \right) \quad \text{as } q \rightarrow \infty$$

uniformly with respect to $s \in J_q$.

Proof. It is easy to see that for any $q \in \mathcal{M}$ and $s \in J_q$, the distance from $\mu_q(s)^2$ to the spectrum of Δ_s can be estimated above by

$$d_{s,q} := |\text{Spec}(\Delta_s) - \mu_q(s)^2| \leq C_M \mu_q(s)^{-M}.$$

Indeed, if $d_{s,q} \neq 0$ the spectral theorem and (8.132) yield

$$\frac{1}{d_{s,q}} \geq \|(\Delta_s - \mu_q(s)^2)^{-1}\| \geq \|(\Delta_s - \mu_q(s)^2)u_{s,q}\|^{-1} \geq \frac{\mu_q(s)^M}{C_M}.$$

Then Theorem 8.2 and (8.127) imply that for any $q \in \mathcal{M}$, $|q| \geq q_0 \gg 1$, and $s \in J_q$ there is $\lambda_{s,q} \in \text{Spec}(\Delta_s)$ such that $\lambda_{s,q} \geq \mu_q(s)^2/4 \geq (2c_0)^{-2}|q|^2$ and

$$|\lambda_{s,q} - \mu_q(s)^2| \leq C' \lambda_{s,q}^{-M/2} \tag{8.138}$$

where $C' = 2^M C_M$. Now using (H₂) we get for any $q \in \mathcal{M}$ with $|q| \geq q_0 \gg 1$ and $s \in J_q$ an integer $k = k(s, q) \geq 1$ such that

$$\lambda_{s,q} \in [a_k, b_k]. \tag{8.139}$$

Fix γ so that $M > 2\gamma > 2d \geq 0$. Then choosing q_0 sufficiently large we obtain from (8.138) and (8.139) that for any $q \in \mathcal{M}$ with $|q| \geq q_0$ and $s \in J_q$ the quasi-eigenvalue $\mu_q(s)^2$ belongs to the interval

$$I_k := \left[a_k - \frac{c}{2} a_k^{-\gamma}, b_k + \frac{c}{2} a_k^{-\gamma} \right], \tag{8.140}$$

where $k = k(s, q)$ and $c > 0$ is the constant of the third assumption of (H₁). In particular,

$$b_{k(q,s)} \geq \mu_q(t)^2 - \frac{c}{2} a_{k(s,q)}^{-\gamma} \geq C_1 |q|^2 - C_2,$$

for some positive constants C_1 and C_2 , which implies that $\lim k(s, q) = \infty$ as $q \rightarrow \infty$ uniformly with respect to $s \in J_q$. On the other hand, using the third assumption of (H₁), the relation

$b_k = a_k(1 + o(1))$ as $k \rightarrow \infty$, which follows from the first two assumptions in (H₁), and the inequality $\gamma > d$, we get

$$(a_{k+1} - \frac{c}{2}a_{k+1}^{-\gamma}) - (b_k + \frac{c}{2}a_k^{-\gamma}) = (a_{k+1} - b_k) - \frac{c}{2}a_{k+1}^{-\gamma} - \frac{c}{2}a_k^{-\gamma} \geq cb_k^{-d} - ca_k^{-\gamma} > 0$$

for any $k \geq k_0$, where $k_0 \gg 1$. This shows that the intervals I_k in (8.140) do not intersect each other for $k \geq k_0$. Choose $q_0 \gg 1$ so that $k(s, q) \geq k_0$ for any $q \in \mathcal{M}$ with $|q| \geq q_0$ and $s \in J_q$ (recall that $k(s, q) \rightarrow \infty$ as $|q| \rightarrow \infty$ uniformly with respect to $s \in J_q$). The function $\mu_q(s)^2$ is continuous on J_q (even C^1), hence, it can not jump from one interval to another when $|q| \geq q_0$. Consequently, $k(s, q)$ does not depend on s for $|q| \geq q_0$. We have proved that for any $q \in \mathcal{M}$ such that $|q| \geq q_0$ there is $k = k(q) \in \mathbb{N}$ independent of s such that

$$\mu_q(s)^2 \in \left[a_k - \frac{c}{2}a_k^{-\gamma}, b_k + \frac{c}{2}a_k^{-\gamma} \right] \quad \forall s \in J_q. \quad (8.141)$$

Moreover, $k(q) \rightarrow \infty$ as $q \rightarrow \infty$ and we obtain

$$\mu_q(s)^2 \geq a_{k(q)} - \frac{c}{2}a_{k(q)}^{-\gamma} \geq \frac{1}{4}a_{k(q)}$$

for $|q| \geq q_0 \gg 1$ and $s \in J_q$. Thus for $|q| \geq q_0 \gg 1$ we obtain

$$|\mu_q(s) - \mu_q(t)| < \mu_q(t)^{-1} |\mu_q(s)^2 - \mu_q(t)^2| \leq 2a_{k(q)}^{-1/2} \left((b_{k(q)} - a_{k(q)}) + ca_{k(q)}^{-\gamma} \right) := \epsilon_q$$

where $C > 0$ is independent of q and of $s \in J_q$. Now (H₁) implies that $\epsilon_q \rightarrow 0$ as $q \rightarrow \infty$. Hence, $\mu_q(s) - \mu_q(t) = o(1)$ as $q \rightarrow \infty$ uniformly with respect to $s \in J_q$. Moreover,

$$\mu_q(s) = \mu_q(t) \left(1 + \frac{o(1)}{\mu_q(t)} \right) = \mu_q(t) \left(1 + o\left(\frac{1}{|q|} \right) \right) \quad \text{as } q \rightarrow \infty$$

uniformly with respect to $s \in J_q$ since $\mu_q(t) \geq \mu_q^0/2 \geq (2c_0)^{-1}|q|$ for $|q| \geq q_0 \gg 1$. \square

Consider the function $t \rightarrow \beta_t(\omega) = \langle \omega, I_t(\omega) \rangle - L_t(I_t(\omega))$.

Lemma 8.4. *Suppose that (H₁) – (H₂) holds. Then $\beta_t(\omega) = \beta_0(\omega)$ for any $t \in [0, \delta]$ and $\omega \in \Omega_\kappa^0$.*

Proof. We are interested in the variation

$$\dot{\beta}_t(\omega) := \frac{d}{dt} \beta_t(\omega).$$

Fix $t \in [0, \delta)$ and choose ω in the set Ξ_κ^t given by Lemma 8.1. Consider the quasi-mode of order $N > 2d + 2 \geq 2$ constructed by Theorem 8.2. Now Lemma 8.3 and Theorem 8.2, (vii), imply together that

$$\zeta_q(s) = \frac{k + \vartheta_0/4}{\mu_q(s)} = \frac{k + \vartheta_0/4}{\mu_q(t)(1 + o(1/|q|))} = \frac{k + \vartheta_0/4}{\mu_q(t)} + o\left(\frac{1}{|q|} \right) = I_t(\omega) + o\left(\frac{1}{|q|} \right) \quad (8.142)$$

as $|q| \rightarrow \infty$ and uniformly with respect to $s \in J_q$. On the other hand Theorem 8.2, (vi), yields

$$L_s(\zeta_q(s)) = 2\pi \frac{k_n - \vartheta/4}{\mu_q(s)} + O(|q|^{-2})$$

uniformly with respect to $s \in J_q$ and using Lemma 8.3 we obtain as in (8.142) that

$$L_s(\zeta_q(s)) = L_t(\zeta_q(t)) + o\left(\frac{1}{|q|}\right) \quad \text{as } q \rightarrow \infty \quad (8.143)$$

uniformly with respect to $s \in J_q$. Then setting $\eta := 1/|q| \rightarrow 0$ we obtain by (8.142) and (8.143) the equality

$$\begin{aligned} L_{t+\eta}(I_t(\omega)) &= L_{t+\eta}(\zeta_q(t+\eta) + o(\eta)) = L_{t+\eta}(\zeta_q(t+\eta)) + o(\eta) \\ &= L_t(\zeta_q(t)) + o(\eta) = L_t(I_t(\omega)) + o(\eta). \end{aligned}$$

We have used also that the map $[0, \delta] \rightarrow L_s \in C^\infty(D)$ is C^1 . Hence,

$$\dot{L}_t(I_t(\omega)) = \frac{d}{ds} L_s(I_t(\omega)) \Big|_{s=t} = 0 \quad \forall \omega \in \Xi_\kappa^t. \quad (8.144)$$

On the other hand, Ξ_κ^t is dense in Ω_κ^0 since any point of Ω_κ^0 is of positive Lebesgue density and $\Omega_\kappa^0 \setminus \Xi_\kappa^t$ has measure zero, and by continuity (the function $I \rightarrow \dot{L}_t(I)$ is smooth) we get (8.144) for any $\omega \in \Omega_\kappa^0$. The point t has been fixed arbitrary in $[0, \delta)$, hence, (8.144) holds true for every $t \in [0, \delta)$ and $\omega \in \Omega_\kappa^0$. Now differentiating $\beta_t(\omega)$ with respect to t we obtain

$$\dot{\beta}_t(\omega) = \langle \omega, \dot{I}_t(\omega) \rangle - \dot{L}_t(I_t(\omega)) - \langle \nabla L_t(I_t(\omega)), \dot{I}_t(\omega) \rangle = 0 \quad \forall \omega \in \Omega_\kappa^0$$

since $\nabla L_t(I_t(\omega)) = \omega$. Hence, $\beta_t(\omega) = \beta_0(\omega)$ for every $t \in [0, \delta)$ and $\omega \in \Omega_\kappa^0$. By continuity we get the last equality for every $t \in [0, \delta]$ as well. \square

Recall that Ω_κ^0 is a set of points of positive Lebesgue density. Differentiating the equality

$$\langle \omega, I_t(\omega) \rangle - L_t(I_t(\omega)) = \beta_t(\omega) = \beta_0(\omega) = \langle \omega, I_0(\omega) \rangle - L_0(I_0(\omega))$$

with respect to $\omega \in \Omega_\kappa^0$ and using Lemma 3.4 we get $I_t(\omega) = I_0(\omega)$. Then plugging it in the expression of $\beta_t(\omega)$ we obtain $E_{\kappa,t} = E_{\kappa,0}$ as well as the equality $L_t(I) = L_0(I)$, $I \in E_{\kappa,0}$. This completes the proof of Theorem 1. \square

Part II

KAM theorems and Birkhoff Normal Forms

9 KAM theorems

In this Section we prove KAM theorems and obtain BNF for C^k smooth families of Hamiltonians $t \rightarrow H_t$ or exact symplectic maps $t \rightarrow P_t$. The main novelty in it can be briefly summarized as follows

- The constant ϵ in the smallness condition depends only on the dimension of the configuration space and on the exponent in the Diophantine condition;

- C^k smooth families of invariant tori $t \rightarrow \Lambda_t(\omega)$ with Diophantine frequencies are obtained;
- C^k smooth with respect to the parameter t BNF is obtained around the union of $\Lambda_t(\omega)$;
- Uniform estimates in the whole scale of Hölder spaces are obtained. To this end a new approach to the iterative schema is proposed. The Modified Iterative Lemma proven in Sect. 11.9 provides in a limit smooth functions in the whole domain Ω (not only on the Cantor set Ω_κ) with a good control of the Hölder norms. In particular, it avoids the Whitney C^∞ extension theorem.

In order to formulate the main results we recall the notion of the Legendre transform. Let $D \subset \mathbb{R}^d$, $d \geq 1$, be an open set. We say that a real valued function $F \in C^\infty(D, \mathbb{R})$ is *non-degenerate* if

$$\nabla F : D \longrightarrow D^* := \nabla F(D) \subset \mathbb{R}^d \text{ is a diffeomorphism.} \quad (9.145)$$

The Legendre transform F^* of F is defined by

$$F^*(\xi) := \text{Crit.val.}_{x \in D} \{ \langle x, \xi \rangle - F(x) \}$$

which is equivalent to

$$F(x) + F^*(\xi) = \langle x, \xi \rangle, \text{ where } x \in D \text{ and } \xi = \nabla F(x) \in D^*. \quad (9.146)$$

It is easy to see that $F^* \in C^\infty(D^*, \mathbb{R})$ and that $\nabla F^* : D^* \longrightarrow D$ is the inverse to the map (9.145). Moreover, $D^{**} = D$ and $F^{**} = F$.

The real valued function F defines a non-degenerate (in Kolmogorov sense) completely integrable Hamiltonian in $\mathbb{T}^d \times D$. Hereafter, $\mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d$. The corresponding Hamiltonian flow is given by $(s, \theta, r) \rightarrow (\theta + s\nabla F(r), r)$. The frequency vector of the restriction of the flow to the invariant torus $\mathbb{T}^d \times \{r\}$ is $\omega = \nabla F(r) \in \Omega := D^*$ and the corresponding rotation vector is $\omega/2\pi$. We work here with frequency vectors instead of rotation vectors because they are more adapted to the Fourier analysis. One can parameterize the invariant tori by their frequency vectors $\omega \in \Omega$ since F is non-degenerate. We are interested below in families of non-degenerate completely integrable Hamiltonians F_t , $t \in [0, \delta]$, with frequency vectors in a fixed open set $\Omega \subset \mathbb{R}^d$. To this end we consider a family of non-degenerate functions $F_t^* \in C^\infty(\Omega)$ and define F_t as the Legendre transform of F_t^* in $D_t := \nabla F_t^*(\Omega)$. The advantage is that the set of frequency vectors is independent of the parameter t . In particular, the Diophantine conditions will be the same for all t . The same discussion holds as well for families of completely integrable exact symplectic maps.

This part is organized as follows. The basic KAM theorem is proved in Sect. 10.

9.1 KAM theorems for C^k families of Hamiltonians

Let $\Omega \subset \mathbb{R}^n$ be an *open convex bounded set*, $k \in \{0; 1\}$, and $\delta > 0$. Denote by $\bar{\Omega}$ the closure of Ω in \mathbb{R}^n . Consider a C^k family H^{0*} of real valued functions

$$[0, \delta] \ni t \rightarrow H^{0*}(\cdot, t) = H_t^{0*}(\cdot) \in C^\infty(\bar{\Omega}, \mathbb{R})$$

satisfying the non-degeneracy condition

$$\nabla H_t^{0*} : \Omega \longrightarrow D_t := \nabla H_t^{0*}(\Omega) \text{ is a diffeomorphism.} \quad (9.147)$$

The corresponding family H^0 of Legendre transforms

$$[0, \delta] \ni t \rightarrow H_t^0 = H_t^{0**} \in C^\infty(D_t, \mathbb{R})$$

is C^k as well and H_t^0 satisfies (9.145) on D_t for each t . Consider a C^k family H of perturbations

$$[0, \delta] \ni t \rightarrow H(\cdot, t) = H_t(\cdot) \in C^\infty(\mathbb{A}_t, \mathbb{R})$$

of H^0 where $\mathbb{A}_t := \mathbb{T}^n \times D_t$.

Let us introduce the arithmetic conditions on the frequency vectors. Fix $\kappa > 0$ and $\tau > n-1$, and denote by $\tilde{D}(\kappa, \tau)$ the set of all $\omega \in \mathbb{R}^n$ satisfying the (κ, τ) -Diophantine condition

$$\forall 0 \neq k \in \mathbb{Z}^n : |\langle \omega, k \rangle| \geq \frac{\kappa}{|k|^\tau}. \quad (9.148)$$

Denote by Ω_κ the set of all (κ, τ) -Diophantine vectors $\omega \in \Omega$ such that the distance from ω to the complement $\mathbb{R}^n \setminus \Omega$ of Ω is $\geq \kappa$. We will often use the following notation

$$\begin{cases} \Omega + \kappa & := \{ \omega \in \mathbb{R}^n : \text{dist}(\omega, \Omega) < \kappa \} \\ \Omega - \kappa & := \{ \omega \in \Omega : \text{dist}(\omega, \mathbb{R}^n \setminus \Omega) > \kappa \}. \end{cases} \quad (9.149)$$

Then $\Omega_\kappa = \tilde{D}(\kappa, \tau) \cap \overline{\Omega - \kappa}$.

In order to formulate the smallness condition we need the following notations. Firstly we define weighted C^ℓ Hölder norms as in [53]. Given $\ell \geq 0$, $0 < \kappa \leq 1$, and a domain $D \subset \mathbb{R}^n$, we denote the weighted (with respect to the small parameter κ) C^ℓ -norm of $u \in C^\ell(\mathbb{T}^n \times D, \mathbb{R}^k)$ evaluated at $\mathbb{T}^n \times D$ by

$$\|u\|_{\ell, \mathbb{T}^n \times D; \kappa} := \|u \circ \sigma_\kappa\|_{C^\ell(\sigma_\kappa^{-1}(\mathbb{T}^n \times D))} \quad (9.150)$$

where $\|\cdot\|_{C^\ell}$ is the corresponding Hölder norm (see Sect. A.1) and $\sigma_\kappa : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n$ is the partial dilation $\sigma_\kappa(\theta, r) := (\theta, \kappa r)$. If $\ell \in \mathbb{N}$ then

$$\|u\|_{\ell, \mathbb{T}^n \times D; \kappa} = \sup_{|\alpha|+|\beta| \leq \ell} \sup_{(\theta, r) \in \mathbb{T}^n \times D} |\partial_\theta^\alpha (\kappa \partial_r)^\beta u(\theta, r)|,$$

where $|\cdot|$ is the Euclidean norm. In the same way we introduce the norm $\|u\|_{\ell, D; \kappa}$ for $u \in C^\ell(D, \mathbb{R}^k)$. We set as well

$$\|u\|_{\ell, D; \kappa} = \sup_{0 \leq m \leq \ell} \|u\|_{\ell-m, D; \kappa}, \quad \text{where } m \in \mathbb{N}. \quad (9.151)$$

If D is *convex*, then $\|u\|_{\ell, D; \kappa} = \|u\|_{\ell, D; \kappa}$. Given $\ell \geq 1$ and a family of functions

$$u := \{u_t \in C^\infty(\Omega, \mathbb{R}) : t \in [0, \delta]\}$$

we set

$$S_\ell(u) := \sup_{0 \leq t \leq \delta} (1 + \|u_t\|_{C^1(\Omega)})^{\ell-1} (1 + \|u_t\|_{C^\ell(\Omega)}). \quad (9.152)$$

This expression arises when one evaluates the C^ℓ -norms of a composition of functions the form $f_t \circ u_t$ with $u_t = \nabla K_t^{0*}$ (see Appendix, Sect. A.4.2). If Ω is *convex* then $\|u_t\|_{C^\ell(\Omega)} \leq \|u_t\|_{C^\mu(\Omega)}$ for $0 \leq \ell \leq \mu$ and the function $\ell \rightarrow S_\ell(u)$ becomes increasing in $[1, +\infty)$.

Fix $\vartheta_0 > 1$ and set

$$\ell_0 := 2\tau + 2 + 2\vartheta_0 \quad \text{and} \quad \ell(m) := 2m(\tau + 1) + \ell_0, \quad m \geq 0. \quad (9.153)$$

Given $0 < \varrho, \kappa \leq 1$ and $m \geq 0$, we denote by \mathcal{A}_m^0 the expression

$$\mathcal{A}_m^0 := \sup_{0 \leq t \leq \delta} (\varrho^2 \|\partial^2 H_t^0\|_{\ell(m), D_t; \kappa} + \|H_t - H_t^0\|_{\ell(m), \mathbb{A}_t; \kappa}) \quad (9.154)$$

and set

$$\mathcal{A}_m = S_{\ell(m)+1}(\nabla H^{0*}) \mathcal{A}_m^0. \quad (9.155)$$

Here $\partial^2 H_t^0(I)$ is the Hessian matrix of H_t^0 at $I \in D_t$. The role of the small parameter ϱ is to compensate the norm of the Hessian matrix which could be very large. The function $m \rightarrow \mathcal{A}_m$, $m \geq 0$, is *increasing* when Ω is convex.

Let $\Phi_t^s := \exp(sX_{H_t})$, $s \in \mathbb{R}$, be the flow of the Hamiltonian vector field X_{H_t} with Hamiltonian H_t in $\mathbb{A}_t = \mathbb{T}^n \times D_t$. Recall that for any $\omega \in \Omega$ the map $R_\omega : \mathbb{T}^n \rightarrow \mathbb{T}^n$ stands for the translation $R_\omega(\varphi) = \varphi + \omega \pmod{2\pi}$. Fix $k \in \{0, 1\}$.

Theorem 9.1. *There exists $\epsilon = \epsilon(n, \tau, \vartheta_0) > 0$ depending only on n , τ and ϑ_0 such that the following holds.*

Let $\Omega \subset \mathbb{R}^n$ be an open convex bounded set, $0 < \varrho \leq \kappa \leq 1$ and $\Omega_\kappa \neq \emptyset$. Let H^{0} be a C^k family of real-valued functions $[0, \delta] \ni t \rightarrow H_t^{0*} \in C^\infty(\overline{\Omega}, \mathbb{R})$ satisfying (9.147), and let $[0, \delta] \ni t \rightarrow H_t \in C^\infty(\mathbb{A}_t, \mathbb{R})$ be a C^k family of Hamiltonians such that*

$$S_{\ell_0}(\nabla H^{0*}) \mathcal{A}_0^0 = \sup_{0 \leq t \leq \delta} (\varrho^2 \|\partial^2 H_t^0\|_{\ell_0, D_t; \kappa} + \|H_t - H_t^0\|_{\ell_0, \mathbb{A}_t; \kappa}) S_{\ell_0}(\nabla H^{0*}) \leq \epsilon \varrho \kappa. \quad (9.156)$$

Then there exists a C^k mapping $[0, \delta] \ni t \rightarrow \Psi_t = (\tilde{U}_t, \tilde{V}_t) \in C^\infty(\mathbb{T}^n \times \Omega; \mathbb{T}^n \times D_t)$ such that

- (i) *for any $\omega \in \Omega_\kappa$, $[0, \delta] \ni s \rightarrow \Lambda_t(\omega) = \Psi_{t, \omega}(\mathbb{T}^n)$ is a C^k family of Kronecker invariant tori of the Hamiltonian vector fields X_{H_t} with a frequency vector ω , where $\Psi_{t, \omega} := \Psi_t(\cdot; \omega)$. Moreover, for any $\omega \in \Omega_\kappa$ and $s \in \mathbb{R}$ the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{R_{s\omega}} & \mathbb{T}^n \\ \downarrow \Psi_{t, \omega} & & \downarrow \Psi_{t, \omega} \\ \Lambda_t(\omega) & \xrightarrow{\Phi_t^s} & \Lambda_t(\omega) \end{array}$$

- (ii) *for any $m \in \{0\} \cup [1, +\infty)$ the following estimates hold*

$$\begin{aligned} |\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta (\tilde{U}_t(\varphi; \omega) - \varphi)| &\leq C_m \frac{\mathcal{A}_m}{\kappa \varrho} \\ |\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta (\tilde{V}_t(\varphi; \omega) - \nabla H_t^{0*}(\omega))| &\leq C_m \frac{\mathcal{A}_m}{\varrho} \left(1 + \frac{\mathcal{A}_1}{\varrho}\right)^m \end{aligned} \quad (9.157)$$

for each $t \in [0, \delta]$, $(\varphi, \omega) \in \mathbb{T}^n \times \Omega$ and $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + 1$, where the constant $C_m > 0$ depends only on n , τ , ϑ_0 and m .

- (iii) *$\text{supp}((\tilde{U}_t, \tilde{V}_t) - (\text{id}, \nabla H_t^{0*})) \subset \mathbb{T}^n \times (\Omega - \kappa/2)$.*

Remark 9.2. If P is analytic with respect to t in the disc $B(0, a) := \{t \in \mathbb{C} : |t| < a\}$ and (9.156) holds for $t \in B(0, a)$, then Ψ and ϕ can be chosen to be analytic with respect to t in $B(0, a)$. Moreover, for any $\alpha, \beta \in \mathbb{N}^n$ of length $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + 1$ and $0 < a_1 < a$, the estimate (9.157) holds true for $t \in B(0, a_1)$, where the supremum with respect to t in the definition of \mathcal{A}_m is taken in $B(0, a)$.

Proof. The idea of the proof is given by Pöschel in [54]. It can be summarized as follows. Let us fix $\omega \in \Omega$ set $r = \nabla H_t^{0*}(\omega) + I$ and apply Taylor's formula up to order two to the function $I \rightarrow H_t^0(\nabla H_t^{0*}(\omega) + I)$ at $I = 0$. Then the affine linear with respect to I term is just $N_t(I; \omega) = e_t(\omega) + \langle I, \omega \rangle$ and we put the quadratic term in the perturbation. Multiplying the perturbation by suitable cut-off functions we obtain a Hamiltonian $\tilde{H}_t(\theta, I; \omega) = N(I; \omega) + P_t(\theta, I; \omega)$, where $t \rightarrow P_t \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n \times \Omega)$ is C^k and P_t are compactly supported with respect to $(I; \omega)$. The smallness condition allows one to apply Theorem 10.1. The main difficulty in the proof is to obtain the corresponding estimates in C^ℓ , $\ell \geq 0$. We divide the proof in several steps.

Step 1. Construction of the Hamiltonian $\tilde{H}_t(\theta, I; \omega)$. Given $\omega \in \Omega - \kappa/4$ and I in the ball $B^n(0, \varrho) \subset \mathbb{R}^n$ of center 0 and radius ϱ , we set $r = \nabla H_t^{0*}(\omega) + I$. Choosing $\epsilon = \epsilon(n, \tau, \vartheta_0) \leq 1/9$ in (9.156) we will show that

$$\omega \in \Omega - \kappa/4, I \in B^n(0, \varrho) \implies \begin{cases} \nabla H_t^{0*}(\omega) + I \in D_t & \text{and} \\ \nabla H_t^0(\nabla H_t^{0*}(\omega) + I) \in \Omega - \frac{1}{8}\kappa \end{cases} \quad (9.158)$$

for each $t \in [0, \delta]$. Fix $t \in [0, \delta]$. The smallness condition (9.156) implies that $\|\partial^2 H_t^0\|_{C^0(D_t)} \leq \epsilon\kappa/\varrho$ since $S_{\ell_0}(\nabla H^{0*}) > 1$. Then for each $\omega \in \Omega - \kappa/4$ there is a positive number $c \leq \varrho$ such that for any $I \in B^n(0, c)$ the following relation holds

$$\begin{cases} \nabla H_t^{0*}(\omega) + I \in D_t & \text{and} \\ |\nabla H_t^0(\nabla H_t^{0*}(\omega) + I) - \omega| \leq \|\partial^2 H_t^0\|_{C^0(D_t)} |I| \leq \epsilon\kappa \frac{c}{\varrho} < \frac{\kappa}{9}. \end{cases} \quad (9.159)$$

Using the notations (9.149) one obtains

$$\nabla H_t^0(\nabla H_t^{0*}(\omega) + I) \in (\Omega - \kappa/4) + \kappa/9 = (\Omega - \kappa/8) - \kappa/72$$

for any $I \in B^n(0, c)$. If $c < \varrho$, then there exists $c < c' \leq \varrho$ such that (9.159) still holds for each $I \in B^n(0, c')$. This proves (9.158). Then Taylor's formula yields

$$H_t^0(r) = e_t(\omega) + \langle I, \omega \rangle + \int_0^1 (1-s) \langle \partial^2 H_t^0(\nabla H_t^{0*}(\omega) + sI) I, I \rangle ds.$$

for $\omega \in \Omega - \kappa/4$ and $I \in B^n(0, \varrho)$, where $e_t(\omega) := H_t^0(\nabla H_t^{0*}(\omega))$.

In order to apply Theorem 10.1 we need suitable cut-off functions.

Lemma 9.3. For any open set $U \subset \mathbb{R}^n$ and $0 < \varepsilon \leq 1$ there exists a smooth cut-off function $\psi_\varepsilon^U \in C_0^\infty(\mathbb{R}^n, [0, 1])$ such that $\psi_\varepsilon^U = 1$ on $U - \varepsilon$, $\psi_\varepsilon^U = 0$ on the complement of $U + \varepsilon$, and

$$\|\psi_\varepsilon^U\|_{\ell; \varepsilon} \leq C_\ell$$

for any $\ell \geq 0$, where the positive constants $C_\ell = C(\ell, n)$ depend only on ℓ and n .

The proof of the Lemma is given in the Appendix, Sect. A.4.1.

Denote by $\psi_\kappa \in C_0^\infty(\mathbb{R}^n, [0, 1])$ the function given by Lemma 9.3 with $U = \Omega - \kappa/2$ and $\varepsilon = \kappa/4$. Then $\psi_\kappa = 1$ on $\Omega - 3\kappa/4$, $\psi_\kappa = 0$ on the complement of $\Omega - \kappa/4$, and

$$\|\psi_\kappa\|_{\ell; \kappa} \leq C_\ell \quad (9.160)$$

for any $\ell \geq 0$, where the constants $C_\ell = C(\ell, n)$ depend only on ℓ and n . Let $\tilde{\psi}_\varrho$ be the cut-off function given by Lemma 9.3 with $U = B^n(0, 3\varrho/4)$ and $\varepsilon = \varrho/4$. Then the support of $\tilde{\psi}_\varrho$ is contained in $B^n(0, \varrho)$, $\tilde{\psi}_\varrho = 1$ on $B^n(0, \varrho/2)$, and

$$\|\tilde{\psi}_\varrho\|_{\ell; \varrho} \leq C_\ell \quad (9.161)$$

for any $\ell \geq 0$, where the constants $C_\ell = C(\ell, n)$ depend only on ℓ and n .

It follows from (9.158) that the function

$$P_t^0(I; \omega) := \int_0^1 (1-s) \langle \partial^2 H_t^0(\nabla H_t^{0*}(\omega) + sI)I, I \rangle \psi_\kappa(\omega) \tilde{\psi}_\varrho(I) ds,$$

is well defined and compactly supported in $B^n(0, \varrho) \times \Omega$. Setting

$$P_t^1(\theta, I; \omega) := (H_t - H_t^0)(\theta, \nabla H_t^{0*}(\omega) + I) \psi_\kappa(\omega) \tilde{\psi}_\varrho(I) \quad \text{and} \quad P_t := P_t^0 + P_t^1,$$

we consider the Hamiltonian

$$\tilde{H}_t(\theta, I; \omega) = e_t(\omega) + \langle I, \omega \rangle + P_t(\theta, I; \omega).$$

We have

$$H_t(\theta, \nabla H_t^{0*}(\omega) + I) = \tilde{H}_t(\theta, I; \omega) \quad (9.162)$$

for $I \in B^n(0, \varrho/2)$ and $\omega \in \Omega - 3\kappa/4$.

Step 2. Hölder estimates of P_t . For any $\ell \geq 1$ we are going to evaluate the weighted norm

$$\|P_t\|_{\ell; r, \kappa} := \|P_t \circ \sigma_{r, \kappa}\|_{C^\ell(U)}, \quad U := \sigma_{r, \kappa}^{-1}(\mathbb{T}^n \times B^n(0, \varrho) \times \Omega),$$

introduced in (10.204). In order to estimate $\|P_t^0\|_{\ell; r, \kappa}$ we set $\Gamma := B^n(0, \varrho) \times (\Omega - 8\kappa/9)$, $\Gamma_t := B^n(0, \varrho) \times D_t$,

$$Q_t^1(I, r) = \int_0^1 (1-s) \partial^2 H_t^0(r + sI) ds$$

and $Q_t^0(I, r) = \langle Q_t^1(I, r)I, I \rangle$ for $(I, r) \in \Gamma_t$. Then we write

$$P_t^0(I, \omega) = \psi_\kappa(\omega) \tilde{\psi}_\varrho(I) Q_t^0(I, \nabla H_t^{0*}(\omega)).$$

The function $Q_t^0 \circ (\text{id}, \nabla H_t^{0*})$ belongs to $C^\infty(\bar{\Gamma})$, where Γ is convex. Then using Remark A.2, (9.160) and (9.161) we obtain

$$\|P_t^0\|_{\ell; \varrho, \kappa} \leq C_\ell \|Q_t^0 \circ (\text{id}, \nabla H_t^{0*})\|_{\ell; \varrho, \kappa}.$$

Proposition A.12, 3, and Remark A.13 imply for any $\ell \geq 1$ the estimate

$$\|Q_t^0 \circ (\text{id}, \nabla H_t^{0*})\|_{\ell; \varrho, \kappa} \leq C_\ell \|Q_t^0\|_{\ell, \Gamma_t; \varrho, \kappa} S_\ell(\nabla H^{0*}),$$

where the norm $\|\cdot\|$ is defined in (9.151). Using (A.22) and the inequality $0 < \varrho \leq \kappa$ we obtain

$$\|Q_t^0\|_{\ell, \Gamma_t; \varrho, \kappa} \leq C_\ell \varrho^2 \|Q_t^1\|_{\ell, \Gamma_t; \varrho, \kappa} \leq C_\ell \varrho^2 \|Q_t^1\|_{\ell, \Gamma_t; \kappa, \kappa} \leq C_\ell \varrho^2 \|\partial^2 H_t^0\|_{\ell, D_t; \kappa}$$

as $|I| \leq \varrho$ for $(I, r) \in \Gamma_t$. This inequality implies

$$\|Q_t^0\|_{\ell, \Gamma_t; \varrho, \kappa} \leq C_\ell \varrho^2 \|\partial^2 H_t^0\|_{\ell, D_t; \kappa}$$

and we obtain

$$\|P_t^0\|_{\ell; \varrho, \kappa} \leq C_\ell \varrho^2 \|\partial^2 H^0\|_{\ell, D_t; \kappa} S_\ell(\nabla H^{0*}).$$

On the other hand $\|P_t^1\|_{\ell; \varrho, \kappa} \leq \|P_t^1\|_{\ell; \kappa, \kappa}$ since $0 < \varrho \leq \kappa \leq 1$ and by the same argument we get

$$\|P_t^1\|_{\ell; \varrho, \kappa} \leq C_\ell \|H_t - H_t^0\|_{\ell, \mathbb{A}_t; \kappa} S_\ell(\nabla H^{0*}).$$

Finally we obtain

$$\|P_t\|_{\ell; \varrho, \kappa} \leq C_\ell (\varrho^2 \|\partial^2 H_t^0\|_{\ell, D_t; \kappa} + \|H_t - H_t^0\|_{\ell, \mathbb{A}_t; \kappa}) S_\ell(\nabla H^{0*}). \quad (9.163)$$

Step 3. Application of Theorem 10.1. The estimate (9.156) gives

$$\|P_t\|_{\ell_0; \varrho, \kappa} \leq C \mathcal{A}_0 S_{\ell_0}(\nabla H_t^{0*}) \leq C \epsilon \varrho \kappa. \quad (9.164)$$

This allows us to apply Theorem 10.1 to the Hamiltonian $(\theta, I) \rightarrow \tilde{H}_t(\theta, I; \omega)$. Set $\Psi_t = (\tilde{U}_t, \tilde{V}_t)$, where $\tilde{U}_t = U_t$ and $\tilde{V}_t = (\nabla H_t^{0*}) \circ \phi_t + V_t$, where (U_t, V_t, ϕ_t) are given by Theorem 10.1. Notice that $\|V_t\|_{C^0} \leq c\epsilon\varrho$ in view of the estimates in (ii), Theorem 10.1, where the constant c depends only on n, τ and ϑ_0 , and taking $\epsilon < \min(1, 1/c)/2$ we obtain $V_t(\theta; \omega) \in B^n(0, \varrho/2)$ for any $\theta \in \mathbb{T}^n$ and $\omega \in \Omega - \kappa$. In the same way we get $\phi_t(\omega) \in \Omega - 3\kappa/4$ for $\omega \in \Omega - \kappa$. In particular, $\tilde{\psi}_\varrho(V_t(\theta; \omega)) = 1$ and $\psi_\kappa(\phi_t(\omega)) = 1$ for any $(\theta, \omega) \in \mathbb{T}^n \times (\Omega - \kappa)$.

By (10.207) and (10.208), we have

$$|d_\theta U_t(\theta; \omega) - \text{Id}| \leq C_1(n, \tau, \vartheta_0)\epsilon \leq 1/2$$

for $(\theta, \omega, t) \in \mathbb{T}^n \times \Omega \times [0, a]$, choosing ϵ sufficiently small. Now Remark 10.3 implies that for any $\omega \in \Omega_\kappa$ and $t \in [0, \delta]$ the Lagrangian manifold $\Lambda_t(\omega) := \Psi_t(\mathbb{T}^n; \omega)$ is a Kronecker invariant torus of H_t of a frequency vector ω satisfying (i) in Theorem 9.1.

Step 4. Estimates of \tilde{U}_t and \tilde{V}_t . The estimates (ii), Theorem 10.1, imply (ii) in Theorem 9.1 using the estimates of $\|P_t\|_{\ell; \varrho, \kappa}$ given above. To estimate the derivatives of

$$\tilde{V}_t - \nabla H_t^{0*} = V_t + (\nabla H_t^{0*}) \circ \phi_t - \nabla H_t^{0*}$$

we use (ii), Theorem 10.1 and the following

Lemma 9.4. *For any $m \in \{0\} \cup [1, +\infty)$ the following estimate holds*

$$\|(\nabla H_t^{0*}) \circ \phi_t - \nabla H_t^{0*}\|_{m, \Omega; \kappa} \leq C_m \frac{\mathcal{A}_m}{\varrho} \left(1 + \frac{\mathcal{A}_1}{\varrho}\right)^m.$$

Proof. The proof of the lemma is based on higher order Hölder estimates of a composition of functions given in the Appendix. We write

$$\nabla H_t^{0*}(\phi_t(\omega)) - \nabla H_t^{0*}(\omega) = u_t(\omega) \cdot v_t(\omega)$$

where

$$u_t(\omega) = \phi_t(\omega) - \omega \in M_{1,n}(\mathbb{R}) \quad \text{and} \quad v_t(\omega) = \int_0^1 (\partial^2 H_t^{0*})(\omega + s(\phi_t(\omega) - \omega)) ds \in M_{n,n}(\mathbb{R}).$$

Theorem 10.1 and (9.163) imply

$$\|\phi_t - \text{id}\|_{m,\Omega;\kappa} \leq C_m \frac{\mathcal{A}_m^0}{\varrho} S_{\ell(m)}(\nabla H^{0*})$$

where

$$S_{\ell(m)}(\nabla H^{0*}) = \sup_{0 \leq t \leq \delta} (1 + \|\nabla H_t^{0*}\|_1)^{\ell(m)-1} (1 + \|\nabla H_t^{0*}\|_{\ell(m)})$$

is increasing with respect to $m \in \mathbb{N}$ since Ω is convex. Moreover, (9.156) yields $\|\phi_t - \text{id}\|_0 \leq C\varepsilon\kappa$. Notice that v_t is well-defined and C^∞ smooth in the convex set $\bar{\Omega}$. Indeed, the image of Ω under the map $\omega \mapsto \omega + s(\phi_t(\omega) - \omega)$ is contained in Ω when $\varepsilon < 1/2C$ since $\text{supp}(\phi_t - \text{id}) \subset \Omega - \kappa/2$ and $\|\phi_t - \text{id}\|_0 < \kappa/2$.

Let $m = 0$. Then

$$\|(\nabla H_t^{0*}) \circ \phi_t - \nabla H_t^{0*}\|_{C^0(\Omega)} \leq C_0 \frac{\mathcal{A}_0^0}{\varrho} S_{\ell_0}(\nabla H^{0*}) \|\nabla H_t^{0*}\|_{C^1(\Omega)} < C_0 \frac{\mathcal{A}_0^0}{\varrho} S_{\ell_0+1}(\nabla H^{0*}) = C_0 \frac{\mathcal{A}_0}{\varrho}$$

since Ω is convex.

Let $m \geq 1$. Using Remark A.2 we get

$$\|\nabla H_t^{0*} \circ \phi_t - \nabla H_t^{0*}\|_{m,\Omega;\kappa} \leq C_m (\|\phi_t - \text{id}\|_{m,\mathbb{R}^n;\kappa} \|v_t\|_{C^0(\Omega)} + \|\phi_t - \text{id}\|_{C^0(\mathbb{R}^n)} \|v_t\|_{m,\Omega;\kappa}).$$

We obtain as above

$$\begin{aligned} \|\phi_t - \text{id}\|_{m,\mathbb{R}^n;\kappa} \|v_t\|_{C^0(\Omega)} &\leq C_m \frac{\mathcal{A}_m^0}{\varrho} S_{\ell(m)}(\nabla H^{0*}) \|\nabla H_t^{0*}\|_{C^1(\Omega)} \\ &< C_m \frac{\mathcal{A}_m^0}{\varrho} S_{\ell(m)+1}(\nabla H^{0*}) = C_m \frac{\mathcal{A}_m}{\varrho} \end{aligned}$$

since Ω is convex. On the other hand, Proposition A.9, 3, applied to $f = (\text{id} + s(\phi_t - \text{id})) \circ \sigma_\kappa$ and $g = \partial^2 H_t^{0*} \in C^\infty(\bar{\Omega}, M_{n,n}(\mathbb{R}))$ yields

$$\begin{aligned} \|v_t\|_{m,\Omega;\kappa} &\leq C_m (1 + \|\phi_t - \text{id}\|_{1;\kappa}^{m-1}) \\ &\quad \times (\|\nabla H^{0*}\|_{m+1} (1 + \|\phi_t - \text{id}\|_1) + \|\nabla H^{0*}\|_2 \|\phi_t - \text{id}\|_{m;\kappa}) \end{aligned}$$

where the corresponding C^ℓ norms of ∇H^{0*} are evaluated on Ω . Then

$$\begin{aligned} \|\phi_t - \text{id}\|_{C^0(\mathbb{R}^n)} \|v_t\|_{m,\Omega;\kappa} &\leq C_m \frac{\mathcal{A}_0^0}{\varrho} \left(1 + \frac{\mathcal{A}_1}{\varrho}\right)^{m-1} S_{\ell_0}(\nabla H^{0*}) \\ &\quad \times \left(\|\nabla H^{0*}\|_{m+1} \left(1 + \frac{\mathcal{A}_1}{\varrho}\right) + \|\nabla H^{0*}\|_2 \frac{\mathcal{A}_m^0}{\varrho} S_{\ell(m)}(\nabla H^{0*}) \right) \end{aligned}$$

The interpolation inequalities (A.8) and Remark A.2 yield for any $r, s \geq 1$ and $f \in C^{r+s-1}(\bar{\Omega})$ the estimate $\|f\|_r \|f\|_s \leq C_{r,s} \|f\|_1 \|f\|_{r+s-1}$, where $C_{r,s} > 0$ depends only on r, s and n . Applying this inequality to $f = \nabla K_t^*$ and using the convexity of Ω we obtain

$$S_{\ell_0}(\nabla H^{0*}) \|\nabla H_t^{0*}\|_{m+1} < C_m S_{\ell_0+m}(\nabla H^{0*}) < C_m S_{\ell(m)+1}(\nabla H^{0*})$$

and

$$S_{\ell(m)}(\nabla H^{0*}) \|\nabla H_t^{0*}\|_2 \leq C_m S_{\ell(m)+1}(\nabla H^{0*}).$$

On the other hand $\mathcal{A}_0^0 S_{\ell_0}(\nabla H^{0*}) \leq \varepsilon \kappa \varrho$ by (9.156) and $\mathcal{A}_0^0 \leq \mathcal{A}_m^0$ for $m \geq 0$ by definition and we obtain

$$\|\phi_t - \text{id}\|_{C^0(\mathbb{R}^n)} \|v_t\|_{m,\Omega;\kappa} \leq C_m \frac{\mathcal{A}_m^0}{\varrho} \left(1 + \frac{\mathcal{A}_1^0}{\varrho}\right)^m$$

This completes the proof of the Lemma. \square

Statement (iii) follows from the definition of \tilde{U} and \tilde{V} using Theorem 10.1. \square

9.2 KAM theorem with parameters for symplectic maps

We are going to prove an analogue of Theorem 10.1 for symplectic maps. More precisely, given a C^k family of “small” exact symplectic perturbations $(\theta, r) \rightarrow P_t(\theta, r; \omega)$ of the translation $(\theta, r) \rightarrow R_\omega(\theta, r) := (\theta + \omega, r)$ with a Diophantine frequency ω , we are going to find a C^k family of Kronecker invariant tori of $P_t(\theta, r; \phi_t(\omega))$, where $t \rightarrow \phi_t$ is a C^k family of diffeomorphisms close to the identity map. Moreover, we will estimate the C^m , $m \in \mathbb{N}$, norm of $\phi_t - \text{id}$, and of the displacement of the Kronecker tori with respect to the corresponding unperturbed tori.

Let $\Omega \subset [0, 2\pi)^{n-1}$ be an open convex set. We identify Ω with an open convex subset of \mathbb{T}^{n-1} . Fix $k \in \{0, 1\}$ and $\varrho_0 > 0$ and consider a C^k family of exact symplectic maps

$$[0, \delta] \ni t \rightarrow P_t(\cdot; \omega) \in C^\infty(\mathbb{T}^{n-1} \times B^{n-1}(0, \varrho_0), \mathbb{T}^{n-1} \times \mathbb{R}^{n-1})$$

depending smoothly on a parameter $\omega \in \Omega$. We suppose that $P_t(\cdot; \omega)$ is defined by a generating functions $\tilde{G}_t(\cdot; \omega)$ of the form

$$\tilde{G}_t(\theta, r; \omega) := \langle \theta, r \rangle - \langle \omega, r \rangle - G_t(\theta, r; \omega) \tag{9.165}$$

i.e.

$$P_t(\theta - \omega - \nabla_r G_t(\theta, r; \omega), r; \omega) = (\theta, r - \nabla_\theta G_t(\theta, r; \omega), r)$$

for any $(\theta, r) \in \mathbb{T}^{n-1} \times B^{n-1}(0, \varrho_0)$ and $\omega \in \Omega$ and that the C^2 norm of $G_t(\cdot; \omega)$ is sufficiently small. If $G_t = 0$ then $P_t(\cdot; \omega)$ becomes a translation with ω , namely, $R_\omega(\theta, r) = (\theta + \omega, r)$. In general we consider G_t as a small perturbation depending smoothly on the frequency ω as well. Thus the perturbation is a real valued function $(\theta, I; \omega, t) \mapsto G(\theta, I; \omega, t)$ defined in $\mathbb{A} \times \Omega \times [0, \delta]$, where $\mathbb{A} := \mathbb{T}^n \times B(0, \varrho_0)$. Hereafter, we assume that

$$G \in C^k([0, \delta]; C^\infty(\mathbb{A} \times \Omega)) \tag{9.166}$$

with $k = 0$ or $k = 1$, i.e. the map $t \rightarrow G_t := G(\cdot, t) \in C^\infty(\mathbb{T}^{n-1} \times B(0, \varrho_0) \times \Omega)$ is C^k -smooth on the interval $[0, \delta]$.

Given $\ell > 0$ and $0 < \varrho, \kappa \leq 1$ we denote by $\|G_t\|_{\ell; \varrho, \kappa}$ the weighted Hölder norm

$$\|G_t\|_{\ell; \varrho, \kappa} := \|G_t \circ \sigma_{\varrho, \kappa}\|_{C^\ell(\sigma_{\varrho, \kappa}^{-1}(\mathbb{A} \times \Omega))} \quad (9.167)$$

where $\sigma_{\varrho, \kappa}$ is the partial dilation $\sigma_{\varrho, \kappa}(\varphi, I; \omega) := (\varphi, \varrho I; \kappa \omega)$ and the Hölder norms are defined in Section A.1. Note that the function $\ell \rightarrow \|G_t\|_{\ell; \varrho, \kappa}$ is *increasing* in the interval $[0, +\infty)$ since the set $B(0, \rho_0) \times \Omega$ is *convex*.

Fix $\tau > n - 1$ and $\kappa \in (0, 1)$, and define Ω_κ as the set of all (κ, τ) -Diophantine vectors satisfying (1.2) and such that $\text{dist}(\omega, \mathbb{T}^{n-1} \setminus \Omega) \geq \kappa$, i.e.

$$\Omega_\kappa = D(\kappa, \tau) \cap \overline{\Omega - \kappa}. \quad (9.168)$$

Recall that ϑ_0 , ℓ_0 and $\ell(m)$ are defined in (9.153). The following statement is a counterpart of Theorem 10.1 in the case of exact symplectic mappings.

Theorem 9.5. *There is a positive constant $\epsilon = \epsilon(n, \tau, \vartheta_0)$ depending only on n , τ and on ϑ_0 such that for any $\delta > 0$, $0 < \kappa < 1$ and $0 < \varrho \leq \rho_0$ the following holds.*

Let P_t be a C^k family of exact symplectic maps with generating functions \tilde{G}_t satisfying (9.165) and (9.166) and such that

$$\sup_{t \in [0, \delta]} \|G_t\|_{\ell_0+1; \varrho, \kappa} \leq \epsilon \varrho \kappa. \quad (9.169)$$

Then there exists a C^k family of maps

$$[0, \delta] \ni t \mapsto \phi_t \in C^\infty(\Omega; \Omega), \quad [0, \delta] \ni t \mapsto \Psi_t = (U_t, V_t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega; \mathbb{T}^{n-1} \times B^{n-1}(0, \varrho))$$

such that

$$(i) \text{ supp}(\phi_t - \text{id}) \subset \Omega - \kappa/2, \text{ supp}((U_t, V_t) - (\text{id}, 0)) \subset \mathbb{T}^{n-1} \times (\Omega - \kappa/2);$$

(ii) For each $\omega \in \Omega_\kappa$ and $t \in [0, \delta]$ the map $\Psi_{t, \omega} := \Psi_t(\cdot, \omega) : \mathbb{T}^{n-1} \rightarrow \mathbb{T}^{n-1} \times B(0, \varrho)$ is a smooth embedding, $\Lambda_t(\omega) := \Psi_{t, \omega}(\mathbb{T}^{n-1})$ is an embedded Lagrangian torus invariant with respect to the exact symplectic map given by $P_{t, \phi_t(\omega)}(\theta, I) := P(\theta, I; \phi_t(\omega), t)$, and

$$P_{t, \phi_t(\omega)} \circ \Psi_{t, \omega} = \Psi_{t, \omega} \circ R_\omega \quad \text{on } \mathbb{T}^n;$$

(iii) For any $m \in \mathbb{N}$ there is $C_m > 0$ depending only on m , n , τ and ϑ_0 such that for any $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + 1$ the following estimate holds

$$\begin{aligned} & \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta (U_t(\theta; \omega) - \theta) \right| + \varrho^{-1} \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta V_t(\theta; \omega) \right| + \kappa^{-1} \left| (\kappa \partial_\omega)^\beta (\phi_t(\omega) - \omega) \right| \\ & \leq C_m (\varrho \kappa)^{-1} \|G_t\|_{\ell(m)+1; \varrho, \kappa} \end{aligned} \quad (9.170)$$

uniformly in $(\theta, \omega, t) \in \mathbb{T}^{n-1} \times \Omega_\kappa \times [0, \delta]$.

If P_t is analytic with respect to t in a disc $B(0, a)$ then so are U , V and ϕ .

Proof. To simplify the notations we fix $k = 1$. The first step in the proof will be to modify P_t by multiplying G_t by suitable cut-off functions without changing the corresponding estimates. This allows us to work with compactly supported functions. To this end we use the cut-off functions $\tilde{\psi}_\varrho$ and ψ_κ constructed in the previous sub-section by means of Lemma 9.3. We set $G_t^1(\theta, I; \omega) := G_t(\theta, I; \omega)\tilde{\psi}_\varrho(I)\psi_\kappa(\omega)$ and denote by P_t^1 the exact symplectic map with generating function

$$\tilde{G}_t^1(\theta, r; \omega) := \langle \theta, r \rangle - \langle \omega, r \rangle - G_t^1(\theta, r; \omega).$$

The function $\ell \rightarrow \|G_t\|_{\ell; \varrho, \kappa}$ is *increasing* in the interval $[0, +\infty)$ since the set $B(0, \rho_0) \times \Omega$ is *convex*. Then using (A.22) one obtains

$$\|G_t^1\|_{\ell; \varrho, \kappa} \leq C_\ell \|G_t\|_{\ell; \varrho, \kappa},$$

where C_ℓ depends only on n and ℓ . In particular, it follows from (9.169) that

$$\|\sigma_\varrho^{-1} \text{sgrad } G_t^1\|_{1; \varrho, \kappa} \leq C_1 \|\sigma_\varrho^{-1} \text{sgrad } G_t\|_{\ell_0-1; \varrho, \kappa} \leq C_1 \epsilon \kappa$$

which allows one to apply Lemma A.15. Hereafter

$$\text{sgrad } G_t^1(\theta, r; \omega) := (\nabla_r G_t^1(\theta, r; \omega), -\nabla_\theta G_t^1(\theta, r; \omega))$$

is the symplectic gradient of G_t^1 . We have

$$P_t^1(\theta, r; \omega) = (\theta + \omega, r) \quad \text{for } (r, \omega) \notin B^{n-1}(0, 7\varrho/8) \times (\Omega - \kappa/4).$$

Moreover,

$$P_t^1(\theta, r; \omega) = P_t(\theta, r; \omega) \quad \text{for } (\theta, r; \omega) \in \mathbb{T}^{n-1} \times B^{n-1}(0, \varrho/2) \times (\Omega - \kappa)$$

since $G_t^1 = G_t$ on that set. This allows us to replace G_t by G_t^1 and P_t by P_t^1 in the theorem. From now on we suppose that

$$\text{supp } G_t \subset \mathbb{A}' := \mathbb{T}^{n-1} \times B^{n-1}(0, \varrho) \times (\Omega - \kappa/4).$$

Set

$$\mathcal{N}_{\omega, \omega_n}(r, r_n) := \langle \omega, r \rangle + \omega_n r_n, \quad (\omega, \omega_n) \in \Omega \times I, \quad I = (\pi, 3\pi),$$

Using an argument of Douady [11] (see also Theorem 1.1 [57] and Theorem 3.1 [58]), we are going to find a C^1 family of Hamiltonians

$$(\theta, \theta_n, r, r_n) \rightarrow H_t(\theta, \theta_n, r, r_n; \omega, \omega_n) = \langle \omega, r \rangle + \omega_n r_n + h_t(\theta, \theta_n, r; \omega) \quad (9.171)$$

in $\mathbb{T}^n \times \mathbb{R}^n$ depending smoothly on parameters $(\omega, \omega_n) \in \Omega \times I$ with the following properties - h_t is “small” and the corresponding Poincaré map is given by P_t at any energy surface. Then we will apply Theorem 10.1 to the family H_t . We set $y = (y', y_n) \in \mathbb{R}^n$, $(\theta, \theta_n) = \text{pr}(y) \in \mathbb{T}^n$, $\eta = (\eta', \eta_n) = (r, r_n)$ and $\tilde{\omega} := (\omega, \omega_n) \in \tilde{\Omega} := \Omega \times I$. We shall denote by Σ_c the energy surface $\Sigma_c := \{H_t = c\} \subset \mathbb{T}^n \times \mathbb{R}^n$ for $c \in \mathbb{R}$ and by $\iota_c : \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \rightarrow \Sigma_c \cap \{\theta_n = 0\}$ the corresponding inclusion map, i.e.

$$\iota_c(\theta, r) = (\theta, 0, r, (c - \langle \omega, r \rangle - h_t(\theta, 0, r; \omega))/\omega_n).$$

Proposition 9.6. *There is a C^1 family of Hamiltonians*

$$H_t(\theta, \eta; \tilde{\omega}) := \mathcal{N}_{\tilde{\omega}}(\eta) + h_t(\theta, \eta'; \omega), \quad h_t \in C^\infty(\mathbb{T}^n \times \mathbb{R}^{n-1} \times \Omega), \quad \tilde{\omega} \in \Omega \times (\pi, 3\pi),$$

such that

- (i) $P_t = \iota_c^{-1} \circ \Phi_t^{\frac{2\pi}{\omega_n}} \circ \iota_c$ for any $c \in \mathbb{R}$, where ι_c is the corresponding inclusion map and Φ_t^x , $x \in \mathbb{R}$, is the Hamiltonian flow of H_t ,
- (ii) $\text{supp}(h_t) \subset A'' := \mathbb{T}^n \times B^{n-1}(0, \varrho) \times \Omega$ and the following estimate holds

$$\|h_t\|_{\ell, A''; \varrho, \kappa} \leq C_\ell \|G_t\|_{\ell+1, A'; \varrho, \kappa}$$

for any $\ell \geq 2$ and $0 < \varrho, \kappa \leq 1$, where $\mathbb{T}^n \times \mathbb{R}^{n-1} \times \Omega$, $C_\ell > 0$ depends only on ℓ and n .

- (iii) $h_t(\theta, \eta'; \omega) = 0$ for $|\theta_n| \leq \pi/2$.

Proof. Choose $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta(s) = 0$ for $|s| \leq 1/4$ and $\eta(s) = 1$ for $|s| \geq 1/2$. Consider the family of exact symplectic maps $s \rightarrow P_t^s$ in $\mathbb{A} = \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ for $s \in \mathbb{R}$ having generating functions of the form

$$\tilde{G}_t^s(\theta, r; \omega) := \langle \theta, r \rangle - s \langle \omega, r \rangle - \eta(s) G_t(\theta, r; \omega), \quad (\theta, r; \omega) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Omega.$$

We have $P_t^s = Q^s \circ W_t^s$, where $Q^s(\theta, r; \omega) = (\theta + s\omega, r)$ and the generating function of W_t^s is $\langle \theta, r \rangle - \eta(s) G_t(\theta, r; \omega)$. Set $G(\theta, r; \omega, t) = G_t(\theta, r; \omega)$ and denote the symplectic gradient of G with respect to (θ, r) by

$$\text{sgrad} G(\theta, r; \omega, t) := (\nabla_r G(\theta, r; \omega, t), -\nabla_\theta G(\theta, r; \omega, t)).$$

Notice that

$$\begin{aligned} P_t^s(\theta, r; \omega) &= (\theta + s\omega, r) \quad \forall s \in [-1/4, -1/4] \\ P_t^s(\theta, r; \omega) &= P_t(\theta, r; \omega) \quad \forall s \in (-\infty, -1/2] \cup [1/2, +\infty). \end{aligned} \tag{9.172}$$

Denote by

$$\xi_t^s := \frac{dP_t^s}{ds} \circ (P_t^s)^{-1}, \quad s \in \mathbb{R}$$

the corresponding vector field and set $v(\theta, s, r; \omega) = (\omega, 0)$. Then

$$\xi_t^s(\theta, r; \omega) = v(\theta, s, r; \omega) = (\omega, 0) \quad \forall s \in (-\infty, -1/2] \cup [-1/4, 1/4] \cup [1/2, +\infty).$$

We set $\xi_t(\theta, s, r; \omega) = \xi_t^s(\theta, r)$ where $(\theta, s) \in \mathbb{T}^{n-1} \times [0, 1]$.

Lemma 9.7. *We have $\xi_t^s(\theta, r; \omega) = (\omega, 0)$ for $s \in [0, 1/4] \cup [1/2, 1]$ and*

$$\text{supp}(\xi_t - v) \subset A' := \mathbb{T}^{n-1} \times (0, 1) \times B^{n-1}(0, \varrho) \times \Omega.$$

Moreover, the following estimates hold

$$\|\sigma_\varrho^{-1}(\xi_t - v)\|_{\ell, A'; \varrho, \kappa} \leq C_\ell \|\sigma_\varrho^{-1} \text{sgrad} G_t(\cdot; \omega)\|_{\ell, A'; \varrho, \kappa}$$

where $\ell \geq 1$ and $C_\ell > 0$ depends only on ℓ and n .

Proof. We have $\xi_t^s = v + dW_t^s/ds \circ (W_t^s)^{-1}$. By Lemma A.14 the support of $W_t^s - \text{id}$ is contained in \mathbb{A}' . The estimates of dW_t^s/ds and $(W_t^s)^{-1}$ follow from Proposition A.12, Lemma A.14 and Lemma A.15 taking into account (9.169). To obtain the corresponding estimates for the composition $dW_t^s/ds \circ (W_t^s)^{-1}$ we use Proposition A.12. \square

Notice that the one-form $\iota(\xi_t^s)d\theta \wedge dr$ is exact, where $d\theta \wedge dr = \sum_{j=1}^{n-1} d\theta_j \wedge dr_j$ is the standard symplectic two-form in $T^*\mathbb{T}^{n-1}$ and $\iota(v)$ is the inner product with the vector field v . This follows from the identity

$$\iota(\xi_t^s)d\theta \wedge dr = (P_t^s)_* \left(\frac{d}{ds} (P_t^s)^*(rd\theta) - d(\iota(\xi_t^s)rd\theta) \right)$$

since $(P_t^s)^*(rd\theta) - rd\theta$ is exact. Let h_t^s be a primitive of

$$\begin{cases} \iota(\xi_t^s - v)d\theta \wedge dr = -dh_t^s \\ h_t^s(0, 0; \omega) = 0. \end{cases} \quad (9.173)$$

The first equality in (9.173) means that

$$\begin{pmatrix} \partial_\varphi h_t^s \\ \partial_r h_t^s \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} (\xi_t^s - v).$$

The second one combined with Lemma 9.7 implies that $\text{supp}(h_t) \subset \mathbb{A}'$ and

$$\begin{aligned} \|h_t\|_{\ell, \mathbb{A}'; \varrho, \kappa} &\leq C(\|\partial_\theta h_t\|_{\ell, \mathbb{A}'; \varrho, \kappa} + \|(\varrho\partial_I)h_t\|_{\ell, \mathbb{A}'; \varrho, \kappa}) \\ &\leq C'_\ell(\|\partial_\theta h_t\|_{\ell, \mathbb{A}'; \varrho, \kappa} + \|(\varrho\partial_I)h_t\|_{\ell-1, \mathbb{A}'; \varrho, \kappa}) \\ &\leq C_\ell\|G_t\|_{\ell+1, \mathbb{A}'; \varrho, \kappa}. \end{aligned}$$

We denote as above $y = (y', y_n) \in \mathbb{R}^n$, where $y_n = 2\pi s$, $(\theta, \theta_n) = \text{pr}(y) \in \mathbb{T}^n$, $\eta = (\eta', \eta_n) = (r, r_n)$, and set

$$h_t(y, \eta'; \omega) := h_t^{y_n/2\pi}(y', \eta'; \omega), \quad H_t(y, \eta; \omega, \omega_n) := \langle \omega, \eta' \rangle + \omega_n \eta_n + h_t(y, \eta'; \omega).$$

Let $\mathcal{N}(\eta; \omega, \omega_n) = \langle \omega, \eta' \rangle + \omega_n \eta_n$ be the corresponding normal form. The Hölder norms of $H_t - \mathcal{N} = h_t$ have been estimated above which proves (ii).

Notice that $h_t^s(\theta, r) = 0$ for $s \in [0, 1/4] \cup [1/2, 1]$ since $\xi_t^s(\theta, r; \omega) = (\omega, 0)$ there. Then

$$h_t(\theta, \eta_n, r, \omega) = 0 \quad \forall \theta_n \in [0, \pi/2] \cup [\pi, 2\pi]$$

which gives (iii). To obtain (i), consider the Hamiltonian vector field X_{H_t} given by

$$\iota(X_{H_t})dy \wedge d\eta = -dH_t(y, \eta).$$

By (9.173) we get

$$(\partial_r H_t(\theta, y_n, r, r_n), -\partial_\theta H_t(\theta, y_n, r, r_n)) = (\partial_r h_t^{y_n/2\pi}(\theta, r) + \omega, -\partial_\theta h_t^{y_n/2\pi}(\theta, r)) = \xi_t^{y_n/2\pi}(\theta, r).$$

Moreover, $\partial_{r_n} H_t(\theta, y_n, r, r_n) = \omega_n$. Setting $P_t^s(\theta, r) = (p_t^s(\theta, r), q_t^s(\theta, r))$ we obtain that the Hamiltonian flow Φ_t^s of H_t is given by

$$\Phi_t^x(\theta, y_n, r, r_n) = (p_t^{x\omega_n/2\pi}(\theta, r), y_n + x\omega_n, q_t^{x\omega_n/2\pi}(\theta, r), q_n^{x\omega_n/2\pi}(\theta, y_n, r, r_n))$$

for $x \in \mathbb{R}$, where $t \rightarrow q_n$ is a C^1 family of smooth functions. This yields

$$\Phi_t^{\frac{2\pi}{\omega_n}}(\theta, 0, r, r_n) = (p_t^1(\theta, r), 2\pi, q_t^1(\theta, r), q_n^1(\theta, 0, r, r_n)).$$

This implies (i) and completes the proof of the proposition. \square

In particular, we have

$$\|h_t\|_{\ell_0, \mathbb{A}^n; \varrho, \kappa} \leq C\epsilon\kappa\varrho$$

for $t \in [0, a]$, where $C > 0$ depends only on ℓ_0 and n . Then Proposition 9.6 allows us to apply Theorem 10.1 to the perturbation H_t of \mathcal{N} in $\mathbb{T}^n \times B^n(0, \varrho) \times \tilde{\Omega}$, where $\tilde{\Omega} := \Omega \times (\pi, 3\pi)$. Denote by

$$\tilde{\Psi}_t = (\tilde{U}_t, \tilde{V}_t, \tilde{\phi}_t) : \mathbb{T}^n \times B^n(0, \varrho) \times \tilde{\Omega} \rightarrow \mathbb{T}^n \times B^n(0, \varrho) \times \tilde{\Omega}$$

the map given by Theorem 10.1. To obtain the map $\tilde{\Psi}_t$ from $\tilde{\Psi}_t$ we use an argument of Douady [11]. Fix $\omega \in \Omega_\kappa$ and $t \in [0, \delta]$. Then $\tilde{\omega} := (\omega, 2\pi) \in \tilde{\Omega}_\kappa := \tilde{D}(\kappa, \tau) \cap (\tilde{\Omega} - \kappa)$, where $\tilde{D}(\kappa, \tau)$ is given by (9.148). By Theorem 10.1 there exists a C^1 family of Kronecker invariant tori $\tilde{\Lambda}_t(\tilde{\omega})$ of the flow $\Phi_t^s(\cdot; \tilde{\phi}_t(\tilde{\omega}))$ of the Hamiltonian $H_t(\cdot; \tilde{\phi}_t(\tilde{\omega}))$ with a frequency vector $\tilde{\omega}$ lying on a certain energy surfaces $\Sigma_t := \{H_t = c_t\}$. Moreover,

$$\left(\tilde{U}_t(y + x\tilde{\omega}; \tilde{\omega}), \tilde{V}_t(y + x\tilde{\omega}; \tilde{\omega}) \right) = \Phi_t^x \left(\tilde{U}_t(y, \tilde{\omega}), \tilde{V}_t(y, \tilde{\omega}); \tilde{\phi}_t(\tilde{\omega}) \right) \quad (9.174)$$

for any $x \in \mathbb{R}$ and $y \in \mathbb{T}^n$. On the other hand, Φ_t^x is of the form

$$\Phi_t^x(y', y_n, \eta; \tilde{\phi}_t(\tilde{\omega})) = (z_t'(x, y, \eta; \tilde{\omega}), y_n + x\omega_n, \zeta_t(x, y, \eta; \tilde{\omega}))$$

since $\partial H_t / \partial \eta_n = \omega_n$, hence, the last coordinate $\tilde{U}_{t,n}$ of \tilde{U}_t satisfies the equality

$$\tilde{U}_{t,n}(y + x\tilde{\omega}; \tilde{\omega}) = \tilde{U}_{t,n}(y; \tilde{\omega}) + x\omega_n$$

for any $y \in \mathbb{T}^n$ and $x \in \mathbb{R}$. Then $\nabla_y \tilde{U}_{t,n}(y + x\tilde{\omega}; \tilde{\omega}) = \nabla_y \tilde{U}_{t,n}(y; \tilde{\omega})$, and since the flow $x \mapsto x\tilde{\omega}$ is ergodic on \mathbb{T}^n (recall that $\tilde{\omega}$ is a Diophantine frequency) we get

$$\nabla_y \tilde{U}_{t,n}(y; \tilde{\omega}) = \nabla_y \tilde{U}_{t,n}(0; \tilde{\omega}) := a_t(\tilde{\omega})$$

for any $y \in \mathbb{T}^n$. The function $\mathbb{T}^n \ni y \mapsto U_{t,n}(y; \tilde{\omega}) \in \mathbb{T}$ is determined up to a translation and we fix it by $U_{t,n}(0; \tilde{\omega}) = 0$. Then $\tilde{U}_{t,n}(y; \tilde{\omega}) = \langle a_t(\tilde{\omega}), y \rangle$ and $a_t(\tilde{\omega}) \in \mathbb{Z}^n$. On the other hand, $|\nabla_y \tilde{U}_t(y; \tilde{\omega}) - \text{Id}| \leq C(n, \tau, \vartheta_0)\epsilon$ in view of (10.208) and (10.207). Taking $\epsilon = \epsilon(n, \tau, \vartheta_0)$ small enough we obtain that $a_t = (0, \dots, 0, 1)$ and $\tilde{U}_{t,n}(y; \tilde{\omega}) = y_n$. Now we set

$$\Psi_t(\theta, r) = (U_t(\theta, \omega), V_t(\theta, \omega)) := p(\tilde{U}_t(\theta, 0, \tilde{\omega}), \tilde{V}_t(\theta, 0, \tilde{\omega})), \quad \phi_t(\omega) = \tilde{\phi}_t(\tilde{\omega}), \quad \tilde{\omega} = (\omega, 2\pi),$$

where p is the projection given by $p(\theta, \theta_n, r, r_n) = (\theta, r)$.

Using Proposition 9.6 one obtains that for each $\omega \in \Omega_\kappa$, the torus

$$\Lambda_t(\omega) := p(\tilde{\Lambda}_t(\tilde{\omega}))$$

is a Kronecker invariant torus of $P_{t,\phi_t(\omega)}$ with a frequency vector ω and we obtain (ii). Moreover, (9.157) implies (9.170). To prove the analyticity with respect to t we use Cauchy theorem at any step of the construction. This completes the proof of Theorem 9.8. \square

9.3 KAM theorems for C^k families of symplectic maps

The aim of this section is to obtain C^k families of Kronecker invariant tori of C^k families of exact maps close to the family $(\theta, r) \rightarrow (\theta + \nabla K_t(r), r)$, $t \in [0, \delta]$.

Let $\Omega \subset [0, 2\pi]^{n-1}$ be an open convex set which we identify with an open convex subset of \mathbb{T}^{n-1} . Fix $k \in \{0, 1\}$ and consider a C^k -family of real-valued functions

$$[0, \delta] \ni t \rightarrow K_t^* \in C^\infty(\overline{\Omega}, \mathbb{R})$$

satisfying the non-degeneracy condition

$$\nabla K_t^* : \Omega \longrightarrow D_t := \nabla K_t^*(\Omega) \text{ is a diffeomorphism.} \quad (9.175)$$

where $\overline{\Omega}$ is the closure of Ω . For any $t \in [0, \delta]$ the Legendre transform $K_t : D_t \rightarrow \Omega$ of K_t^* is in $C^\infty(D_t, \mathbb{R})$ and it satisfies the non-degenerate condition (9.145). Moreover, the corresponding family of functions $[0, \delta] \ni t \rightarrow K_t \in C^\infty(D_t, \mathbb{R})$ is C^k smooth. We set $\mathbb{A}_t := \mathbb{T}^{n-1} \times D_t$ and denote by

$$Q_t : \mathbb{A}_t \rightarrow \mathbb{A}_t, \quad Q_t(\theta, r) = (\theta + \nabla K_t(r), r) \quad (9.176)$$

the corresponding family of exact symplectic maps on \mathbb{A}_t . The frequency vector of Q_t on the invariant torus $\mathbb{T}^{n-1} \times \{r\}$ is $\omega = \nabla K_t(r) \in \Omega$.

We consider a C^k -family of exact symplectic maps

$$[0, \delta] \ni t \rightarrow P_t \in C^\infty(\mathbb{A}_t, \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}) \quad (9.177)$$

close Q_t . We suppose that P_t is defined by a generating function \tilde{G}_t of the form

$$\tilde{G}_t(\theta, r) := \langle \theta, r \rangle - K_t(r) - G_t(\theta, r) \quad (9.178)$$

which means that

$$P_t(\theta - \nabla K_t(r) - \nabla_r G_t(\theta, r), r) = (\theta, r - \nabla_\theta G_t(\theta, r))$$

for any $(\theta, r) \in \mathbb{A}_t$. We suppose as well that the map

$$[0, \delta] \ni t \rightarrow G_t \in C^\infty(\mathbb{T}^{n-1} \times D_t, \mathbb{R})$$

is C^k smooth with $k = 0$ or $k = 1$.

We assume as well that the C^2 norm of G_t is sufficiently small. Then the inverse function theorem implies that the map $\theta \rightarrow \theta - \nabla_r G_t(\theta, r)$ is a diffeomorphism of \mathbb{T}^{n-1} for any fixed $r \in D_t$ and P_t is well defined.

Denote by R_t the exact symplectic map with generating function $(\theta, r) \rightarrow \langle \theta, r \rangle - G_t(\theta, r)$, i.e. $R_t(\theta - \nabla_r G_t(\theta, r), r) = (\theta, r - \nabla_\theta G_t(\theta, r))$. One can show that $P_t = R_t \circ Q_t$ on \mathbb{A}_t (see Lemma A.16).

Given $\varrho, \kappa \in (0, 1)$ and $m \geq 0$ we set

$$\mathcal{B}_m^0 := \sup_{0 \leq t \leq \delta} (\varrho^2 \|\partial^2 K_t\|_{\ell(m)+1, D_t; \kappa} + \|G_t\|_{\ell(m)+1, \mathbb{A}_t; \kappa}), \quad (9.179)$$

where $\|G_t\|_{\ell, \mathbb{A}_t; \kappa} := \|G_t \circ \sigma_\kappa\|_{C^\ell(\sigma_\kappa^{-1}(\mathbb{A}_t))}$, σ_κ is defined by $\sigma_\kappa(x, \xi) = (x, \kappa\xi)$, and

$$\|u\|_{\ell, D; \kappa} = \sup_{0 \leq m \leq \ell} \|u\|_{\ell-m, D; \kappa}.$$

Recall that ϑ_0 , ℓ_0 and $\ell(m)$ are defined in (9.153). The sequence \mathcal{B}_m^0 , $m \geq 0$, is increasing by definition.

To formulate the smallness condition in the KAM theorem below we need as well the notation

$$S_\ell(\nabla K^*) := \sup_{0 \leq t \leq \delta} (1 + \|\nabla K_t^*\|_{C^1(\Omega)})^{\ell-1} (1 + \|\nabla K_t^*\|_{C^\ell(\Omega)}) \quad (9.180)$$

introduced in (9.152). This is an increasing sequence with respect to $\ell \in [1, +\infty)$ since Ω is convex. For any $m \geq 0$ we set

$$\mathcal{B}_m := \mathcal{B}_m^0 S_{\ell(m)+2}(\nabla K^*). \quad (9.181)$$

Theorem 9.8. *There exists $\epsilon = \epsilon(n, \tau, \vartheta_0) > 0$ depending only on n , τ and ϑ_0 such that the following holds.*

Let $\Omega \subset \mathbb{T}^{n-1}$ be an open convex set, $[0, \delta] \ni t \rightarrow K_t^* \in C^\infty(\overline{\Omega}, \mathbb{R})$ a C^k family satisfying (9.175) and let $\varrho > 0$ and $\kappa > 0$ be such that $0 < \varrho \leq \kappa \leq 1$ and $\Omega_\kappa \neq \emptyset$. Consider a C^k family of exact symplectic maps $[0, \delta] \ni P_t \in C^\infty(\mathbb{A}_t, \mathbb{T}^{n-1} \times \mathbb{R}^{n-1})$ with generating functions \tilde{G}_t of the form (9.178) such that

$$\mathcal{B}_0^0 S_{\ell_0+1}(\nabla K^*) = \sup_{0 \leq t \leq \delta} (\varrho^2 \|\partial^2 K_t\|_{\ell_0+1, D_t; \kappa} + \|G_t\|_{\ell_0+1, \mathbb{A}_t; \kappa}) S_{\ell_0+1}(\nabla K^*) \leq \epsilon \varrho \kappa. \quad (9.182)$$

Then there is a C^k family $[0, \delta] \ni t \rightarrow f_t = (u_t, v_t) \in C^\infty(\mathbb{T}^{n-1} \times \Omega; \mathbb{T}^{n-1} \times D_t)$ such that

- (i) for any $\omega \in \Omega_\kappa$, $[0, \delta] \ni s \rightarrow \Lambda_t(\omega) = f_{t, \omega}(\mathbb{T}^{n-1})$ is a C^k family of Kronecker invariant tori of P_t with a frequency vector ω , where $f_{t, \omega} := f_t(\cdot; \omega)$, and the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}^{n-1} & \xrightarrow{R_\omega} & \mathbb{T}^{n-1} \\ \downarrow f_{t, \omega} & & \downarrow f_{t, \omega} \\ \Lambda_t(\omega) & \xrightarrow{P_t} & \Lambda_t(\omega) \end{array}$$

- (ii) for any $m \in \{0\} \cup [1, +\infty)$ the following estimates hold

$$\begin{aligned} |\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta (u_t(\varphi; \omega) - \varphi)| &\leq C_m \frac{\mathcal{B}_m}{\kappa \varrho} \\ |\partial_\varphi^\alpha (\kappa \partial_\omega)^\beta (v_t(\varphi; \omega) - \nabla K_t^*(\omega))| &\leq C_m \frac{\mathcal{B}_m}{\varrho} \left(1 + \frac{\mathcal{B}_1}{\varrho}\right)^m \end{aligned} \quad (9.183)$$

for any $(\varphi, \omega) \in \mathbb{T}^{n-1} \times \Omega_\kappa$, $t \in [0, \delta]$, and multi-indices $\alpha, \beta \in \mathbb{N}^{n-1}$ with $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + 1$, where the constant $C_m > 0$ depends only on n , τ , ϑ_0 and m ,

(iii) $\text{supp}((u_t, v_t) - (\text{id}, \nabla K_t^*)) \subset \mathbb{T}^{n-1} \times (\Omega - \kappa/2)$.

If P is analytic with respect to t in a disc $B(0, a)$ then so are u and v .

We note that the C^1 families of invariant tori $\Lambda_t(\omega)$, $t \in [0, \delta]$, given by the theorem are uniquely defined.

Proof. The proof is similar to that of Theorem 9.1 and we give only the main steps in it. We are going to apply Theorem 9.5. To this end we will firstly construct the function $\tilde{G}_t(\theta, r; \omega)$ in (9.165).

Step 1. Construction of the generating function $\tilde{G}_t(\theta, r; \omega)$. Given $\omega \in \Omega - \kappa/4$ and I the ball $B^{n-1}(0, \varrho)$, we set $r = \nabla K_t^*(\omega) + I$. Choosing $\epsilon = \epsilon(n, \tau, \vartheta_0) < 1/9$ in (9.182) we obtain as in the proof of of Theorem 9.1 the following relation

$$\forall \omega \in \Omega - \kappa/4, \forall I \in B^{n-1}(0, \varrho), \quad \begin{cases} \nabla K_t^*(\omega) + I \in D_t & \text{and} \\ \nabla K_t(\nabla K_t^*(\omega) + I) \in \Omega - \frac{1}{8}\kappa. \end{cases} \quad (9.184)$$

By Taylor's formula we obtain

$$\tilde{G}_t(\theta, r) = \langle \theta, \nabla K_t^*(\omega) \rangle + \langle \theta, I \rangle - K_t(\nabla K_t^*(\omega)) - \langle \omega, I \rangle - G_t^0(I; \omega) - G_t^1(\theta, I; \omega), \quad (9.185)$$

where

$$G_t^0(I; \omega) = \int_0^1 (1-s) \langle \partial^2 K_t(\nabla K_t^*(\omega) + sI)I, I \rangle ds$$

and

$$G_t^1(\theta, I; \omega) := G_t(\theta, \nabla K_t^*(\omega) + I).$$

It follows from (9.184) that the functions G_t^0 and G_t^1 are well defined for $I \in B^{n-1}(0, \varrho)$ and $\omega \in \Omega - \kappa/4$. Denote by $P_{t,\omega} : \mathbb{A}_t \rightarrow \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ the exact symplectic map defined by means of the generating function

$$\begin{aligned} (\theta, I) \rightarrow \tilde{G}_{t,\omega}(\theta, I) &:= \langle \theta, I \rangle - \langle \omega, I \rangle - G_t^0(I; \omega) - G_t^1(\theta, I; \omega) \\ &= \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I) - \langle \theta, \nabla K_t^*(\omega) \rangle - K_t(\nabla K_t^*(\omega)) \end{aligned}$$

and set $\psi_\omega(\theta, I) = (\theta, \nabla K_t^*(\omega) + I)$.

Lemma 9.9. *For any $\omega \in \Omega - \kappa/4$ the map $P_{t,\omega} : \mathbb{A}_t \rightarrow \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ is well defined and*

$$P_{t,\omega} = \psi_\omega^{-1} \circ P_t \circ \psi_\omega$$

on $\mathbb{T}^{n-1} \times B^{n-1}(0, \varrho)$ provided that the constant $\epsilon = \epsilon(n, \tau, \vartheta_0)$ in (9.182) is sufficiently small.

Proof. The smallness condition (9.182) implies that $\|\nabla_\theta \nabla_I \tilde{G}_{t,\omega}(\theta, I) - \text{Id}_n\| \leq C\epsilon$ for $(\theta, I; \omega) \in \mathbb{T}^{n-1} \times B^{n-1}(0, \varrho) \times (\Omega - \kappa/4)$, where $C = C(n)$ depends only on the dimension n and $\text{Id}_n \in M_n(\mathbb{R}^{n-1})$ is the identity matrix. Choosing $\epsilon = \epsilon(n, \tau, \vartheta_0)$ sufficiently small we obtain that the map $\theta \rightarrow \varphi = \nabla_I \tilde{G}_{t,\omega}(\theta, I)$ is a diffeomorphism on \mathbb{T}^{n-1} for any fixed $I \in B^{n-1}(0, \varrho)$

and $\omega \in \Omega - \kappa/4$, hence, $P_{t,\omega}$ is well-defined. Notice that $\nabla_I \tilde{G}_{t,\omega}(\theta, I) = \nabla_r \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I)$ and $\nabla_\theta \tilde{G}_{t,\omega}(\theta, I) = \nabla_\theta \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I) - \nabla K_t^*(\omega)$. Then

$$(\psi_\omega \circ P_{t,\omega})(\nabla_r \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I), I) = (\theta, \nabla_\theta \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I)).$$

On the other hand,

$$\begin{aligned} (P_t \circ \psi_\omega)(\nabla_r \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I), I) &= P_t(\nabla_r \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I), \nabla K_t^*(\omega) + I) \\ &= (\theta, \nabla_\theta \tilde{G}_t(\theta, \nabla K_t^*(\omega) + I)) \end{aligned}$$

and we obtain that $\psi_\omega \circ P_{t,\omega} = P_t \circ \psi_\omega$ since the map $\theta \rightarrow \nabla_r \tilde{G}_t(\theta, r)$ is a diffeomorphism. \square

Step 2. Hölder estimates of \tilde{G}_t . We are going to apply Theorem 9.5 to the family of exact symplectic maps $P_t(\cdot; \omega) := \tilde{P}_{t,\omega}(\cdot)$. To this end we evaluate the weighted norms of G_t^0 and G_t^1 . We have $G_t^0 = Q_t^0 \circ (\text{id}, \nabla K_t^*)$, where

$$Q_t^0(I; r) = \int_0^1 (1-s) \langle \partial^2 K_t(r+sI)I, I \rangle ds$$

is well defined and smooth in $\bar{\Gamma}$, $\Gamma := B^{n-1}(0, \varrho) \times (\Omega - \kappa/4)$. As in the proof of Theorem 9.1 we obtain that

$$\|Q_t^0\|_{\ell, \Gamma; \varrho, \kappa} \leq C_\ell \varrho^2 \|\partial^2 K_t\|_{\ell, D_t; \kappa}.$$

Since Ω is convex and $K_t^* \in C^\infty(\bar{\Omega})$ using Proposition A.12, 3., and Remark A.13 as in the proof of Theorem 9.1 we obtain for any $\ell \geq 1$ the estimate

$$\|G_t^0\|_{\ell; \varrho, \kappa} \leq C_\ell \varrho^2 \|\partial^2 K_t\|_{\ell, D_t; \kappa} S_\ell(\nabla K^*).$$

Moreover, $\|G_t^1\|_{\ell; \varrho, \kappa} \leq \|G_t^1\|_{\ell; \kappa, \kappa} \leq C_\ell \|G_t\|_{\ell; \kappa} S_\ell(\nabla K^*)$ since $0 < \varrho \leq \kappa \leq 1$ and we obtain

$$\|G_t^0\|_{\ell; \varrho, \kappa} + \|G_t^1\|_{\ell; \varrho, \kappa} \leq C_\ell (\varrho^2 \|\partial^2 K_t\|_{\ell, D_t; \kappa} + \|G_t\|_{\ell, \mathbb{A}_t; \kappa}) S_\ell(\nabla K^*). \quad (9.186)$$

Step 3. Applying Theorem 9.5. Now (9.182) gives

$$\|G_t^0\|_{\ell_0+1; \varrho, \kappa} + \|G_t^1\|_{\ell_0+1; \varrho, \kappa} \leq \mathcal{B}_0 S_{\ell_0+1}(\nabla K^*) \leq \epsilon \varrho \kappa.$$

This allows us to apply Theorem 9.5 to the family of exact symplectic maps $P_t(\cdot; \omega) = \tilde{P}_{t,\omega}(\cdot)$. Set $\Psi_t = (u_t, v_t)$, where $u_t = U_t$, $v_t = (\nabla K_t^*) \circ \phi_t + V_t$ and (U_t, V_t, ϕ_t) are given by Theorem 9.5. Notice that $\|V_t\|_{C^0} \leq c\epsilon\varrho$ in view of the estimates in (ii), Theorem 9.5, where the constant c depends only on n, τ and ϑ_0 , and taking $\epsilon < \min(1, 1/c)$ we obtain $V_t(\theta; \omega) \in B^{n-1}(0, \varrho)$ for any $\theta \in \mathbb{T}^{n-1}$ and $\omega \in \Omega - \kappa$. In the same way we get $\phi_t(\omega) \in \Omega - \kappa/4$ for $\omega \in \Omega - \kappa$.

Now Lemma 9.9 implies that for any $\omega \in \Omega_\kappa$ and $t \in [0, \delta]$ the Lagrangian manifold $\Lambda_t(\omega) := \Psi_t(\mathbb{T}^{n-1}; \omega)$ is a Kronecker invariant torus of P_t of a frequency vector ω satisfying (i) in Theorem 9.8.

Step 4. Estimates of u_t and v_t .

The estimates (ii), Theorem 9.5, imply (ii) in Theorem 9.8. To estimate the derivatives of

$$v_t - \nabla K_t^* = V_t + (\nabla K_t^*) \circ \phi_t - \nabla K_t^*$$

we use (ii), Theorem 9.5 and the following Lemma which is an analogue of Lemma 9.4.

Lemma 9.10. *For any $m \in \mathbb{N}$ the following estimate holds*

$$\|(\nabla K_t^*) \circ \phi_t - \nabla K_t^*\|_{m, \Omega; \kappa} \leq C_m \frac{\mathcal{B}_m}{\varrho} \left(1 + \frac{\mathcal{B}_1}{\varrho}\right)^m.$$

To prove (iii) we use suitable cut-off functions in ω given by Lemma 9.3. □

9.4 Birkhoff Normal Forms for C^k -families of symplectic maps

Let $\Omega \subset [0, 2\pi]^{n-1}$ be an open convex set which we identify with an open convex subset of \mathbb{T}^{n-1} . Fix $\tau > n - 1$ and denote by $\kappa_0(\Omega)$ the supremum of all $0 < \kappa \leq 1$ such that the set $\Omega_\kappa = D(\kappa, \tau) \cap \overline{\Omega} - \kappa$ is of positive Lebesgue measure. Given $0 < \kappa < \kappa_0(\Omega)$ we denote by Ω_κ^0 the set of points of Ω_κ of *positive Lebesgue density*. Recall that $\omega \in \Omega_\kappa^0$ if the Lebesgue measure of $\Omega_\kappa \cap U$ is positive for any neighborhood U of ω . Then $\Omega_\kappa \setminus \Omega_\kappa^0$ is a set of measure zero. Recall that ℓ_0 and $\ell(m)$ defined in (9.153), i. e. $\ell_0 := 2\tau + 2 + 2\vartheta_0$ and $\ell(m) := 2m(\tau + 1) + \ell_0$ for $m \geq 0$.

We are going to use as well the notations \mathcal{B}_m and $S_\ell(\nabla K_t^*)$ introduced in (9.181) and (9.180).

To construct a BNF we have to deal with the second differential $d^2 K_t$ of the Legendre transform K_t of K_t^* . We denote by $\partial^2 K_t(I)$ its Hessian matrix. Its norm could be very large, as in the case of the billiard ball map close to the boundary, and to measure it we introduce a parameter $\lambda \geq 1$. More precisely, we suppose below that

$$\sup_{t \in [0, \delta]} \|\partial^2 K_t\|_{2, D_t; \kappa} \leq \lambda < \infty, \quad (9.187)$$

where $\lambda \geq 1$.

Theorem 9.11. *There exists $\epsilon = \epsilon(n, \tau, \vartheta_0) > 0$ depending only on n , τ , and ϑ_0 such that the following holds.*

Let $\Omega \subset \mathbb{T}^{n-1}$ be an open convex set and $0 < \varrho < \kappa < \kappa_0(\Omega)$. Let $[0, \delta] \ni t \rightarrow K_t^* \in C^\infty(\overline{\Omega}, \mathbb{R})$ be a C^k family satisfying the non-degeneracy condition (9.175) and suppose that its Legendre transform K_t satisfies (9.187). Let $[0, \delta] \ni P_t \in C^\infty(\mathbb{A}_t, \mathbb{A})$, be a C^k family of exact symplectic maps defined by generating functions $\tilde{G}_t(\theta, r) = \langle \theta, r \rangle - K_t(r) - G_t(\theta, r)$ such that

$$\mathcal{B}_2 \leq \epsilon \varrho \kappa \lambda^{-4}. \quad (9.188)$$

Then

- (i) *there exist C^k -smooth with respect to $t \in [0, \delta]$ families of exact symplectic maps $\chi_t : \mathbb{A}_t \rightarrow \mathbb{A}_t$ and of real valued functions $L_t \in C^\infty(D_t)$ and $R_t^0 \in C^\infty(\mathbb{A}_t)$ such that*

- (a) $(\varphi, I) \mapsto \langle \varphi, I \rangle - L_t(I) - R_t^0(\varphi, I)$ is a generating function of $P_t^0 := \chi_t^{-1} \circ P_t \circ \chi_t$;
- (b) $\nabla L_t : D_t \rightarrow \Omega$ is a diffeomorphism, $L_t = K_t$ outside $D_t^1 := \nabla K_t^*(\Omega - \kappa/2)$, and $\nabla L_t^*(\omega) = I_t(\omega)$ is given by (1.6) for each $\omega \in \Omega_\kappa^0$;
- (c) R_t^0 is flat at $\mathbb{T}^{n-1} \times \nabla L_t(\Omega_\kappa^0)$;

(ii) For any $t \in [0, \delta]$ and $m \in \mathbb{N}$ the following estimates hold

$$\begin{aligned} & \|\sigma_\kappa^{-1}(\chi_t - \text{id})\|_{m, \mathbb{A}_t; \kappa} + \|\sigma_\kappa^{-1}(\chi_t^{-1} - \text{id})\|_{m, \mathbb{A}_t; \kappa} \\ & \leq \frac{C_m}{\varrho \kappa} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa}) \end{aligned} \quad (9.189)$$

and

$$\begin{aligned} & \|\nabla L_t - \nabla K_t\|_{m, D_t; \kappa} + \|\sigma_\kappa \nabla R_t^0\|_{m, D_t; \kappa} \\ & \leq \frac{C_m}{\varrho} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa}), \end{aligned} \quad (9.190)$$

where the constant $C_m > 0$ depends only on n, τ, ϑ_0 and m .

If \tilde{G} is analytic with respect to t , then so are χ, L and R^0 .

Before proving the Theorem we observe that

Remark 9.12. (Birkhoff Normal Form) Let $k = 1$. Then for any $\omega \in \Omega_\kappa^0$ the map

$$[0, \delta] \ni t \rightarrow \Lambda_t(\omega) := \chi_t(\mathbb{T}^{n-1} \times \{I_t(\omega)\})$$

provides a C^1 family of invariant tori of P_s with a frequency vector ω and taking into account Lemma 3.4 we obtain

$$P_t^0(\varphi, I) = (\varphi + \nabla L_t(I), I) + O_N(|I - \nabla L_t^*(\omega)|^N)$$

for any $N \in \mathbb{N}$. Moreover, the last formula can be differentiated $N - 1$ times with respect to (φ, I) . Hence, Theorem 9.11 gives a simultaneous Birkhoff Normal Form of P_t on the invariant tori $\Lambda_t(\omega)$, where $t \in [0, \delta]$ and the frequency vectors ω are in Ω_κ^0 . Recall as well that the complement of Ω_κ^0 in Ω_κ is of Lebesgue measure zero.

Proof of Theorem 9.11. Without loss of generality we consider only the case when $k = 1$. We devide the proof in several steps.

Step 1. Writing $\Lambda_t(\omega)$ as graphs. For any $\omega \in \Omega_\kappa$ and $t \in [0, \delta]$ the Lagrangian manifold

$$\Lambda_t(\omega) := \{(u_t(\theta; \omega), v_t(\theta; \omega)) : \theta \in \mathbb{T}^n\}$$

given by Theorem 9.8 is a Kronecker invariant torus of P_t of a frequency vector ω satisfying (i) of Theorem 9.8. Firstly we will solve the equation $\varphi = u_t(\theta, \omega)$ with respect to θ and get the respective estimates of the solution. To this end we consider the map $w_t : \mathbb{A} \rightarrow \mathbb{A}$ defined by $w_t(\theta, \omega) = (u_t(\theta, \omega), \omega)$, where $\mathbb{A} = \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$. Recall from Theorem 9.8, (iii), that $\text{supp}(w_t - \text{id}) \subset \mathbb{T}^{n-1} \times (\Omega - \kappa/2)$. It follows from (9.183) and (9.188) that

$$\|\sigma_\kappa^{-1}(w_t - \text{id})\|_{1; \kappa} = \|w_t - \text{id}\|_{1; \kappa} \leq C_1 \epsilon < (2n - 2)^{-1}$$

choosing $\epsilon = \epsilon(n, \tau, \vartheta_0)$ small enough (recall that C_1 depends only on n, τ and ϑ_0). Then applying Proposition A.11 we obtain a solution $\theta_t(\varphi, \omega) = \varphi + \psi_t(\varphi, \omega)$ of the equation $\varphi = u_t(\theta, \omega)$, where $(\varphi, \omega) \in \mathbb{T}^{n-1} \times \Omega$. Then $\text{supp } \psi_t \subset \mathbb{T}^{n-1} \times (\Omega - \kappa/2)$ and

$$\|\psi_t\|_{m;\kappa} \leq \frac{C_m}{\kappa\varrho} \mathcal{B}_m \quad (9.191)$$

for any $m \in \mathbb{N}$, where the constant $C_m > 0$ depends only on n, τ, ϑ_0 and m . Setting $\tilde{F}_t(\varphi, \omega) = v_t(\theta_t(\varphi, \omega), \omega)$ and

$$F_t(\varphi, \omega) := -\nabla K_t^*(\omega) + \tilde{F}_t(\varphi, \omega) \quad (9.192)$$

for $(\varphi, \omega) \in \mathbb{T}^{n-1} \times \Omega$ we write

$$\Lambda_t(\omega) = \{(\varphi, \tilde{F}_t(\varphi, \omega)) : \varphi \in \mathbb{T}^{n-1}\} = \{(\varphi, \nabla K_t^*(\omega) + F_t(\varphi, \omega)) : \varphi \in \mathbb{T}^{n-1}\}, \quad \omega \in \Omega_\kappa. \quad (9.193)$$

Notice that $\text{supp } F_t \subset \mathbb{T}^{n-1} \times (\Omega - \kappa/2)$. We are going to prove that

$$\|F_t\|_{m;\kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m, \quad (9.194)$$

for any $m \in \mathbb{N}$, where the constant $C_m > 0$ depends only on n, τ, ϑ_0 and m . To this end we write

$$F_t(\varphi, \omega) = (v_t(\varphi, \omega) - \nabla K_t^*(\omega)) + (v_t(\theta_t(\varphi, \omega), \omega) - v_t(\varphi, \omega)).$$

The estimate of $v_t(\varphi, \omega) - \nabla K_t^*(\omega)$ follows directly from (9.183) using the inequality $\mathcal{B}_1 \leq \mathcal{B}_2 \leq \epsilon\kappa\varrho \leq \varrho$. To obtain the estimate of the second term of (11.221) we write

$$v_t(\theta_t(\varphi, \omega), \omega) - v_t(\varphi, \omega) = \int_0^1 (d_\theta v_t)(\varphi + s\psi_t(\varphi, \omega)) \psi_t(\varphi, \omega) ds.$$

Then one uses (A.9), Proposition A.12, 2, and (9.188) as well.

Denote by $p : \mathbb{R}^{n-1} \rightarrow \mathbb{T}^{n-1}$ the natural projection.

Lemma 9.13. *There is a C^1 family of real-valued functions $h_t \in C^\infty(\mathbb{R}^{n-1} \times \Omega)$ and $I_t \in C^\infty(\Omega)$ in $t \in [0, \delta]$ such that $h_t^0(x, \omega) := h_t(x, \omega) - \langle x, I_t(\omega) \rangle$ is 2π -periodic with respect to x and*

- (i) $\forall (x, \omega) \in \mathbb{R}^{n-1} \times \Omega_\kappa, \nabla_x h_t(x, \omega) = \tilde{F}_t(p(x), \omega),$
- (ii) $\nabla_x h_t^0(x, \omega) = 0$ and $I_t(\omega) = \nabla K_t^*(\omega)$ for $\omega \notin \Omega - \kappa/2,$
- (iii) $\|h_t^0\|_{m;\kappa} + \|I_t - \nabla K_t^*\|_{m;\kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m$

for $m \in \mathbb{N}$, where C_m is a positive constant depending only on n, τ, ϑ_0 and m .

Proof. To obtain h_t we consider the function

$$\tilde{h}_t(x, \omega) = \int_{\gamma_x} \sigma = \int_0^1 \langle \tilde{F}_t(p(sx), \omega), x \rangle ds = \langle \nabla K_t^*(\omega), x \rangle + \int_0^1 \langle F_t(p(sx), \omega), x \rangle ds$$

for $(x, \omega) \in \mathbb{R}^{n-1} \times \Omega$, where $\gamma_x = \{(sx, \tilde{F}_t(p(sx), \omega)) : 0 \leq s \leq 1\}$ and $\sigma = \xi dx$ is the canonical one-form on $T^*\mathbb{R}^{n-1}$. In view of (9.194), the function $Q_t^0(x, \omega) := \tilde{h}_t(x, \omega) - \langle \nabla K_t^*(\omega), x \rangle$ satisfies the estimate

$$\|Q_t^0\|_{m;\kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m \quad (9.195)$$

for $m \in \mathbb{N}$. We set

$$I_{tj}(\omega) = \tilde{h}_t(2\pi e_j, \omega)/2\pi, \quad \omega \in \Omega,$$

where $\{e_1, \dots, e_{n-1}\}$ is the canonical basis in \mathbb{R}^{n-1} . Then (9.195) implies

$$\|I_t - \nabla K_t^*\|_{m;\kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m \quad (9.196)$$

for $m \in \mathbb{N}$ and $t \in [0, \delta]$.

As $\Lambda_t(\omega)$ is a Lagrangian torus for $\omega \in \Omega_\kappa$, we get

$$\forall y \in \mathbb{R}^{n-1}, \quad \tilde{h}_t(x+y, \omega) - \tilde{h}_t(x, \omega) = \int_{l_t(x,y)} \sigma \quad (9.197)$$

where $l_t(x, y) = \{(x+sy, \tilde{F}_t(p(x+sy), \omega)) : 0 \leq s \leq 1\}$ and σ is the pull-back to $\Lambda_t(\omega)$ of the fundamental one-form Idx . The integral in (9.197) is equal to

$$\int_{l_t(x,y)} \sigma = \int_0^1 \langle \tilde{F}_t(p(x+sy), \omega), y \rangle ds = \langle \tilde{F}_t(p(x), \omega), y \rangle + O(y^2),$$

and we obtain $\nabla_x \tilde{h}(x, \omega) = \tilde{F}_t(p(x), \omega)$ for any $\omega \in \Omega_\kappa$. In particular, the function $\nabla_x \tilde{h}_t(x, \omega)$ is 2π -periodic with respect to x and we get

$$\forall \omega \in \Omega_\kappa, \forall \alpha \in \mathbb{Z}^{n-1}, \quad \tilde{h}_t(x+2\pi\alpha, \omega) - \tilde{h}_t(x, \omega) = \tilde{h}_t(2\pi\alpha, \omega) - \tilde{h}_t(0, \omega) = \langle 2\pi\alpha, I_t(\omega) \rangle.$$

Consider the function

$$\tilde{h}_t^0(x, \omega) = \tilde{h}_t(x, \omega) - \langle x, I_t(\omega) \rangle.$$

It is 2π -periodic with respect to x for $\omega \in \Omega_\kappa$ and \tilde{h}_t^0 satisfies the estimates (9.195) in $[0, \delta] \times \mathbb{R}^{n-1} \times \Omega$. We are going to average \tilde{h}_t^0 on \mathbb{T}^{n-1} using the following

Lemma 9.14. *There exists $f \in C^\infty(\mathbb{R}^{n-1})$ with $\text{supp } f \subset [\pi, 7\pi]^{n-1}$ such that*

$$\sum_{k \in \mathbb{Z}^{n-1}} f(x - 2\pi k) = 1$$

for each $x \in \mathbb{R}^{n-1}$.

Consider the function

$$h_t^0(x, \omega) = \sum_{k \in \mathbb{Z}^{n-1}} (f\tilde{h}_t^0)(x - 2\pi k, \omega).$$

It is 2π -periodic with respect to x by construction and $h_t^0(x, \omega) = \tilde{h}_t^0(x, \omega)$ for $(x, \omega) \in \mathbb{R}^{n-1} \times \Omega_\kappa$. Moreover, h_t^0 satisfies (9.195) in $[0, \delta] \times \mathbb{R}^{n-1} \times \Omega$. We set

$$h_t(x, \omega) = h_t^0(x, \omega) + \langle x, I_t(\omega) \rangle.$$

Recall that

$$\text{dist}(\Omega_\kappa, \mathbb{R}^{n-1} \setminus \Omega) \geq \kappa.$$

Then multiplying h_t^0 and $I_t - \nabla K_t^*$ by a suitable cut-off function which is equal to one on $\Omega - 3\kappa/4$ and zero outside $\Omega - \kappa/2$, we can assume that $h_t(x, \omega) = \langle x, \nabla K_t^*(\omega) \rangle$ and $I_t(\omega) = \nabla K_t^*(\omega)$ outside $\Omega - \kappa/2$ (see Lemma 9.3). This proves (ii). The statement (iii) follows from (9.195), the definition of h_t^0 and (9.196). \square

Step 2. Inverting I_t .

Lemma 9.15. *Choosing $\epsilon = \epsilon(n, \tau, \vartheta_0) > 0$ small enough one has the following for each $t \in [0, \delta]$.*

1. *The map $I_t : \Omega \rightarrow D_t$ is a diffeomorphism and its inverse $\omega_t : D_t \rightarrow \Omega$ satisfies the estimates $\|\omega_t - \nabla K_t\|_{1, D_t; \kappa} \leq C_0 \epsilon \kappa$ and*

$$\|\omega_t - \nabla K_t\|_{m, D_t; \kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m \lambda^m (\lambda + \|\partial^2 K_t\|_{m, D_t; \kappa}) \quad (9.198)$$

for any $m \in \mathbb{N}_$. Moreover, $\omega_t = \nabla K_t$ outside the set $\overline{D}_t^1 := \nabla K_t^*(\Omega - \kappa/2)$.*

2. *For any $x \in \mathbb{R}^{n-1}$ the map $\Omega \ni \omega \rightarrow \nabla_x h_t(x, \omega) \in D_t$ is a diffeomorphism.*

Proof. We are going to show that the map $\Omega \ni \omega \rightarrow I_t(\omega)$ a diffeomorphism. To this end we write

$$I_t = (\text{id} + \varphi_t) \circ \nabla K_t^*, \quad \varphi_t := (I_t - \nabla K_t^*) \circ \nabla K_t.$$

Moreover, Lemma 9.13, (9.187) and (9.188) imply

$$\|\varphi_t\|_1 \leq \|I_t - \nabla K_t^*\|_0 + \|I_t - \nabla K_t^*\|_1 (1 + \|\nabla K_t\|_1) \leq C \epsilon \lambda^{-2}$$

where $C = C(n, \tau, \vartheta_0) > 0$. Recall that $I_t = \nabla K_t^*$ outside $\Omega - \kappa/2$, hence $\text{supp } \varphi_t \subset D_t$. Then choosing $0 < \epsilon < 1/(2C)$ and applying Proposition A.11 we obtain that $\text{id} + \varphi_t : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ is invertible and that $(\text{id} + \varphi_t)^{-1} = \text{id} + \psi_t$, where $\text{supp } \psi_t \subset D_t$. Hence, $I_t(\Omega) = D_t$ and $I_t : \Omega \rightarrow D_t$ is a diffeomorphism with inverse

$$\omega_t = \nabla K_t \circ (\text{id} + \psi_t) : D_t \rightarrow \Omega.$$

Lemma 9.16. *For any $m \in \mathbb{N}_*$ there exists $C_m > 0$ depending only on m, n, τ and ϑ_0 such that*

$$\|\varphi_t\|_{m, \kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m \lambda^{m-1} (\lambda + \|\partial^2 K_t\|_{m-1, D_t; \kappa}).$$

Proof. The support of φ_t is contained in the closure of $D_t^1 := \nabla K_t^*(\Omega - \kappa/2)$. Set $2r_t := \text{dist}(D_t^1, \mathbb{R}^{n-1} \setminus D_t)$ and fix I^0 in \overline{D}_t^1 . Applying Remark A.2 to the restriction of K_t to the closed ball $\overline{B}(I^0, r_t)$ as well as Proposition A.12, (9.187) and (9.188) one obtains

$$\begin{aligned} |(\kappa \partial_I)^\beta \varphi_t(I)| &\leq C'_m \left(1 + \|\partial^2 K_t\|_{C^0(D_t)}^{m-1}\right) \\ &\times \left(\|I_t - \nabla K_t^*\|_{m, \kappa} \|\partial^2 K_t\|_{C^0(D_t)} + \|I_t - \nabla K_t^*\|_{C^1} \|\partial^2 K_t\|_{m-1, D_t; \kappa}\right) \\ &\leq \frac{C_m}{\varrho} \mathcal{B}_m \lambda^{m-1} (\lambda + \|\partial^2 K_t\|_{m-1, D_t; \kappa}), \end{aligned}$$

for $(I, t) \in B(I^0, r_t) \times [0, \delta]$ and $\beta \in \mathbb{N}^{n-1}$, $|\beta| = m \in \mathbb{N}_*$, where $C'_m > 0$ depends only on m and n and $C_m > 0$ depends only on m, n, τ and ϑ_0 . On the other hand, $\varphi_t = 0$ outside D_t^1 which completes the proof of the Lemma. This argument will be used many times in the sequel. \square

Proposition A.11 applied to $\text{id} + \kappa^{-1}\varphi_t \circ \sigma_\kappa$ implies that $\text{supp } \psi_t \subset \overline{D}_t^1$, $\|\psi_t\|_{C^0} \leq C_0\epsilon/\lambda$ and

$$\|\psi_t\|_{m,\kappa} \leq \frac{C_m}{\varrho} \mathcal{B}_m \lambda^{m-1} (\lambda + \|\partial^2 K_t\|_{m-1, D_t; \kappa})$$

for $m \in \mathbb{N}^*$. In particular, $\|d\psi_t\|_{C^0} \leq C_1\epsilon$. Consider

$$\omega_t(I) - \nabla K_t(I) = \int_0^1 d(\nabla K_t)(I + s\psi_t(I)) \psi_t(I) ds, \quad I \in D_t.$$

The support of $\omega_t - \nabla K_t$ is contained in \overline{D}_t^1 . Moreover, $\|\omega_t - \nabla K_t\|_{C^0(D_t)} \leq C_0\epsilon\kappa$, and using Remark A.2 and Proposition A.12 one obtains as in the proof of Lemma 9.16 the estimate

$$\begin{aligned} & \|(\kappa\partial)^\alpha(\omega_t - \nabla K_t)\|_{C^0} \leq C_m \|\partial^2 K_t\|_{C^0(D_t)} \|\psi_t\|_{m,\kappa} \\ & + C_m \|\psi_t\|_{C^0} (1 + \|d\psi_t\|_{C^0}^{m-1}) (\|\partial^2 K_t\|_{1, D_t; \kappa} \|d\psi_t\|_{m-1, \kappa} + \|\partial^2 K_t\|_{m, D_t; \kappa} \|d\psi_t\|_{C^0(D_t)}) \end{aligned}$$

for any $m \in \mathbb{N}_*$ and $\alpha \in \mathbb{N}^{n-1}$ with $|\alpha| = m$. Then using (9.187), (9.188) and the previous estimates we obtain (9.198). The estimate $\|d\omega_t - d\nabla K_t\|_{C^0(D_t)} \leq C\epsilon\lambda$ follows from (9.198) with $m = 1$ and (9.188). We are going to prove that for each $x \in \mathbb{R}^{n-1}$ the map $\Omega \ni \omega \rightarrow \nabla_x h_t(x, \omega)$ is a diffeomorphism. To this end we fix x and we write the map $\omega \rightarrow \nabla_x h_t(x, \omega)$ as follows

$$\nabla_x h_t = (\text{id} + \varphi_t^1) \circ \nabla K_t^*, \quad \varphi_t^1 := (\nabla_x h_t^0 + I_t - \nabla K_t^*) \circ \nabla K_t.$$

Then $\text{supp } \varphi_t^1 \subset \overline{D}_t^1$ and

$$\|\varphi_t^1\|_{C^1} \leq (\|\nabla_x h_t^0\|_{C^1} + \|I_t - \nabla K_t^*\|_{C^1}) (1 + \|\partial^2 K_t\|_{C^0}) \leq C_1\epsilon/\lambda$$

and we complete the proof of \mathcal{Q} as above. \square

Step 3. Construction of χ_t . The second statement of the Lemma implies that there is a C^∞ -foliation of $\mathbb{T}^{n-1} \times D_t$ by Lagrangian tori

$$\Lambda_t(\omega) = \{(p(x), \nabla_x h_t(x, \omega)) : x \in \mathbb{R}^{n-1}\}, \quad \omega \in \Omega,$$

which is a smooth extension of the family of the Kronecker invariant tori (9.193) of P_t . Notice that $I_t(\omega)$ is the action along the basis of cycles $[\gamma_{t,j}(\omega)], \dots, [\gamma_{t,n-1}(\omega)]$ of $H_1(\Lambda_t(\omega), \mathbb{R})$, where $\gamma_{t,j}(\omega) = \{(p(s2\pi e_j), \nabla_x h_t(s2\pi e_j, \omega)) : 0 \leq s \leq 1\}$. Indeed, it follows from the definition of h_t that $I_t(\omega) = \nabla_x h_t(x, \omega) - \nabla_x h_t^0(x, \omega)$, where h_t^0 is 2π -periodic in x and we obtain

$$I_t(\omega) = \left(\int_{\gamma_{t,1}(\omega)} \sigma, \dots, \int_{\gamma_{t,n-1}(\omega)} \sigma \right) \quad (9.199)$$

for $\omega \in \Omega$. Now we set $\Phi_t(x, I) = h_t(x, \omega_t(I))$. Then

$$\Phi_t^0(x, I) := \langle x, I \rangle - \Phi_t(x, I) = -h_t^0(x, \omega_t(I))$$

is 2π -periodic with respect to x , and it has a compact support in $\mathbb{T}^{n-1} \times D_t$. Moreover, it follows from Lemma 9.13, Lemma 9.15, Remark A.2 and Proposition A.9 that

$$\begin{aligned} \|\Phi_t^0\|_{m, \mathbb{A}_t; \kappa} &\leq C_m \left(1 + \|d\omega_t\|_{C^0(D_t)}^{m-1}\right) \left(\|h_t^0\|_{m, \kappa} \|d\omega_t\|_{C^0(D_t)}\right) \\ &+ \|h_t^0\|_{1, \kappa} \|\partial^2 K_t\|_{m-1, D_t; \kappa} + \|h_t^0\|_{1, \kappa} \|(d\omega_t - d\nabla K_t)\|_{m-1, D_t; \kappa} \\ &\leq \frac{C_m}{\varrho} \mathcal{B}_m \lambda^{2m-2} (\lambda + \|\partial^2 K_t\|_{m, D_t; \kappa}) \end{aligned}$$

for $t \in [0, \delta]$ and $m \in \mathbb{N}_*$, where C_m depends only on m, n, τ and ϑ_0 . This implies

$$\|\sigma_\kappa^{-1} \text{sgrad} \Phi_t^0\|_{m, \mathbb{A}_t; \kappa} \leq \frac{C_m}{\kappa \varrho} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa}) \quad (9.200)$$

for $t \in [0, \delta]$ and $m \in \mathbb{N}$, where C_m depends only on m, n, τ and ϑ_0 . In particular, one obtains by means of (9.187) and (9.188) that

$$\|\sigma_\kappa^{-1} \text{sgrad} \Phi_t^0\|_{1, \mathbb{A}_t; \kappa} \leq c\epsilon/\lambda \quad (9.201)$$

for $t \in [0, \delta]$, where $c = c(n, \tau, \vartheta_0) > 0$. Using Lemma A.14 we obtain

Lemma 9.17. *Choosing $\epsilon = \epsilon(n, \tau, \vartheta_0) > 0$ small enough we have the following*

1. Φ_t is a generating function of a symplectic transformation $\chi_t : \mathbb{T}^{n-1} \times D_t \rightarrow \mathbb{T}^{n-1} \times D_t$ and the map $[0, \delta] \ni t \rightarrow \chi_t \in C^\infty(\mathbb{A}_t, \mathbb{A}_t)$ is C^1 ;
2. $\chi_t(\Lambda_t(\omega)) = \mathbb{T}^{n-1} \times \{I_t(\omega)\}$ for any $\omega \in \Omega$ and $t \in [0, \delta]$;
3. $\chi_t - \text{id}$ and $\chi_t^{-1} - \text{id}$ are compactly supported in $\mathbb{T}^{n-1} \times \overline{D}_t^1$, where $D_t^1 = \nabla K_t^*(\Omega - \kappa/2)$ and they satisfy the estimates (9.189). Moreover,

$$\|\sigma_\kappa^{-1}(\chi - \text{id})\|_{1, \mathbb{A}_t; \kappa} + \|\sigma_\kappa^{-1}(\chi^{-1} - \text{id})\|_{1, \mathbb{A}_t; \kappa} \leq c\epsilon/\lambda.$$

Proof. Using Lemma A.14 one obtains a symplectic transformation $\chi_t : \mathbb{T}^{n-1} \times D_t \rightarrow \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}$ defined by

$$\chi_t(\nabla_I \Phi_t(\theta, I), I) = (\theta, \nabla_\theta \Phi_t(\theta, I)), \quad (\theta, I) \in \mathbb{T}^{n-1} \times \mathbb{R}^{n-1}.$$

Notice that the map

$$D_t \ni I \rightarrow \nabla_\theta \Phi_t(\theta, I) = I + \nabla_\theta h_t^0(\theta, \omega_t(I)) \in D_t$$

is a diffeomorphism since the map $\Omega \ni \omega \rightarrow I_t(\omega) + \nabla_\theta h_t^0(\theta, \omega) = \nabla_\theta h_t(\theta, \omega) \in D_t$ is a diffeomorphism in view of Lemma 9.15, 2, hence, $\chi_t(\mathbb{A}_t) = \mathbb{A}_t$. For any $\omega \in \Omega$ and any θ we have

$$(\theta, \nabla h_t(\theta, \omega)) = (\theta, \nabla_\theta \Phi_t(\theta, I_t(\omega))) = \chi_t(\nabla_I \Phi_t(\theta, I_t(\omega)), I_t(\omega)),$$

hence, $\Lambda_t(\omega) = \chi_t(\mathbb{T}^{n-1} \times \{I_t(\omega)\})$. Moreover, $\chi_t(\varphi, I_t(\omega)) = (\varphi, I_t(\omega)) = (\varphi, \nabla K_t^*(\omega))$ if $\text{dist}(\omega, \mathbb{R}^{n-1} \setminus \Omega) \leq \kappa/2$, hence, the support of both $\chi_t - \text{id}$ and $\chi_t^{-1} - \text{id}$ is contained in $\mathbb{T}^{n-1} \times \overline{D}_t^1$. The estimate (9.189) follows from (9.200) and Lemma A.14. \square

Step 4. Estimates. Consider the exact symplectic map $P_t^0 = \chi_t^{-1}P_t\chi_t$. Using Lemma A.16 we write P_t as a composition $P_t = W_tQ_t$, where W_t is the exact symplectic map defined by the generating function $(x, r) \rightarrow \langle x, r \rangle - G_t(x, r)$ and $Q_t(\theta, r) = (\theta + \nabla K(r), r)$. Then

$$P_t^0 = W_t^0Q_t,$$

where

$$W_t^0 = \chi_t^{-1}W_t + \chi_t^{-1}W_tQ_t(\chi_t^{-1} - \text{id})Q_t^{-1}.$$

Lemma 9.18. *The exact symplectic map W_t^0 , $t \in [0, \delta]$, admits a generating function of the form*

$$(x, I) \rightarrow \langle x, I \rangle - G_t^0(x, I)$$

such that the map $[0, \delta] \ni t \rightarrow G_t^0 \in C^\infty(\mathbb{A})$ is C^1 , $\text{supp}(dG_t^0) \subset \mathbb{T}^{n-1} \times \overline{D}_t^1$ and

$$\|\sigma_\kappa^{-1} \text{sgrad } G_t^0\|_{m, \mathbb{A}; \kappa} \leq \frac{C_m}{\rho\kappa} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa})$$

for any $m \in \mathbb{N}$, where C_m depends only on m , n , τ and ϑ_0 .

Proof. We have

$$\begin{aligned} W_t^0 - \text{id} &= (W_t - \text{id} + (\chi_t^{-1} - \text{id})W_t) + Q_t(\chi_t - \text{id})Q_t^{-1} \\ &+ (W_t - \text{id} + (\chi_t^{-1} - \text{id})W_t) \circ (Q_t(\chi_t - \text{id})Q_t^{-1}). \end{aligned} \tag{9.202}$$

We estimate the C^m norms of it term by term. Notice that the support of each term is contained in $\mathbb{T}^{n-1} \times \overline{D}_t^1$.

Lemma A.14 and (9.188) imply $\|\sigma_\kappa^{-1}(W_t - \text{id})\|_{1, \mathbb{A}; \kappa} < C\epsilon/\lambda$ and

$$\|\sigma_\kappa^{-1}(W_t - \text{id})\|_{m, \mathbb{A}; \kappa} \leq \frac{C_m}{\rho\kappa} \mathcal{B}_m.$$

The last estimate, Lemma 9.17, 3, and Lemma A.12, 2, imply

$$\|\sigma_\kappa^{-1}(\chi_t^{-1} - \text{id})W_t\|_{m, \mathbb{A}; \kappa} \leq \frac{C_m}{\rho\kappa} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa}).$$

and

$$\|\sigma_\kappa^{-1}(\chi_t^{-1} - \text{id})W_t\|_{1, \mathbb{A}; \kappa} \leq c\epsilon/\lambda.$$

Using the argument in the proof of Lemma 9.16 first to $(\chi_t - \text{id})Q_t^{-1}$ and then to $Q_t(\chi_t - \text{id})Q_t^{-1}$ we obtain the estimate

$$\|\sigma_\kappa^{-1}Q_t(\chi_t - \text{id})Q_t^{-1}\|_{m, \mathbb{A}; \kappa} \leq \frac{C_m}{\rho\kappa} \mathcal{B}_{m+1} \lambda^{2m} (\lambda + \|\partial^2 K_t\|_{m+1, D_t; \kappa}).$$

This yields the estimate of the first line of (9.202). To obtain the estimates for the composition in the second line one uses the preceding estimates and the argument of Lemma 9.16. It remains to apply Lemma A.14 in order to complete the proof of the Lemma. \square

Step 4. Proof of (i). Now Lemma A.16 implies that the function

$$\tilde{G}_t^0(x, I) = \langle x, I \rangle - K_t(I) - G_t^0(\varphi, I)$$

is a generating function of P_t^0 . The function \tilde{G}_t^0 is uniquely defined modulo a constant depending only on t which is chosen appropriately in order to obtain a C^1 -smooth the map $t \rightarrow \tilde{G}_t^0$. Set $L_t(I) := K_t(I) - G_t^0(0, I)$ and $R_t(\theta, I) = G_t^0(\theta, I) - G_t^0(0, I)$. We have

$$P_t^0(\mathbb{T}^{n-1} \times \{I_t(\omega)\}) = (\chi_t^{-1} \circ P_t)(\Lambda_t(\omega)) = \chi_t^{-1}(\Lambda_t(\omega)) = \mathbb{T}^{n-1} \times \{I_t(\omega)\}$$

for any $\omega \in \Omega_\kappa$, which implies

$$I_t(\omega) - \nabla_x R_t^0(x, I_t(\omega)) = \nabla_x \tilde{G}_t^0(x, I_t(\omega)) = I_t(\omega)$$

for any such $x \in \mathbb{R}^{n-1}$. On the other hand, $R_t^0(0, I) = 0$, hence, $R_t^0(\theta, I) = 0$ on $\mathbb{T}^{n-1} \times E_{t,\kappa}$, where $E_{t,\kappa} := I_t(\Omega_\kappa)$. Now Lemma 3.4 implies that $\partial_I^\beta R_t^0(\theta, I) = 0$ for any $(\theta, I) \in \mathbb{T}^{n-1} \times E_{t,\kappa}^0$, where $E_{t,\kappa}^0 := I_t(\Omega_\kappa^0)$ is the set of points of positive Lebesgue density in $E_{t,\kappa}$. Then

$$P_t^0(\theta + \nabla L_t(I), I) = (\theta, I) \quad \text{on} \quad \mathbb{T}^{n-1} \times E_{t,\kappa}^0,$$

and we obtain that $\nabla_I L_t(I_t(\omega)) = \omega$ for each $\omega \in \Omega_\kappa^0$. Hence $\nabla L_t^*(\omega) = I_t(\omega)$ for each $\omega \in \Omega_\kappa^0$, where $I_t(\omega)$ is given by (1.6) for such ω , according to (9.199). The estimates of the derivatives of L_t and R_t^0 follow from that of Lemma 9.18. \square

10 KAM theorem with parameters

The theorems formulated above follow from a KAM theorem with parameters. A complete and very comprehensive proof of it has been given by Pöschel [54] and Kuksin [41] in the analytic case. It can be extended to the case of smooth Hamiltonians using suitable approximation lemma. In the case of Gevrey Hamiltonians this has been done in [59]. The advantage of this approach is that frequencies are separated from action variables which makes it easier to obtain smoothness with respect to them. Moreover, it allows one to prove Hölder estimates of the transformations putting the Hamiltonian to a normal form. Here, the normal form of the Hamiltonian is $N(I; \omega) := \langle \omega, I \rangle$. The perturbation is a real valued function $(\theta, I; \omega, t) \mapsto P(\theta, I; \omega, t)$ defined in $\mathbb{A}^n \times \Omega \times [0, a]$, where $\mathbb{A}^n := \mathbb{T}^n \times B(0, \rho_0)$, $B(0, \rho_0) \subset \mathbb{R}^n$ is the ball centered at $I = 0$ with radius $\rho_0 \in (0, 1]$ and Ω is a bounded domain in \mathbb{R}^n . Hereafter, we assume that

$$P \in C^k([0, a]; C_0^\infty(\mathbb{A}^n \times \Omega)), \quad k \in \{0; 1\}, \quad (10.203)$$

i.e. the map $t \rightarrow P_t := P(\cdot, t) \in C_0^\infty(\mathbb{T}^n \times B(0, \rho_0) \times \Omega)$ is C^k -smooth on the interval $[0, a]$. This means that the support of the function $(I, \omega) \rightarrow P(\theta, I; \omega, t)$ is contained in a fixed compact subset of $B(0, \rho_0) \times \Omega$ independent of $(\theta, t) \in \mathbb{T}^n \times [0, a]$ and that the maps

$$t \rightarrow P_t := \partial_t^q P(\cdot, t) \in C^j(\mathbb{T}^n \times B(0, \rho_0) \times \Omega), \quad 0 \leq q \leq k,$$

are continuous in $t \in [0, a]$ for $j \in \mathbb{N}$. Given $\ell > 0$ and $0 < r, \kappa \leq 1$, $r \leq \rho_0$, we denote by $\|P_t\|_{\ell; r, \kappa}$ the weighted Hölder norm

$$\|P_t\|_{\ell; r, \kappa} := \|P_t \circ \sigma_{r, \kappa}\|_{C^\ell(\sigma_{r, \kappa}^{-1}(\mathbb{A}^n \times \Omega))} \quad (10.204)$$

where $\sigma_{r, \kappa}$ is the partial dilation $\sigma_{r, \kappa}(\varphi, I; \omega) := (\varphi, rI; \kappa\omega)$. The Hölder norms are defined in Section A.1 (see also [53]).

Fix $1 \leq \vartheta_1 < \vartheta_0 < \tau + 1$ and set

$$\ell_0 := 2\tau + 2 + \vartheta_0 \quad \text{and} \quad \ell(m) := 2m(\tau + 1) + \ell_0, \quad m \in \mathbb{N}. \quad (10.205)$$

Denote the Hamiltonian vector field associated to the Hamiltonian $(\theta, I) \rightarrow N(I; \omega) = \langle \omega, I \rangle$ by $\mathcal{L}_\omega := X_N(\cdot, \omega) = \langle \omega, \partial/\partial\theta \rangle$. We consider C^k -families, $k \in \{0, 1\}$, of Hamiltonians $t \rightarrow H_{t, \omega}$, $t \in [0, \delta]$, where

$$H_{t, \omega}(\theta, I) := H(\theta, I; \omega, t) = N(I; \omega) + P(\theta, I; \omega, t)$$

and P satisfies (10.203). Recall that for given $0 < \kappa \leq 1$ and $\tau > n - 1$, the set $\Omega_\kappa = \tilde{D}(\kappa, \tau) \cap \overline{\Omega - \kappa}$ consists of all (κ, τ) -Diophantine frequencies ω in Ω (ω satisfies (9.148)) such that the distance from ω to the complement of Ω in \mathbb{R}^n is greater or equal to κ . Set

$$\begin{aligned} \langle P \rangle_{\ell(m); r, \kappa}^{(0)} &= \sup_{t \in [0, a]} \|P_t\|_{\ell(m); r, \kappa} \\ \langle P \rangle_{\ell(m); r, \kappa}^{(1)} &= \frac{\langle P \rangle_{\ell(m); r, \kappa}^{(0)}}{\langle P \rangle_{\ell(0); r, \kappa}^{(0)}} \sum_{0 \leq p \leq 1} \sup_{t \in [0, a]} \|\partial_t^p P_t\|_{\ell(m); r, \kappa}. \end{aligned} \quad (10.206)$$

The following result is an analogue of Theorem A in [54].

Theorem 10.1. *There exists a positive constant $\epsilon = \epsilon(n, \tau, \vartheta_0, \vartheta_1) > 0$ depending only on n , τ , ϑ_0 and ϑ_1 such that, for any $a > 0$, $0 < \kappa < 1$, $0 < r < \rho_0$ and any real valued Hamiltonian $H = N + P$, where $N(I; \omega) = \langle \omega, I \rangle$ and P satisfies (10.203) and the smallness hypothesis*

$$\sup_{t \in [0, a]} \|P_t\|_{\ell_0; r, \kappa} \leq \epsilon \kappa r, \quad (10.207)$$

the following holds.

There exist C^k families of maps

$$[0, a] \ni t \mapsto \phi_t \in C^\infty(\Omega; \Omega) \quad \text{and} \quad [0, a] \ni t \mapsto \Psi_t = (U_t, V_t) \in C^\infty(\mathbb{T}^n \times \Omega; \mathbb{T}^n \times B(0, r))$$

such that $\text{supp}(\phi_t - \text{id}) \subset \Omega - \kappa/2$, $\text{supp}((U_t, V_t) - (\text{id}, 0)) \subset \mathbb{T}^n \times (\Omega - \kappa/2)$, and

- (i) For each $\omega \in \Omega_\kappa$ and $t \in [0, a]$ the map $\Psi_{t, \omega} := \Psi_t(\cdot, \omega) : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B(0, r)$ is a smooth embedding, $\Lambda_t(\omega) := \Psi_{t, \omega}(\mathbb{T}^n)$ is an embedded Lagrangian torus invariant with respect to the Hamiltonian flow of $H_{t, \phi_t(\omega)}(\theta, I) := H(\theta, I; \phi_t(\omega), t)$, and

$$X_{H_{t, \phi_t(\omega)}} \circ \Psi_{t, \omega} = D\Psi_{t, \omega} \cdot \mathcal{L}_\omega \quad \text{on } \mathbb{T}^n,$$

(ii) For any $m \geq 0$ there is $C_m > 0$ depending only on $n, \tau, \vartheta_0, \vartheta_1$, and m , such that for any $\alpha, \beta \in \mathbb{N}^n$ of length $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + \vartheta_1$ and $0 \leq q \leq k$ the following estimate holds

$$\begin{aligned} & \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta \partial_t^q (U_t(\theta; \omega) - \theta) \right| + r^{-1} \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta \partial_t^q V_t(\theta; \omega) \right| \\ & + \kappa^{-1} \left| (\kappa \partial_\omega)^\beta \partial_t^q (\phi_t(\omega) - \omega) \right| \leq C_m (\kappa r)^{-1} \langle P \rangle_{\ell(m); r, \kappa}^{(q)} \end{aligned} \quad (10.208)$$

uniformly in $(\theta, \omega, t) \in \mathbb{T}^n \times \Omega \times [0, a]$.

Remark 10.2. If P is analytic with respect to t in the disc $B(0, a) := \{t \in \mathbb{C} : |t| < a\}$ and (10.207) holds for $t \in B(0, a)$, then Ψ and ϕ can be chosen to be analytic with respect to t in $B(0, a)$. Moreover, for any $\alpha, \beta \in \mathbb{N}^n$ of length $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + \vartheta_1$ and $0 < \delta < a$, the following estimate holds

$$\begin{aligned} & \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta (U_t(\theta; \omega) - \theta) \right| + r^{-1} \left| \partial_\theta^\alpha (\kappa \partial_\omega)^\beta V_t(\theta; \omega) \right| + \kappa^{-1} \left| (\kappa \partial_\omega)^\beta (\phi_t(\omega) - \omega) \right| \\ & \leq C_{m, \delta} (\kappa r)^{-1} \sup_{t \in B(0, a)} \|\partial_t^p P_t\|_{\ell(m); r, \kappa} \end{aligned}$$

uniformly in $(\theta, \omega, t) \in \mathbb{T}^n \times \Omega \times B(0, a - \delta)$, with $C_{m, \delta} > 0$ depending only on $n, \tau, \vartheta_0, \vartheta_1, \delta, m$.

Before starting the proof of Theorem 10.1 and Remark 10.2 we are going to list several comments. For each $t \in [0, \delta]$ and $\omega \in \Omega$ denote by

$$\Phi_{t, \omega}^s := \exp(s X_{H_{t, \omega}}), \quad s \in \mathbb{R},$$

the flow of the Hamiltonian vector field $X_{H_{t, \omega}}$ of the Hamiltonian $H_{t, \omega}$ and set

$$g_\omega^s(\theta) = \theta + p(s\omega), \quad \theta \in \mathbb{T}^n, \quad s \in \mathbb{R}, \quad \omega \in \Omega,$$

where $p: \mathbb{R}^n \rightarrow \mathbb{T}^n$ is the canonical projection. By (10.207) and (10.208), we have

$$|d_\theta U_t(\theta; \omega) - \text{Id}| \leq C_1(n, \tau, \vartheta_0) \epsilon \leq 1/2$$

for $(\theta, \omega, t) \in \mathbb{T}^n \times \Omega \times [0, a]$, choosing ϵ sufficiently small and we obtain

Remark 10.3. The assertion (i) of Theorem 10.1 means that for each $\omega \in \Omega_\kappa$ the family $[0, \delta] \ni t \rightarrow \Lambda_t(\omega)$ is a C^k family of Kronecker invariant tori with respect to the flow $\Phi_{t, \tilde{\omega}}^s$, where $\tilde{\omega} = \phi_t(\omega)$. More precisely, for each $t \in [0, \delta]$, $\omega \in \Omega_\kappa$, and $s \in \mathbb{R}$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{g_\omega^s} & \mathbb{T}^n \\ \downarrow \Psi_{t, \omega} & & \downarrow \Psi_{t, \omega} \\ \Lambda_t(\omega) & \xrightarrow{\Phi_{t, \tilde{\omega}}^s} & \Lambda_t(\omega) \end{array}$$

Remark 10.4. -

1. The Theorem could be obtained for any $k \in \mathbb{N}$ (then C_m depends on k as well). We suppose here that $k \in \{0; 1\}$ to simplify the proof.

2. We point out that the parameter $\varepsilon > 0$ does not depend on the parameters κ and r , the domain Ω , the annulus $\mathbb{A}^n = \mathbb{T}^n \times B(0, \rho_0)$, nor on the interval $[0, a]$.

3. (ii) still holds if $P \in C^k([0, 1]; C_0^{\ell(M)}(\mathbb{A}^n \times \Omega))$ with $M \geq 0$ (see Theorem 11.22).

Remark 10.5. Without loss of generality one can assume that $\kappa = r = 1$. Indeed, consider the C^k -family of Hamiltonians

$$\tilde{H}_t = (\kappa r)^{-1}(N + P_t) \circ \sigma_{\kappa, r} = N + \tilde{P}_t,$$

where

$$\tilde{P}_t(\theta, I; \omega) = (\kappa r)^{-1}P_t(\theta, rI; \kappa\omega), \quad (\theta, I; \omega) \in \mathbb{T}^n \times \mathbb{R}^n \times (\kappa^{-1}\Omega).$$

If P_t satisfy (10.207), then so do \tilde{P}_t with $\kappa = r = 1$. Let $\tilde{\phi}_t$ and $\tilde{\Psi}_t = (\tilde{U}_t, \tilde{V}_t)$ be the family of maps obtained by Theorem 10.1 for the family of Hamiltonians \tilde{H}_t with $\kappa = \rho = 1$. Then taking

$$\phi_t := \kappa \tilde{\phi}_t \circ \sigma_{\kappa}^{-1}, \quad \Psi_t := (\tilde{U}_t, r\tilde{V}_t) \circ \sigma_{\kappa, r}^{-1}$$

we obtain items (i)-(iii) in Theorem 10.1 for H_t and for $0 < \kappa \leq 1, 0 < r \leq 1$.

In order to avoid the repeating use of the parameters κ and ρ , we suppose from now on that

$$\kappa = \rho = 1. \tag{10.209}$$

Idea of the Proof. The proof of Theorem 10.1 and Remark 10.2 is organized as follows. In Sect. 11.1 we prove the KAM Lemma and choose the parameters for the next iteration. The KAM Lemma is close to that of Pöschel in [54] but one needs additional arguments to estimate the derivatives with respect t . To this end we give a complete prove of it skipping some details. In Sect. 11.2 we iterate the KAM Step infinitely many times. The choice of the parameters leads to an exponentially converging scheme. Additional efforts are needed to get convergence for the derivatives with respect to t and to obtain the corresponding estimates in the Iterative Lemma. The iteration procedure is convergent in a Whitney sense only on the Cantor set Ω_{κ} and one can not hope to get the global (in Ω) estimates (10.208) using Whitney's extension theorem for C^{∞} jets. For this reason we propose a new method in Sect. 11.2.4. Using suitable almost analytic extensions in Gevrey classes, we prove a Modified Iterative Lemma which provides a convergent scheme over the whole domain Ω and yields the desired estimates. The almost analytic extensions is obtained in Sect. A.2.

11 Proof of Theorem 10.1

11.1 The KAM Step

11.1.1 The KAM Lemma

Given two domains $D_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, we denote by $\mathcal{A}(D_1, D_2)$ the space of analytic maps $f : D_1 \rightarrow D_2$ equipped by the inductive topology generated by sup-norms on compact sets of D_1 , and by $C^k([0, a], \mathcal{A}(D_1, D_2))$, $k \in \mathbb{N}$, the corresponding space of the C^k functions. If $D_2 = \mathbb{C}$ we write $\mathcal{A}(D_1) := \mathcal{A}(D_1, \mathbb{C})$. Recall that an analytic function $f \in \mathcal{A}(D_1)$ is said to be real analytic if $D_1 \cap \mathbb{R}^{n_1} \neq \emptyset$ and $f(D_1 \cap \mathbb{R}^{n_1}) \subset \mathbb{R}$. Introduce the complex domains

$$D_{s, r} = \{\theta \in \mathbb{C}^n / 2\pi\mathbb{Z}^n : |\operatorname{Im} \theta| < s\} \times \{I \in \mathbb{C}^n : |I| < r\},$$

$$O_h = \{\omega \in \mathbb{C}^n : |\omega - \Omega_1| < h\}.$$

Hereafter, $|v| = |(v_1, \dots, v_n)| = \sup_j |v_j|$ is the sup-norm of $v \in \mathbb{C}^n$. The sup-norm of functions in $\mathcal{V} := D_{s,r} \times O_h$ will be denoted by $|\cdot|_{s,r,h}$ and the corresponding space of analytic functions in \mathcal{V} by $\mathcal{A}(\mathcal{V})$. We state below a variant of the KAM Lemma following Pöschel [54]. It involves a small parameter $\varepsilon > 0$ and several parameters σ, s, r, η, K such that

$$0 < s, r < 1, \quad 0 < \eta < 1/8, \quad 0 < 5\sigma < s < 1, \quad K \geq 1, \quad (11.210)$$

as well as a positive $c_0 = c_0(n, \tau) \leq 1$ depending only on n and τ . We suppose that the following inequalities are satisfied

- (a) $\varepsilon \leq c_0 \eta r \sigma^{\tau+1}$,
- (b) $\varepsilon \leq c_0 h r$,
- (c) $h \leq \frac{1}{2K^{\tau+1}}$.

Moreover, we will require below the inequality

- (d) $2h \leq \sigma^{\tau+1}$

which follows from (c) provided that $K\sigma \geq 1$. Fix $k \in \{0; 1\}$.

Proposition 11.1 (KAM Step Lemma). *There is a positive $c_0 = c_0(n, \tau) < 1$ depending only on n and τ such that, for any $\sigma, s, h, r, \eta, K, a > 0$ and $\varepsilon > 0$ satisfying (11.210) and (a)-(c) and for every real valued Hamiltonian $H = N + P$, where*

$$N(I; \omega, t) = e(\omega, t) + \langle \omega, I \rangle \quad \text{and} \quad P \in C^k([0, a], \mathcal{A}(D_{s,r} \times O_h))$$

satisfies the estimate

$$\sup_{0 \leq p \leq k} \sup_{t \in [0, a]} |\partial_t^p P_t|_{s,r,h} \leq \varepsilon, \quad (11.211)$$

the following holds.

- (1) *There exists a C^k family of real analytic transformation $\mathcal{F} = (\Phi, \phi)$, where*

$$\Phi \in C^k([0, a], \mathcal{A}(D_{s-5\sigma, \eta r} \times O_{h/4}, D_{s,r})) \quad \text{and} \quad \phi \in C^k([0, a], \mathcal{A}(O_{h/4}, O_h))$$

such that $H \circ \mathcal{F} = N_+ + P_+$ with

$$N_+(I; \omega, t) = e_+(\omega, t) + \langle \omega, I \rangle \quad \text{and} \quad P_+ \in C^k([0, a], \mathcal{A}(D_{s-5\sigma, \eta r} \times O_{h/4}))$$

satisfying the estimate

$$|\partial_t^p P_+(\cdot, t)|_{s-5\sigma, \eta r, h/4} \leq C_0 \left(\frac{\varepsilon^2}{r\sigma^{\tau+1}} + (\eta^2 + \sigma^{-n} e^{-K\sigma}) \varepsilon \right) \quad (11.212)$$

for any $t \in [0, a]$ and $0 \leq p \leq k$, where $C_0 = C_0(n, \tau) > 0$ depends only on n and τ ;

(2) $\Phi(\theta, I; \omega, t) = (U(\theta; \omega, t), V(\theta, I; \omega, t))$, where V is affine linear with respect to I and the transformation $(\theta, I) \rightarrow \Phi(\theta, I; \omega, t)$ is canonical for each (ω, t) fixed. Moreover, for any $0 \leq p \leq k$, $\alpha, \beta \in \mathbb{N}^n$, and $|\gamma| \leq 1$ the maps Φ and ϕ satisfy the estimates

$$|W \partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta (\Phi(\theta, I; \omega, t) - (\theta, I))| \leq C_{\alpha, \beta} \frac{\varepsilon}{r \sigma^{\tau+1}},$$

$$|(h \partial_\omega)^\gamma \partial_t^p (\phi_t - id)| \leq C \frac{\varepsilon}{r},$$

uniformly on $D_{s-5\sigma, \eta r} \times O_h \times [0, a]$ and $O_{h/4} \times [0, a]$, respectively, where $W = \text{diag}(\sigma^{-1} \text{Id}, r^{-1} \text{Id})$, $C_{\alpha, \beta} > 0$ depends only on n, τ, α, β , and $C > 0$ depends only on n and τ .

Remark 11.2. Set $\bar{W} = \text{diag}(\sigma^{-1} \text{Id}, r^{-1} \text{Id}, h^{-1} \text{Id})$ and suppose that (d) holds, i.e. $2h \leq \sigma^{\tau+1}$. Then

$$|\bar{W} \partial_t^p (D\mathcal{F}(\cdot, t) - \text{Id}) \bar{W}^{-1}| \leq C_0 \frac{\varepsilon}{rh}$$

on $D_{s-5\sigma, \eta r} \times O_{h/4} \times [0, a]$, where $D\mathcal{F}(\cdot, t)$ stands for the Jacobian of $\mathcal{F}(\cdot, t)$. Moreover, (2) and the Cauchy estimate of the derivatives of \mathcal{F} with respect to ω yield for $0 \leq p \leq 1$ and any $\alpha, \beta, \gamma \in \mathbb{N}^n$ the estimate

$$|\bar{W} \partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta (h \partial_\omega)^\gamma (\mathcal{F}(\theta, I; \omega, t) - (\theta, I; \omega))| \leq C_{\alpha, \beta, \gamma} \frac{\varepsilon}{rh}$$

on $D_{s-5\sigma, \eta r} \times O_{h/6} \times [0, a]$, where $C_{\alpha, \beta, \gamma} > 0$ depends only on n, τ, α, β and γ .

Remark 11.3. If P is analytic with respect to t in the disc $B(0, a) \subset \mathbb{C}$ and (11.211) holds in $B(0, a)$ for $k = 0$, then Ψ is analytic with respect to t in $B(0, a)$ and items (1) and (2) hold for $t \in B(0, a)$ with $p = 0$.

Remark 11.4. Hereafter we use the Cauchy estimates for analytic functions in \mathbb{C}^n (see for example Theorem 2.2.7, [32] and Appendix A in [54]). More precisely, let D be a domain in \mathbb{C}^n and $D_r := \{z \in \mathbb{C}^n : |z - D| < r\}$ the corresponding polydisc. Then for any analytic function f in D_r with a bounded sup-norm $|f|_r := \sup_{z \in D_r} |f(z)|$ and any $0 \leq \rho < r$ and $\alpha \in \mathbb{N}^n$ one has

$$|\partial^\alpha f|_\rho \leq \alpha! (r - \rho)^{-|\alpha|-1} |f|_r. \quad (11.213)$$

We recall as well the standard estimates of the Fourier coefficients

$$f_k = \int_{\mathbb{T}^n} f(\theta) e^{-2\pi i \langle k, \theta \rangle} d\theta, \quad k \in \mathbb{Z}^n,$$

of an analytic function f in a strip $\mathbb{T}^n + s := \{\theta \in \mathbb{C}^n / \mathbb{Z}^n : |\text{Im } \theta| < s\}$, $s > 0$, with a bounded sup-norm $|f|_s$, namely,

$$|f_k| \leq e^{-|k|s} |f|_s, \quad (11.214)$$

where $|k| = \sum_{j=1}^n |k_j|$.

Proof of Proposition 11.1. For Hamiltonians independent of t the proposition is formulated and proved in [54]. It follows easily from [54] in the case $k = 0$. The proof of the corresponding estimates of $\partial_t P_t$ requires additional efforts. For this reason we give a complete proof in the case $k = 1$.

Step 1. Truncation. Consider the linear part of P with respect to I

$$Q(\theta, I; \omega, t) := P(\theta, 0; \omega, t) + \langle \nabla_I P(\theta, 0; \omega, t), I \rangle.$$

Given a positive integer K we denote by

$$R(\theta, I; \omega, t) := \sum_{|k| \leq K} R_k(I; \omega, t) e^{i\langle k, \theta \rangle}$$

the trigonometric polynomial of degree K in the Fourier series expansion of Q with respect to θ . By (11.211) and the Cauchy inequalities (11.213) one obtains the following estimates

$$|\partial_t^p Q|_{s,r} < C_0 \varepsilon, \quad |\partial_t^p (P - Q)|_{s, 2\eta r} < C_0 \eta^2 \varepsilon$$

for $0 \leq p \leq 1$ uniformly with respect to $(\omega, t) \in O_h \times [0, a]$ (recall that $0 < \eta < 1/8$). Moreover, estimating the Fourier coefficients of $\partial_t^p Q$ by (11.214) one obtains

$$|\partial_t^p (Q - R)|_{s-\sigma, r} < C_0 \sigma^{-n} e^{-K\sigma} \varepsilon \quad \text{and} \quad |\partial_t^p R|_{s-\sigma, r} < C_0 \varepsilon. \quad (11.215)$$

The Cauchy estimates imply

$$\begin{aligned} |\partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta R|_{s-2\sigma, r/2} &\leq C_{\alpha, \beta} \varepsilon \\ |\partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta (P - R)|_{s-2\sigma, 2\eta r} &\leq C_{\alpha, \beta} (\eta^2 + \sigma^{-n} e^{-K\sigma}) \varepsilon \end{aligned} \quad (11.216)$$

for $0 \leq p \leq 1$ uniformly in $(\omega, t) \in O_h \times [0, a]$. Hereafter $C_0 \geq 1$ stands for a constant depending only on n and τ and we denote by $C_{\alpha, \beta}$ a positive constant depending only on n, τ, α and β .

Step 2. Homological equation. The idea is to put $\partial_t^p (P - R)$ in the error term and to kill $\partial_t^p R$ by means of a canonical transformation Φ which is the time-one-map of a Hamiltonian vector field $X_F = (\nabla_I F, -\nabla_\theta F)$. More precisely, consider the Hamiltonian flow

$$(x, \theta, I) \rightarrow \exp(x X_F)(\theta, I) = (u(x, \theta, I), v(x, \theta, I))$$

and set

$$\Phi = (U, V) := \exp(X_F), \quad \text{where } U(\cdot) = u(1, \cdot) \text{ and } V(\cdot) = v(1, \cdot).$$

The corresponding Hamiltonian system is

$$\begin{cases} \frac{du}{dx} = \nabla_v F(u, v; \omega, t) \\ \frac{dv}{dx} = -\nabla_u F(u, v; \omega, t) \\ u(0) = \theta, \quad v(0) = I. \end{cases} \quad (11.217)$$

The Lie method is based on the identity

$$\frac{d}{dx} (f \circ \exp(x X_F)) = \{f, F\} \circ \exp(x X_F),$$

where $\{f, F\} = \langle \nabla_I f, \nabla_\theta F \rangle - \langle \nabla_\theta f, \nabla_I F \rangle$ is the Poisson bracket. Using Taylor's formula with respect to x at $x = 0$ and the above identity one gets

$$\begin{aligned} (N + R) \circ \Phi &= N \circ \exp(xX_F)|_{x=1} + R \circ \exp(xX_F)|_{x=1} \\ &= N + \{N, F\} + R + \int_0^1 \{(1-x)\{N, F\} + R, F\} \circ \exp(xX_F) dx. \end{aligned} \quad (11.218)$$

We are looking for a trigonometric polynomial F of degree K and for a function \hat{N} depending only on (I, ω, t) solving the homological equation

$$\{N, F\} + R = \hat{N}. \quad (11.219)$$

Recall that $N(I; \omega, t) = e(\omega, t) + \langle \omega, I \rangle$. Then (11.219) becomes $\mathcal{L}_\omega F = \hat{N} - R$, where $\mathcal{L}_\omega = \langle \omega, \partial/\partial\theta \rangle$. Take

$$\hat{N}(I; \omega, t) := R_0(I; \omega, t) = \int_{\mathbb{T}^n} R(\theta, I; \omega, t) d\theta,$$

which is affine linear in I . Then the zero order term of the trigonometric polynomial $\hat{N} - R$ is zero which is a necessary condition for solving the above equation. On the other hand, the Diophantine condition (9.148) with $\kappa = 1$ and (c) imply

$$|\langle \omega, k \rangle| \geq \frac{1}{2|k|^\tau} \quad \text{for all } \omega \in O_h \text{ and } 0 \neq |k| \leq K, \quad (11.220)$$

where $|k| = \sum_{j=1}^n |k_j|$. Denote by \mathcal{H}_K the space of trigonometric polynomials in $\theta \in \mathbb{T}^n$ of degree $\leq K$ with zero order terms equal to 0. This space is generated by the functions $\exp(i\langle k, \theta \rangle)$, where $k \in \mathbb{Z}^n$ and $0 < |k| = |k_1| + \dots + |k_n| \leq K$. It follows from (11.220) that the map $\mathcal{L}_\omega : \mathcal{H}_K \rightarrow \mathcal{H}_K$ is an automorphism. Denote by $\mathcal{L}_\omega^{-1} : \mathcal{H}_K \rightarrow \mathcal{H}_K$ the inverse map and set

$$F := \mathcal{L}_\omega^{-1}(R - \hat{N}). \quad (11.221)$$

The Fourier coefficients of F are $F_0 = 0$, $F_k = (i\langle \omega, k \rangle)^{-1} R_k$ for $0 < |k| \leq K$ and $F_k = 0$ for $|k| > K$. Hence, F is well defined, it solves (11.219) and is affine linear in I . Moreover, it is uniquely defined by

$$\int_{\mathbb{T}^n} F(\theta, I; \omega, t) d\theta = F_0(I; \omega, t) = 0. \quad (11.222)$$

Now (11.218) reads

$$(N + R) \circ \Phi = N + \hat{N} + \int_0^1 \{(1-x)\hat{N} + xR, F\} \circ \exp(xX_F) dx. \quad (11.223)$$

Moreover, (11.214), (11.215) and (11.220) imply

$$|\partial_t^p F|_{s-2\sigma, r} \leq C_0 \frac{|\partial_t^p R|_{s-\sigma, r}}{\sigma^\tau} < C_0 \frac{\varepsilon}{\sigma^\tau}. \quad (11.224)$$

Using the Cauchy estimates one gets

$$|\partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta F| \leq C_{\alpha, \beta} \frac{\varepsilon}{\sigma^\tau} \quad (11.225)$$

uniformly in $D_{s-3\sigma,r/2} \times O_h \times [0, a]$ and for any $0 \leq p \leq 1$ and $\alpha, \beta \in \mathbb{N}^n$, $|\beta| \leq 1$. By (11.214), (11.215) and Cauchy one has as well

$$|\partial_t^p \hat{N}|_r = |\partial_t^p R_0|_r \leq |\partial_t^p R|_{s-\sigma,r} \leq C_0 \varepsilon \quad \text{and} \quad |\partial_t^p (r \partial_I) \hat{N}|_{r/2} \leq C_0 \varepsilon \quad (11.226)$$

for $0 \leq p \leq 1$ uniformly with respect to $(\omega, t) \in O_h \times [0, a]$. The derivatives of F and \hat{N} with respect I of order bigger than one are all zeros since the functions are affine linear in I .

Step 3. Canonical transformation. The solution (u, v) of the Cauchy problem (11.217) are real analytic in (x, θ, I, ω) and C^1 in t . Consider the canonical transformation $\Phi = (U, V)$, where $U(\cdot) = u(1, \cdot)$ and $V(\cdot) = v(1, \cdot)$ are defined in Step 2. Since F is affine linear in v one observes that u is independent of I and v is affine linear in I . In particular, U is independent of I and V is affine linear as a function of I . Moreover, (11.225) and condition (a) imply for $p \in \{0; 1\}$ the inequality

$$|\partial_t^p \nabla_\theta F| \leq \eta r \leq r/8 \quad \text{and} \quad |\partial_t^p \nabla_I F| \leq \sigma \quad (11.227)$$

in $D_{s-3\sigma,r/2} \times O_h \times [0, a]$ choosing the constant $c_0 = c_0(n, \tau) < 1$ in (a) sufficiently small. Then

$$\exp(x X_H) : D_{s-4\sigma,r/4} \rightarrow D_{s-3\sigma,r/2} \quad (11.228)$$

for every $(\omega, t) \in O_h \times [0, a]$ and $0 \leq x \leq 1$. In particular, $\Phi(\cdot; \omega, t) = (U(\cdot; \omega, t), V(\cdot; \omega, t))$ is a well defined real analytic map

$$\Phi(\cdot; \omega, t) : D_{s-4\sigma,r/4} \rightarrow D_{s-3\sigma,r/2}$$

for every $(\omega, t) \in O_h \times [0, a]$ and we get $\Phi \in C^1([0, a], \mathcal{A}(D_{s-4\sigma,r/4} \times O_h, D_{s-3\sigma,r/2}))$. We are going to show that

$$\begin{cases} \frac{1}{\sigma} |\partial_t^p (u(x, \theta, I; \omega, t) - \theta)| \leq C_0 \frac{\varepsilon}{r \sigma^{\tau+1}} \\ \frac{1}{r} |\partial_t^p (v(x, \theta, I; \omega, t) - I)| \leq C_0 \frac{\varepsilon}{r \sigma^{\tau+1}} \end{cases} \quad (11.229)$$

in $[0, 1] \times D_{s-4\sigma,r/4} \times O_h \times [0, a]$. For $p = 0$ it follows directly from (11.217) and (11.225). Let $p = 1$. Set

$$\bar{u}(\theta, I; \omega, t) := \frac{1}{\sigma} \sup_{0 \leq x \leq 1} |\partial_t u(x, \theta, I; \omega, t)| \quad \text{and} \quad \bar{v}(\theta, I; \omega, t) := \frac{1}{r} \sup_{0 \leq x \leq 1} |\partial_t v(x, \theta, I; \omega, t)|.$$

Differentiating (11.217) with respect to t and using (11.225) one gets

$$\begin{aligned} \bar{v} &\leq \frac{1}{\sigma r} \sup_{0 \leq x \leq 1} |\partial_t (\sigma \nabla_\theta) F| + \frac{1}{\sigma r} \sup_{0 \leq x \leq 1} |(\sigma \partial_\theta) (\sigma \nabla_\theta) F| \bar{u} + \frac{1}{\sigma r} \sup_{0 \leq x \leq 1} |(r \partial_I) (\sigma \nabla_\theta F)| \bar{v} \\ &\leq C \left(\frac{\varepsilon}{r \sigma^{\tau+1}} + \frac{\varepsilon}{r \sigma^{\tau+1}} \bar{u} + \frac{\varepsilon}{r \sigma^{\tau+1}} \bar{v} \right) \end{aligned}$$

where $C = C(n, \tau) > 0$ and by (a) one obtains

$$\bar{v} \leq C \frac{\varepsilon}{r \sigma^{\tau+1}} + C c_0 (\bar{u} + \bar{v})$$

in $D_{s-4\sigma, r/4} \times O_h \times [0, a]$. Choosing $c_0 \leq (4C)^{-1}$ this gives

$$\bar{v} \leq C \frac{\varepsilon}{r\sigma^{\tau+1}} + \frac{1}{4}(\bar{u} + \bar{v}).$$

The same estimate holds for \bar{u} and we get (11.229). By Cauchy this implies

$$\begin{cases} \frac{1}{\sigma} |\partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta (u(x, \theta, I; \omega, t) - \theta)| \leq C_{\alpha, \beta} \frac{\varepsilon}{r\sigma^{\tau+1}} \\ \frac{1}{r} |\partial_t^p (\sigma \partial_\theta)^\alpha (r \partial_I)^\beta (v(x, \theta, I; \omega, t) - I)| \leq C_{\alpha, \beta} \frac{\varepsilon}{r\sigma^{\tau+1}} \end{cases} \quad (11.230)$$

in $[0, 1] \times D_{s-5\sigma, r/8} \times O_h \times [0, a]$. Since $\eta < 1/8$ this proves the estimates of Φ in statement (2) of the KAM step. By (11.227) we get

$$\begin{aligned} |U(\theta, I; \omega, t) - I| &\leq \sup_{0 \leq x \leq 1} |\nabla_I F(\cdot; \omega, t)|_{s-3\sigma, r/2} \leq \sigma, \\ |V(\theta, I; \omega, t) - I| &\leq \sup_{0 \leq x \leq 1} |\nabla_\theta F(\cdot; \omega, t)|_{s-3\sigma, r/2} \leq \eta r \end{aligned}$$

on $D_{s-5\sigma, \eta r} \times O_h \times [0, a]$. This implies that $\Phi(\cdot; \omega, t)$ maps $D_{s-5\sigma, \eta r}$ to $D_{s-4\sigma, 2\eta r}$, and that

$$\Phi \in C^1([0, a], \mathcal{A}(D_{s-5\sigma, \eta r} \times O_h, D_{s-4\sigma, 2\eta r})). \quad (11.231)$$

Step 4. New error term. The identity (11.223) yields

$$H \circ \Phi = (N + R) \circ \Phi + (P - R) \circ \Phi = N_+ + P_+ \quad (11.232)$$

where the Hamiltonian $N_+ = N + \hat{N}$ is independent of θ and affine linear in I and

$$P_+ = \int_0^1 \{(1-x)\hat{N} + xR, F\} \circ \exp(xX_F) dx + (P - R) \circ \Phi \quad (11.233)$$

is the new error term.

We are going to prove (11.212). In the case $p = 0$ it follows from the corresponding estimates in [54]. Take $p = 1$ and consider firstly

$$\partial_t((P - R) \circ \Phi) = (\partial_t(P - R)) \circ \Phi + (D(P - R) \circ \Phi) \partial_t \Phi,$$

where D stands for the differential with respect to (θ, I) . By (11.231) and (11.216) we have

$$|\partial_t^p (P - R) \circ \Phi|_{s-5\sigma, \eta r} \leq |\partial_t^p (P - R)|_{s-4\sigma, 2\eta r} \leq C_0(\eta^2 + \sigma^{-n} e^{-K\sigma})\varepsilon.$$

Moreover, (11.216) implies

$$|D(P - R)W^{-1}|_{s-4\sigma, 2\eta r} \leq C_0(\eta^2 + \sigma^{-n} e^{-K\sigma}),$$

while (11.230) gives $|W \partial_t \Phi| \leq C_0 \varepsilon / r \sigma^{\tau+1}$ on $D_{s-5\sigma, r/8}$, and we get

$$\begin{aligned} |(D(P - R) \circ \Phi) \partial_t \Phi|_{s-5\sigma, \eta r} &\leq |D(P - R)W^{-1}|_{s-4\sigma, r/2} |W \partial_t \Phi|_{s-5\sigma, r/8} \\ &\leq C_0(\eta^2 \varepsilon + \sigma^{-n} e^{-K\sigma} \varepsilon) \frac{\varepsilon}{r\sigma^{\tau+1}} \end{aligned}$$

uniformly with respect to $(\omega, t) \in O_h \times [0, a]$. To evaluate the derivative with respect to t of the first term in (11.233), we consider

$$G := \partial_t(\{R, F\} \circ \exp(xX_F)) = G_1 + G_2 + G_3,$$

where

$$G_1 := \{\partial_t R, F\} \circ \exp(xX_F), \quad G_2 := \{R, \partial_t F\} \circ \exp(xX_F)$$

and

$$G_3 := (D\{R, F\} \circ \exp(xX_F)) \cdot \partial_t \exp(xX_F).$$

Using (11.228) one obtains

$$|\{\partial_t^p R, \partial_t^q F\} \circ \exp(xX_F)|_{s-5\sigma, \eta r} \leq |\{\partial_t^p R, \partial_t^q F\}|_{s-3\sigma, r/2}$$

for $0 \leq p, q \leq 1$. Now (11.216) and (11.225) imply

$$\begin{aligned} |\{\partial_t^p R, \partial_t^q F\}|_{s-3\sigma, r/2} &\leq |\partial_t^p \nabla_I R| |\partial_t^q \nabla_\theta F| + |\partial_t^p \nabla_I F| |\partial_t^q \nabla_\theta R| \\ &\leq C_0 \left(\frac{\varepsilon}{r} \cdot \frac{\varepsilon}{\sigma^{\tau+1}} + \frac{\varepsilon}{r\sigma^\tau} \cdot \frac{\varepsilon}{\sigma} \right) = 2C_0 \frac{\varepsilon^2}{r\sigma^{\tau+1}} \end{aligned}$$

uniformly with respect to $(\omega, t) \in O_h \times [0, a]$, which gives the desired estimate for G_1 and G_2 . By the same argument one obtains

$$|(\sigma \partial_\theta)^\alpha (r \partial_I)^\beta \{R, F\}|_{s-3\sigma, r/2} \leq C_{\alpha, \beta} \left(\frac{\varepsilon}{r} \cdot \frac{\varepsilon}{\sigma^{\tau+1}} + \frac{\varepsilon}{r\sigma^\tau} \cdot \frac{\varepsilon}{\sigma} \right) = 2C_{\alpha, \beta} \frac{\varepsilon^2}{r\sigma^{\tau+1}}.$$

Using (11.230) and the preceding estimate one gets

$$\begin{aligned} |G_3|_{s-5\sigma, \eta r} &\leq |(\sigma D_\theta)\{R, F\}|_{s-3\sigma, r/2} |\sigma^{-1} \partial_t u|_{s-5\sigma, \eta r} \\ &\quad + |(r D_I)\{R, F\}|_{s-3\sigma, r/2} |r^{-1} \partial_t v|_{s-5\sigma, \eta r} \leq C_0 \frac{\varepsilon^2}{r\sigma^{\tau+1}} \end{aligned}$$

where D_θ and D_I are the partial differentials with respect to θ and I respectively. The function $\partial_t(\{\hat{N}, F\} \circ \exp(xX_F))$ can be evaluated in the same way using (11.226). This proves (11.212).

Step 5. Transforming the frequencies. Consider

$$N(I; \omega, t) + \hat{N}(I; \omega, t) = e_+(\omega, t) + \langle \omega + (\nabla_I \hat{N})(\omega, t), I \rangle = e_+(\omega, t) + \langle \omega + (\nabla_I R_0)(\omega, t), I \rangle.$$

Following Pöschel [54], Sec. 4, we obtain a real analytic inverse $\phi_t : O_{h/4} \rightarrow O_{h/2}$ of the map

$$\omega \rightarrow \omega_+ := \omega + (\nabla_I R_0)(\omega, t)$$

i.e.

$$\phi_t(\omega) + (\nabla_I R_0)(\phi_t(\omega), t) = \omega, \quad \omega \in O_{h/4}, \quad t \in [0, a]. \quad (11.234)$$

Moreover, the following estimate is true

$$|\phi_t - id| + h|D\phi_t - \text{Id}| \leq C \frac{\varepsilon}{r},$$

on $O_{h/4}$. We set $\phi(\cdot, t) = \phi_t$ and $N_+ = (N + \hat{N}) \circ \phi$.

We are going to estimate $\partial_t \phi$ on $O_{h/4}$. Using (11.226) and Cauchy we obtain the estimate

$$|(h\partial_\omega)^\gamma \partial_t^p \nabla_I R_0(\omega, t)| \leq C_\gamma \frac{\varepsilon}{r} \quad \text{in } (\omega, t) \in O_{h/4} \times [0, a] \quad (11.235)$$

for each $0 \leq p \leq 1$ and $\gamma \in \mathbb{N}^n$ (recall that R_0 is affine linear in I). In particular, using (b) we obtain

$$|D\nabla_I R_0(\omega, t)| \leq C(n, \tau) \frac{\varepsilon}{hr} \leq C(n, \tau) c_0(n, \tau) < \frac{1}{2} \quad \text{in } (\omega, t) \in O_{h/4} \times [0, a] \quad (11.236)$$

for c_0 small enough. Differentiating (11.234) we get

$$\partial_t \phi_t(\omega) + (D\nabla_I R_0)(\phi_t(\omega), t) \cdot \partial_t \phi_t(\omega) + (\nabla_I \partial_t R_0)(\phi_t(\omega), t) = 0 \quad (11.237)$$

Using (11.235) and (11.236) we get the estimate

$$|\partial_t \phi_t(\omega)| \leq C(n, \tau) \frac{\varepsilon}{r} \leq C_0(n, \tau) h \quad \text{in } (\omega, t) \in O_{h/4} \times [0, a]$$

Differentiating (11.237) with respect to ω we obtain

$$(h\partial_{\omega_j}) \partial_t \phi_t(\omega) + (D\nabla_I R_0)(\phi_t(\omega), t) \cdot (h\partial_{\omega_j}) \partial_t \phi_t(\omega) = Q_t(\omega)$$

where

$$\begin{aligned} Q_t = & - (hD)^2 (\nabla_I \partial_t^p R_0)(\phi_t, t) [\partial_{\omega_j} \phi_t, h^{-1} \partial_t \phi_t] - (h\partial_{\omega_j}) (hD) \nabla_I R_0(\phi_t, t) \cdot h^{-1} \partial_t \phi_t \\ & - (h\partial_{\omega_j}) (\nabla_I \partial_t R_0)(\phi_t, t) - hD (\nabla_I \partial_t R_0)(\phi_t, t) \cdot \partial_{\omega_j} \phi_t \end{aligned}$$

Hereafter, $D^2 f[\cdot, \cdot]$ stands for the quadratic form representing the second differential of f . Using (11.235), (11.236) and (b), we obtain

$$|(h\partial_{\omega_j}) \partial_t^p (\phi_t - id)| \leq C \frac{\varepsilon}{r} \quad \text{in } O_{h/4}.$$

This completes the proof of the KAM Step Lemma. \square

The analyticity with respect to t in Remark 11.3 follows from the theorem of Cauchy. \square

Does the transformation \mathcal{F} obtained by the KAM Step Lemma depend on the choice of the parameters K, σ, h, r, η and how? Following the construction of \mathcal{F} we obtain the following

Remark 11.5 (Uniqueness by construction in the KAM Lemma). *The transformation \mathcal{F} , the new normal form N_+ and the error term P_+ depend on the choice of K via the truncation in Step 1. If K is fixed, then they do not depend on the choice of the other parameters σ, h, r and η in the following sense. Let σ', h', r' and η' be another admissible choice of the parameters and \mathcal{F}', N'_+ and P'_+ , be the corresponding transformation, normal form and error term. Then $\mathcal{F}' = \mathcal{F}$, $N'_+ = N_+$ and $P'_+ = P_+$ on the intersection of their domains of definition.*

11.1.2 Preparing next iteration.

We are going to prepare the next iteration. Choose a “weighted error” E satisfying

$$0 < E \leq \eta^2 < 1/64 \quad (11.238)$$

fix $0 < \hat{\varepsilon} \leq 1$ and set

$$\varepsilon = \hat{\varepsilon} r \sigma^{\tau+1} E. \quad (11.239)$$

where $0 < \sigma < 1/5$. Define K and h by

$$K = \sigma^{-1} \ln^2(\sigma), \quad 2h = \frac{1}{K^{\tau+1}} = (\sigma / \ln^2(\sigma))^{\tau+1}. \quad (11.240)$$

Lemma 11.6. *There exists $E^0 = E^0(n, \tau) > 0$ depending only on n and τ such that the hypothesis (a)-(d) of Proposition 11.1 are satisfied for $0 < \sigma < 1/5$ and $0 < E \leq E^0(n, \tau)$ provided that*

$$2 \ln^{2\tau+2}(\sigma) E \leq c_0. \quad (11.241)$$

Proof. Firstly, a) follows from the definition of ε choosing $E \leq E^0 \leq c_0(n, \tau)^2$ and c) follows from the definition h , while (d) follows from the inequality $K\sigma = \ln^2(\sigma) > \ln^2(5) > 1$. The hypothesis (b) follows from the inequality

$$\frac{\varepsilon}{hr} \leq \frac{r\sigma^{\tau+1}E}{hr} = 2 \ln^{2\tau+2}(\sigma) E \leq c_0 \quad (11.242)$$

in view of (11.239) and (A.64), which yields (b) in Proposition 11.1. \square

We are going to fix η and determine the parameters s_+ , σ_+ , r_+ , η_+ , K_+ , h_+ , ε_+ , and the weighted error E_+ for the next iteration. Suppose that

$$\sigma^{-n} \exp(-K\sigma) = \sigma^{-n} \exp(-\ln^2(\sigma)) \leq \eta^2. \quad (11.243)$$

Then using (11.238) and (11.239), one obtains from (11.212) the following inequality

$$|\partial_t^p P_+|_{s-5\sigma, \eta r, h/4} \leq C_0 \hat{\varepsilon} r \sigma^{\tau+1} E (E + \eta^2 + \sigma^{-n} e^{-K\sigma}) < 3C_0 \hat{\varepsilon} \eta^2 r \sigma^{\tau+1} E := \frac{1}{2} \varepsilon_+, \quad (11.244)$$

where $0 \leq p \leq k$ and $C_0 = C_0(n, \tau) > 1$ depends only on n and τ . Set

$$r_+ = \eta r, \quad s_+ = s - 5\sigma, \quad \sigma_+ = \delta \sigma, \quad s = \frac{5}{1-\delta} \sigma, \quad (11.245)$$

where $0 < \delta < 1/6$ will be fixed below and put $\varepsilon_+ := \hat{\varepsilon} r_+ \sigma_+^{\tau+1} E_+$. Plugging the expression of ε_+ in (11.244) and using (11.245) we get

$$E_+ = (6C_0(n, \tau) \delta^{-\tau-1}) \eta E$$

which leads to an exponentially converging iteration scheme if $6C_0(n, \tau) \delta^{-\tau-1} \eta < 1$. Now we fix

$$0 < \vartheta < \min(\vartheta_0/4, 1), \quad \delta := (6C_0(n, \tau))^{-\frac{1}{\vartheta}}, \quad \eta := \delta^{\tau+1+\vartheta+\nu}, \quad (11.246)$$

where ν is a positive number which will be determined in Sect. 11.2 and $\vartheta_0 > 1$ is fixed in (10.205). In particular, (11.246) implies that

$$0 < \delta < 1/(6C_0) < 1/6 \quad \text{and} \quad \eta < \delta^2 < 1/6,$$

since $C_0 > 1$ and $\tau > n - 1 \geq 1$. Moreover,

$$E_+ = \delta^\nu E. \tag{11.247}$$

Set $\eta_+ = \eta\delta^{\nu_+ - \nu}$ with certain $\nu_+ \geq \nu$ which will be determined by the next iteration and put $h_+ = (1/2)K_+^{-\tau-1}$, where $K_+ = \sigma_+^{-1} \ln^2(\sigma_+)$. Notice that

$$s_+ = s - 5\sigma = \delta s, \quad s_+ - 5\sigma_+ = \delta(s - 5\sigma) > 0,$$

and one obtains that $\sigma_+, s_+, r_+, \eta_+$ and K_+ satisfy (11.210). Moreover,

$$\frac{h_+}{h} < \left(\frac{\sigma_+}{\sigma}\right)^{(\tau+1)} < \delta^{\tau+1} < \frac{1}{6}, \tag{11.248}$$

and (11.244) implies

$$|\partial_t^p P_+|_{s_+, r_+, h_+} \leq \frac{1}{2}\varepsilon_+ = \frac{1}{2}\hat{\varepsilon}r_+\sigma_+^{\tau+1}E_+ \tag{11.249}$$

for $0 \leq p \leq k$. We have prepared the next iteration.

11.2 Iteration

11.2.1 Choice of the small parameters.

As in [54] we are going to iterate the KAM step infinitely many times choosing appropriately the parameters $0 < s, r, \sigma, h, \eta < 1$ and so on. Our goal is to get a convergent scheme. We are going to define suitable strictly decreasing sequences of positive numbers $\{s_j\}_{j=0}^\infty$, $\{r_j\}_{j=0}^\infty$ and $\{h_j\}_{j=0}^\infty$, tending to 0. Set

$$s_j = s_0\delta^j, \quad \sigma_j = \sigma_0\delta^j, \quad s_0 = 5\sigma_0(1 - \delta)^{-1} \in (0, 1), \tag{11.250}$$

where $\delta = \delta(n, \tau, \vartheta) < 1/6$ is given by (11.246).

Given $m \geq 0$, we define an increasing sequence $\nu(m) := (\nu_j(m))_{j \in \mathbb{N}}$ as follows. We set

$$\nu_j(m) = \begin{cases} \vartheta_0 - \vartheta & \text{for } j < J(m) \\ m(\tau + 1) + \vartheta_0 - \vartheta & \text{for } j \geq J(m), \end{cases} \tag{11.251}$$

where $0 < \vartheta < \min(\vartheta_0/4, 1)$, and

$$J(m) \geq m(\tau + 1)\vartheta^{-1} \tag{11.252}$$

is an integer which will be determined in Sect. 11.2.5. If $m = 0$, we have $\nu_j(0) = \vartheta_0 - \vartheta$ for any $j \in \mathbb{N}$ and we set $J(0) = 0$.

Taking into account (11.245), (11.246) and (11.247), we define the sequences $\{r_j(m)\}_{j \in \mathbb{N}}$, $\{\eta_j(m)\}_{j \in \mathbb{N}}$ and $\{E_j(m)\}_{j \in \mathbb{N}}$ as follows. Fix

$$r_0 = s_0 < 1, \quad \eta_0 = \delta^{\tau+1+\vartheta+\nu_0} = \delta^{\tau+1+\vartheta_0},$$

and set for $j \geq 1$

$$\begin{cases} \eta_j = \eta_j(m) := \delta^{\nu_j - \nu_{j-1}} \eta_{j-1} = \delta^{\nu_j - \nu_0} \eta_0 = \delta^{\nu_j + \tau + 1 + \vartheta}, \\ r_j = r_j(m) := \eta_{j-1} r_{j-1} = \delta^{p_j} r_0, \quad p_j = j(\tau + 1 + \vartheta) + (\nu_0 + \cdots + \nu_{j-1}), \end{cases} \quad (11.253)$$

and

$$E_j = E_j(m) := \delta^{\nu_j - 1} E_{j-1} = \delta^{\nu_0 + \cdots + \nu_{j-1}} E_0. \quad (11.254)$$

Take the positive number $E_0 = E_0(n, \tau, \vartheta_0)$ sufficiently small so that

$$E_0 < \eta_0^2 = \delta^{2\tau + 2 + 2\vartheta_0}.$$

The inequality (11.252) implies that

$$2\nu_j - 2\nu_0 \leq \nu_0 + \cdots + \nu_{j-1}, \quad j \geq 1. \quad (11.255)$$

Indeed, if $m = 0$ then $\nu_j = \nu_0 = \vartheta_0 - \vartheta > 0$ for each j . Let $m \geq 1$. For $j < J(m)$ have $\nu_j = \nu_0 = \vartheta_0 - \vartheta > 0$ and for $j \geq J(m)$ we get

$$2\nu_j - 2\nu_0 = 2m(\tau + 1) \leq J(m)\vartheta \leq 2j\vartheta < j(\vartheta_0 - \vartheta) \leq \nu_0 + \cdots + \nu_{j-1}$$

which yields the inequality for any $m \in \mathbb{N}$ and $j \in \mathbb{N}$. Now, (11.253), (11.254) and (11.255) yield

$$0 < E_j < \eta_j^2 \leq \eta_0^2 < 1/64, \quad j \in \mathbb{N}. \quad (11.256)$$

Taking into account (11.239), we put

$$\varepsilon_j := \hat{\varepsilon} r_j \sigma_j^{\tau+1} E_j = \hat{\varepsilon} r_0 \sigma_0^{\tau+1} E_0 \delta^{q_j}, \quad (11.257)$$

where $q_0 = 0$ and $q_j = q_j(m)$ is given for $j \geq 1$ by

$$\begin{aligned} q_j &:= p_j + j(\tau + 1) + (\nu_0 + \cdots + \nu_{j-1}) \\ &= j(2\tau + 2 + \vartheta) + 2(\nu_0 + \cdots + \nu_{j-1}). \end{aligned} \quad (11.258)$$

The parameter $0 < \hat{\varepsilon} \leq 1$ will be chosen later. Finally, taking into account (A.64) we set

$$K_j = \sigma_j^{-1} \ln^2(\sigma_j) \quad \text{and} \quad 2h_j = K_j^{-\tau-1} = (\sigma_j / \ln^2(\sigma_j))^{\tau+1}, \quad j \in \mathbb{N}. \quad (11.259)$$

We have

$$s_{j+1} = s_j - 5\sigma_j, \quad \sigma_j = 5^{-1}(1 - \delta)s_j \quad \text{and} \quad h_{j+1}/h_j < \delta^{\tau+1} < 1/6 \quad (11.260)$$

in view of (11.248) and (11.250). Moreover, $s_{j+1} - 5s_{j+1} = \delta(s_j - 5s_j)$ and (11.210) holds for each $j \in \mathbb{N}$. We are going to show that (11.243) and hypothesis (a) - (d) of Proposition 11.1 are satisfied for any $j \in \mathbb{N}$.

Lemma 11.7. *There exist constants*

$$0 < \tilde{\sigma}_0 = \tilde{\sigma}_0(n, \tau, \vartheta_0, \vartheta) < (1 - \delta)/5, \quad 0 < \tilde{E}_0 = \tilde{E}_0(n, \tau, \vartheta_0, \vartheta) < 1/64,$$

depending only on n, τ, ϑ_0 and ϑ , such that (11.210), (11.243), (11.256), and the hypothesis (a)-(d) are satisfied for any $j \in \mathbb{N}$, provided that

$$0 < \sigma_0 \leq \tilde{\sigma}_0, \quad 0 < E_0 \leq \tilde{E}_0, \quad 2 \ln^{2\tau+2}(\sigma_0) E_0 \leq c_0.$$

Proof. We have already obtained (11.210) and (11.256) for $j \in \mathbb{N}$. Choosing $E_0 \leq \tilde{E}_0(n, \tau, \vartheta_0) \leq c_0^2$ we get (a) for any $j \in \mathbb{N}$, while (11.259) implies (c). Moreover, (d) holds since $K_j \sigma_j = \ln^2(\sigma_j) > 1$. On the other hand, (b) holds if E_j and σ_j verify (11.241). By (11.251) and since $\vartheta_0 > 4\vartheta$, we obtain

$$2 \ln^{2\tau+2}(\sigma_j) E_j = 2 \ln^{2\tau+2}(\sigma_0 \delta^j) \delta^{\nu_0 + \dots + \nu_{j-1}} E_0 < 2 \ln^{2\tau+2}(\sigma_0 \delta^j) \delta^{2j\vartheta} E_0 = f(\delta^j),$$

where the function

$$x \mapsto f(x) = 2 \ln^{2\tau+2}(\sigma_0 x) x^{2\vartheta} E_0$$

is increasing in the interval $(0, 1]$, provided that $0 < \sigma_0 \leq \tilde{\sigma}'_0 := \exp(-(\tau+1)\vartheta^{-1})$. Then we have

$$2 \ln^{2\tau+2}(\sigma_j) E_j < f(\delta^j) \leq f(1) = 2 \ln^{2\tau+2}(\sigma_0) E_0 \leq c_0$$

for $0 < \sigma_0 \leq \tilde{\sigma}'_0$.

We are going to prove (11.243). For $j = 0$ this means that

$$\sigma_0^{-n} \exp(-\ln^2(\sigma_0)) \delta^{-2\tau-2-2\vartheta_0} \leq 1.$$

The function $x \mapsto x^{-n} \exp(-\ln^2 x)$ is increasing in the interval $(0, e^{-\sqrt{n}}]$ and δ depends only on n, τ and ϑ , hence, there exists a positive constant $\tilde{\sigma}''_0 = \tilde{\sigma}''_0(n, \tau, \vartheta) \leq e^{-\sqrt{n}}$ such that the inequality is satisfied for any $0 < \sigma_0 \leq \tilde{\sigma}''_0$.

Suppose now that $j \geq 1$. Notice that

$$\nu_j(m) \leq j\vartheta + \vartheta_0 - \vartheta$$

in view of (11.251) and (11.252), hence,

$$\eta_j \geq \delta^{j\vartheta + \tau + 1 + \vartheta_0} = \delta^{j\vartheta} \eta_0.$$

This implies

$$\sigma_j^{-n} \exp(-\ln^2(\sigma_j)) \eta_j^{-2} \leq \sigma_0^{-n} \delta^{-jn} \exp(-\ln^2(\sigma_0 \delta^j)) \delta^{-2j\vartheta} \eta_0^2 := g(\delta_j^{-n}).$$

The function

$$x \mapsto g(x) := x^{-n-2\vartheta} \exp(-\ln^2(\sigma_0 x)) \sigma_0^{-n} \eta_0^2$$

is increasing in the interval $0 < x \leq 1$ for $0 < \sigma_0 \leq \tilde{\sigma}'''_0 := e^{-\sqrt{n+2\vartheta}}$ and we get

$$\sigma_j^{-n} \exp(-\ln^2(\sigma_j)) \eta_j^{-2} \leq g(\delta_j^{-n}) \leq g(1) = \sigma_0^{-n} \exp(-\ln^2(\sigma_0)) \eta_0^{-2} \leq 1.$$

for $0 < \sigma_0 \leq \tilde{\sigma}'''_0(n, \tau, \vartheta)$. This yields (11.243) for any $j \in \mathbb{N}$. □

We fix $0 < \sigma_0 < (1 - \delta)/5$ ones forever by

$$0 < \sigma_0 = \sigma_0(n, \tau, \vartheta_0, \vartheta) := \min\left(\frac{1}{37}, \tilde{\sigma}'_0, \tilde{\sigma}''_0, \tilde{\sigma}'''_0\right), \quad (11.261)$$

and then choose \tilde{E}_0 in Lemma 11.7 such that $2\tilde{E}_0 \leq c_0 \ln^{-2\tau-2}(\sigma_0)$. Then (b) holds for any $0 < E_0 \leq \tilde{E}_0$ and $j \in \mathbb{N}$. The choice of σ_0 is motivated by the previous Lemma and by (11.264).

Using the proof of (b) in Lemma 11.7 we obtain the inequality

$$\frac{\varepsilon_j}{r_j h_j} \leq 2 \ln^{2\tau+2}(\sigma_j) E_j \leq 3\sigma_0^{-3\vartheta} \ln^{2\tau+2}(\sigma_j) \sigma_j^{3\vartheta} E_0 \leq C(n, \tau, \vartheta_0, \vartheta) \sigma_j^{2\vartheta} E_0, \quad (11.262)$$

since $\vartheta < \vartheta_0/4$.

Remark 11.8. The sequences of $\eta_j = \eta_j(m)$, $r_j = r_j(m)$ and weighted errors $E_j = E_j(m)$ depend on the choice of $m \in \mathbb{N}$, but σ_j , h_j and K_j do not depend on m .

11.2.2 Analytic smoothing of P_t .

The Hamiltonian P_t is not analytic and one can not apply directly the KAM step to it. We are going to approximate it by real analytic functions. To this end we recall some facts about the analytic smoothing technique in Section A.1. We are going to apply the Approximation Lemmas A.1 to the real valued Hamiltonian $P \in C^k([0, a]; C_0^L(\mathbb{A}^n \times \Omega))$, where $a > 0$ and $0 < L \leq \infty$. Set

$$u_j = u_0 \delta^j, \quad j \in \mathbb{N}, \quad (11.263)$$

where

$$0 < u_0 = 6s_0 = 30(1 - \delta)^{-1} \sigma_0 \leq 36\sigma_0 < 1, \quad (11.264)$$

the small parameter $0 < \delta = \delta(n, \tau, \vartheta) < 1/6$ is given by (11.246) and σ_0 is fixed in (11.261). Let us denote by \mathcal{U}_j the complex strips in $\mathbb{C}^n/2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^n$ consisting of all $(\theta, I; \omega)$ such that

$$|\operatorname{Im} \theta|, |\operatorname{Im} I|, |\operatorname{Im} \omega| < u_j, \quad (11.265)$$

and by $A(\mathcal{U}_j)$ the set of all real-analytic bounded functions in \mathcal{U}_j equipped with the sup-norm $|\cdot|_{u_j}$. Define

$$P_t^j := S_{u_j} P_t, \quad j \in \mathbb{N}, \quad (11.266)$$

by means of the Approximation Lemma A.1. This is a C^k family with respect to $t \in [0, a]$ of real analytic in $\mathbb{C}^n/2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^n$ functions. In view of (A.7), for each finite $\ell \leq L$ and $0 \leq \ell' \leq \ell$, the following inequality is true

$$\|P_t^j - P\|_{\ell'} \leq C(n, \ell) u_j^{\ell - \ell'} \|P\|_{\ell} \quad (11.267)$$

in the corresponding Hölder norms on $\mathbb{T}^n \times \mathbb{R}^n \times \Omega$. On the other hand, the inequality (A.6) with $\rho = u_j$ and $\tilde{\rho} = u_{j-1} = \delta^{-1} u_j$, yields the estimate

$$\left| \partial_t^p (P_t^j - P_t^{j-1}) \right|_{u_j} \leq C_0 u_{j-1}^{\ell} \|\partial_t^p P_t\|_{\ell} = C u_j^{\ell} \|\partial_t^p P_t\|_{\ell}$$

for each finite ℓ , $0 \leq \ell \leq L$ and $0 \leq p \leq k$, where $C = C(\ell, n, \tau, \vartheta_0) = C_0(n, \ell) \delta^{-\ell}$ is a positive constant depending only on $\ell, n, \tau, \vartheta_0$. Moreover,

$$\left| \partial_t^p P_t^0 \right|_{u_0} \leq C_0 \|\partial_t^p P_t\|_0 \leq \tilde{C} u_0^{\ell} \|\partial_t^p P_t\|_{\ell}$$

where $\tilde{C} = C_0(n) u_0^{-\ell}$. The positive constant $C_{\ell} := \max(c, \tilde{c}) = C_{\ell}(n, \tau, \vartheta_0)$ depends only on ℓ, n, τ and ϑ_0 . Hence,

$$\left| \partial_t^p P_t^0 \right|_{u_0} \leq \tilde{\varepsilon}_{\ell, 0, p} \quad \text{and} \quad \left| \partial_t^p P_t^j - \partial_t^p P_t^{j-1} \right|_{u_j} \leq \tilde{\varepsilon}_{\ell, j, p} \quad \text{for } j \geq 1 \quad (11.268)$$

for any finite ℓ , $0 \leq \ell \leq L \leq \infty$, where

$$\tilde{\varepsilon}_{\ell, j, k} := C_{\ell} u_j^{\ell} \sum_{p=0}^k \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_{\ell} \quad (11.269)$$

and $C_\ell = C(\ell, n, \tau, \vartheta_0) > 0$ depends only on ℓ, n, τ and ϑ_0 .

We would like to deal with P^j at the j -th iteration putting $P^j - P^{j-1}$ in the error term. To this end we need for $0 \leq p \leq k$ the following inequalities

$$|\partial_t^p P_t^0|_{u_0} \leq \frac{\varepsilon_1}{2} \quad \text{and} \quad \left| \partial_t^p P_t^j - \partial_t^p P_t^{j-1} \right|_{u_j} \leq \frac{\varepsilon_{j+1}}{4} \quad \text{for } j \geq 1. \quad (11.270)$$

These inequalities will be obtained in Sect. 11.2.5, choosing appropriately the sequence ν and the small constants ϵ and $\hat{\varepsilon}$.

Using the notations introduced in the beginning of Sect. 11.1.1 we set

$$D_j := D_{s_j, r_j}, \quad O_j := O_{h_j}, \quad \mathcal{V}_j := D_j \times O_j. \quad (11.271)$$

Moreover, given an integer $1 \leq q \leq 3$ we set

$$D_j^q := D_{\frac{q}{4}s_j, \frac{q}{4}r_j}, \quad O_j^q := O_{\frac{q}{4}h_j} \quad \text{and} \quad \mathcal{V}_j^q := D_j^q \times O_j^q. \quad (11.272)$$

We have

$$D_{j+1} \times O_{j+1} \subset D_j^1 \times O_j^1$$

since $\sup \{s_{j+1}/s_j, r_{j+1}/r_j, h_{j+1}/h_j\} \leq \delta < 1/6$.

11.2.3 Iterative Lemma.

We are ready to make the iterations. Consider the real analytic in \mathcal{U}_j Hamiltonian

$$H_t^j(\varphi, I; \omega) = H^j(\varphi, I; \omega, t) := N_0(I; \omega) + P_t^j(\varphi, I; \omega),$$

where $N_0(I; \omega) := \langle \omega, I \rangle$ and \mathcal{U}_j is defined by (11.265). Let us denote by \mathcal{U}_j^0 the subset of $\mathbb{C}^n / 2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^n$ consisting of all $(\theta, I; \omega)$ such that

$$|\operatorname{Im} \theta|, \quad |\operatorname{Im} I|, \quad |\operatorname{Im} \omega| < \frac{1}{2}u_j.$$

We have $2s_j < u_j$, which yields $D_j \times O_j \subset \mathcal{U}_j^0$. Using the notations introduced in (11.272) we obtain

$$D_{j+1} \times O_{j+1} \subset D_j^2 \times O_j^2 \subset D_j \times O_j \subset \mathcal{U}_j^0 \subset \mathcal{U}_j \quad (11.273)$$

since $\sup \{s_{j+1}/s_j, r_{j+1}/r_j, h_{j+1}/h_j\} \leq \delta < 1/6$. For any $j \in \mathbb{N}$, let us denote by \mathcal{D}_j the class of real-analytic diffeomorphisms

$$\mathcal{F}_j : D_{j+1} \times O_{j+1} \rightarrow D_j^2 \times O_j^2$$

of the form

$$\mathcal{F}_j(\theta, I; \omega) = (\Phi_j(\theta, I; \omega), \phi_j(\omega)), \quad \Phi_j(\theta, I; \omega) = (U_j(\theta; \omega), V_j(\theta, I; \omega)), \quad (11.274)$$

where $V_j(\theta, I; \omega)$ is affine linear with respect to I , and $(\theta, I) \rightarrow \Phi_j(\theta, I; \omega)$ is a canonical transformation for any fixed ω . To simplify the notations we denote the sup-norm of functions $f : D_j \times O_j \rightarrow \mathbb{C}$ by $|f|_j = |f|_{s_j, r_j, h_j}$. Fix $k \in \{0; 1\}$.

Proposition 11.9 (Iterative Lemma). *Let $P^j \in C^k([0, a], \mathcal{A}(\mathcal{U}_j))$, $j \in \mathbb{N}$, be a C^k family of real analytic Hamiltonians in \mathcal{U}_j satisfying (11.270) and $H^j = N_0 + P^j$. Then for each $j \in \mathbb{N}$ there is a normal form $N_j(I; \omega, t) = e_j(\omega, t) + \langle \omega, I \rangle$ and a C^k family of real analytic transformations*

$$\mathcal{F}^j \in C^k([0, a], \mathcal{A}(D_j \times O_j, (D_0 \times O_0) \cap \mathcal{U}_j^0)), \quad \mathcal{F}_t^j = \mathcal{F}^j(\cdot, t), \quad (11.275)$$

such that

1. $\mathcal{F}^0 = \text{id}$ and $\mathcal{F}_t^{j+1} = \mathcal{F}_{t,0} \circ \cdots \circ \mathcal{F}_{t,j}$, for $j \geq 0$, where

$$\mathcal{F}_j \in C^k([0, a], \mathcal{A}(D_{j+1} \times O_{j+1}, D_j^2 \times O_j^2)) \quad \text{and} \quad \mathcal{F}_{t,j}(\cdot, t) := \mathcal{F}_j(\cdot, t) \in \mathcal{D}_j; \quad (11.276)$$

2. $H^j \circ \mathcal{F}^{j+1} = N_{j+1} + R_{j+1}$ and $|\partial_t^p R_{j+1}|_{j+1} \leq \varepsilon_{j+1}/2$ for $0 \leq p \leq k$;

3. The following estimates hold

$$|\overline{W}_j \partial_t^p (\mathcal{F}_{t,j} - \text{id})|_{j+1} + |\overline{W}_j \partial_t^p (D\mathcal{F}_{t,j} - \text{Id}) \overline{W}_j^{-1}|_{j+1} < \frac{C_0 \varepsilon_j}{r_j h_j}, \quad (11.277)$$

$$|\partial_t^p (\mathcal{F}_t^{j+1} - \mathcal{F}_t^j)|_{j+1} < \frac{C_0 \varepsilon_j}{r_j h_j}, \quad (11.278)$$

for $0 \leq p \leq k$ and uniformly with respect to $t \in [0, a]$, where $C_0 = C_0(n, \tau, \vartheta_0, \vartheta) > 0$, $D\mathcal{F}_t^j$ stands for the Jacobian of \mathcal{F}_t^j with respect to $(\theta, I; \omega)$, and $\overline{W}_j = \text{diag}(\sigma_j^{-1} \text{Id}, r_j^{-1} \text{Id}, h_j^{-1} \text{Id})$.

Proof. For $k = 0$ the proof is similar to that of the Iterative Lemma in [54] and it is done in [59] in the case of Gevrey Hamiltonians independent of t . Additional efforts are required for the proof of the estimates (11.277) and (11.278) in the case when $p = k = 1$.

Consider firstly the Hamiltonian $H^0 = N_0 + P^0$. It satisfies the hypothesis of Proposition 11.1 in $D_0 \times O_0$ for $t \in [0, a]$ in view of (11.270) and Lemma 11.7. Hence, applying the KAM Step Lemma to the Hamiltonian H^0 we find $\mathcal{F}^1 = \mathcal{F}_0$ such that $H^0 \circ \mathcal{F}^1 = N_1 + R_1$, where $R_1(\cdot, t)$ is real analytic in $D_1 \times O_1$ and $|\partial_t^p R_1(\cdot, t)|_1 \leq \varepsilon_1/2$. Moreover, (11.277) holds for $j = 0$.

Given $j \geq 1$ we suppose that the Proposition holds for all indexes $0 \leq l \leq j - 1$. We are going to prove it for $l = j$. We are looking for a transformation $\mathcal{F}^{j+1} = \mathcal{F}^j \circ \mathcal{F}_j$, where \mathcal{F}_j belongs to \mathcal{D}_j . By the inductive assumption we have

$$H^{j-1} \circ \mathcal{F}^j = N_j + R_j,$$

where $N_j(I; \omega, t) = e_j(\omega, t) + \langle \omega, I \rangle$, $R_j(\cdot, t)$ is real analytic in $D_j \times O_j$, and $|\partial_t^p R_j(\cdot, t)|_j \leq \varepsilon_j/2$. Then we write

$$\begin{aligned} H^j \circ \mathcal{F}^{j+1} &= (N_0 + P^{j-1}) \circ \mathcal{F}^{j+1} + (P^j - P^{j-1}) \circ \mathcal{F}^{j+1} \\ &= (H^{j-1} \circ \mathcal{F}^j) \circ \mathcal{F}_j + (P^j - P^{j-1}) \circ \mathcal{F}^{j+1} \\ &= (N_j + R_j + (P^j - P^{j-1}) \circ \mathcal{F}^j) \circ \mathcal{F}_j. \end{aligned}$$

Consider the Hamiltonian $\tilde{H}_j = N_j + R_j + (P^j - P^{j-1}) \circ \mathcal{F}^j$ in $D_j \times O_j$ for $t \in [0, a]$ and set $R_j^1 = (P^j - P^{j-1}) \circ \mathcal{F}^j$. Using (11.270) we get

$$|(P_t^j - P_t^{j-1}) \circ \mathcal{F}^j(\cdot, t)|_j \leq |P_t^j - P_t^{j-1}|_{\mathcal{U}_j^0} \leq \frac{\varepsilon_j}{4}.$$

On the other hand, by the inductive assumptions (11.275) we obtain

$$|\partial_t((P_t^j - P_t^{j-1}) \circ \mathcal{F}^j(\cdot, t))|_j \leq |\partial_t P_t^j - \partial_t P_t^{j-1}|_{\mathcal{U}_j^0} + |(D(P_j - P_t^{j-1}) \circ \mathcal{F}^j(\cdot, t)) \cdot \partial_t \mathcal{F}^j(\cdot, t)|_j.$$

The first term of the right hand side is estimated by $\frac{\varepsilon_j}{4}$ in view of (11.270). Using (11.275) we estimate the second one by

$$|D(P_t^j - P_t^{j-1})W_j^{-1}|_{\mathcal{U}_j^0} |W_j \partial_t \mathcal{F}^j(\cdot, t)|_j$$

(here we consider \overline{W}_j as a linear operator acting on \mathbb{C}^{3n}). Now Cauchy estimates (see Remark 11.4) and (11.270) yield

$$\begin{aligned} |D(P_t^j - P_t^{j-1})W_j^{-1}|_{\mathcal{U}_j^0} &= |(s_j \nabla_\theta, r_j \nabla_I, h_j \nabla_\omega)(P_t^j - P_t^{j-1})|_{\mathcal{U}_j^0} \\ &\leq \sup\{s_j, r_j, h_j\} \frac{2}{u_j} |P_t^j - P_t^{j-1}|_{\mathcal{U}_j} \leq 2 \times \frac{1}{6} \times \frac{\varepsilon_j}{4} < \frac{\varepsilon_j}{4}. \end{aligned} \quad (11.279)$$

Moreover, (11.277) and (11.262) imply that $|W_j \partial_t \mathcal{F}^j(\cdot, t)|_j \leq 1$ for $E_0 = E_0(n, \tau, \vartheta_0, \vartheta) > 0$ small enough. Finally, we obtain

$$|\partial_t^p R_j + \partial_t^p R_j^0|_j < \varepsilon_j, \quad p \in \{0; 1\}.$$

We apply the KAM Step Lemma - Proposition 11.1 - to the C^k family of Hamiltonians \tilde{H}_j . Using Remark 11.4, (11.249) and (11.262) as well, we find a C^k family of real-analytic maps

$$\mathcal{F}_j(\cdot, t) : D_{j+1} \times O_{j+1} \rightarrow D_j^2 \times O_j^2$$

which belong to the class \mathcal{D}_j , satisfy (11.277) and such that $(N_j + R_j) \circ \mathcal{F}_j = N_{j+1} + R_{j+1}$, where

$$|\partial_t^p R_{j+1}|_{j+1} \leq \frac{1}{2} \hat{\varepsilon}_j r_{j+1} \sigma_{j+1}^{\tau+1} E_{j+1} \leq \frac{1}{2} \hat{\varepsilon}_{j+1} r_{j+1} \sigma_{j+1}^{\tau+1} E_{j+1} = \frac{\varepsilon_{j+1}}{2}.$$

We are going to show that

$$\mathcal{F}^{j+1} : D_{j+1} \times O_{j+1} \longrightarrow \mathcal{U}_j^0. \quad (11.280)$$

To prove (11.280) we estimate the norm of the linear operator $W_q \overline{W}_{q+1}^{-1}$. We have

$$|\overline{W}_q \overline{W}_{q+1}^{-1}| = \sup\{s_{q+1}/s_q, r_{q+1}/r_q, h_{q+1}/h_q\} = s_{q+1}/s_q = \delta,$$

since $r_{q+1}/r_q \leq \delta^{\tau+1} < \delta$ and $h_{q+1}/h_q \leq \delta^{\tau+1} < \delta$ for any $q \in \mathbb{N}$ by (11.253) and (11.260). Recall that δ and E_0 depend only on n, τ, ϑ_0 and ϑ . Then using (11.262) and the inductive assumption (11.277), we estimate the Jacobian of \mathcal{F}^{j+1} in $D_{j+1} \times O_{j+1}$ as follows (see also [54])

$$\begin{aligned} |\overline{W}_0 D \mathcal{F}^{j+1} \overline{W}_j^{-1}|_{j+1} &= |\overline{W}_0 D(\mathcal{F}_0 \circ \dots \circ \mathcal{F}_j) \overline{W}_j^{-1}|_{j+1} \\ &\leq \prod_{q=0}^{j-1} \left(|\overline{W}_q D \mathcal{F}_q \overline{W}_q^{-1}|_{q+1} |\overline{W}_q \overline{W}_{q+1}^{-1}| \right) |\overline{W}_j D \mathcal{F}_j \overline{W}_j^{-1}|_{j+1} \\ &\leq \delta^j \prod_{k=0}^{\infty} \left(1 + \frac{C \varepsilon_k}{r_k h_k} \right) \leq \delta^j \exp \left(\sum_{k=0}^{\infty} \frac{C \varepsilon_k}{r_k h_k} \right) \\ &< \delta^j \exp \left(C(1 - \delta^{2\vartheta})^{-1} E_0 \right), \end{aligned}$$

where $C = C(n, \tau, \vartheta_0, \vartheta)$ stands for different positive constants depending only on n, τ, ϑ_0 and ϑ . Choosing the parameter $E_0 = E_0(n, \tau, \vartheta_0, \vartheta) > 0$ sufficiently small we obtain

$$\left| \overline{W}_0 D\mathcal{F}^{j+1} \overline{W}_j^{-1} \right|_{j+1} < \delta^j, \quad j \in \mathbb{N}. \quad (11.281)$$

Set

$$z = (\theta, I, \omega) = x + iy \in D_{j+1} \times O_{j+1},$$

where x and y are respectively the real and the imaginary part of z . Then $|\overline{W}_{j+1} y| \leq 1$, where $|\cdot|$ stands for the sup-norm. We have

$$\mathcal{F}^{j+1}(x + iy) = \mathcal{F}^{j+1}(x) + i\overline{W}_0^{-1} T_{j+1}(x, y) \overline{W}_j y,$$

$$T_{j+1}(x, y) = \int_0^1 \overline{W}_0 D\mathcal{F}^{j+1}(x + ity) \overline{W}_j^{-1} dt$$

(we consider \overline{W}_j as a linear operator acting in $(\mathbb{R}^{3n}, |\cdot|)$). Moreover, $|T_{j+1}(x, y)| < \delta^j$ and since $|\overline{W}_j y| \leq \delta |\overline{W}_{j+1} y| \leq \delta \leq 1/6$, we get

$$|T_{j+1}(x, y) \overline{W}_j y| < \frac{1}{2} \delta^j, \quad x + iy \in D_{j+1} \times O_{j+1}.$$

Denote by $Z_{j+1}(x, y)$ the imaginary part of $\mathcal{F}^{j+1}(x + iy)$. Since $\mathcal{F}^{j+1}(x)$ is real valued, $Z_{j+1}(x, y)$ is equal to the real part of $\overline{W}_0^{-1} T_{j+1}(x, y) \overline{W}_j y$. Then we get

$$u_j^{-1} |Z_{j+1}(x, y)| \leq \delta^{-j} u_0^{-1} |\overline{W}_0^{-1}| |T_{j+1}(x, y) \overline{W}_j y| < \frac{1}{2}, \quad x + iy \in D_{j+1} \times O_{j+1},$$

and we obtain (11.280).

It remains to prove (11.278). In the case when $p = 0$ it follows from the arguments in [54]. Suppose now that $p = k = 1$. Denote by $D\mathcal{F}^j(z)$ the differential of \mathcal{F}^j with respect to $z = (\theta, I, \omega)$ acting on vectors $\eta \in \mathbb{C}^{3n}$ by $\eta \rightarrow D\mathcal{F}^{j-1}(z) \cdot \eta$. Consider

$$\begin{aligned} \partial_t(\mathcal{F}^{j+1} - \mathcal{F}^j) &= \partial_t(\mathcal{F}^j \circ \mathcal{F}_j - \mathcal{F}^j) \\ &= (D\mathcal{F}^j \circ \mathcal{F}_j) \cdot \partial_t \mathcal{F}_j + (\partial_t \mathcal{F}^j) \circ \mathcal{F}_j - \partial_t \mathcal{F}^j = \Sigma_1 + \Sigma_2 \end{aligned}$$

where

$$\Sigma_1 := (D\mathcal{F}^j \circ \mathcal{F}_j) \cdot \partial_t \mathcal{F}_j,$$

$$\Sigma_2 := \int_0^1 (D\partial_t \mathcal{F}^j)(x\mathcal{F}_j + (1-x)id) \cdot (\mathcal{F}_j - id) dx.$$

We are going to estimate $\Sigma_l, 1 \leq l \leq 2$. By (11.276), (11.277) and (11.281) we get for any $j \geq 1$

$$|\Sigma_1|_{j+1} < h_0^{-1} \left| \overline{W}_0 D\mathcal{F}^j \overline{W}_j^{-1} \right|_j |\overline{W}_j \partial_t \mathcal{F}_j|_{j+1} < C_0 \frac{\varepsilon_j}{r_j h_j}. \quad (11.282)$$

Consider Σ_2 now. Set $\mathcal{F}^0 = \mathcal{F}_{-1} = id$ and put $\mathcal{F}^{q,j} = \mathcal{F}_q \circ \dots \circ \mathcal{F}_{j-1}$ for $q \leq j-1$ and $\mathcal{F}^{j,j} = id$. For $j \geq 1$ we have

$$\partial_t \mathcal{F}^j = \partial_t (\mathcal{F}_0 \circ \dots \circ \mathcal{F}_{j-1}) = \sum_{q=0}^{j-1} (D\mathcal{F}^q \circ \mathcal{F}^{q,j}) \cdot ((\partial_t \mathcal{F}_q) \circ \mathcal{F}^{q+1,j}).$$

Using (11.276), (11.277), (11.281) and (11.262) we get as above

$$\begin{aligned} |\partial_t \mathcal{F}^j|_j &< h_0^{-1} \sum_{q=0}^{j-1} |\overline{W}_0 D \mathcal{F}^q W_q^{-1}|_q |\overline{W}_q \partial_t \mathcal{F}^q|_{q+1} \\ &\leq C \sum_{q=0}^j \frac{\varepsilon_q}{r_q h_q} < C(1 - \delta^{2\vartheta})^{-1} E_0 \end{aligned}$$

where C stands for different constants depending only on n, τ, ϑ_0 and ϑ . By Cauchy this implies

$$|D \partial_t \mathcal{F}^j \overline{W}_j| \leq C \quad (11.283)$$

uniformly on $D_j^2 \times O_j^2 \times [0, a]$, and we get

$$|\Sigma_2|_{j+1} \leq \sup_{D_j^2 \times O_j^2 \times [0, a]} \left| \overline{W}_0 D \partial_t \mathcal{F}^j \overline{W}_j^{-1} \right| |\overline{W}_j (\mathcal{F}_j - id)|_{j+1} \leq C \frac{\varepsilon_j}{r_j h_j} \quad (11.284)$$

where $C = C(n, \tau, \vartheta_0, \vartheta)$ stands for different positive constants depending only on n, τ, ϑ_0 and ϑ .

This proves (11.278) for $p = k = 1$. In the case when $p = 0$ we use the same arguments. This completes the proof of Proposition 11.9. \square

Remark 11.10 (Uniqueness by construction in the Iterative Lemma). *The transformations $\mathcal{F}_{t,j}$, the normal forms $N_{t,j} := N_j(\cdot, t)$ and the error terms $P_{t,j} := P_j(\cdot, t)$ do not depend on the choice of $m \geq 0$ in (11.251) in the following sense. Let $m' \geq 0$ and let $\mathcal{F}'_{t,j}$, $N'_{t,j}$ and $P'_{t,j}$, be the corresponding transformations, normal forms and error terms. Then $\mathcal{F}'_{t,j} = \mathcal{F}_{t,j}$, $N'_{t,j} = N_{t,j}$ and $P'_{t,j} = P_{t,j}$ on the intersection of their domains of definition.*

Remark 11.10 follows from Remark 11.5 and Remark 11.8 by induction with respect to $j \in \mathbb{N}$.

The Iterative Lemma provides a convergent schema giving in a limit a C^∞ function on $\mathbb{T}^n \times \Omega_1$ in a Whitney sense. To avoid inconveniences arising from the Whitney extension theorem, we propose a modified Iterative Lemma in the next section.

11.2.4 Modified Iterative Lemma.

We are going to modify \mathcal{F}_t^j multiplying $\mathcal{F}_t^j - id$ by a suitable almost analytic cut-off function in $\omega \in \mathbb{C}^n$.

1. Construction of almost analytic cut-off functions.

We say that a function $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, given by $x + iy \mapsto f(x + iy) := f(x, y)$ for $x, y \in \mathbb{R}^n$, is \mathbb{R} -smooth, or C^∞ in a real sense, if the function $\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto f(x, y)$ is C^∞ -smooth. As usually we denote by $\bar{\partial}_l$, $1 \leq l \leq n$, the operators

$$\bar{\partial}_l = \frac{\partial}{\partial \bar{z}_l} = \frac{1}{2} \left(\frac{\partial}{\partial x_l} + i \frac{\partial}{\partial y_l} \right),$$

and we set $\bar{\partial} = (\bar{\partial}_1, \dots, \bar{\partial}_n)$. A \mathbb{R} -smooth function $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is called almost-analytic if the vector-function $(x, y) \mapsto \bar{\partial} f(x + iy)$ is flat at $\mathbb{R} \times \{0\}$, in the sense that

$$\partial_y^\beta \bar{\partial} f(x, y)|_{y=0} = 0 \quad \text{for any } \beta \in \mathbb{N}^n.$$

Such a function is “very small” for y small. If f is an almost analytic Gevrey functions, then it is even exponentially small. Given $\rho > 1$ and $L \geq 1$, we say that f belongs to the Gevrey class $\mathcal{G}_L^\rho(\mathbb{C}^n)$ if it is \mathbb{R} -smooth and

$$\|f\|_L := \sup_{\alpha, \beta \in \mathbb{N}^n} \sup_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n} \left(|\partial_x^\alpha \partial_y^\beta f(x, y)| L^{-|\alpha| - |\beta|} \alpha!^{-\rho} \beta!^{-\rho} \right) < \infty,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We say that f is Gevrey- \mathcal{G}^ρ function. If the function $f \in \mathcal{G}_L^\rho(\mathbb{C}^n)$ is almost-analytic, then there exist positive constants $C = C(n, \rho)$ and $c = c(n, \rho)$ depending only on n and ρ , such that

$$|\partial_x^\alpha \partial_y^\beta \bar{\partial}_l f(x + iy)| \leq C \|f\|_L L^{|\alpha| + |\beta|} \alpha!^\rho \beta!^\rho \exp\left(-c(L|y|)^{-\frac{1}{\rho-1}}\right)$$

for any $\alpha, \beta \in \mathbb{N}^n$ and $1 \leq l \leq n$.

Proposition 11.11. *For any $n \geq 2$ and $\rho > 1$ there exist positive constants $C = C(n, \rho)$, $L = L(n, \rho)$ and $c = c(n, \rho)$, and a family of almost-analytic functions $\chi_j \in \mathcal{G}_{L/h_{j+1}}^\rho(\mathbb{C}^n)$, $j \in \mathbb{N}$, with the following properties*

- (i) $\text{supp } (\chi_j) \subset O_{j+1}^3$ and $\chi_j = 1$ on O_{j+1}^2 ;
- (ii) $\|\chi_j\|_{L/h_{j+1}} \leq C$ for $j \in \mathbb{N}$;
- (iii) the following estimate holds

$$|\partial_x^\alpha \partial_y^\beta \bar{\partial} \chi_j(x + iy)| \leq C(L/h_{j+1})^{|\alpha| + |\beta| + 1} (\alpha! \beta!)^\rho \exp\left(-c(|y|/h_{j+1})^{-\frac{1}{\rho-1}}\right)$$

on \mathbb{C}^n for any $j \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^n$.

The proposition will be proved in Section A.2.

2. Modified transformations.

From now on we take $\rho = 2$ in Proposition 11.11. We define the modified transformations

$$\mathcal{H}_{t,j} : D_{j+1} \times \mathbb{C}^n \rightarrow \mathbb{C}^n / 2\pi\mathbb{Z}^n \times \mathbb{C}^n \times \mathbb{C}^n$$

by

$$\mathcal{H}_{t,j}(z) := z + \chi_j(\omega) (\mathcal{F}_{t,j}(z) - z), \quad z = (\theta, I; \omega) \in D_{j+1} \times \mathbb{C}^n. \quad (11.285)$$

Setting $\mathcal{H}_{t,j} = (\tilde{\Phi}_{t,j}, \tilde{\phi}_{t,j})$, this means that

$$\begin{cases} \tilde{\Phi}_{t,j}(\theta, I; \omega) &= (\theta, I) + \chi_j(\omega) (\Phi_{t,j}(\theta, I; \omega) - (\theta, I)), \\ \tilde{\phi}_{j,t}(\omega) &= \omega + \chi_j(\omega) (\phi_{t,j}(\omega) - \omega), \end{cases}$$

for $(\theta, I; \omega) \in D_{j+1} \times \mathbb{C}^n$.

Lemma 11.12. *The following relations hold for any $j \in \mathbb{N}$ provided $E_0 = E_0(n, \tau, \vartheta_0, \vartheta)$ is sufficiently small*

- (1) $\mathcal{H}_{t,j} : D_{j+1} \times \mathbb{C}^n \rightarrow D_j^2 \times \mathbb{C}^n$,

(2) $\tilde{\phi}_{t,j}(O_{j+1}) \subset O_j^2$ and $\mathcal{H}_{t,j} : D_{j+1} \times O_{j+1} \rightarrow D_j^2 \times O_j^2$.

Proof. (1) Recall from (11.277) that

$$|W_j(\Phi_{t,j} - \text{id})|_{D_{j+1}} < \frac{C_0 \varepsilon_j}{r_j h_j} < C_0 C(n, \tau, \vartheta_0, \vartheta) E_0$$

by (11.277) and (11.262), where $W_j = \text{diag}(\sigma_j^{-1} \text{Id}, r_j^{-1} \text{Id})$. Moreover,

$$|\chi_j|_{\mathbb{C}^n} < \|\chi_j\|_{1/h_{j+1}} \leq C(n)$$

in view of Proposition 11.11, (ii). This yields

$$|W_j(\tilde{\Phi}_{t,j} - \text{id})|_{D_{j+1}} < C_1(n, \tau, \vartheta_0, \vartheta) E_0 \leq \frac{1}{8}$$

choosing $E_0 = E_0(n, \tau, \vartheta_0, \vartheta)$ sufficiently small, and we obtain (1).

(2) Let $\omega \in O_{j+1}$. Then there exists $\omega' \in \Omega_1$ such that $|\omega - \omega'| \leq h_{j+1}$ and we get as above

$$\begin{aligned} |\tilde{\phi}_{t,j}(\omega) - \omega'| &\leq |\omega - \omega'| + \|\chi_j\|_{1/h_{j+1}} |\phi_{t,j}(\omega) - \omega'| \leq h_{j+1} + C(n) C_0 \frac{\varepsilon_j}{r_j} \\ &< h_{j+1} + C(n, \tau, \vartheta_0, \vartheta) E_0 h_j < h_{j+1} + \frac{1}{3} h_j < \frac{1}{2} h_j, \end{aligned}$$

for $E_0 = E_0(n, \tau, \vartheta_0, \vartheta) > 0$ sufficiently small, hence, $\tilde{\phi}_{t,j}(\omega) \in O_j^2$. \square

Let us define $\mathcal{H}_j = (\tilde{\Phi}_j, \tilde{\phi}_j)$ and \mathcal{H}^j by $\mathcal{H}_j(\cdot, t) = \mathcal{H}_{t,j}$ and $\mathcal{H}^j(\cdot, t) = \mathcal{H}_t^j$ for $t \in [0, a]$, where

$$\mathcal{H}_t^0 = \text{id}, \quad \mathcal{H}_t^{j+1} := \mathcal{H}_{t,0} \circ \cdots \circ \mathcal{H}_{t,j} : D_{j+1} \times \mathbb{C}^n \rightarrow D_0 \times \mathbb{C}^n.$$

We set $\bar{\partial}_l = \frac{\partial}{\partial \bar{\omega}_l}$ for $1 \leq l \leq n$ and $\bar{\partial} = (\bar{\partial}_1, \dots, \bar{\partial}_n)$. We are going to use as well the convention $\frac{1}{\mp 0} = +\infty$ and $\exp(-\infty) = 0$.

Proposition 11.13 (Modified Iterative Lemma). *Under the assumptions of Proposition 11.9, the transformations \mathcal{H}^j are well defined on $D_j \times \mathbb{C}^n \times [0, a]$ and have the following properties*

(i) $\mathcal{H}^j \in C^k([0, a], C^\infty(D_j \times \mathbb{C}^n, D_0 \times \mathbb{C}^n))$ and $\mathcal{H}_t^j = \mathcal{F}_t^j$ on $D_j \times O_j^2$ for $t \in [0, a]$. Moreover,

$$\text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j) \subset D_{j+1} \times O_{j+1}^3 \quad \text{and} \quad \mathcal{H}_t^{j+1} - \mathcal{H}_t^j = \mathcal{F}_t^j \circ \mathcal{H}_{t,j} - \mathcal{F}_t^j;$$

(ii) $|\partial_t^p(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j)(z)| < \frac{C_0 \varepsilon_j}{r_j h_j}$ for $z = (\theta, I; \omega) \in D_{j+1} \times \mathbb{C}^n$, $t \in [0, a]$, and $0 \leq p \leq k$, where $C_0 = C_0(n, \tau, \vartheta_0, \vartheta) > 0$;

(iii) $\mathcal{H}^{j+1} - \mathcal{H}^j$ is analytic with respect to $(\theta, I) \in D_j$ and almost analytic and Gevrey- \mathcal{G}^2 with respect to ω . Moreover, for any $0 \leq p \leq k$ the following estimate holds

$$|\bar{\partial}_t^p(\mathcal{H}^{j+1} - \mathcal{H}^j)(\theta, I; \omega, t)| \leq C h_{j+1}^{-1} \exp\left(-c \frac{h_{j+1}}{|\text{Im}(\omega)|}\right) \frac{\varepsilon_j}{r_j h_j}, \quad (11.286)$$

$$\text{for } (\theta, I; \omega) \in D_{j+1}^3 \times \mathbb{C}^n, \quad t \in [0, a],$$

where $C = C(n, \tau, \vartheta_0, \vartheta)$ and $c = c(n)$ are positive constants;

(iv) the following estimate is true

$$\left| \partial_t^p \partial_\theta^\beta \partial_\omega^\gamma (\mathcal{H}^{j+1} - \mathcal{H}^j) (\theta, I; \omega, t) \right| \leq C_{\beta, \gamma} \frac{\varepsilon_j}{r_j h_j} \sigma_{j+1}^{-|\beta|} h_{j+1}^{-|\gamma|} \ln^{2|\gamma|+2}(\sigma_{j+1}) \quad (11.287)$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$, $t \in [0, a]$, and $0 \leq p \leq k$, $\beta, \gamma \in \mathbb{N}^n$, where $C_{\beta, \gamma} = C_{\beta, \gamma}(n, \tau, \vartheta_0, \vartheta) > 0$.

Proof. (i) Recall that $\chi_{j-1} = 1$ on O_j^2 , hence, $\mathcal{H}_{t, j-1} = \mathcal{F}_{t, j-1}$ on $D_j \times O_j^2$. Moreover, $\mathcal{F}_{t, j-1} : D_j \times O_j \rightarrow D_{j-1}^2 \times O_{j-1}^2$ by Proposition 11.9 and $\chi_{j-2} = 1$ on O_{j-1}^2 , which implies

$$(\mathcal{H}_t^{j-1} \circ \mathcal{H}_{t, j-1})(z) = (\mathcal{H}_t^{j-1} \circ \mathcal{F}_{t, j-1})(z) = (\mathcal{F}_t^{j-1} \circ \mathcal{F}_{t, j-1})(z)$$

for any $z \in D_j \times O_j^2$. Repeating this argument we obtain the equality

$$\mathcal{H}_t^j(z) = (\mathcal{H}_t^{j-1} \circ \mathcal{F}_{t, j-1})(z) = \cdots = (\mathcal{F}_{t, 0} \circ \cdots \circ \mathcal{F}_{t, j-1})(z) = \mathcal{F}_t^j(z).$$

If $\omega \notin O_{j+1}^3$, then $\chi_j(\omega) = 0$, $\mathcal{H}_{t, j}(\theta, I; \omega) = (\theta, I; \omega)$ and $\mathcal{H}_t^{j+1}(\theta, I; \omega) = \mathcal{H}_t^j(\theta, I; \omega)$, hence, $\text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j) \subset D_{j+1} \times O_{j+1}^3$.

Let $z = (\theta, I; \omega) \in D_{j+1} \times O_{j+1}^3 \subset D_{j+1} \times O_{j+1}$. Lemma 11.12 implies that $\mathcal{H}_{t, j}(z) \in D_j^2 \times O_j^2$, hence, $\chi_{j-1}(\tilde{\phi}_j(\omega)) = 1$ and we obtain

$$(\mathcal{H}_{t, j-1} \circ \mathcal{H}_{t, j})(z) = (\mathcal{F}_{t, j-1} \circ \mathcal{H}_{t, j})(z) \in D_{j-1}^2 \times O_{j-1}^2.$$

On the other hand $\chi_{j-2} = 1$ on O_{j-1}^2 and repeating this argument we get

$$\begin{aligned} (\mathcal{H}_t^j \circ \mathcal{H}_{t, j})(z) &= (\mathcal{H}_{t, 0} \circ \cdots \circ \mathcal{H}_{t, j-1} \circ \mathcal{H}_{t, j})(z) = (\mathcal{H}_{t, 0} \circ \cdots \circ \mathcal{F}_{t, j-1} \circ \mathcal{H}_{t, j})(z) \\ &= \cdots = (\mathcal{F}_{t, 0} \circ \cdots \circ \mathcal{F}_{t, j-1} \circ \mathcal{H}_{t, j})(z) = (\mathcal{F}_t^j \circ \mathcal{H}_{t, j})(z). \end{aligned}$$

Moreover, $\mathcal{H}_t^j(z) = \mathcal{F}_t^j(z)$ since $z \in D_{j+1} \times O_{j+1} \subset D_j^2 \times O_j^2$, and we obtain the equality

$$\mathcal{H}_t^{j+1} - \mathcal{H}_t^j = \mathcal{F}_t^j \circ \mathcal{H}_{t, j} - \mathcal{F}_t^j$$

on $D_{j+1} \times O_{j+1}^3$. On the other hand, both sides of it vanish at any $z \in D_{j+1} \times (\mathbb{C}^n \setminus O_{j+1}^3)$. Hence, the above equality is true on $D_{j+1} \times \mathbb{C}^n$. This completes the proof of (i).

(ii) Let $p = 0$. Using (i) we obtain

$$\mathcal{H}^{j+1} - \mathcal{H}^j = \mathcal{F}^j \circ \mathcal{H}_j - \mathcal{F}^j = \chi_j \int_0^1 (D\mathcal{F}^j)(x\mathcal{H}_j + (1-x)id) \cdot (\mathcal{F}_j - id) dx.$$

Moreover, for any $z \in \text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j) \subset D_{j+1} \times O_{j+1} \subset D_j^2 \times O_j^2$ we have

$$x\mathcal{H}_j(z) + (1-x)z \in D_j^2 \times O_j^2 \quad \text{for } 0 \leq x \leq 1, \quad (11.288)$$

and (11.277) and (11.278) imply

$$|\mathcal{H}^{j+1} - \mathcal{H}^j|_{j+1} \leq C(n)h_0^{-1} \left| \overline{W}_0 D\mathcal{F}^j \overline{W}_j^{-1} \right|_j |\overline{W}_j(\mathcal{F}_j - id)|_{j+1} < C_0 \frac{\varepsilon_j}{r_j h_j}.$$

Let $p = 1$. By the chain rule we get

$$\begin{aligned}\partial_t(\mathcal{H}^{j+1} - \mathcal{H}^j) &= \partial_t(\mathcal{F}^j \circ \mathcal{H}_j - \mathcal{F}^j) \\ &= (D\mathcal{F}^j \circ \mathcal{H}_j) \cdot \partial_t \mathcal{F}_j + (\partial_t \mathcal{F}^j) \circ \mathcal{H}_j - \partial_t \mathcal{F}^j = \Sigma_1 + \Sigma_2\end{aligned}$$

where

$$\begin{aligned}\Sigma_1 &:= (D\mathcal{F}^j \circ \mathcal{H}_j) \cdot \partial_t \mathcal{F}_j, \\ \Sigma_2 &:= \int_0^1 (D\partial_t \mathcal{F}^j)(x\mathcal{H}_j + (1-x)id) \cdot (\mathcal{F}_j - id) dx.\end{aligned}$$

Using (11.288), we estimate Σ_1 and Σ_2 as in (11.282) and (11.284).

(iii) Let $p = 0$. Recall that \mathcal{F}^j is analytic on $D_j \times O_j$, \mathcal{F}_j is analytic with respect to $z = (\theta, I; \omega) \in D_{j+1} \times O_{j+1}$, and that $\mathcal{H}_j(z, t) = z + \chi_j(\omega)(\mathcal{F}_j(z, t) - z)$.

Differentiating the identity in (i) we obtain for any $1 \leq l \leq n$ the following one

$$\begin{aligned}\bar{\partial}_l(\mathcal{H}^{j+1} - \mathcal{H}^j)(z, t) &= \frac{\partial}{\partial \omega_l}(\mathcal{F}^j \circ \mathcal{H}_j - \mathcal{F}^j)(z, t) \\ &= \bar{\partial}_l \chi_j(\omega) D\mathcal{F}_t^j(\mathcal{H}_j(z, t)) \cdot (\mathcal{F}_j(z, t) - z)\end{aligned}\tag{11.289}$$

for each $z \in \text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j)$ and $t \in [0, a]$. According to Proposition 11.11 we have

$$\sup_{\omega \in \mathbb{C}^n} |\bar{\partial}_l \chi_j(\omega)| \leq Ch_{j+1}^{-1} \exp\left(-c \frac{h_{j+1}}{|\text{Im}(\omega)|}\right).\tag{11.290}$$

Morreover, $z \in D_{j+1} \times O_{j+1}^3$ and $\mathcal{H}_j(z, t) \in D_j^2 \times O_j^2$ in view of item (i), and arguing as in the proof of (11.282) we obtain (iii) for $p = 0$.

Let $p = k = 1$. Differentiating the identity (11.289) with respect to t we obtain

$$\begin{aligned}\partial_t \bar{\partial}_l(\mathcal{H}^{j+1} - \mathcal{H}^j)(z, t) &= \bar{\partial}_l \chi_j(\omega) \partial_t D\mathcal{F}_t^j(\mathcal{H}_j(z, t)) \cdot (\mathcal{F}_j(z, t) - z) \\ &\quad + \bar{\partial}_l \chi_j(\omega) D\mathcal{F}_t^j(\mathcal{H}_j(z, t)) \cdot \partial_t \mathcal{F}_j(z, t) \\ &\quad + \chi_j(\omega) \bar{\partial}_l \chi_j(\omega) D^2 \mathcal{F}_t^j(\mathcal{H}_j(z, t)) [\partial_t \mathcal{F}_j(z, t), \mathcal{F}_j(z, t) - z]\end{aligned}$$

for $z \in \text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j)$.

Consider for any $z = (\theta, I; \omega) \in D_j \times O_j$ the symmetric bilinear form

$$(\xi, \eta) \rightarrow D^2 \mathcal{F}_t^j(z, t)[\xi, \eta], \quad \xi, \eta \in \mathbb{C}^{3n},$$

representing the second differential of \mathcal{F}_t^j at z . We have

Lemma 11.14. *There exists $C = C(n, \tau, \vartheta_0, \vartheta) > 0$ such that*

$$\left| D^2 \partial_t^p \mathcal{F}_t^j(z) [\bar{W}_j^{-1} \xi, \bar{W}_j^{-1} \eta] \right| \leq C |\xi|_{\mathbb{C}^{3n}} |\eta|_{\mathbb{C}^{3n}}, \quad \xi, \eta \in \mathbb{C}^{3n},\tag{11.291}$$

for any $z \in D_j^2 \times O_j^2$, $t \in [0, a]$, $0 \leq p \leq k$, and $j \in \mathbb{N}$.

Proof. The expression $D^2 \partial_t^p \mathcal{F}^j(z, t) [\overline{W}_j^{-1} \xi, \overline{W}_j^{-1} \eta]$ is a sum of monomials of the form

$$(\sigma_j \partial_\theta)^\alpha (r_j \partial_I)^\beta (h_j \partial_\omega)^\gamma \partial_t^p \mathcal{F}^j(\theta, I; \omega, t) \xi_l \eta_m,$$

where $|\alpha| + |\beta| + |\gamma| = 2$ and $1 \leq l, m \leq n$. The estimate follows from (11.278) and the Cauchy inequalities. \square

For each $z \in \text{supp}(\mathcal{H}_t^{j+1} - \mathcal{H}_t^j) \subset D_{j+1} \times O_{j+1}$ we have $\mathcal{H}_t^j(z) \in D_j^2 \times O_j^2$ in view of Lemma 11.12. Then the estimate (11.287) follows from (11.290), Lemma 11.14, and (11.277) as in the case $p = 0$. This proves (iii).

(iv) We are going to use Cauchy formula for almost analytic functions. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a \mathbb{R} -smooth almost analytic function. Denote by $D_r(x)$ the open disc $\{z \in \mathbb{C} : |z - x| < r\}$ of radius $r > 0$ and by $\partial D_r(x)$ its boundary oriented counter clockwise. For any $x \in \mathbb{R}$ and $\beta \in \mathbb{N}$, the following Cauchy integral formula is valid

$$\frac{d^\beta f}{dx^\beta}(x) = \frac{\beta!}{2\pi i} \left\{ \int_{\partial D_r(x)} (z - x)^{-\beta-1} f(z) dz + \iint_{D_r(x)} (z - x)^{-\beta-1} \bar{\partial} f(z) dz \wedge d\bar{z} \right\} \quad (11.292)$$

(see e.g. [25], Proposition 1.1). Notice that the second integral is well defined since the function $\bar{\partial} f$ is flat at \mathbb{R} .

Set

$$R_j = h_{j+1} \ln^{-2}(\sigma_{j+1})$$

and

$$I_{\beta, \gamma_1}(\theta, I; \omega, t) := \partial_t^\beta \partial_\theta^\beta \partial_{\omega_1}^{\gamma_1} (\mathcal{H}^{j+1} - \mathcal{H}^j)(\theta, I; \omega, t)$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$, $t \in [0, a]$. Recall that \mathcal{H}^j is analytic with respect to $(\theta, I) \in D_{j+1}$. Applying first the Cauchy inequality with respect to θ , we get

$$|I_{\beta, \gamma_1}(\theta, I; \omega, t)| \leq (2\sigma_{j+1})^{-|\beta|-1} |I_{0, \gamma_1}(\theta, I; \omega, t)|.$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$, $t \in [0, a]$. We are going to estimate $|I_{0, \gamma}(\theta, I; \omega, t)|$.

Applying Cauchy formula (11.292) to the variable $x = \omega_1 \in \mathbb{R}$, keeping $\omega' = (\omega_2, \dots, \omega_n)$ fixed in \mathbb{R}^{n-1} , we obtain

$$\begin{aligned} I_{0, \gamma_1}(\theta, I; \omega, t) &= \frac{\gamma_1!}{2\pi i} \int_{D_{R_j}(\omega_1)} \frac{\partial_t^p \bar{\partial}_1 (\mathcal{H}^{j+1} - \mathcal{H}^j)(\theta, I; z, \omega', t)}{(z - \omega_1)^{\gamma_1+1}} dz \\ &+ \frac{\gamma_1!}{2\pi i} \int_{D_{R_j}(\omega_1)} \frac{\partial_t^p \bar{\partial}_1 (\mathcal{H}^{j+1} - \mathcal{H}^j)(\theta, I; z, \omega', t)}{(z - \omega_1)^{\gamma_1+1}} dz \wedge d\bar{z} \end{aligned} \quad (11.293)$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$ and $t \in [0, a]$.

Using (ii) we estimate the first integral by

$$C_{\beta, \gamma_1} \frac{\varepsilon_j}{r_j h_j} h_{j+1}^{-\gamma_1-1} \ln^{2\gamma_1+2}(\sigma_{j+1})$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$ and $t \in [0, a]$.

In order to estimate the second integral we are going to use the following estimate

$$\begin{aligned} & \frac{1}{|z - \omega_1|^{\gamma_1+1}} \exp\left(-c \frac{h_{j+1}}{|\operatorname{Im}(z)|}\right) h_{j+1}^{-1} \\ & \leq \frac{1}{|\operatorname{Im}(z)|^{\gamma_1+1}} \exp\left(-\frac{c}{2} \frac{h_{j+1}}{|\operatorname{Im}(z)|}\right) 2^{\tau+1} \sigma_{j+1}^{-\tau-1} \ln^{2\tau+2}(1/\sigma_{j+1}) e^{-\frac{\varepsilon}{2} \ln^2(\sigma_{j+1})} \\ & \leq C_{\gamma_1} \frac{1}{h_j^{\gamma_1+1}} \end{aligned}$$

for $z \neq \omega_1$. Using (iii) and the estimate above, we estimate the second integral by

$$C_{\gamma} \frac{\varepsilon_j}{r_j h_j} h_{j+1}^{-\gamma_1-1}$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$ and $t \in [0, a]$.

This implies

$$|I_{0, \gamma_1}(\theta, I; \omega, t)| \leq C_{\gamma_1} \frac{\varepsilon_j}{r_j h_j} h_{j+1}^{-\gamma_1-1} \ln^{2\gamma_1+2}(1/\sigma_{j+1})$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$ and $t \in [0, a]$.

Finally we obtain

$$|I_{\beta, \gamma_1}(\theta, I; \omega, t)| \leq C_{\beta, \gamma_1} \frac{\varepsilon_j}{r_j h_j} h_{j+1}^{-\gamma_1-1} \ln^{2\gamma_1+2}(1/\sigma_{j+1}) \sigma_{j+1}^{-|\beta|}$$

for $(\theta, I; \omega) \in D_{j+1}^2 \times \mathbb{R}^n$ and $t \in [0, a]$. This proves (iv) in the case when $\gamma = (\gamma_1, 0, \dots, 0)$. By a permutation of the indexes, we obtain it as well for $\gamma = (0, \dots, 0, \gamma_l, 0, \dots, 0)$. It remains to prove the estimate for the mixed derivatives with respect to ω . To this end we shall use the following

Lemma 11.15. *For any $\gamma \in \mathbb{N}^n$ of length $N = |\gamma| \geq 2$ there exist $(N+1)^{n-1}$ vectors \vec{v}_m and constants c_m such that*

$$\partial_{\omega}^{\gamma} = \sum c_m \mathcal{L}_{\vec{v}_m}^N$$

where $\mathcal{L}_{\vec{v}_m}$ stands for the directional derivative $\mathcal{L}_{\vec{v}_m} f(\omega) = \frac{d}{ds} \Big|_{s=0} f(\omega + s\vec{v}_m)$.

Proof. We proceed by induction with respect to $n \geq 2$. Let $n = 2$ and $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_2 \neq 0$. Denote by \vec{e}_1 and \vec{e}_2 the canonical basis of \mathbb{R}^2 and set $\vec{v} = \vec{e}_1 + \lambda \vec{e}_2$, where $\lambda > 0$. We have

$$\mathcal{L}_{\vec{v}}^N = \sum_{l=0}^N \lambda^l L_l, \quad L_l := \frac{N!}{l!(N-l)!} \partial_1^{N-l} \partial_2^l.$$

Choosing $\lambda_m = m/(N+1)$ and $\vec{v}_m = \vec{e}_1 + \lambda_m \vec{e}_2$ for $1 \leq m \leq N+1$, we obtain the linear system

$$\sum_{l=0}^N \lambda_m^l L_l = \mathcal{L}_{\vec{v}_m}^N, \quad m = 0, \dots, N.$$

This system has a unique solution with respect to L_l , $0 \leq l \leq N$, since the corresponding determinant is just the Vandermonde determinant. Then we use induction with respect to n . \square

Applying the preceding argument for each derivative $\mathcal{L}_{\vec{v}_m}^N$, we complete the proof of (iv). \square

Remark 11.16 (Uniqueness in the Modified Iterative Lemma). *The transformations \mathcal{H}^j do not depend on the choice of $m \geq 0$ in (11.251) in the sense of Remark 11.10.*

11.2.5 *Choice of the sequence ν and the small parameters ϵ and $\hat{\epsilon}$.*

Given $m \in \mathbb{N}$ we consider the sequence $(\nu_j(m))_{j \in \mathbb{N}}$ introduced in (11.251) and set

$$\ell_j(m) = \begin{cases} \ell_0 = 2\tau + 2 + 2\vartheta_0 & \text{for } j < J(m), \\ \ell_m = 2m(\tau + 1) + \ell_0 & \text{for } j \geq J(m), \end{cases} \quad (11.294)$$

where $J(0) = 0$ and $J(m)$, $m > 0$, will be a suitable integer satisfying (11.252).

Consider the family of functions P_t^j , $j \in \mathbb{N}$, defined by (11.266). In order to apply the Iterative Lemma and the Modified Iterative Lemma to that family, we have to show that it satisfies (11.270). To this end we will choose appropriately the small constants ϵ and $\hat{\epsilon}$ as well as the integer $J(m)$ for each $m > 0$. By (11.268) and (11.269), it suffices to prove for each $j \in \mathbb{N}$ that

$$C_\ell u_j^\ell \sum_{p=0}^k \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_\ell \leq \frac{\epsilon_{j+1}}{4} = \frac{1}{4} \hat{\epsilon} r_{j+1}(m) \sigma_{j+1}^{\tau+1} E_{j+1}(m) \quad \text{with } \ell = \ell_j(m). \quad (11.295)$$

Here $r_j(m)$ and $E_j(m)$ are given by (11.253) and (11.254), respectively, $\hat{\epsilon} \in (0, 1]$. In view of (11.263) and (11.253)-(11.258), the relation (11.295) becomes

$$\sum_{p=0}^k \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_\ell \leq \hat{\epsilon} r_0 \sigma_0^{\tau+1} \frac{(\delta/6s_0)^\ell}{4C_\ell} \delta^{M_j(m)} \quad \text{with } \ell = \ell_j(m), \quad (11.296)$$

where

$$\begin{aligned} M_j(m) &:= q_{j+1} - (j+1)\ell_j(m) \\ &= (j+1)(2\tau + 2 + \vartheta - \ell_j(m)) + 2(\nu_0(m) + \dots + \nu_j(m)). \end{aligned} \quad (11.297)$$

Since $r_0 = s_0 r > \sigma_0 r$, the inequality (11.296) will follow from the following one

$$\sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_{\ell_j(m)} \leq \hat{\epsilon} \epsilon_j(m) \delta^{M_j(m)}, \quad 0 \leq p \leq k, \quad (11.298)$$

where

$$\epsilon_j(m) := \sigma_0^{\tau+2} E_0 \frac{(\delta/6s_0)^{\ell_j(m)}}{8C_{\ell_j(m)}}. \quad (11.299)$$

We have

$$\epsilon_j(m) = \begin{cases} \epsilon_0 = \sigma_0^{\tau+2} E_0 \frac{(\delta/6s_0)^{\ell_0}}{8C_{\ell_0}} & \text{for } j < J(m); \\ \epsilon_m = \sigma_0^{\tau+2} E_0 \frac{(\delta/6s_0)^{\ell_m}}{8C_{\ell_m}} & \text{for } j \geq J(m). \end{cases}$$

For $m = 0$, taking into account (11.251) and (11.294), we obtain

$$M_j(0) = -(j+1)\vartheta.$$

For $m > 0$ we obtain in the same way

$$M_j(m) = \begin{cases} -(j+1)\vartheta, & j < J(m), \\ -j(2m(\tau+1) + \vartheta) - \vartheta, & j = J(m), \\ M_{J(m)}(m) - (j - J(m))\vartheta, & j \geq J(m). \end{cases} \quad (11.300)$$

Our aim now is to satisfy (11.298) for each $m \in \mathbb{N}$, $j \in \mathbb{N}$, and $0 \leq p \leq k \leq 1$, choosing appropriately $\hat{\epsilon} \in (0, 1]$ and $J(m)$.

(1) *The case when $p = 0$.*

Suppose firstly that $m = 0$. Then (11.298) becomes

$$\sup_{0 \leq t \leq a} \|P_t\|_{\ell_0} \leq \hat{\epsilon} \epsilon_0 \delta^{-\vartheta j}, \quad \forall j \in \mathbb{N}. \quad (11.301)$$

Let us set

$$\begin{cases} \epsilon & := & \epsilon_0 = \sigma_0^{\tau+2} E_0 \frac{(4s_0)^{-\ell_0}}{4C_{\ell_0}}, \\ \hat{\epsilon} & := & \epsilon^{-1} \sup_{0 \leq t \leq a} \|P_t\|_{\ell_0}. \end{cases} \quad (11.302)$$

Then (11.301) holds for any $j \in \mathbb{N}$. Moreover, (10.207) just means that

$$0 < \hat{\epsilon}_0 \leq 1.$$

Notice that ϵ depends only on n , τ , ϑ_0 and ϑ since c_{ℓ_0} , σ_0 , s_0 and E_0 depend only on n , τ , ϑ_0 and ϑ by Lemma 11.7, (11.260) and (11.261). Hence, one can apply the Iterative Lemma.

Suppose now that $m > 0$. If $j < J(m)$, then $\ell_j(m) = \ell_0$ and $M_j(m) = -(j+1)\vartheta$, and (11.298) for $p = 0$ reduces to (11.301) with ϵ and $\hat{\epsilon}$ given by (11.302).

On the other hand, for $j = J(m)$ and any $k \in \{0; 1\}$, the inequality (11.298) becomes

$$A_j^k(m) := C_0(m) \delta^{j(2m(\tau+1) + \vartheta)} \sum_{p=0}^k \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_{\ell(m)} \leq \hat{\epsilon}, \quad (11.303)$$

where $C_0(m) = \delta^\vartheta \epsilon_m^{-1}$.

The sequence $(A_j^0(m))_{j \in \mathbb{N}}$ is decreasing and it tends to zero. Let $J(m)$ be the smallest integer

$$j \geq m(\tau+1)\vartheta^{-1}$$

such that $A_j^0(m) \leq \hat{\epsilon}$. Then (11.298) holds for $j = J(m)$ and $p = 0$. Moreover, $J(m)$ satisfies (11.252) by definition. For $j \geq J(m)$ we have

$$\sup_{0 \leq t \leq a} \|P_t\|_{\ell_j(m)} = \sup_{0 \leq t \leq a} \|P_t\|_{\ell_m} \leq \hat{\epsilon} \epsilon_m \delta^{M_{J(m)}(m)} \leq \hat{\epsilon} \epsilon_m \delta^{M_j(m)}$$

in view of (11.300), hence, (11.298) is satisfied for each $j \in \mathbb{N}$ when $p = 0$.

Lemma 11.17. *There exist*

$$\tilde{C}_m = \tilde{C}_m(n, \tau, \vartheta, \vartheta_0) > 0,$$

depending only on $n, \tau, \vartheta, \vartheta_0$ and m , such that

$$\tilde{C}_m \hat{\varepsilon} \leq A_{J(m)}^0(m) \leq \hat{\varepsilon}. \quad (11.304)$$

Proof. If $J(m) - 1 \geq m(\tau + 1)\vartheta^{-1}$, then $A_{J(m)-1}^0(m) > \hat{\varepsilon}$, and we get

$$\hat{\varepsilon} \geq A_{J(m)}^0(m) = \delta^{2m(\tau+1)+\vartheta} A_{J(m)-1}^0(m) > \hat{\varepsilon} \delta^{2m(\tau+1)+\vartheta}.$$

If $J(m) < m(\tau + 1)\vartheta^{-1} + 1$, then

$$\begin{aligned} \hat{\varepsilon} &\geq A_{J(m)}^0(m) \geq C_0(m) \delta^{J(m)(2m(\tau+1)+\vartheta)} \sup_{0 \leq t \leq a} \|P_t\|_{\ell_0} \\ &\geq C_0(m) \delta^{b(m)} \epsilon \hat{\varepsilon}, \end{aligned}$$

where $b(m) = (m(\tau + 1)\vartheta^{-1} + 1)(2m(\tau + 1) + \vartheta)$, and we obtain (11.304) since $C_0(m)$, ϵ and δ depend only on $n, \tau, \vartheta, \vartheta_0$ and m . \square

(2) *The case when $p = k = 1$.*

Choosing $\hat{\varepsilon}$ and $J(m)$ as in the case (1), we obtain that P_t satisfies (11.298) for $p = 0$. To satisfy (11.298) for $p = 1$, we need an additional argument. We rescale t by setting $\tilde{t} = tT(m) \in [0, \tilde{a}(m)]$, where $\tilde{a}(m) = aT(m)$ and

$$T(m) := \frac{1}{\epsilon \hat{\varepsilon}} \sup_{p \in \{0;1\}} \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_{\ell_m} = \frac{\sup_{p \in \{0;1\}} \sup_{0 \leq t \leq a} \|\partial_t^p P_t\|_{\ell_m}}{\sup_{0 \leq t \leq a} \|P_t\|_{\ell_0}} \geq 1. \quad (11.305)$$

Then $\tilde{P}_{\tilde{t}}$ defined by $\tilde{P}(\cdot, \tilde{t}) := P(\cdot, \tilde{t}/T(m))$ satisfies (11.298) for $p = 0$. Moreover,

$$\|\partial_{\tilde{t}} \tilde{P}_{\tilde{t}}\|_{\ell_m} \leq \frac{1}{T(m)} \sup_{0 \leq t \leq a} \|\partial_t P_t\|_{\ell_m} \leq \sup_{0 \leq t \leq a} \|P_t\|_{\ell_0}$$

and we obtain (11.298) for $p = 1$. Replacing P_t with $\tilde{P}_{\tilde{t}}$, we can apply Proposition 11.1 to $\tilde{P}_{\tilde{t}}^j$ for $\tilde{t} \in [0, \tilde{a}]$ at each iteration (recall that the constants in Proposition 11.1 do not depend on a).

We summarize the above construction by the following

Lemma 11.18. *Fix the positive constants ϵ and $\hat{\varepsilon} \leq 1$ by (11.302). Then*

- (i) *If $k = 0$ then for each $m \in \mathbb{N}$ the sequence $(\tilde{\varepsilon}_{\ell_j(m), j, 0})_{j \in \mathbb{N}}$ defined in (11.269) satisfies (11.295) and the Iteration Lemma as well as the Modified Iteration Lemma hold for any $m \in \mathbb{N}$;*
- (ii) *If $k = 1$, then $\tilde{P}_{\tilde{t}}$, $\tilde{t} \in [0, \tilde{a}(m)]$, satisfies (11.295) and the Iteration Lemma as well as the Modified Iteration Lemma hold for any $m \in \mathbb{N}$.*

How do the maps $\mathcal{F}_t^j, j \in \mathbb{N}$, constructed by the Iteration Lemma and \mathcal{H}_t^j given by the Modified Iteration Lemma, depend on $m \in \mathbb{N}$? The answer of this question is given in Remark 11.10 and Remark 11.16 and we summarize it by the following

Lemma 11.19 (Uniqueness by construction). *We have the following:*

- (i) *The transformations \mathcal{F}_t^j and \mathcal{H}_t^j , $j \in \mathbb{N}$, do not depend on m in the following sense. If $m' \in \mathbb{N}$ and $\mathcal{F}_t^{l_j}$ and $\mathcal{H}_t^{l_j}$ are the corresponding transformations, then $\mathcal{F}_t^j = \mathcal{F}_t^{l_j}$ and $\mathcal{H}_t^j = \mathcal{H}_t^{l_j}$ on the intersection of their domains of definition.*
- (ii) *Let $k = 1$ and $\tilde{\mathcal{F}}_{\tilde{t}}^j$ and $\tilde{\mathcal{H}}_{\tilde{t}}^j$, $j \in \mathbb{N}$, be the transformations corresponding to $\tilde{P}_{\tilde{t}}$, where $\tilde{t} = tT(m) \in [0, T(m)a]$. Let \mathcal{F}_t^j and \mathcal{H}_t^j be the transformations corresponding to P_t , $t \in [0, a]$. Then $\tilde{\mathcal{F}}_{\tilde{t}}^j = \mathcal{F}_t^j$ and $\tilde{\mathcal{H}}_{\tilde{t}}^j = \mathcal{H}_t^j$.*

Item (ii) means that the map $[0, a] \rightarrow \mathcal{H}_t^j$ is C^1 , if $k = 1$, and that

$$\partial_t \mathcal{H}_t^j = T(m) \partial_{\tilde{t}} \tilde{\mathcal{H}}_{\tilde{t}}^j \Big|_{\tilde{t}=tT(m)}.$$

In order to prove (ii), Theorem 10.1, we need the following

Lemma 11.20. *For each $m \geq 0$ there exists $C_m = C_m(n, \tau, \vartheta, \vartheta_0) > 0$ depending only on $m, n, \tau, \vartheta, \vartheta_0$ such that*

$$\hat{\varepsilon} E_j(m) \leq C_m \sigma_{j+1}^{m(\tau+1)+\vartheta_0-\vartheta} \sup_{0 \leq t \leq a} \|P_t\|_{\ell(m)}.$$

Proof. Let $m = 0$. We have

$$E_j(0) = \delta^{\nu_0(0)+\dots+\nu_{j-1}(0)} E_0 = \delta^{j(\vartheta_0-\vartheta)} E_0 < (\sigma_0 \delta)^{-\vartheta_0} \sigma_{j+1}^{\vartheta_0-\vartheta} E_0$$

and the estimate holds in view of the choice of $\hat{\varepsilon}$ in (11.302).

Suppose now that $m > 0$. Using Lemma 11.17 and (11.303) we obtain

$$\begin{aligned} \hat{\varepsilon} E_j(m) &\leq \tilde{C}_m^{-1} A_{J(m)}^0(m) E_j(m) \\ &= C_m^{-1} C_0(m) \delta^{F_m(j)} \sup_{0 \leq t \leq a} \|P_t\|_{\ell(m)}, \end{aligned}$$

where

$$F_m(j) := J(m)(2m(\tau+1) + \vartheta) + \nu_0(m) + \dots + \nu_{j-1}(m).$$

Let $j \leq J(m)$. Using (11.251), we get

$$\begin{aligned} F_m(j) &= J(m)(2m(\tau+1) + \vartheta) + j(\vartheta_0 - \vartheta) \\ &\geq j(2m(\tau+1) + \vartheta_0). \end{aligned}$$

If $j \geq J(m) + 1$, we obtain by (11.251) the inequality

$$\begin{aligned} F_m(j) &= J(m)(2m(\tau+1) + \vartheta) + J(m)(\vartheta_0 - \vartheta) \\ &\quad + (j - J(m))(m(\tau+1) + \vartheta_0 - \vartheta) \\ &\geq j(m(\tau+1) + \vartheta_0 - \vartheta). \end{aligned}$$

Choosing $C_m := \tilde{C}_m^{-1} C_0(m) (\sigma_0 \delta)^{-m(\tau+1)-\vartheta_0}$ we complete the proof of the Lemma. \square

We are ready to prove Theorem 10.1. Fix the parameter m and set

$$\langle P \rangle_{\ell(m)}^{(p)} := \langle P \rangle_{\ell(m);1,1}^{(p)}, \quad 0 \leq p \leq k, \quad (11.306)$$

using the notations in (10.206). If $p = 1$ we scale back with respect to t by $T(m)$. Combining (11.287) and Lemma 11.20 and using Lemma 11.19, (ii), in the case when $p = 1$, we obtain the estimate

$$\begin{aligned} & \left| \partial_t^p \partial_\theta^\beta \partial_\omega^\gamma (\mathcal{H}^{j+1} - \mathcal{H}^j) (\theta, I; \omega, t) \right| \\ & \leq C_{\beta,\gamma} \frac{\varepsilon_j}{r_j h_j} \sigma_{j+1}^{-|\beta|} h_{j+1}^{-|\gamma|} \ln^{2|\gamma|} (1/\sigma_{j+1}) T(m)^p \\ & \leq C_{m,\beta,\gamma} \sigma_{j+1}^{m(\tau+1)+\vartheta_0-\vartheta} \sigma_{j+1}^{-|\beta|} h_{j+1}^{-|\gamma|} \ln^{2|\gamma|} (1/\sigma_{j+1}) \langle P \rangle_{\ell(m)}^{(p)} \\ & \leq C_{m,\beta,\gamma} \sigma_{j+1}^{(m-|\gamma|)(\tau+1)-|\beta|+\vartheta_0-\vartheta} \ln^{4|\gamma|} (1/\sigma_{j+1}) \langle P \rangle_{\ell(m)}^{(p)} \\ & \leq C_{m,\beta,\gamma} \sigma_{j+1}^{(m-|\gamma|)(\tau+1)-|\beta|+\vartheta_0-2\vartheta} \langle P \rangle_{\ell(m)}^{(p)} \end{aligned}$$

on $D_{j+1} \times \mathbb{R}^n \times [0, a]$, where $C_{m,\beta,\gamma}$ stands for possibly different positive constants. Let us fix $\vartheta = (\vartheta_0 - \vartheta_1)/4$. Then

$$\begin{aligned} \left| \partial_t^p \partial_\theta^\beta \partial_\omega^\gamma (\mathcal{H}^{j+1} - \mathcal{H}^j) (\theta, I; \omega, t) \right| & \leq C_{m,\beta,\gamma} \sigma_{j+1}^{(m-|\gamma|)(\tau+1)-|\beta|+\vartheta_1+2\vartheta} \langle P \rangle_{\ell(m)}^{(p)} \\ & \leq C_{m,\beta,\gamma} \sigma_{j+1}^{2\vartheta} \langle P \rangle_{\ell(m)}^{(p)} \end{aligned} \quad (11.307)$$

on $D_{j+1} \times \mathbb{R}^n \times [0, a]$ provided that $|\beta| + |\gamma|(\tau + 1) \leq m(\tau + 1) + \vartheta_1$. Set

$$\mathcal{H}^j(\theta, I; \omega, t) = (\Phi^j(\theta, I; \omega, t), \phi^j(\omega, t))$$

where $\Phi^j(\theta, I; \omega, t) = (U^j(\theta; \omega, t), V^j(\theta, I; \omega, t))$ and V^j is affine linear in I by construction. Set

$$\begin{cases} \Psi(\theta; \omega, t) = \Psi_t(\theta; \omega) = (U(\theta; \omega, t), V(\theta; \omega, t)) = \lim_{j \rightarrow \infty} \Phi^j(\theta, 0; \omega, t), \\ \phi(\omega, t) = \phi_t(\omega) = \lim_{j \rightarrow \infty} \phi^j(\omega, t), \quad (\theta, \omega, t) \in \mathbb{T}^n \times \mathbb{R}^n \times [0, a]. \end{cases} \quad (11.308)$$

Lemma 11.19 implies that the transformations Ψ and ϕ do not depend on the choice of m . Then it follows from (11.307) that the function $[0, a] \ni t \mapsto (\partial_\theta^\beta \partial_\omega^\gamma \Psi_t, \partial_\omega^\gamma \phi_t) \in C^{r\vartheta_1}$ is C^k for any $m \geq 0$ and $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta|(\tau + 1) \leq m(\tau + 1) + \vartheta_1$. Moreover, the estimates in Theorem 10.1, (ii), hold (here $\kappa = \rho = 1$).

We are going to prove (i). To this end we use the identity

$$\mathcal{H}^j = \mathcal{F}^j \text{ on } D_j^2 \times O_j^2 \times [0, a]$$

given in Proposition 11.13, (i). As in Sect. 5.d, [54], we obtain that

$$\left| X_{H^j} \circ \mathcal{F}^j - D\Phi^j \cdot X_N \right| \leq \frac{c\varepsilon_j}{r_j h_j}$$

on $\mathbb{T}^n \times \{0\} \times \Omega_1$ for all $j \geq 0$, where X_{H^j} and $X_N = \mathcal{L}_\omega$ stand for the Hamiltonian vector fields of $H^j(\theta, I; \omega, t)$ and $N(\theta, I; \omega) = \langle \omega, I \rangle$, respectively. On the other hand, $\nabla_{(\theta, I)} H^j$ converges uniformly to $\nabla_{(\theta, I)} H$ as $j \rightarrow \infty$ in view of the estimate (11.267), with $\ell = \ell_0$ and $\ell' = \vartheta_1$ hence,

$$X_{H(\cdot; \phi(\omega, t), t)} \circ \Psi(\cdot; \omega, t) = D\Psi(\cdot; \omega, t) \cdot \mathcal{L}_\omega$$

on $\mathbb{T}^n \times \{0\} \times \Omega_1$. Moreover, (11.307) implies that

$$U(\cdot, \omega, \cdot) = \lim U_j(\cdot, \omega, \cdot) \in C^k([0, a], C^{1+\vartheta}(\mathbb{T}^n)), \quad \text{for each } \omega \in \Omega,$$

and

$$\|U(\cdot; \omega, t) - \text{id}\|_{1+\vartheta} < C(n, \tau, \vartheta_0, \vartheta_1) \langle P \rangle_{\ell(m)}^{(0)} \leq C(n, \tau, \vartheta_0, \vartheta_1) \epsilon < 1/2,$$

choosing ϵ small enough in a function of n, τ, ϑ_0 and ϑ_1 , hence, $U(\cdot, \omega, t)$ is an embedding. Then

$$t \rightarrow \{\Psi(\theta; \omega, t) : \theta \in \mathbb{T}^n\}$$

is a C^k family of embedded invariant tori of the Hamiltonians $(\theta, I) \rightarrow H(\theta, I; \phi(\omega, t), t)$ with frequency $\omega \in \Omega_1$. They are Lagrangian by construction (see also [26], Sect. I.3.2).

Using Remark 11.3 and Cauchy one obtains

Remark 11.21. *If P^j are analytic with respect to t in $B(0, a)$ and satisfy (11.270) for $t \in B(0, a)$, then \mathcal{F}_j are analytic with respect to t in $B(0, a)$ and the estimates (11.277) and (11.278) hold for $p = 0$ and $t \in B(0, a)$. Moreover, Ψ and ϕ are analytic in t in $B(0, a)$.*

11.3 KAM theorem with parameters in Hölder classes.

Better Hölder estimates of the transformations Ψ_t and ϕ_t than those in (ii) Theorem 10.1 can be obtained by means of the anisotropic Hölder spaces $C^{\rho(m)}(\mathbb{T}^n \times \Omega)$ introduced by Pöschel [53], where

$$\rho(m) = ((m(\tau + 1) + \vartheta_1, m + \vartheta_2), \quad m \geq 0, \quad 1 < \vartheta_1 < \vartheta_0, \quad \vartheta_2 = (\vartheta_0 - \vartheta_1)/(4\tau + 4).$$

Denote the corresponding weighted Hölder norms by $\|\cdot\|_{\rho(m); \kappa}$.

Theorem 11.22. *There exists a positive constant $\epsilon = \epsilon(n, \tau, \vartheta_0, \vartheta_1) > 0$ depending only on n, τ, ϑ_0 and ϑ_1 such that, for any $a > 0, 0 < \kappa < 1, 0 < r < \rho_0$ and $M \geq 0$, and any real valued Hamiltonian $H = N + P$, where the perturbation $P \in C^k([0, a]; C_0^{\ell(M)}(\mathbb{A} \times \Omega))$ satisfies the smallness condition*

$$\sup_{t \in [0, a]} \|P_t\|_{\ell_0; r, \kappa} \leq \epsilon \kappa r, \quad (11.309)$$

and $N(I; \omega) = \langle \omega, I \rangle$ is the normal form, the following holds.

There exist families of maps

$$[0, a] \ni t \mapsto \phi_t \in C^{M+\vartheta_2}(\Omega; \Omega), \quad [0, a] \ni t \mapsto \Psi_t = (U_t, V_t) \in C^{\rho(M)}(\mathbb{T}^n \times \Omega; \mathbb{T}^n \times B(0, r)) \quad (11.310)$$

such that $\text{supp}(\phi_t - \text{id}) \subset \Omega - \kappa/2$, $\text{supp}((U_t, V_t) - (\text{id}_{\mathbb{T}^n}, 0)) \subset \mathbb{T}^n \times (\Omega - \kappa/2)$ and item (i) of Theorem 10.1 holds true.

Moreover, for any $0 \leq m \leq M$ there is $C_m > 0$ depending only on $n, \tau, \vartheta_0, \vartheta_1$, and m , such that

$$\begin{aligned} & \|\partial_t^q(U_t - \text{id}_{\mathbb{T}^n})\|_{\rho(m); \kappa} + r^{-1} \|\partial_t^q V_t\|_{\rho(m); \kappa} + \kappa^{-1} \|\partial_t^q(\phi_t - \text{id})\|_{m+\vartheta_2; \kappa} \\ & \leq C_m (\kappa r)^{-1} \sup_{0 \leq p \leq q} \sup_{t \in [0, a]} \|\partial_t^p P_t\|_{\ell(m); r, \kappa} \end{aligned} \quad (11.311)$$

for each $m \in [0, M]$ and $t \in [0, a]$. These estimates hold for each $m \in [0, +\infty)$ if $P \in C^k([0, a]; C_0^\infty(\mathbb{A} \times \Omega))$.

If P is analytic with respect to t in an open disc $B(0, a) \subset \mathbb{C}$ of radius a and (11.309) holds for any $t \in B(0, a)$, then ϕ and Ψ are analytic in $t \in B(0, a)$, and the inequalities (11.311) hold uniformly in $t \in B(0, a')$, $0 < a' < a$, where $q = k = 0$, the interval $[0, a]$ is replaced by the disc $B(0, a)$ in the right hand side of (11.311), and the constant C_m depends on a' as well.

The estimates follow from (11.307), the properties of the norms $\|\cdot\|_{\rho, \kappa}$ for anisotropic Hölder spaces obtained in [53] and the Inverse Approximation Lemma obtained by Pöschel in [53].

Remark 11.23. Can $\ell(m) = 2m(\tau + 1) + \ell_0$ be replaced by $m(\tau + 1) + \ell_0$? The loss of $m(\tau + 1)$ derivatives in the estimates (ii) is due to the fact that we take only the affine linear approximation Q of P with respect to I in the KAM Step Lemma below. Using the approximation proposed by Rüssmann in Theorem 7.2 [64] as Bounemora [3], one could prove (ii) with $\ell(m)$ replaced by $m(\tau + 1) + \ell_0$. This needs additional efforts and will be done elsewhere.

A Appendix.

A.1 Approximation Lemma

The Hamiltonian P_t is not analytic and one can not apply directly the KAM step to it. We are going to approximate it by real analytic functions. To this end we recall some facts about the analytic smoothing technique invented by Moser [49], [50], and developed in different situations by Zehnder [73], Pöschel [53], Salamon [67] and Salamon and Zehnder [68]. The Approximation Lemma and the Inverse Approximation Lemma characterize Hölder classes of differentiable functions in terms of quantitative estimates of approximating sequences of analytic functions.

Let $m \in \mathbb{N}$, $0 < \mu \leq 1$, and let $U \subset \mathbb{R}^n$ be an open set. The Hölder space $C^{m, \mu}(U)$ consists of all $f \in C^m(U)$ such that

$$\|f\|_{C^{m, \mu}(U)} := \sup(\|f\|_{C^m(U)}, H_{m, \mu}(f)) < \infty, \quad (\text{A.1})$$

where

$$\|f\|_{C^m(U)} := \sup_{|\alpha| \leq m} \sup_{x \in U} |\partial^\alpha f(x)| \quad (\text{A.2})$$

is the C^m norm of f and

$$H_{m, \mu}(f) := \sup \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\mu} \quad (\text{A.3})$$

where the supremum is taken over all $x, y \in U$ such that $x \neq y$ and all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n = m$.

Given a non negative number $\ell = m + \mu \notin \mathbb{N}$, where $m = [\ell] \in \mathbb{N}$ is the entire part of ℓ and $0 \leq \mu = \{\ell\} < 1$ the residual one, we set $C^\ell(U) = C^{m, \mu}(U)$. To simplify the notations we set

$$\|f\|_\ell = \|f\|_{\ell, U} = \|f\|_{C^\ell(U)}.$$

Denote by

$$(T_x^m f)(y) := \sum_{|\alpha| \leq m} \partial^\alpha f(x) y^\alpha / \alpha!$$

the Taylor polynomial of f up to order m . Given $0 < \rho \leq \infty$ and an open set $U \subset \mathbb{R}^n$ we denote by U_ρ the strip of all $x + iy \in \mathbb{C}^n$ such that $x, y \in \mathbb{R}^n$, $x \in U$ and $|y| < \rho$. Recall that $\mathcal{A}(U_\rho)$ is

the space of analytic functions on U_ρ . We denote by $|\cdot|_\rho$ the sup-norm on U_ρ . The function f is said to be real analytic in U_ρ if f is analytic on U_ρ and real valued on U . In this Section we take $U = \mathbb{R}^n$. The space of entire functions $\mathcal{A}(\mathbb{C}^n)$ is endowed by the inductive topology generated by the sup-norms on compact sets of \mathbb{C}^n .

Lemma A.1. (Approximation Lemma) ([67],[73]). *There exists an entire function $K \in \mathcal{A}(\mathbb{C}^n)$ generating a family of convolution operators*

$$S_\rho f(x) := \rho^{-n} \int_{\mathbb{R}^n} K(\rho^{-1}(x-y)) f(y) dy, \quad 0 < \rho \leq 1, \quad (\text{A.4})$$

from $C^0(\mathbb{R}^n)$ to $\mathcal{A}(\mathbb{C}^n)$ with the following properties.

1. For any $\ell = m + \mu \geq 0$, where $m \in \mathbb{N}$ and $0 \leq \mu \leq 1$, there is a constant $C = C(n, \ell) > 0$ such that, for every $f \in C^{m, \mu}(\mathbb{R}^n)$, any $\alpha \in \mathbb{N}^n$ of length $|\alpha| \leq \ell$ and any $x = u + iv \in \mathbb{C}^n$, $u, v \in \mathbb{R}^n$ with $|v| < \rho$, we have

$$\left| \partial^\alpha S_\rho f(u + iv) - (T_u^{[\ell] - |\alpha|} \partial^\alpha f)(iv) \right| \leq C \rho^{\ell - |\alpha|} \|f\|_{C^{m, \mu}}, \quad (\text{A.5})$$

and in particular for any $0 < \rho < \tilde{\rho} \leq 1$, and $f \in C^\ell(\mathbb{R}^n)$,

$$|\partial^\alpha S_\rho f - \partial^\alpha S_{\tilde{\rho}} f|_\rho \leq C \tilde{\rho}^{\ell - |\alpha|} \|f\|_\ell. \quad (\text{A.6})$$

2. The restriction of $S_\rho f$ to \mathbb{R}^n satisfies

$$\|S_\rho f - f\|_s \leq C \rho^{\ell - s} \|f\|_{C^{m, \mu}}, \quad 0 \leq s < \ell. \quad (\text{A.7})$$

3. $K(\mathbb{R}^n) \subset \mathbb{R}$ and in particular the function $S_\rho f$ is real analytic whenever f is real valued. Moreover, if f is periodic in some variables then so is $S_\rho f$ in the same variables.
4. If $[0, 1] \ni t \mapsto f_t \in C^\ell(\mathbb{R}^n)$ is a C^k family, then for any $\rho > 0$ fixed, the family

$$[0, 1] \ni t \mapsto S_\rho f_t \in \mathcal{A}(\mathbb{C}^n)$$

is C^k as well and $\left(\frac{d}{dt}\right)^p S_\rho f_t = S_\rho \left(\frac{d}{dt}\right)^p f_t$ for $0 \leq p \leq k$. Moreover, if $t \rightarrow f_t(x)$ is analytic in a complex neighborhood V of $t = 0$ for each $x \in \mathbb{R}^n$, then so is $S_\rho f_t(x)$, and (A.5)-(A.7) are satisfied for $t \in V$.

A complete proof of the claims 1.-3. is given for example in [67], Lemma 3, and in [73]. In the case of anisotrop Hölder spaces the lemma has been obtained by Pöschel in [53]. The claim 4. follows easily from the properties of K . In order to obtain item 1, one uses Taylor's formula with integral remainder, which yields the estimate

$$|\partial^\alpha f(u + v) - (T_u^{[\ell] - |\alpha|} \partial^\alpha f)(v)| \leq c \|f\|_{C^{m, \mu}} |v|^{\ell - |\alpha|}$$

for $|\alpha| \leq m$ (see (3.4) in [67]). □

Using item 3. one obtains as in [67], Lemma 5, the interpolation and product estimates. More precisely, let r, s, ℓ be positive numbers such that $0 \leq r < s < \ell$, and $\ell = m + \mu$, $0 \leq \mu \leq 1$.

If $\mu < 1$, then $C^{m,\mu} = C^\ell$. Set $\nu := (\ell - s)/(\ell - r)$. Then there is a constant $c = c_{r,\ell} > 0$ depending only on ℓ and r such that for any compactly supported $f \in C_0^{m,\mu}(\mathbb{R}^n)$ the following estimate holds

$$\|f\|_{C^s} \leq c_{r,\ell} \|f\|_{C^r}^\nu \|f\|_{C^{m,\mu}}^{1-\nu}. \quad (\text{A.8})$$

Moreover, given $f, g \in C_0^{m,\mu}(\mathbb{T}^n \times \mathbb{R}^n)$ one can estimate the C^ℓ -norm of the product

$$\|fg\|_{C^{m,\mu}} \leq \|f\|_{C^0} \|g\|_{C^{m,\mu}} + \|f\|_{C^{m,\mu}} \|g\|_{C^0}. \quad (\text{A.9})$$

Remark A.2. Let $D \subset \mathbb{R}^m$ be an open bounded convex set. Then

$$\|f\|_{C^s} \leq c_{r,\ell,m} \|f\|_{C^r}^\nu \|f\|_{C^\ell}^{1-\nu}$$

for each $f \in C^\infty(\overline{D})$ or $f \in C^\infty(\mathbb{T}^n \times \overline{D})$. Moreover, there exists C_ℓ depending only on ℓ and on the dimensions n and m such that if $f, g \in C^\infty(\overline{D})$ or $f, g \in C^\infty(\mathbb{T}^n \times \overline{D})$ then

$$\|fg\|_\ell \leq C_\ell (\|f\|_0 \|g\|_\ell + \|f\|_\ell \|g\|_0).$$

Proof. Firstly we apply Whitney's extension theorem (see e.g. [71], Chapter VI, Theorem 4) to $f \in C^{k,\mu}(\overline{D})$, where \overline{D} is compact, $k \in \mathbb{N}$, $0 < \mu \leq 1$. We obtain an extension $\tilde{f} \in C^{k,\mu}(\mathbb{R}^m)$ of f such that

$$\|\tilde{f}\|_{C^{k,\mu}(\mathbb{R}^m)} \leq C_\ell \|f\|_{C^{k,\mu}(D)},$$

where $C_\ell = C_\ell(m) > 0$ depends only on $\ell = k + \mu$ and m . Moreover,

$$\|f\|_{C^{k,1}(D)} \leq \|f\|_{C^{k+1}(D)},$$

since f is C^∞ -smooth on the compact \overline{D} and D is convex. Then we apply the interpolation inequalities (A.8) to the extension $\tilde{f} \in C^{k,\mu}(\mathbb{R}^m)$ of f . In the same way we prove the product estimate. \square

Consider a subdivision $x = (x^{(1)}, \dots, x^{(p)}) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_p}$, $n = n_1 + \dots + n_p$, where $1 \leq p \leq n$. Given $a = (a_1, \dots, a_p)$, where a_j are positive numbers for $1 \leq j \leq p$, we denote by $\sigma_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the dilation $\sigma_a(x) = (a_1 x^{(1)}, \dots, a_p x^{(p)})$. More generally, for any $d \in \mathbb{N}$ we denote by $\sigma_a : \mathbb{T}^d \times \mathbb{R}^n \rightarrow \mathbb{T}^d \times \mathbb{R}^n$ the partial dilation $\sigma_a(\theta, x) = (\theta, a_1 x^{(1)}, \dots, a_p x^{(p)})$ (by convention $\mathbb{T}^0 = \{0\}$) and define the “ a -weighted” Hölder norm of $f \in C^\ell(\mathbb{T}^d \times \mathbb{R}^n)$ by

$$\|f\|_{\ell;a} := \|f \circ \sigma_a\|_{C^\ell}. \quad (\text{A.10})$$

A.2 Almost analytic Gevrey extensions

Proof of Proposition 11.11. Fix $\rho > 1$. Let φ be a real valued compactly supported Gevrey function belonging to the class $\mathcal{G}_\lambda^\rho(\mathbb{R}^n)$ for some $\lambda > 0$, which means that

$$\|\varphi\|_\lambda := \sup_{\alpha \in \mathbb{N}^n} \sup_{x \in \mathbb{R}^n} \left(|\partial_x^\alpha \varphi(x)| \lambda^{-|\alpha|} \alpha!^{-\rho} \right) < \infty, \quad (\text{A.11})$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\alpha! = \alpha_1! \dots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We suppose as well that the support of φ is contained in the unit ball $B_1^n(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ in \mathbb{R}^n and that

$$\int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Set $U_j^0 := O_j^2 \cap \mathbb{R}^n$ and $U_j^q := \{x \in \mathbb{R}^n : \text{dist}(x, U_j^0) < q/16\}$, $q \in \{0; 1; 2; 3; 4\}$, in particular, $U_j^4 = O_j^3 \cap \mathbb{R}^n$. Denote by $\mathbf{1}_j$ the characteristic function of the set U_j^2 in \mathbb{R}^n and consider for any $j \in \mathbb{N}$ the function f_j defined by the convolution

$$f_j(x) = (16/h_j)^n \int_{\mathbb{R}^n} \mathbf{1}_j(x-u) \varphi(16u/h_j) du = (16/h_j)^n \int_{\mathbb{R}^n} \mathbf{1}_j(u) \varphi(16(x-u)/h_j) du.$$

These functions have the following properties

- (1) $f_j \in \mathcal{G}_{\lambda_j}^\rho(\mathbb{R}^n)$ with $\lambda_j = 16\lambda/h_j$ and $\|f_j\|_{\lambda_j} \leq \text{vol}(B_1^n(0))\|\varphi\|_\lambda$;
 - (2) $\text{supp } f_j \subset U_j^3$ and $f_j = 1$ on U_j^1 ,
- (A.12)

where the positive constant λ is given in (A.11). We are going to obtain a Gevrey- \mathcal{G}^ρ almost analytic extension of f_j in \mathbb{C}^n which is equal to one on O_j^2 and has a support in O_j^3 .

To this end we introduce a family of the compact sets in $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$K_j := (\mathbb{R}^n \times \{0\}) \cup \overline{O_j^2} \cup (\overline{O_j} \setminus O_j^3),$$

where the set $O_j \subset \mathbb{C}^n$ is identified with the corresponding open set in $\mathbb{R}^n \times \mathbb{R}^n$ via the map $\mathbb{C}^n \ni x + iy \mapsto (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\overline{O_j}$ stands for the closure of O_j . Let us extend f_j to a continuous function with support in K_j by

$$\tilde{f}_j(x, y) := \begin{cases} f_j(x) & \text{if } x \in \mathbb{R}^n, y = 0; \\ 1 & \text{if } (x, y) \in \overline{O_j^2}; \\ 0 & \text{if } (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus O_j^3. \end{cases}$$

It is easy to see that a formal almost analytic extension of \tilde{f}_j is given by the power series

$$\sum_{\gamma \in \mathbb{N}^n} (iy)^\gamma \partial_x^\gamma f_j(x) / \beta!, \quad (\text{A.13})$$

which means that the operators $\bar{\partial}_k$, $k = 1, \dots, n$, annihilate it. The corresponding Taylor series centered at $(x, y) \in K_j$ is

$$\sum_{(\beta, \beta') \in \mathbb{N}^n \times \mathbb{N}^n} (x' - x)^\beta (iy' - iy)^{\beta'} \partial^{\beta + \beta'} f_j(x) / (\beta! \beta'!).$$

The family of jets $F_j = \left(f_j^{(\beta, \beta')} \right)_{\beta, \beta' \in \mathbb{N}^n}$ corresponding to the power series given above is defined for any $(\beta, \beta') \in \mathbb{N}^n \times \mathbb{N}^n$ and $(x, y) \in K_j$ by

$$f_j^{(\beta, \beta')}(x, y) = i^{|\beta'|} \partial^{\beta + \beta'} f_j(x). \quad (\text{A.14})$$

Remark A.3. If $(x, y) \in K_j$ and $y \neq 0$, then either $x \in U_j^0$ or $x \notin U_j^4$. On the other hand, $f_j = 1$ on U_j^1 and $f_j = 0$ on $\mathbb{R}^n \setminus U_j^3$. Then $f_j^\alpha(x, y) = 0$ for $\alpha = (\beta, \beta') \neq 0$ and $f_j^0(x, y) = 1$ if $(x, y) \in U_j^0 = O_j^2$, $f_j^0(x, y) = 0$ if $(x, y) \notin U_j^4 = O_j^3$. In particular,

$$f_j^\alpha(x, y) = f_j^\alpha(x, 0) \text{ for each } \alpha.$$

We are going to extend the jet F_j to a Gevrey- \mathcal{G}^ρ function using a Whitney extension theorem in Gevrey classes.

Let us first recall the notion of Gevrey smoothness of Whitney jets. Let K be a compact set in \mathbb{R}^d , $d \geq 1$, and $F = (f^\beta)_{\beta \in \mathbb{N}^d}$ a jet of continuous functions $f^\beta \in C(K)$. For each $N \in \mathbb{N}$ we denote by $T_u^N F$ the formal Taylor polynomial of order N centered at $u \in K$, i.e.

$$T_u^N F(z) := \sum_{|\beta| \leq N} f^\beta(u)(z-u)^\beta / \beta!, \quad z \in \mathbb{R}^d.$$

Given $\alpha \in \mathbb{N}^d$ we denote by $F^{(\alpha)}$ the jet $(f^{\alpha+\beta})_{\beta \in \mathbb{N}^d}$. Then for $|\alpha| \leq N$, the partial derivative ∂^α of the Taylor polynomial is given by

$$\partial_z^\alpha T_u^N F(z) = T_u^{N-|\alpha|} F^{(\alpha)}(z) = \sum_{|\beta| \leq N-|\alpha|} f^{\alpha+\beta}(u)(z-u)^\beta / \beta!.$$

For each $\alpha \in \mathbb{N}^d$, the corresponding Taylor remainder is defined by

$$\begin{aligned} R_u^N F^{(\alpha)}(z) &= f^\alpha(z) - \partial_z^\alpha T_u^N(F)(z) \\ &= f^\alpha(z) - T_u^{N-|\alpha|} F^{(\alpha)}(z) \\ &= f^\alpha(z) - \sum_{|\beta| \leq N-|\alpha|} f^{\alpha+\beta}(u)(z-u)^\beta / \beta!. \end{aligned}$$

Recall from Stein [71] p. 177 the following identity

$$\partial_z^\alpha T_v^N F(z) - \partial_z^\alpha T_u^N F(z) = \sum_{|\beta| \leq N} (z-v)^\beta R_u F^{(\alpha+\beta)}(v) / \beta! \quad (\text{A.15})$$

for any $N \in \mathbb{N}$, $|\alpha| \leq N$, $u, v \in K$ and $z \in \mathbb{R}^d$.

Let $L > 0$. The jet $F = (f^\beta)_{\beta \in \mathbb{N}^d}$ is said to belong to the Whitney space $W\mathcal{G}_L^\rho(K)$ of Gevrey jets if there exists $A > 0$ such that

$$\begin{aligned} (1) \quad & |f^\beta(u)| \leq AL^{|\beta|}(\beta!)^\rho \quad \text{for } \beta \in \mathbb{N}^d, u \in K; \\ (2) \quad & |R_u^N F^{(\gamma)}(z)| \leq AL^{N+1}((N+1)!)^\rho |z-u|^{N-|\gamma|+1} / (N-|\gamma|+1)! \\ & \text{for } |\gamma| \leq N, u, z \in K. \end{aligned} \quad (\text{A.16})$$

The corresponding norm of F is defined by $\|F\|_L := \inf A$. The space $W\mathcal{G}_L^\rho(K)$ equipped with this space is a Banach space. We recall the Whitney extension theorem of Bruna [4] as it has been presented in [59], Theorem 3.8.

Theorem A.4. *There exist positive constants $A_0 = A_0(d, \rho)$ and $C_0 = C_0(d, \rho)$ such that the following holds.*

For any compact subset K of \mathbb{R}^d and jet $F = (f^\beta)_{\beta \in \mathbb{N}^d} \in W\mathcal{G}_L^\rho(K)$, satisfying (A.16) on K with some $L > 0$, there exists $f \in \mathcal{G}_{C_0 L}^\rho(\mathbb{R}^d)$ such that

$$(i) \quad \partial^\beta f = f^\beta \text{ on } K \text{ for any } \beta;$$

(ii) $\|f\|_{C_0L} \leq A_0\|F\|_L$.

We are going to prove that F_j are Whitney jets. Using (A.12) we obtain

Lemma A.5. *For each $j \in \mathbb{N}$ the jet F_j belongs to the Whitney space $W\mathcal{G}_{L_j}^\rho(K_j)$ with $L_j = 385n^22^{n\rho}\lambda/h_j$ and $\|F_j\|_{L_j} \leq \text{vol}(B_1^n(0))\|\varphi\|_\lambda$.*

Proof. Set $A = \text{vol}(B_1^n(0))\|\varphi\|_\lambda$. Using item (1) of (A.12) we obtain for each $j \in \mathbb{N}$ the estimate

$$|f_j^{(\beta, \beta')}(x, y)| < A\lambda_j^{|\beta|+|\beta'|}((\beta + \beta')!)^\rho \leq A(2^{n\rho}\lambda_j)^{|\beta|+|\beta'|}(\beta!)^\rho(\beta')^\rho$$

for $(\beta, \beta') \in \mathbb{N}^n \times \mathbb{N}^n$, $(x, y) \in K_j$.

We are going to prove (ii). For any $u = (x, y) \in K_j$ the formal Taylor polynomial of F_j of order N which is centered at u and evaluated at $z = (x', y')$, is given by

$$T_u^N F_j(z) = \sum_{|\beta|+|\beta'| \leq N} i^{|\beta'|} \partial^{\beta+\beta'} f_j(x) (x' - x)^\beta (y' - y)^{\beta'} (\beta! \beta'!).$$

We consider separately the following two cases.

1. Let $z = (x', 0)$. Suppose at first that $u = (x, 0)$. Then, setting $M = N + 1 - |\alpha| - |\alpha'|$, we obtain by Taylor's formula

$$\begin{aligned} R_u^N F_j^{(\alpha, \alpha')}(z) &= f_j^{(\alpha, \alpha')}(z) - \partial_z^{(\alpha, \alpha')} T_u^N F_j(z) \\ &= i^{|\alpha'|} \partial^{\alpha+\alpha'} f_j(x') - \sum_{|\beta| \leq N - |\alpha| - |\alpha'|} i^{|\alpha'|} \partial^{\alpha+\alpha'+\beta} f_j(x) (x' - x)^\beta / (\beta!) \\ &= i^{|\alpha'|} M \sum_{|\beta|=M} \frac{(x' - x)^\beta}{\beta!} \int_0^1 (1-t)^{M-1} \partial^{\alpha+\alpha'+\beta} f_j(x + t(x' - x)) dt. \end{aligned}$$

Now item (1) of (A.12) yields

$$|R_u^N F_j^{(\alpha, \alpha')}(z)| \leq AL_j^{N+1} ((N+1)!)^\rho |x' - x|^M \sum_{|\beta|=M} \frac{M}{\beta!}.$$

On the other hand

$$\sum_{|\beta|=M} \frac{M}{\beta!} = \frac{M}{M!} \sum_{|\beta|=M} \frac{(\beta_1 + \dots + \beta_n)!}{\beta_1! \dots \beta_n!} = \frac{M}{M!} n^M < \frac{(2n)^M}{M!}.$$

Setting $\gamma = (\alpha, \alpha') \in \mathbb{N}^n \times \mathbb{N}^n$ and $\tilde{\lambda}_j = 2n\lambda_j = 32n\lambda/h_j$ we obtain

$$|R_u^N F_j^{(\gamma)}(z)| < A\tilde{\lambda}_j^{N+1} ((N+1)!)^\rho |z - u|^{N-|\gamma|+1} / (N - |\gamma| + 1)! \quad (\text{A.17})$$

for $z = (x', 0)$, $u = (x, 0) \in K_j$.

Let $u = (x, y) \in K_j$ and $y \neq 0$. By Remark A.3 we have $f^\gamma(x, y) = f^\gamma(x, 0)$ for each $\gamma \in \mathbb{N}^n \times \mathbb{N}^n$, hence, $R_u^N F^{(\gamma)}(z) = R_{(x,0)}^N F^{(\gamma)}(z)$. Then using (A.17) we obtain the same estimate since $|z - (x, 0)| < |z - u|$. Hence (A.17) is true for any $z = (x', 0)$ and $u = (x, y)$ in K_j .

2. Let $z = (x', y') \in K_j$, $y' \neq 0$ and $u \in K_j$. Set $v = (x', 0)$. Remark A.3 implies

$$f_j^\gamma(z) = \partial_z^\gamma T_v^N F_j(z)$$

and by means of (A.15) we obtain

$$R_u^N F_j^{(\gamma)}(z) = \partial_z^\gamma T_v^N F_j(z) - \partial_z^\gamma T_u^N F_j(z) = \sum_{|\beta| \leq N} (z-v)^\beta R_u^N F_j^{(\gamma+\beta)}(v) / \beta!.$$

Now applying (A.17) to $R_u^N F_j^{(\gamma+\beta)}(v)$ we get

$$|R_u^N F_j^{(\gamma)}(z)| < A \tilde{\lambda}_j^{N+1} ((N+1)!)^\rho \sum_{|\beta| \leq N} \frac{|y'|^{|\beta|} |v-u|^{N-|\beta|-|\gamma|+1}}{(N-|\beta|-|\gamma|+1)! \beta!}.$$

Remark A.3 implies that $|x' - x| \geq h_j/16$ on the support of the function $u \mapsto R_u^N F^{(\gamma)}(z)$ and we get

$$|y'| \leq \frac{2}{3} h_j \leq \frac{32}{3} |x' - x| \leq \frac{32}{3} |z - u| \quad \text{and} \quad |v - u| \leq |x' - x| + |y'| \leq \frac{35}{3} |z - u|.$$

Setting $L_j = 385n^2 > 35\tilde{\lambda}_j/3$ we obtain as above

$$|R_u^N F^{(\gamma)}(z)| < A L_j^{N+1} ((N+1)!)^\rho |z-u|^{N-|\gamma|+1} / (N-|\gamma|+1)!.$$

This completes the proof of the lemma. \square

Lemma A.5 enables us to apply Theorem A.4 and we denote by χ_j the corresponding extension of the jet $F_{j+1} \in W\mathcal{G}_{L_{j+1}}^\rho(K_{j+1})$. By construction, the Taylor series of χ_{j-1} at $(x, 0)$ coincides with the power series (A.13), which implies that χ_j is almost analytic. The function χ_j satisfies item (i) of Proposition 11.11 since it coincides with \tilde{f}_{j+1} on K_{j+1} . It satisfies (ii), taking $L = 385n^2 2^{n\rho} \lambda C_0$, where λ is introduced in (A.11) and C_0 is the constant in Theorem A.4. Fixing for any n the function φ and the constants λ and C_0 , we may suppose that $L = L(n, \rho)$ depends only on n and ρ .

It remains to prove (iii). Expanding $\partial_x^\alpha \partial_y^\beta \chi_j(x, y)$ in Taylor series at $y = 0$, we obtain for any $\alpha, \beta \in \mathbb{N}^n$ and $m \in \mathbb{N}$

$$|\partial_x^\alpha \partial_y^\beta \chi_j(x, y)| \leq A (L/h_{j+1})^{|\alpha|+|\beta|} ((L/h_{j+1}))^m \alpha!^\rho \beta!^\rho m!^{\rho-1} |y|^m, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Using Stirling's formula we minimize the right-hand side with respect to $m \in \mathbb{N}$. An optimal choice for m is given by

$$m \sim (L|y|/h_{j+1})^{-\frac{1}{\rho-1}},$$

which leads to

$$|\partial_x^\alpha \partial_y^\beta \chi_j(x, y)| \leq C_0 A (L/h_{j+1})^{|\alpha|+|\beta|} \alpha!^\rho \beta!^\rho \exp\left(-\frac{1}{2} (L|y|/h_{j+1})^{-\frac{1}{\rho-1}}\right)$$

for any $\alpha, \beta \in \mathbb{N}^n$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $0 < |y| \leq 1$, with $C_0 \geq 1$. \square

A.3 Borel's Theorem

Proof of Proposition 7.1. We follow the standard proof of the Borel's theorem (see [80], Theorem 4.15 and [45], Proposition 2.3.2). To simplify the notations set $z = (x, \xi) \in T^*\mathbb{R}^d$. Consider the increasing sequence $(\eta_j)_{j \in \mathbb{N}}$ where

$$\eta_j := 1 + \sup_{0 \leq k \leq 1} \sup_{|\alpha| \leq j} \sup_{m \leq j} \sup_{(t,z) \in I \times T^*\mathbb{R}^d} |\partial_t^k \partial_z^\alpha a_{t,m}(z)|.$$

Choose $\chi \in C^\infty(\mathbb{R})$ such that $\chi = 1$ on $(-\infty, 1/2]$, $0 < \chi < 1$ on $(1/2, 1)$, and $\chi = 0$ on the interval $[1, +\infty)$ and set

$$a_t(z, \lambda) = \sum_{j \geq 0} \chi\left(\frac{\eta_j}{|\lambda|}\right) a_{t,j}(z) \lambda^{-j}$$

where $\lambda \in \mathcal{D}$. Notice the the sum is finite for any $\lambda \in \mathcal{D}$ fixed. Moreover, $\frac{\eta_j}{|\lambda|} \chi\left(\frac{\eta_j}{|\lambda|}\right) \leq 1$, hence, for any $k \in \{0, 1\}$ and $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| \leq j$ we have

$$\chi\left(\frac{\eta_j}{|\lambda|}\right) |\partial_t^k \partial_z^\alpha a_{t,j}(z)| < \chi\left(\frac{\eta_j}{|\lambda|}\right) \eta_j \leq |\lambda|. \quad (\text{A.18})$$

On the other hand $|\lambda| \leq 2\eta_j$ whenever $\chi\left(\frac{\eta_j}{|\lambda|}\right) < 1$, hence,

$$\left(1 - \chi\left(\frac{\eta_j}{|\lambda|}\right)\right) |\partial_t^k \partial_z^\alpha a_{t,j}(z)| \leq C_{p,j,\alpha} |\lambda|^{-p} \quad (\text{A.19})$$

for any $p \in \mathbb{N}$. For any $N \in \mathbb{N}$ we have

$$\begin{aligned} & \left| \partial_t^k \partial_x^\alpha \left(a_t(z, \lambda) - \sum_{j=0}^{N-1} a_{t,j}(z) \lambda^{-j} \right) \right| \\ & \leq \sum_{j=0}^{N-1} \left(1 - \chi\left(\frac{\eta_j}{|\lambda|}\right)\right) |\partial_t^k \partial_x^\alpha a_{t,j}(z)| |\lambda|^{-j} + \sum_{j \geq N} \chi\left(\frac{\eta_j}{|\lambda|}\right) |\partial_t^k \partial_x^\alpha a_{t,j}(z)| |\lambda|^{-j}. \end{aligned}$$

Using (A.19) we estimate the first sum by $C_{N,\alpha} |\lambda|^{-N}$. If $N \geq |\alpha|$ taking into account (A.18) we estimate the second sum by $\eta_N |\lambda|^{-N} + \sum_{j \geq N+1} |\lambda|^{-j+1} \leq C_{N,\alpha} |\lambda|^{-N}$ for $|\lambda| \geq 2$. Finally, if $N < |\alpha|$ we estimate the first $|\alpha| - N$ terms of the second sum by $C_{N,\alpha} |\lambda|^{-N}$ and for the other terms we apply the preceding argument. This proves (7.65). The proof shows as well that for any $k \in \{0, 1\}$ and $\alpha \in \mathbb{N}^{2d}$ the function $(t, z) \rightarrow \partial_t^k \partial_x^\alpha a_t(z, \lambda)$ is a sum of a normally convergent series of functions, hence, it is a continuous for any λ fixed. Thus the map $J \ni t \rightarrow a_t$ is a C^1 family of symbols of order zero. \square

A.4 Higher order Hölder estimates of a composition and of the inverse function.

A.4.1 Estimates of the composition and the inverse function.

The aim of this section is to obtain estimates of the composition and the inverse function in certain Hölder norms with constants depending only on the dimension of the spaces and the

Hölder exponent. We start by introducing certain semi-norms as follows. Hereafter we denote the Euclidean norm of $x \in \mathbb{R}^n$ by $|x|$. Recall that for any $\ell \in \mathbb{R}$, $[\ell] \in \mathbb{R}$ stands for the entire part of ℓ and $\{\ell\} := \ell - [\ell] \in [0, 1)$ for the residual one. Given $f \in C^\ell(U)$, where $U \subset \mathbb{R}^n$ is an open set, we introduce a ℓ -semi-norm of f by

$$|f|_{\ell,U} := \begin{cases} \sup_{|\alpha|=\ell, x \in U} |\partial^\alpha f(x)| & , \quad \text{if } \ell \in \mathbb{N}; \\ H_{[\ell],\{\ell\}}(f) & , \quad \text{if } \ell \notin \mathbb{N}, \end{cases} \quad (\text{A.20})$$

where the semi-norm $H_{k,\mu}(f)$ is defined by (A.3). To simplify the notations we often write $|f|_\ell := |f|_{\ell,U}$. This notation should not be confused with the sup-norm in Sect. A.1. The μ -semi-norm of the product of two functions $f, g \in C^\mu(U)$ with $0 < \mu < 1$ can be estimated by

$$|fg|_\mu \leq |f|_\mu |g|_0 + |f|_0 |g|_\mu. \quad (\text{A.21})$$

More generally, for any $m \in \mathbb{N}$, $0 \leq \mu < 1$, and $f, g \in C^{m+\mu}(U)$, the Leibniz formula implies

$$|fg|_{m+\mu} \leq C_m \sum_{k=0}^m \left(|f|_{k+\mu} |g|_{m-k} + |f|_k |g|_{m-k+\mu} \right), \quad (\text{A.22})$$

where C_m depends only on m and on the dimension n .

Consider now the composition of Hölder functions.

Lemma A.6. *Let $U_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$, be open sets. Suppose that U_1 is convex. Let $0 < \mu < 1$ and $f_1 \in C^1(U_1)$, $f_2 \in C^\mu(U_1)$. Then $|f_2 \circ f_1|_{\ell,U_1} \leq |f_2|_{\ell,U_2} |f_1|_{1,U_1}^\ell$.*

To estimate higher Hölder norms of the composition of two functions we apply the Faa di Bruno formula. To this end we introduce the following notations. Given $\ell \geq 1$, $f \in C^\ell(U)$ and an integer $1 \leq m \leq \ell$ we set

$$\mathcal{P}_U^{\ell,m}(f) := \sum_{\Delta(\ell,m)} \sum_{j=1}^m |f|_{k_1} \cdots |f|_{k_{j-1}} |f|_{k_j+\{\ell\}} |f|_{k_{j+1}} \cdots |f|_{k_m} \quad (\text{A.23})$$

where the $|f|_\alpha := |f|_{\alpha,U}$ and the index set $\Delta(\ell, m)$ consists of all $(k_1, \dots, k_m) \in \mathbb{N}^m$ such that

$$k_1 + \cdots + k_m = [\ell], \quad \min_{1 \leq j \leq m} k_j \geq 1.$$

For any $\ell \geq 1$ we set

$$\mathcal{P}_U^\ell(f) := \sum_{m=1}^{[\ell]} \mathcal{P}_U^{\ell,m}(f).$$

Proposition A.7. *Let $f_j \in C^\ell(U_j, U_{j+1})$, $j = 1, 2$, where $\ell \geq 1$ and $U_j \subset \mathbb{R}^{n_j}$ are open sets such that $f_1(U_1) \subset U_2$. Then the following holds:*

1. *If $\ell \in \mathbb{N}_*$ then*

$$|f_2 \circ f_1|_{\ell,U_1} \leq C_\ell \sum_{m=1}^{\ell} |f_2|_{m,U_2} \mathcal{P}_{U_1}^{\ell,m}(f_1)$$

where the constant C_ℓ depends only on ℓ and on the dimensions n_j , $j = 1, 2, 3$.

2. If $\ell \in [1, \infty)$ and U_1 is convex, then

$$|f_2 \circ f_1|_{\ell, U_1} \leq C_\ell \left(1 + |f_1|_{1, U_1}^{\{\ell\}}\right) \\ \times \sum_{m=1}^{[\ell]} \left(|f_2|_{m+\{\ell\}, U_2} \mathcal{P}_{U_1}^{[\ell], m}(f_1) + |f_2|_{m, U_2} \mathcal{P}_{U_1}^{\ell, m}(f_1)\right)$$

where the constant C_ℓ depends only on $[\ell]$ and on the dimensions n_j , $j = 1, 2, 3$.

Proof. Statement 1. follows directly from the Faa di Bruno formula

$$\partial^\alpha (f_2 \circ f_1) = \sum \frac{(\partial^\beta f_2) \circ f_1}{\beta!} (\partial^{\alpha_1} f_1) \cdots (\partial^{\alpha_m} f_1) \frac{\alpha_1! \cdots \alpha_m!}{\alpha!}$$

where the summation is over all the indices

$$m \in \mathbb{N}_*, \quad \beta \in \mathbb{N}^{n_2}, \quad (\alpha_1, \dots, \alpha_m) \in \underbrace{\mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_1}}_m = \mathbb{N}^{n_1 m}$$

such that

$$1 \leq |\beta| = m \leq |\alpha|, \quad \alpha_1 + \dots + \alpha_m = \alpha, \quad \min_{1 \leq j \leq m} |\alpha_j| \geq 1.$$

Here $|\beta| = \beta_1 + \dots + \beta_{n_2}$ stands for the length of $\beta \in \mathbb{N}^{n_2}$.

We are going to prove 2. Suppose now that U_1 is convex and that $\ell = m + \mu$, where $m \in \mathbb{N}_*$ and $0 < \mu < 1$. Firstly we apply (A.21) to the product in the Faa di Bruno formula. Then we estimate $|(\partial^\beta f_2) \circ f_1|_{\mu, U_1}$ by means of Lemma A.6, which yields

$$|(\partial^\beta f_2) \circ f_1|_{\mu, U_1} \leq |\partial^\beta f_2|_{\mu, U_2} |f_1|_{1, U_1}^\mu.$$

This implies 2. □

Similar inequalities can be proven for compensated domains U_1 ([42], Theorem 5.4) but then the constants depend on U_1 .

Proposition A.8. *Let $\ell > 0$ and let $f \in C^{\ell+1}(U, V)$ be a diffeomorphism with inverse $g = f^{-1}$, where U and $V = f(U)$ are open subsets of \mathbb{R}^n , $n \geq 1$. Then the following holds:*

1. If $\ell \in \mathbb{N}_*$ then

$$\|dg\|_{\ell, V} \leq C_\ell \left(1 + \|(df)^{-1}\|_{C^0(U)}^{3\ell}\right) \mathcal{P}_U^\ell(df)$$

where $C_\ell > 0$ depends only on ℓ and on the dimension n ;

2. Let $0 < \ell < 1$ and let $V = f(U)$ be convex. Then

$$\|dg\|_{C^\ell(V)} \leq \|(df)^{-1}\|_{C^0(U)}^\ell \|(df)^{-1}\|_{C^\ell(U)} \leq \|(df)^{-1}\|_{C^0(U)}^{\ell+2} \|df\|_{C^\ell(U)};$$

3. Let $\ell = m + \mu$, where $m \in \mathbb{N}_*$ and $0 < \mu < 1$. Let $V = f(U)$ be convex. Then

$$\|dg\|_{C^\ell(V)} \leq C_\ell \left(1 + \|(df)^{-1}\|_{C^0(U)}^{3\ell+2}\right) \left(\mathcal{P}_U^\ell(df) + \|df\|_{C^\mu(U)} \mathcal{P}_U^m(df)\right),$$

where $C_\ell > 0$ depends only on ℓ and on the dimension n .

Proof. To prove the statement one needs a sort of Lagrange inversion formula for higher derivatives of the Jacobian matrix $Dg \in C^\infty(U, M_n(\mathbb{R}))$. Hereafter $M_n(\mathbb{R}) = M_{n,n}(\mathbb{R})$ is the space of real $n \times n$ matrices equipped with the corresponding sup-norm and we denote by $A \cdot B$ the product of two matrices A and B . Denote by $L_p : M_n(\mathbb{R}) \rightarrow M_{1,n}(\mathbb{R})$ the linear operator which assigns to each matrix $A \in M_n(\mathbb{R})$ its p^{th} -line and by $C_q : M_n(\mathbb{R}) \rightarrow M_{n,1}(\mathbb{R})$ the linear operator which assigns to each matrix $A \in M_n(\mathbb{R})$ its q^{th} -column. We identify L_p and C_q with the corresponding matrices in $M_{1,n}(\mathbb{R})$ and $M_{n,1}(\mathbb{R})$, respectively. Since $(Dg)(f(x)) = Df(x)^{-1}$, where $Df(x)^{-1}$ is the inverse of the matrix $Df(x)$, we get for any $1 \leq p, q \leq n$ the equality

$$\left(\frac{\partial}{\partial y_q} g_p \right) (y) = L_p \cdot Df(x)^{-1} \cdot C_q \in \mathbb{R} \quad \text{as } y = f(x). \quad (\text{A.24})$$

Then differentiating the identity $Dg(y) = (Df)^{-1}(g(y))$ with respect to y_q we obtain

$$\left(\frac{\partial}{\partial y_q} Dg \right) (y) = \sum_{p=1}^n \frac{\partial}{\partial x_p} (Df(x)^{-1}) \cdot (L_p \cdot Df(x)^{-1} \cdot C_q) I_n \quad \text{as } x = g(y).$$

Denote by \mathcal{A} the set of automorphisms of $M_n(\mathbb{R})$ generated under composition by the identity map and the automorphisms of the form

$$A \mapsto (L_p \cdot A \cdot C_q) I_n, \quad A \in M_n(\mathbb{R}).$$

Using (A.24) we obtain by induction with respect to $m \in \mathbb{N}$ the following relation

$$\partial_y^\alpha Dg(y) = \sum_{\gamma \in \Delta_\alpha} c_\gamma \prod_{1 \leq j \leq m+1} \mathcal{L}_j(\partial_x^{\gamma_j} (Df(x)^{-1})) \quad \text{as } x = g(y), \quad (\text{A.25})$$

for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$. The index set Δ_α consists of all $\gamma = (\gamma_1, \dots, \gamma_{m+1}) \in (\mathbb{N}^n)^{m+1}$ such that

$$|\gamma_1| + \dots + |\gamma_{m+1}| = m = |\alpha|,$$

$c_\gamma \in \mathbb{R}$ are universal constants, and $\mathcal{L}_j \in \mathcal{A}$. Consider the derivatives of the inverse matrix $Df(x)^{-1}$ of $Df(x)$. One can easily show that

$$\frac{\partial}{\partial x_p} (Df(x)^{-1}) = -Df(x)^{-1} \cdot \left(\frac{\partial}{\partial x_p} Df(x) \right) \cdot Df(x)^{-1}. \quad (\text{A.26})$$

This equality implies by induction that for any $0 \neq \alpha \in \mathbb{N}^n$,

$$\partial_x^\alpha (Df(x)^{-1}) = \sum_{\beta \in \Delta_\alpha^1} c_\beta \prod_{1 \leq j \leq |\alpha|} Df(x)^{-1} \cdot \partial_x^{\beta_j} Df(x) \cdot Df(x)^{-1}, \quad (\text{A.27})$$

where $c_\beta \in \mathbb{R}$ are universal constants and the index set Δ_α^1 consists of all

$$\beta = (\beta_1, \dots, \beta_{|\alpha|}) \in \mathbb{N}^n \times \dots \times \mathbb{N}^n = (\mathbb{N}^n)^{|\alpha|}$$

such that

$$|\beta_1| + \dots + |\beta_{|\alpha|}| = |\alpha|.$$

Now statement 1. follows easily from (A.25) and (A.27).

We are going to prove 2 and 3. Since V is convex, we have in view of Lemma A.6

$$\|h \circ g\|_{C^\mu(V)} \leq \|h\|_{C^\mu(U)} \|Dg\|_{C^0(V)}^\mu = \|h\|_{C^\mu(U)} \|(Df)^{-1}\|_{C^0(U)}^\mu$$

for any $h \in C^\mu(U)$ and $0 < \mu < 1$. Moreover,

$$\|(Df)^{-1}\|_{C^\mu(U)} \leq \|Df\|_{C^\mu(U)} \|(Df)^{-1}\|_{C^0(U)}^2.$$

We have $Dg = (Df)^{-1} \circ g$, hence, taking $h = (Df)^{-1}$ we obtain 2. Statement 3 follows from (A.25) and (A.27) as in the proof of Proposition A.7. \square

Proof of Lemma 9.3. One can take the convolution $\psi_\varepsilon^U = \mathbf{1}_U * \chi_\varepsilon$, where $\mathbf{1}_U$ is the characteristic function of U , $\chi_\varepsilon(x) = \varepsilon^{-n} \chi(x/\varepsilon)$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ is a test-function such that

$$\chi(x) > 0 \text{ if } |x| < 1, \chi(x) = 0 \text{ if } |x| \geq 1 \text{ and } \int_{\mathbb{R}^n} \chi(x) dx = 1.$$

More precisely, we first define a smooth function $\tilde{\chi}$ by $\tilde{\chi}(x) = \exp(-(1 - |x|^2)^{-1})$ for $|x| < 1$ and by $\tilde{\chi}(x) = 0$ for $|x| \geq 1$ and then we set $\chi(x) = \tilde{\chi}(x)/\|\tilde{\chi}\|_{L^1}$. For any $\ell \geq 0$ we have $\|\chi_\varepsilon\|_{\ell;\varepsilon} = \varepsilon^{-n} \|\chi\|_\ell$. Set $\phi := \psi_1^B$ and $\phi_\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon)$, where $B = B^n(0, 2)$. Then $\phi(x) = 1$ for $|x| \leq 1$, hence, $\phi(x/\varepsilon) = 1$ for x in the support of χ_ε and we easily obtain the inequality

$$|\partial_x^\alpha \chi_\varepsilon(x - y) - \partial_z^\alpha \chi_\varepsilon(z - y)| \leq (\phi((x - y)/\varepsilon) + \phi((z - y)/\varepsilon)) |\partial_x^\alpha \chi_\varepsilon(x - y) - \partial_z^\alpha \chi_\varepsilon(z - y)|$$

for any $x, y, z \in \mathbb{R}^n$ and $0 < \varepsilon \leq 1$. This implies

$$\|\psi_\varepsilon^U\|_{\ell;\varepsilon} \leq 2\varepsilon^n \|\mathbf{1}_U * \phi_\varepsilon\|_{L^1} \|\chi_\varepsilon\|_{\ell;\varepsilon} \leq 2\|\phi_\varepsilon\|_{L^1} \|\chi\|_\ell = 2\|\phi\|_{L^1} \|\chi\|_\ell = C_\ell$$

which proves the Lemma. \square

A.4.2 Higher order Hölder estimates and Interpolation inequalities.

The above estimates can be simplified considerably if the domain of definition of the functions is the whole space or an open convex bounded set. We set $\mathbb{A} := \mathbb{T}^n \times D$, where D is an open set in \mathbb{R}^d . We shall use the convention $\mathbb{A} := \mathbb{T}^n$ if $d = 0$ and $\mathbb{A} := D$ if $n = 0$.

Using the interpolation inequalities as in [42], Proposition 5.5, one obtains

Proposition A.9. *Let $f \in C^\infty(\mathbb{A}_1, \mathbb{A}_2)$ and $g \in C^\infty(\mathbb{A}_2, \mathbb{R})$, where $\mathbb{A}_1 = \mathbb{T}^{n_1} \times \mathbb{R}^{d_1}$, $\mathbb{A}_2 = \mathbb{T}^{n_2} \times D_2$ and $D_2 \subset \mathbb{R}^{d_2}$ is an open set. Then the following holds*

1. For any $\ell \geq 1$,

$$\begin{aligned} |g \circ f|_{\ell, \mathbb{A}_1} &\leq C_\ell (1 + \|df\|_{0, \mathbb{A}_1}^\ell) \\ &\times \sum_{m=1}^{[\ell]} (|g|_{m, \mathbb{A}_2} \|df\|_{\ell-m, \mathbb{A}_1} + |g|_{m+\{\ell\}, \mathbb{A}_2} \|df\|_{[\ell]-m, \mathbb{A}_1}) \end{aligned}$$

where $C_\ell > 0$ depends only on $\ell \geq 1$ and on the dimensions n_j and d_j .

2. Let $\mathbb{A}_j = \mathbb{T}^{n_j} \times \mathbb{R}^{d_j}$, $j = 1, 2$. Then

$$\begin{aligned} |g \circ f|_{\ell, \mathbb{A}_1} &\leq C_\ell (1 + \|df\|_{0, \mathbb{A}_1}^\ell) \\ &\times (\|g\|_{\ell, \mathbb{A}_2} \|df\|_{0, \mathbb{A}_1} + \|g\|_{1, \mathbb{A}_2} \|df\|_{\ell-1, \mathbb{A}_1}) \end{aligned} \quad (\text{A.28})$$

where $C_\ell > 0$ depends only on $\ell \geq 1$ and on the dimensions n_j and d_j .

3. Let D_j , $j = 1, 2$, be open convex subsets of \mathbb{R}^{d_j} and $\overline{\mathbb{A}}_j = \mathbb{T}^{n_j} \times \overline{D}_j$. Then (A.28) holds for any $f \in C^\infty(\overline{\mathbb{A}}_1, \overline{\mathbb{A}}_2)$ and $g \in C^\infty(\overline{\mathbb{A}}_2, \mathbb{R})$ with a constant C_ℓ depending only on $\ell \geq 1$ and the dimensions n_j and d_j .

Proof. To prove the first statement we make use of Proposition A.7 and of the interpolation inequalities. Consider a typical term of (A.23) given by

$$A := |f|_{k_1} \cdots |f|_{k_{j-1}} |f|_{k_j + \{\ell\}} |f|_{k_{j+1}} \cdots |f|_{k_m}$$

where $k_1 + \cdots + k_m = [\ell]$ and $k_p \geq 1$, for any p . Set $r = 0$, $s = s_p = k_p + \delta_p - 1$ and $t = \ell - m$, where $\delta_p = 0$ if $p \neq j$ and $\delta_j = \{\ell\}$. By means of the interpolation inequalities we get

$$|f|_{k_p + \delta_p} \leq \|df\|_s \leq c_\ell \|df\|_0^{\frac{t-s}{t}} \|df\|_t^{\frac{s}{t}}$$

which implies

$$A \leq c_\ell \|df\|_0^{m-1} \|df\|_{\ell-m},$$

where $c_\ell > 0$ depends only on ℓ . Hence,

$$\mathcal{P}_U^{\ell, m}(f) \leq C_\ell \sum_{m=1}^{[\ell]} \|df\|_0^{m-1} \|df\|_{\ell-m}. \quad (\text{A.29})$$

Notice also that $|f|_{1, \mathbb{A}_1}^{\{\ell\}} \leq 1 + |df|_0$. Using Proposition A.7 one obtains 1.

To prove 2 one uses the interpolation inequalities with respect to both functions f and g . Namely, given $0 < s < t$ and $u, v \in C^t$ one obtains

$$\|u\|_s \|v\|_{t-s} \leq c_t \|u\|_0^{\frac{t-s}{t}} \|u\|_t^{\frac{s}{t}} \|v\|_0^{\frac{s}{t}} \|v\|_t^{\frac{t-s}{t}} < c_t (\|u\|_t \|v\|_0 + \|v\|_t \|u\|_0) \quad (\text{A.30})$$

by means of (A.8) and Young's inequality

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q < x^p + y^q,$$

where x and y are non-negative and p and q are positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Putting $s = m$, $t = \ell$, $u = g$, $v = dg$, and then $s = m - 1 + \{\ell\}$, $t = \ell - 1$, $u = dg$, $v = dg$, and using 1, we obtain 2. The inequality in 2 has been proven for more general domains in ([42], Proposition 5.5) but the constants there depend on the domains. Statement 3 follows from 2 and Remark A.2. \square

In the same way we obtain

Proposition A.10. *Let $f \in C^\infty(\overline{\mathbb{A}}_1, \overline{\mathbb{A}}_2)$ be a diffeomorphism with inverse $g = f^{-1} \in C^\infty(\overline{\mathbb{A}}_2, \overline{\mathbb{A}}_1)$, where $\overline{\mathbb{A}}_j = \mathbb{T}^n \times \overline{D}_j$ and $D_j \subseteq \mathbb{R}^d$, $j = 1, 2$, are open sets. Then the following holds.*

1. *Let D_1 be convex. Then for any positive integer $\ell \in \mathbb{N}_*$,*

$$\|dg\|_{\ell, \mathbb{A}_2} \leq C_\ell \left(1 + \|(df)^{-1}\|_{0, \mathbb{A}_1}^{3\ell+1}\right) \sum_{m=1}^{[\ell]} \|df\|_{0, \mathbb{A}_1}^{m-1} \|df\|_{\ell-m, \mathbb{A}_1} \quad (\text{A.31})$$

where C_ℓ depends only on ℓ and on the dimensions n and d .

2. *Let both D_1 and D_2 be convex. Then (A.31) holds for any $\ell \geq 1$.*

Proof. The inequality (A.31) follows for $\ell \in \mathbb{N}_*$ from Proposition A.8 and Remark A.2 using (A.29) as in the proof of Proposition A.9. To prove it for any $\ell \geq 1$ we use (A.30) as well. \square

Given $n, d \in \mathbb{N}$ we set $\mathbb{A} = \mathbb{T}^n \times \mathbb{R}^d$. Using the interpolation inequalities and Proposition A.8 we obtain

Proposition A.11. *Let $u = \text{id} + \phi$, where $\phi \in C^\infty(\mathbb{A}, \mathbb{A})$, $\mathbb{A} = \mathbb{T}^n \times \mathbb{R}^d$, and*

$$(n + d)\|\phi\|_1 \leq \varepsilon_0 < 1.$$

Then u is a diffeomorphism homotope to the identity with inverse $u^{-1} = \text{id} + \psi$, where $\psi \in C^\infty(\mathbb{A}, \mathbb{A})$ and for any $\ell \geq 0$ we have

$$\|\psi\|_\ell \leq C_\ell \|\phi\|_\ell$$

where $C_\ell = C_\ell(\varepsilon_0, n, d) > 0$ depends only on ℓ , ε_0 , and on the dimensions n and d . If $\text{supp } \phi \subset \mathbb{T}^n \times K$ then $\text{supp } \psi \subset \mathbb{T}^n \times K$ as well. Moreover, if the map $[0, \delta] \ni t \rightarrow \phi_t \in C^\infty(\mathbb{A}, \mathbb{A})$ is C^k then so is the map $t \rightarrow \psi_t$. The same holds if $\mathbb{A} = \mathbb{T}^n$ ($d = 0$) or $\mathbb{A} = \mathbb{R}^d$ ($n = 0$).

Proof. Notice that $\|d\phi\|_0 < (n + d)\|\phi\|_1 \leq \varepsilon_0 < 1$. The inverse function theorem implies that u is a diffeomorphism with inverse $u^{-1} = \text{id} + \psi$, where $\psi \in C^\infty(\mathbb{A}, \mathbb{A})$. Moreover,

$$\psi = -\phi \circ (\text{id} + \psi)^{-1}, \quad (\text{A.32})$$

which implies $\|\psi\|_0 = \|\phi\|_0$. If $0 \leq \ell < 1$, then

$$\|\psi\|_\ell \leq \|\phi\|_0 + \|\phi\|_{C^\ell} \|(\text{Id} + d\phi)^{-1}\|_0^\ell \leq C(1 - \varepsilon_0)^{-1} \|\phi\|_\ell.$$

The estimate of ψ in the C^ℓ norms with $\ell \geq 1$ follows from (A.32) using Proposition A.9 and Proposition A.10. To prove the assertion about the support notice that $u^{-1} = \text{id}$ on $\mathbb{T}^n \times (\mathbb{R}^d \setminus K)$. \square

A.4.3 Weighted Hölder norms and interpolation inequalities.

Given $0 < \kappa \leq 1$, $\ell \geq 0$, and $f \in C^\ell(\mathbb{T}^n \times D)$, where $D \subset \mathbb{R}^d$ is an open set we define the corresponding weighted C^ℓ norm by

$$\|f\|_{\ell, \mathbb{T}^n \times D; \kappa} := \|f \circ \sigma_\kappa\|_{\ell, \sigma_\kappa^{-1}(\mathbb{T}^n \times D)},$$

where $\sigma_\kappa(\theta, r) = (\theta, \kappa r)$. We set as well

$$|f|_{\ell, \mathbb{T}^n \times D; \kappa} := |f \circ \sigma_\kappa|_{\ell, \sigma_\kappa^{-1}(\mathbb{T}^n \times D)}.$$

In particular, if $\ell \in \mathbb{N}$, then

$$\|f\|_{m, \mathbb{T}^n \times D; \kappa} = \sup_{0 \leq m \leq \ell} |f|_{m, \mathbb{T}^n \times D; \kappa},$$

where

$$|f|_{m, \mathbb{T}^n \times D; \kappa} := \sup_{|\alpha|+|\beta|=m} \|\partial_\theta^\alpha (\kappa \partial_r)^\beta f\|_{C^0(\mathbb{T}^n \times D)}.$$

Applying (A.8) to $f = u \circ \sigma_\kappa$ with $0 < \kappa \leq 1$ one gets the interpolating inequalities for $\|u\|_{t; \kappa} := \|u\|_{t, \mathbb{A}; \kappa}$, where $\mathbb{A} = \mathbb{T}^n \times \mathbb{R}^d$.

We list below several estimates which follow directly from Proposition A.9. Set

$$\|u\|_{\ell, D; \kappa} = \sup_{0 \leq m \leq \ell} \|u\|_{\ell-m, D; \kappa},$$

where m are integers.

Proposition A.12. *Fix $0 < \kappa \leq 1$.*

1. *Let $u \in C^\infty(\mathbb{A}_1, \mathbb{A}_2)$ and $v \in C^\infty(\mathbb{A}_2, \mathbb{R})$, where $\mathbb{A}_1 = \mathbb{T}^{n_1} \times \mathbb{R}^{d_1}$, $\mathbb{A}_2 = \mathbb{T}^{n_2} \times D_2$ and $D_2 \subset \mathbb{R}^{d_2}$ is an open set. Then for any $\ell \geq 1$,*

$$\begin{aligned} |v \circ u|_{\ell, \mathbb{A}_1; \kappa} &\leq C_{[\ell]} \left(1 + \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^0}^{\ell-1} \right) \\ &\times \sum_{m=1}^{[\ell]} \left(\|v\|_{m, \mathbb{A}_2; \kappa} \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^{\ell-m}(\mathbb{A}_1)} + \|v\|_{m+\{\ell\}, \mathbb{A}_2; \kappa} \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^{[\ell]-m}(\mathbb{A}_1)} \right) \end{aligned}$$

where $C_\ell > 0$ depends only on ℓ and on the dimensions n_j and d_j .

2. *Let $\mathbb{A}_1 = \mathbb{T}^{n_2} \times D_1$, where D_1 is an open convex subset of \mathbb{R}^{d_1} , $\mathbb{A}_2 = \mathbb{T}^{n_2} \times \mathbb{R}^{d_2}$, and $u \in C^\infty(\overline{\mathbb{A}_1}, \mathbb{A}_2)$. Then*

$$\begin{aligned} |v \circ u|_{\ell, \mathbb{A}_1; \kappa} &\leq C_\ell \left(1 + \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^0}^{\ell-1} \right) \\ &\times (\|v\|_{\ell; \kappa} \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^0} + \|v\|_{1; \kappa} \|d(\sigma_\kappa^{-1} \circ u \circ \sigma_\kappa)\|_{C^{\ell-1}}) \end{aligned}$$

for any $\ell \geq 1$, where $C_\ell > 0$ depends only on ℓ and on the dimensions n_j and d_j . In particular, if $\mathbb{A}_2 = \mathbb{T}^{n_2}$ ($d_2 = 0$) then

$$\|v \circ u\|_{\ell; \kappa} \leq \|v \circ u\|_{C^0} + C_\ell \left(1 + \|u\|_{1; \kappa}^{\ell-1} \right) (\|v\|_{\ell; \kappa} \|u\|_{1; \kappa} + \|v\|_{1; \kappa} \|u\|_{\ell; \kappa}).$$

3. Let $D_j \subset \mathbb{R}^{d_j}$, $j = 1, 2$, be open sets in \mathbb{R}^{d_j} and let D_1 be convex. Then for any $u \in C^\infty(\overline{D_1}, \mathbb{R}^{d_2})$ with $u(D_1) \subset D_2$ and $v \in C^\infty(\mathbb{A}_2)$ with $\mathbb{A}_2 = \mathbb{T}^{n_2} \times D_2$ we have

$$\begin{aligned} |v \circ u|_{\ell, D_1; \kappa} &\leq C_\ell \left(1 + \|du\|_{C^0(D_1)}^{\ell-1} \right) \\ &\times \sum_{m=1}^{[\ell]} \left(|v|_{m, \mathbb{A}_2; \kappa} (1 + \|du\|_{\ell-m, D_1; \kappa}) + |v|_{m+\{\ell\}, \mathbb{A}_2; \kappa} (1 + \|du\|_{[\ell]-m, D_1; \kappa}) \right) \\ &\leq C_\ell \left(1 + \|du\|_{C^0(D_1)}^{\ell-1} \right) (1 + \|du\|_{C^\ell(D_1)}) \|v\|_{\ell, \mathbb{A}_2; \kappa} \end{aligned}$$

for any $\ell \geq 1$, where the constant $C_{[\ell]} > 0$ depends only on $[\ell]$ and on the dimensions d_j and n_2 .

Proof. To prove 1 we put $f = \sigma_\kappa^{-1} \circ u \circ \sigma_\kappa$ and $g = v \circ \sigma_\kappa$ in Proposition A.9. The statement 2 follows from Proposition A.9, 3. To prove 3 we use Proposition A.7, 3, and apply Remark A.2 to $u \in C^\infty(\overline{D_1})$ using the interpolation inequalities for the extension of u as in the proof of Proposition A.9. To prove the second inequality in 3 notice that $\|du\|_{\mu, D_1; \kappa} \leq \|du\|_{\ell, D_1; \kappa}$ for $\mu \leq \ell$ since D_1 is convex. \square

Remark A.13. The estimates in Proposition A.12 hold when $\kappa = (\kappa', \kappa'')$ with $0 < \kappa', \kappa'' \leq 1$, $D_j = D'_j \times D''_j$, $\sigma_\kappa(\theta, x', x'') = (\theta, \kappa' x', \kappa'' x'')$ and $|u|_{\ell, \mathbb{A}_1; \kappa} := |u \circ \sigma_\kappa|_{C^\ell(\sigma_\kappa^{-1}(\mathbb{A}_1))}$.

A.4.4 Symplectic transformations and generating functions.

Consider a C^1 family of exact symplectic maps $W_t : \mathbb{A} \rightarrow \mathbb{A}$, $t \in [0, \delta]$, where $\mathbb{A} := \mathbb{T}^d \times \mathbb{R}^d$. Suppose that $W_t - \text{id}$ is compactly supported for any t . We are looking for a C^1 family of generating functions

$$\tilde{G}_t(\varphi, r) = \langle \varphi, r \rangle - G_t(\text{pr}(\varphi), r), \quad (\varphi, r) \in \mathbb{R}^d \times \mathbb{R}^d, \quad (\text{A.33})$$

of W_t such that the function $G_t \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is compactly supported with respect to r and

$$W_t \left(\nabla_r \tilde{G}_t(\varphi, r), r \right) = \left(\varphi, \nabla_\varphi \tilde{G}_t(\varphi, r) \right), \quad (\varphi, r) \in \mathbb{A} \quad (\text{A.34})$$

(see Definition 3.1). Slightly abusing the notations, we will identify below a 2π -periodic function with the corresponding functions on \mathbb{T}^d . Given a smooth function G in $\mathbb{A} := \mathbb{T}^d \times D$, we denote by

$$\text{sgrad } G(\theta, r) := (\nabla_r G(\theta, r), -\nabla_\theta G(\theta, r))$$

its symplectic gradient. Notice that $\|\sigma_\kappa^{-1} \text{sgrad } G_t\|_{\ell, \mathbb{A}; \kappa} \leq \kappa^{-1} \|G_t\|_{\ell+1, \mathbb{A}; \kappa}$.

Lemma A.14. 1. Let $[0, \delta] \ni t \rightarrow W_t : \mathbb{A} \rightarrow \mathbb{A}$ be a C^1 family of exact symplectic mappings. Suppose that

$$2d \|\sigma_\kappa^{-1}(W_t - \text{id})\|_{1, \mathbb{A}; \kappa} \leq \varepsilon_0 < 1 \quad (\text{A.35})$$

for $t \in [0, \delta]$. Then there exists a C^1 family of generating functions \tilde{G}_t of W_t given by (A.33) such that for any $\ell \geq 0$ the following estimate hold true

$$\|\sigma_\kappa^{-1} \text{sgrad } G_t\|_{\ell, \mathbb{A}; \kappa} \leq C_\ell \|\sigma_\kappa^{-1}(W_t - \text{id})\|_{\ell, \mathbb{A}; \kappa} \quad (\text{A.36})$$

for $t \in [0, \delta]$, where $C_\ell = C_\ell(\varepsilon_0, d) > 0$ depends only on ℓ , ε_0 and d . Moreover, the relation $\text{supp}(W_t - \text{id}) \subset \mathbb{T}^d \times K$ implies $\text{supp}(\text{sgrad } G_t) \subset \mathbb{T}^d \times K$ as well.

2. Conversely, let G_t be a C^1 family of functions such that

$$2d \|\sigma_\kappa^{-1} \text{sgrad } G_t\|_{1, \mathbb{A}; \kappa} \leq \varepsilon_0 < 1 \quad (\text{A.37})$$

for $t \in [0, \delta]$. Then \tilde{G}_t given by (A.33) is a C^1 family of generating functions of symplectic maps $W_t : \mathbb{A} \rightarrow \mathbb{A}$ and for any $\ell \geq 0$ we have

$$\begin{aligned} & \|\sigma_\kappa^{-1}(W_t - \text{id})\|_{\ell, \mathbb{A}; \kappa} + \|\sigma_\kappa^{-1}(W_t^{-1} - \text{id})\|_{\ell, \mathbb{A}; \kappa} \\ & \leq C_\ell \|\sigma_\kappa^{-1} \text{sgrad } G_t\|_{\ell, \mathbb{A}; \kappa} \end{aligned} \quad (\text{A.38})$$

where $t \in [0, \delta]$ and $C_\ell = C_\ell(\varepsilon_0, d) > 0$ depends only on ℓ , ε_0 and d . Moreover, if $\text{supp}(\text{sgrad } G_t) \subset \mathbb{T}^d \times K$ then $\text{supp}(W_t - \text{id}) \subset \mathbb{T}^d \times K$ as well.

Proof. 1. Set $W_t = (U_t, V_t) : \mathbb{A} \rightarrow \mathbb{A}$. It follows from (A.35) that the map $\theta \rightarrow U_t(\theta, r) - \theta$ can be identified with a 2π -periodic vector function on \mathbb{R}^d . Consider the map

$$f_t = \text{id} + g_t : \mathbb{A} \rightarrow \mathbb{A}, \quad \text{where } g_t(\theta, r) = (U_t(\theta, r) - \theta, 0).$$

By (A.35) one obtains

$$2d \|g_t\|_{1, \mathbb{A}; \kappa} < 2d \|\sigma_\kappa^{-1}(W_t - \text{id})\|_{1, \mathbb{A}; \kappa} \leq \varepsilon_0 < 1$$

for any $t \in [0, \delta]$. The inverse function theorem (Proposition A.11) implies that $f_t : \mathbb{A} \rightarrow \mathbb{A}$ is a diffeomorphism homotope to the identity. In particular, the equation $\varphi = U_t(\theta, r)$ has a unique smooth solution $\theta = \varphi + \phi_t(\varphi, r)$, where ϕ_t can be identified with a 2π -periodic with respect to $\varphi \in \mathbb{R}^d$ function and the map $[0, \delta] \ni t \rightarrow \phi_t \in C^\infty(\mathbb{A}, \mathbb{A})$ is C^1 . Then

$$f_t^{-1} = \text{id} + h_t, \quad \text{where } h_t = (\phi_t, 0).$$

Proposition A.11 applied to

$$\sigma_\kappa^{-1} \circ f_t \circ \sigma_\kappa = \text{id} + g_t \circ \sigma_\kappa \quad \text{and} \quad \sigma_\kappa^{-1} \circ f_t^{-1} \circ \sigma_\kappa = \text{id} + h_t \circ \sigma_\kappa$$

yields

$$\|\phi_t\|_{\ell; \kappa} \leq C_\ell \|g_t\|_{\ell; \kappa} \leq C_\ell \|\sigma_\kappa^{-1}(W_t - \text{id})\|_{\ell, \mathbb{A}; \kappa} \quad (\text{A.39})$$

where $C_\ell > 0$ depends only on ℓ , ε_0 and d . On the other hand, the map W_t is exact symplectic and close to the identity and there exists a C^1 -family of generating functions \tilde{G}_t such that G_t is compactly supported and

$$\nabla_r G_t(\varphi, r) = -\phi_t(\varphi, r), \quad \nabla_\varphi G_t(\varphi, r) = V_t(\varphi + \phi_t(\varphi, r), r) - r.$$

We are going to prove (A.36). The estimate of $\nabla_r G_t$ follows from (A.39). To prove the estimate of $\kappa^{-1} \nabla_\varphi G_t$ we write

$$\nabla_\varphi G_t(\varphi, r) = (V_t(\varphi, r) - r) + \int_0^1 d_\theta V_t(\varphi + s\phi_t(\varphi, r), r) \phi_t(\varphi, r) ds,$$

where d_θ is the partial differential with respect to the first variables θ . Notice that $\|\phi_t\|_{1;\kappa} \leq C_1\varepsilon_0$ in view of (A.35) and (A.39). Then using (A.9), Proposition A.12, 2, and (A.35) we complete the proof of (A.36). Suppose now that $\text{supp}(W_t - \text{id}) \subset \mathbb{T}^d \times K$. Then $\text{supp}(f_t - \text{id}) \subset \mathbb{T}^d \times K$ which implies that $\text{supp}(f_t^{-1} - \text{id}) \subset \mathbb{T}^d \times K$. Hence, $\phi_t(\varphi, r) = 0$ for $\varphi \in \mathbb{T}^n$ and $r \notin K$ and $\text{supp}(\text{sgrad } G_t) \subset \mathbb{T}^d \times K$.

2. In the same way we prove the second part of the Lemma. Suppose that (A.37) holds. Using the inverse function theorem given by Proposition A.11 one solves as above the equation

$$\theta = \varphi - \nabla_r G_t(\varphi, r)$$

with respect to $\varphi \in \mathbb{T}^d$. The corresponding solution has the form $\varphi = \theta + \psi_t(\theta, r)$ and

$$W_t(\theta, r) = (\theta + \psi_t(\theta, r), r + \nabla_\theta G_t(\theta + \psi_t(\theta, r), r)).$$

Moreover, Proposition A.11 yields as above the estimate

$$\|\psi_t\|_{\ell, \mathbb{A}; \kappa} \leq C_\ell \|\sigma_\kappa^{-1} \text{sgrad } G_t\|_{\ell, \mathbb{A}; \kappa} \quad (\text{A.40})$$

for any $t \in [0, \delta]$, where $\ell \geq 0$ and $C_\ell > 0$ depends only on ℓ , ε_0 and n . Then using (A.34) and Proposition A.12 we estimate of $W_t - \text{id}$. We get the same estimates for $(W_t)^{-1} - \text{id}$, where

$$(W_t)^{-1}(\varphi, r - \nabla_\varphi G_t(\varphi, r)) = (\varphi - \nabla_r G_t(\varphi, r), r).$$

To this end we first solve the equation $r - \nabla_\varphi G_t(\varphi, r) = I$ with respect to r and then we proceed as above. \square

The estimates (A.36) and (A.38) are still valid if we add additional parameters $s \in \mathbb{T}^p$ and $\omega \in \mathbb{R}^q$. Set $\mathbb{A} := \mathbb{A}_1 \times \mathbb{A}_2$, where $\mathbb{A}_1 = \mathbb{T}^d \times \mathbb{R}^d$ and $\mathbb{A}_2 = \mathbb{T}^p \times \mathbb{R}^q$. Given $\mu = (\varrho, \kappa)$ with $0 < \varrho, \kappa \leq 1$ and $f \in C^\ell(\mathbb{A}_1 \times \mathbb{A}_2)$ we set $\|f\|_{\ell, \mathbb{A}; \mu} = \|f \circ \sigma_\mu\|_{C^\ell(\mathbb{A})}$, where $\sigma_\mu(\theta, r; s, \omega) = (\theta, \varrho r; s, \kappa \omega)$. We consider the symplectic gradient of the function $(\theta, r) \rightarrow G_t(\theta, r; s, \omega)$ for (s, ω) fixed. Following the proof of Lemma A.14 we obtain

Lemma A.15. *Suppose that the map $[0, \delta] \ni t \rightarrow G_t \in C^\infty(\mathbb{A}, \mathbb{R})$ is C^1 and*

$$(2d + p + q) \|\sigma_\varrho^{-1}(\text{sgrad } G_t(\cdot; s, \omega) - \text{id}_{\mathbb{A}_1}(\cdot))\|_{1, \mathbb{A}_1; \varrho} \leq \varepsilon_0 < 1,$$

for $t \in [0, \delta]$ and $(s, \omega) \in \mathbb{A}_2$, where $\text{id}_{\mathbb{A}_1}$ is the identity map on \mathbb{A}_1 . Then for any $(s, \omega) \in \mathbb{A}_2$ fixed, the function $(\theta, r) \rightarrow \tilde{G}_t(\theta, r; s, \omega) = \langle \theta, r \rangle - G_t(\theta, r; s, \omega)$ is a generating function of an exact symplectic map $W_t(\cdot; s, \omega)$ in \mathbb{A}_1 , the map $[0, \delta] \ni t \rightarrow W_t \in C^\infty(\mathbb{A}, \mathbb{A}_1)$ is C^1 and

$$\|\sigma_\varrho^{-1}(W_t - \text{id}_{\mathbb{A}_1})\|_{\ell, \mathbb{A}; \mu} + \|\sigma_\varrho^{-1}(W_t^{-1} - \text{id}_{\mathbb{A}_1})\|_{\ell, \mathbb{A}; \mu} \leq C_\ell \|\sigma_\varrho^{-1} \text{sgrad } G_t\|_{\ell, \mathbb{A}; \mu} \quad (\text{A.41})$$

where $W_t^{-1}(\cdot; s, \omega)$ is the inverse of $W_t(\cdot; s, \omega)$ in \mathbb{A}_1 with $(s, \omega) \in \mathbb{A}_2$ fixed and $C_\ell > 0$ depends only on ℓ , ε_0 and on the dimensions d, p and q .

Let $D \subset \mathbb{R}^d$ be an open set and $\mathbb{A} = \mathbb{T}^d \times D$. Consider a function $\tilde{G} \in C^\infty(\mathbb{A}, \mathbb{A})$ of the form

$$\tilde{G}(\theta, r) = \langle \theta, r \rangle - K(r) - G(\theta, r),$$

Recall that the map Q defined by $Q(\theta, r) = (\theta + \nabla K(r), r)$ is a symplectic map with generating function $(\theta, r) \rightarrow \langle \theta, r \rangle - K(r)$.

Lemma A.16. *Let $\text{supp } G \subset \mathbb{T}^{n-1} \times F$, where $F \subset D$ is a compact and let G satisfy (A.37). Then the function $(\theta, r) \rightarrow \langle \theta, r \rangle - G(\theta, r)$ is a generating functions of a symplectic transformation $W : \mathbb{A} \rightarrow \mathbb{A}$, G is a generating functions of a symplectic transformations $P : \mathbb{A} \rightarrow \mathbb{A}$, the support of $W - \text{id}$ is contained in $\mathbb{T}^{n-1} \times F$ and $P = W \circ Q$.*

Proof. The assertions about W follow from Lemma A.14. As in the proof of Lemma A.14 one obtains that the map $\theta \rightarrow \theta - \nabla_r G(\theta, r)$ is a diffeomorphism of \mathbb{T}^d homothope to the identity mapping. Then comparing the identities

$$P(\theta - \nabla_r K(r) - \nabla_r G(\theta, r), r) = (\theta, r - \nabla_\theta G(\theta, r))$$

and

$$(W \circ Q)(\theta - \nabla_r K(r) - \nabla_r G(\theta, r), r) = W(\theta - \nabla_r G(\theta, r), r) = (\theta, r - \nabla_\theta G(\theta, r))$$

we obtain the relation $P = W \circ Q$. □

B Appendix.

B.1 Invariant characterization of Liouville billiards

Here we prove the following invariant characterization of Liouville billiard tables defined in [60, Sec. 2].

Theorem 7. *Let (X, g) be a smooth oriented compact and connected Riemannian manifold of dimension two with connected boundary $\Gamma \equiv \partial X$. Assume that*

- (a) *There exists a smooth quadratic in velocities integral of the geodesic flow $I : TX \rightarrow \mathbb{R}$ that is invariant with respect to the reflection at the boundary $TM|_\Gamma \rightarrow TM|_\Gamma$, $\xi \mapsto \xi - 2g(\nu, \xi)$, where ν is the outward unit normal to Γ . In addition, we assume that the metric g does not allow global Killing symmetries;*
- (b) *There is no point $x_0 \in \Gamma$ and a constant $c \in \mathbb{R}$ such that $g_{x_0}(\xi, \xi) = cI_{x_0}(\xi, \xi)$ for any $\xi \in T_{x_0}X$.*

Then (X, g) is isometric to a Liouville billiard table.¹ Conversely, any Liouville billiard table satisfies the properties stated above.

Remark B.1. *The assumption that g does not allow global Killing symmetries is needed for excluding the case when (X, g) is a surface of revolution. Condition (b) can be replace by a similar condition but can not be avoided. One can easily see this by considering the billiard table on the surface of the ellipsoid $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$, $0 < a < b < c$, defined by the condition $y \geq 0$. This billiard table is completely integrable but it is not a Liouville billiard table. Its boundary is the geodesic that corresponds to the intersection of the coordinate plane O_{xz} with the ellipsoid. In particular, this curve is not locally geodesically convex and it contains the four umbilics of the ellipsoid. One can easily see that the billiard table defined this way satisfies all conditions of Theorem 7 except (b). Condition (b) is also needed to ensure that the integral is non-trivial, i.e. $I \not\equiv cg$ where c is a real constant.*

¹In particular, X is diffeomorphic to the unit disk \mathbb{D}^2 in \mathbb{R}^2 .

As a consequence of Theorem 7 we see that there exists a double covering map with two branched points,

$$\tau : C \rightarrow X,$$

where C denotes the cylinder $(\mathbb{R}/\mathbb{Z}) \times [-N, N]$, $N > 0$, coordinatized by the variables x and y respectively, so that the metric $\tau^*(g)$ and the integral $\tau^*(I)$ have the following form on C ,

$$\begin{aligned} dg^2 &= (f(x) - q(y))(dx^2 + dy^2) \\ dI^2 &= \alpha dF^2 + \beta dg^2 \end{aligned} \quad (\text{A.42})$$

where $\alpha \neq 0$ and β are real constants and

$$dF^2 := (f(x) - q(y))(q(y) dx^2 + f(x) dy^2). \quad (\text{A.43})$$

In other words, the integral dI^2 belongs to the pencil of dg^2 and dF^2 . Here $f \in C^\infty(\mathbb{R})$ is 1-periodic, $q \in C^\infty([-N, N])$, and

- (i) f is even, $f > 0$ if $x \notin \frac{1}{2}\mathbb{Z}$, and $f(0) = f(1/2) = 0$;
- (ii) q is even, $q < 0$ if $y \neq 0$, $q(0) = 0$ and $q''(0) < 0$;
- (iii) $f^{(2k)}(l/2) = (-1)^k q^{(2k)}(0)$, $l = 0, 1$, for every natural $k \in \mathbb{N}$.

In particular, if $f \sim \sum_{k=1}^{\infty} f_k x^{2k}$ is the Taylor expansion of f at 0, then, by (iii), the Taylor expansion of q at 0 is $q \sim \sum_{k=1}^{\infty} (-1)^k f_k x^{2k}$.

Remark B.2. *The branched points of the covering correspond to the points $(0, 0)$ and $(1/2, 0)$ of the cylinder C . The metric (A.42) and the integral (A.43) on C vanish at these points.*

Proof of Theorem 7. Consider a tubular neighborhood $V \equiv V(\Gamma) \subseteq X$ of the boundary Γ in X that is diffeomorphic to the strip $(\mathbb{R}/\mathbb{Z}) \times [-\epsilon, 0]$, $\epsilon > 0$, and assume that the boundary Γ corresponds to the circle $(\mathbb{R}/\mathbb{Z}) \times \{0\}$. By gluing two 2-dimensional closed disks along the boundaries of this strip and then by extending the Riemannian metric g to a smooth Riemannian metric g on the corresponding 2-sphere, we obtain an isometrical embedding of our tubular neighborhood V of the boundary Γ into a Riemannian manifold diffeomorphic to the unit 2-sphere \mathbb{S}^2 in \mathbb{R}^3 . Using the metric g on \mathbb{S}^2 and passing to isothermal charts we obtain a complex atlas on \mathbb{S}^2 , that transforms \mathbb{S}^2 into a Riemann surface. Then, by the Riemann mapping theorem, this Riemann surface is biholomorphically equivalent to the standard Riemann sphere that we identify with the complex projective plane $\mathbb{C}P^1$. Taking a point N on $\mathbb{C}P^1$ that does not lie in the image of the strip V and then applying stereographic projection $\mathbb{C}P^1 \setminus \{N\} \rightarrow \mathbb{C}$ we obtain an embedding of the strip V into the complex plane. By construction, the push-forward of the metric g is conformally equivalent to the Euclidean metric on \mathbb{C} . Let $\{(x, y)\}$ and $z = x + iy$ be the coordinates in \mathbb{C} . For simplicity, we will identify the metric g , the integral I , and the neighborhood V and Γ with their corresponding push-forward images. Then we have,

$$dg^2 = \frac{1}{2} \lambda(x, y) (dx^2 + dy^2)$$

and V is a closed domain in \mathbb{C} diffeomorphic to the annulus $(\mathbb{R}/\mathbb{Z}) \times [-\epsilon, 0]$. By construction, g is extended to a smooth Riemannian metric $d\tilde{g}^2 = \frac{\lambda(x, y)}{2} (dx^2 + dy^2)$ on the whole of \mathbb{C} . Let

$\{(p_1, p_2, x, y)\}$ be the standard coordinates on $T^*\mathbb{R}^2$ where \mathbb{R}^2 is identified with \mathbb{C} . Applying the Legendre transform, then passing to complex notations and introducing the complex impulses $p := \frac{1}{2}(p_1 - ip_2)$, $\bar{p} := \frac{1}{2}(p_1 + ip_2)$, and $\partial_p := \partial_{p_1} + i\partial_{p_2}$, $\partial_{\bar{p}} := \partial_{p_1} - i\partial_{p_2}$, we obtain complex coordinates $\{(p, z)\}$ on $T^*\mathbb{R}^2$ so that

$$H = \frac{2p\bar{p}}{\lambda(z, \bar{z})} \quad (\text{A.44})$$

and

$$I = A(z, \bar{z})p^2 + B(z, \bar{z})p\bar{p} + \overline{A(z, \bar{z})}\bar{p}^2 \quad (\text{A.45})$$

with $\overline{B(z, \bar{z})} = B(z, \bar{z})$.² In view of condition (b) of the theorem, the coefficient $A(z, \bar{z})$ does *not* vanish on Γ , i.e.,

$$A(z, \bar{z}) \neq 0 \quad \forall z \in \Gamma. \quad (\text{A.46})$$

In the coordinates $\{(z, p)\}$ on $T^*\mathbb{R}^2$ the canonical symplectic structure ω takes the form $\omega = dp \wedge dz + d\bar{p} \wedge d\bar{z}$. Hence,

$$\{H, I\} = (H_p I_z - H_z I_p) - (H_{\bar{p}} I_{\bar{z}} - H_{\bar{z}} I_{\bar{p}}) = 2\operatorname{Re}(H_p I_z - H_z I_p). \quad (\text{A.47})$$

As I is a first integral of the geodesic flow of g , we have

$$\{H, I\}|_V \equiv 0. \quad (\text{A.48})$$

Using (A.47) one sees that equation (A.48) is equivalent to the following system of equations

$$\begin{cases} A_{\bar{z}} = 0, \\ \lambda A_z + 2\lambda_z A + B\lambda_{\bar{z}} + \lambda B_{\bar{z}} = 0. \end{cases} \quad (\text{A.49})$$

In particular, we see that the coefficient $A(z, \bar{z})$ in front of p^3 in the formula for the integral (A.45) is holomorphic in $z \in V \subseteq \mathbb{C}$, $A = A(z)$. Take z_0 in the interior of V and consider a biholomorphic change of the variable $w = w(z)$ in an open neighborhood of z_0 in the interior of V . Then the expression for the integral (A.45) implies that,

$$\tilde{A}(w) = A(z) \left(\frac{dw}{dz} \right)^2, \quad (\text{A.50})$$

where $\tilde{A}(w)$ is the coefficient in front of $(\tilde{p})^3$ in the expression for the integral I in the chart corresponding to w . Here \tilde{p} is the complex impulse in the chart corresponding to w .³

Remark B.3. *In fact, (A.50) implies that the bivector field,*

$$\Omega_{\{z\}} := A(z) \partial_z \otimes \partial_z,$$

when written in an isothermal atlas, will correspond to a globally defined holomorphic section of the bundle $T_{\mathbb{C}}^{2,0} X \subseteq T_{\mathbb{C}} X$. As the integral I is non-trivial (condition (b)), the holomorphic bivector Ω vanishes only at finitely many points in the interior of X . If X were a closed surface, then by Hopf theorem, $\deg(\Omega) = 2\chi(X)$, where $\chi(X) = 2 - 2g$ is the Euler characteristic of X and $\deg(\Omega)$ is the number of zeros (counted with multiplicities) of Ω . This would imply that $g = 0, 1$, and therefore X would be diffeomorphic to the 2-sphere or the 2-torus ([39, 40]).

² $\bar{I} = I$ as I is real-valued.

³Note that $p = \tilde{p} \frac{dw}{dz}$.

Next, we want to simplify (A.49) by passing to a new complex variable $w = w(z)$, with $w(z)$ holomorphic in the interior of V , so that $\tilde{A}(w) \equiv 1$. In view of (A.50), this amounts to solving the differential equation $1 = A(z)\left(\frac{dw}{dz}\right)^2$. In the case when $A(z_0) \neq 0$ and z_0 lies in the interior of V , the later equation can be solved explicitly in a sufficiently small open disk centered at z_0 ,

$$w(z) = w(z_0) + \int_{z_0}^z \frac{d\lambda}{\sqrt{A(\lambda)}},$$

where the path of integration connecting z_0 with z is C^1 -smooth and lies in the small disk centered at z_0 . The square root $\sqrt{A(\lambda)}$ is holomorphic in the considered disk and is defined up to the choice of the sign. As it was mentioned above, condition (b) of the theorem implies (A.46). Hence, by shrinking the strip V if necessary, we can ensure that $\Gamma \subseteq V$ and $A(z) \neq 0$ for any $z \in V$. Now, take $z_0 \in \Gamma$ and consider the map,

$$\Phi : z \mapsto w(z) := \int_{z_0}^z \frac{d\lambda}{\sqrt{A(\lambda)}}, \quad V \rightarrow \mathbb{C}, \quad (\text{A.51})$$

where the path of integration connecting z_0 with z is C^1 -smooth and is contained in V . Clearly, the map Φ above is well-defined on V and holomorphic in the interior of V . Moreover, it follows from (A.51) that the directional derivatives of Φ of all orders exist and are continuous up to the boundary of V . This allows us to extend Φ to a smooth map defined in some open set $\tilde{V} \supseteq V$. Next, let us consider the image $\Phi(\Gamma)$ of the boundary Γ . Take $z_1 \in \Gamma$. By the inverse function theorem, there exist an open neighborhood $U(z_1)$ of z_1 in \mathbb{C} and an open neighborhood $W(w_1)$ of $w_1 := \Phi(z_1)$ in \mathbb{C} so that $\Phi|_{U(z_1)} : U(z_1) \rightarrow W(w_1)$ is a diffeomorphism. Let $w = u + iv$. Then, as $\tilde{A}(w) = 1$ for all $w \in \Phi(V)$ we conclude from (A.44) and (A.45) that $\tilde{g} := \Phi_*(g)$ and $\tilde{I} := \Phi_*(I)$ are diagonal in the coordinates $\{(u, v)\}$ on $W(w_1) \cap \Phi(V)$ and non-proportional at all points of $\Phi(V)$. In other words, the coordinate vector fields ∂_v and ∂_u on $W(w_1) \cap \Phi(V)$ coincide with the principle directions of the quadratic forms \tilde{g} and \tilde{I} . As by assumption the integral \tilde{I} is invariant with respect to the reflections at the boundary $\Phi(\Gamma)$ we conclude that $\Phi(\Gamma) \cap W(w_1)$ is a coordinate line. As $z_1 \in \Gamma$ was chosen arbitrarily, we see that $\Phi(\Gamma)$ is a straight line. By shrinking the strip V so that $\Gamma \subseteq V$ once more if necessary and by rotating the target copy of \mathbb{C} we get that for some $\delta > 0$,

$$\Phi(V) = \{w = u + iv \mid -\delta \leq v \leq 0\},$$

and $\Phi : V \rightarrow \Phi(V)$ is a smooth covering map. The boundary $\Phi(\Gamma)$ coincides with the real line. This proves that there exist a tubular neighborhood V of Γ in X and $\delta, l > 0$ such that V is diffeomorphic to the cylinder

$$Z := \{z = x + iy \mid x \in \mathbb{R}/l\mathbb{Z}, -\delta \leq y \leq 0\},$$

with Γ corresponding to $(\mathbb{R}/l\mathbb{Z}) \times \{0\}$,

$$H = 2p\bar{p}/\lambda, \quad I = p^2 + Bp\bar{p} + \bar{p}^2, \quad \bar{B} = B, \quad (\text{A.52})$$

and

$$B\lambda_{\bar{z}} + \lambda B_{\bar{z}} = -2\lambda_z. \quad (\text{A.53})$$

Note that (A.53) is equivalent to the vanishing of the Poisson bracket $\{H, I\}$. Equation (A.53) is also equivalent to $(\lambda B)_{\bar{z}} = -2\lambda_z$. By separating its real and the imaginary parts we get

$$\begin{cases} (\lambda B)_x = -2\lambda_x, \\ (\lambda B)_y = 2\lambda_y. \end{cases}$$

This implies that

$$\lambda(B+2) = (\lambda(B+2))|_{(y,0)} = q(y)/4, \quad \lambda(B-2) = (\lambda(B-2))|_{(0,x)} = f(x)/4$$

where $f \in C^\infty(\mathbb{R})$ is 1-periodic and $q \in C^\infty([-\delta, 0])$. By subtracting these two equations we see that,

$$\lambda(x, y) = q(y) - f(x) > 0.$$

This, together with (A.52) and $p = (p_1 - ip_2)/2$ implies that

$$H = \frac{1}{2} \frac{p_1^2 + p_2^2}{q(y) - f(x)} \quad (\text{A.54})$$

and

$$\begin{aligned} 2I &= (B+2)p_1^2 + (B-2)p_2^2 \\ &= \frac{q(y)p_1^2 + f(x)p_2^2}{q(y) - f(x)}. \end{aligned} \quad (\text{A.55})$$

Remark B.4. *Our arguments also show that for any choice of functions $f \in C^\infty(\mathbb{R})$ 1-periodic and $q \in C^\infty([a, b])$, $a, b \in \mathbb{R}$, so that $q(y) - f(x) > 0$ the functions H and I defined by (A.54) and (A.65) are in involution with respect to the canonical symplectic structure $\omega = dp_1 \wedge dx + dp_2 \wedge dy$ on the cotangent bundle to the cylinder $(\mathbb{R}/l\mathbb{Z}) \times [a, b]$.*

This Remark allows us to extend the metric g and its first integral I to a larger cylinder,

$$\tilde{Z} := \{z = x + iy \mid x \in \mathbb{R}/l\mathbb{Z}, -\delta \leq y \leq \delta\},$$

that contains the boundary $\Gamma = (\mathbb{R}/l\mathbb{Z}) \times \{0\}$ in its interior. In order to do this we extend the function q to a function $q \in C^\infty[-\delta, \delta]$ so that $q(y) - f(x) > 0$ on \tilde{Z} and

$$\forall k \geq 1, \quad q^{(k)}(\delta) = 0. \quad (\text{A.56})$$

Then we use (A.54) and (A.65) to extend the metric g and I to smooth quadratic forms on \tilde{Z} . By Remark B.4, I continues to be a quadratic integral of the Riemannian metric g on \tilde{Z} . In this way we extend the Riemannian manifold (X, g) to a smooth Riemannian manifold (\tilde{X}, \tilde{g}) with connected boundary $\tilde{\Gamma}$, so that $X \subseteq \tilde{X}$, Γ is in the interior of \tilde{X} , $\tilde{g}|_X = g$, $\tilde{I}|_X = I$, and \tilde{I} is a quadratic first integral of \tilde{g} . In addition, a collar neighborhood of $\tilde{\Gamma}$ in \tilde{X} can be coordinatized by the cylinder \tilde{Z} so that the Legendre transforms of the metric and the integral are given by (A.54) and (A.65).

Our final step is to take two copies of (\tilde{X}, \tilde{g}) and glue them along their boundaries by a diffeomorphism that, in the coordinates $\{(x, y)\}$, corresponds to the identity,

$$(x, \delta) \mapsto (x, \delta), \quad (\mathbb{R}/l\mathbb{Z}) \times \{\delta\} \rightarrow (\mathbb{R}/l\mathbb{Z}) \times \{\delta\}.$$

In this way we obtain a closed Riemannian manifold (\hat{X}, \hat{g}) . In view of the flatness condition (A.56) on q , the metric \hat{g} and the corresponding quadratic form \hat{I} are smooth, the Riemannian manifold (X, g) is isometrically embedded into (\hat{X}, \hat{g}) , and $\hat{I}|_X = I$. Moreover, by construction, \hat{I} is a quadratic integral of the geodesic flow of \hat{g} . Finally, Theorem 7 follows from the classification theorem for Liouville surfaces (see e.g. [39]). \square

B.2 Kolmogorov Nondegeneracy of the bouncing ball map for Liouville billiards

In this Appendix we show that the Poincaré map of the Liouville billiard tables on the surfaces of constant curvature is non-degenerate at the elliptic fixed point.

Let (X, g) be a Liouville billiard table of classical type. Then there exists a double covering with two branched points

$$\tau : C \rightarrow X \tag{A.57}$$

where C denotes the cylinder $(\mathbb{R}/\mathbb{Z}) \times [-N, N]$, $N > 0$, coordinatized by the variables x and y respectively, so that the pull-back of the Riemannian metric on X and the corresponding quadratic in velocities first integral take the form

$$dg^2 = (f(x) - q(y))(dx^2 + dy^2) \tag{A.58}$$

$$dF^2 = (f(x) - q(y))(q(y)dx^2 + f(x)dy^2) \tag{A.59}$$

where $f \in C^\infty(\mathbb{R})$ is 1-periodic, $q \in C^\infty([-N, N])$, and the hypotheses (i)–(v) in the definition of Liouville billiard tables of classical type hold. In addition, we will assume that f has a *Morse singularity* at $x = 1/4$ which amounts to $f''(1/4) < 0$. Note that the line (taken twice) on the cylinder C corresponding to $x = 1/4$ is an elliptic closed broken geodesic of (X, g) with two vertices. Let

$$f(x) = \alpha_0 + \alpha_1(x - x_0)^2 + \alpha_2(x - x_0)^4 + O((x - x_0)^6) \tag{A.60}$$

where $\alpha_0 > 0$ and $\alpha_1 < 0$ be the Taylor's expansion of f at $x_0 = 1/4$. Let $\{(I, \theta)\}$ be action-angle variables in an open neighborhood in $\mathbf{B}^*\Gamma$ of the elliptic fixed point of the billiard ball map of (X, g) normalized so that $I = 0$ at the elliptic point. We have the following

Theorem B.5. *Denote by K the Hamiltonian that generates the billiard ball map in the action-angle coordinates $\{(I, \theta)\}$. Then*

$$\frac{dH}{dI}(0) = -\frac{\sqrt{-\alpha_1}}{\pi} \int_{-N}^N \frac{dy}{\sqrt{\alpha_0 - q(y)}}$$

and

$$\frac{d^2K}{dI^2}(0) = \frac{\alpha_1}{4\pi^2} \left(2 \int_{-N}^N \frac{dy}{(\alpha_0 - q(y))^{3/2}} - \frac{3\alpha_2}{\alpha_1^2} \int_{-N}^N \frac{dy}{\sqrt{\alpha_0 - q(y)}} \right).$$

Integrable billiard tables on surfaces of constant curvature are examples of Liouville billiard tables of classical type – see [60, §3]. In the case of elliptic billiard tables we have that

$$f(x) = 4\epsilon^2\pi^2 \sin^2 2\pi x \quad \text{and} \quad q(y) = -4\epsilon^2\pi^2 \sinh^2 2\pi y \tag{A.61}$$

where $\epsilon > 0$ is the distance between the center of the ellipse and one of the foci (see [60, §3.1]). As a consequence we obtain

Corollary B.6. *For any $\epsilon > 0$ and for any $N > 0$ we have that $-1 < \frac{dK}{dI}(0) < 0$ and $\frac{d^2K}{dI^2}(0) < 0$. In particular, the Poincaré map of the elliptic billiard ball map is non-degenerate (twisted) at the elliptic fixed point. Moreover, it is 4-elementary except for five different values of the parameter $N > 0$.*

Remark B.7. *Similar results can be proved for the Liouville billiard tables on the surfaces of constant curvature.*

Proof of Theorem B.5. Let $\{(x, y, p_1, p_2)\}$ be the standard coordinates on the cotangent bundle T^*C . By the Legendre transform

$$H = \frac{p_1^2 + p_2^2}{f(x) - q(y)}, \quad F = \frac{q(y)p_1^2 + f(x)p_2^2}{f(x) - q(y)}$$

where

$$p_1 = (f(x) - q(y))\dot{x}, \quad p_2 = (f(x) - q(y))\dot{y} \quad (\text{A.62})$$

and \dot{x} and \dot{y} denote the components of the velocity vectors in TC . For

$$0 < h \leq \max f = \alpha_0$$

consider the invariant with respect to the geodesic flow on T^*C surface

$$Q_h := \{H = 1, F = h\} \subseteq T^*C.$$

Since the variables separate one easily sees that Q_h is characterized by the set of equations

$$p_1^2 = f(x) - h, \quad p_2^2 = h - q(y). \quad (\text{A.63})$$

One concludes from (A.63) and the hypothesis (i)÷(v) on the functions f and q that for $0 < h < \alpha_0$ the surface Q_h consists of two copies of $(\mathbb{R}/\mathbb{Z}) \times [-N, N]$. The billiard reflection map at the boundary of C preserves the boundary of Q_h and can be used to “glue” the two components of Q_h into a single Liouville torus \tilde{Q}_h of the broken geodesic flow on C . By integrating the Liouville form $\kappa = p_1 dx + p_2 dy$ along the two cycles on \tilde{Q}_h that are “parallel” to the coordinate lines on C , we obtain from (A.63) that

$$K(h) = 2 \int_{-N}^N \sqrt{h - q(y)} dy \quad (\text{A.64})$$

and

$$I(h) = 2 \int_{x'_h}^{x''_h} \sqrt{f(x) - h} dx \quad (\text{A.65})$$

where $0 < x'_h \leq 1/4 \leq x''_h < 1/2$ are the two zeros of the equation $f(x) = h$. Note that $x'_h = x''_h = 1/4$ if and only if $h = \alpha_0$. Since f has a Morse singularity at $x_0 = 1/2$ there exists an orientation preserving change of variables $p : U(0) \rightarrow V(0)$ from an open neighborhood of zero $U(0)$ onto an open neighborhood of zero $V(0)$ such that

$$x - x_0 = p(u), \quad (\text{A.66})$$

and

$$f(x) - h = (\alpha_0 - h) - u^2. \quad (\text{A.67})$$

It follows directly from (A.60), (A.66), and (A.67) that

$$p(u) = \frac{1}{\sqrt{-\alpha_1}} y + \frac{\alpha_2}{2\alpha_1^2 \sqrt{-\alpha_1}} y^3 + O(y^5). \quad (\text{A.68})$$

In view of (A.65), (A.66), (A.67), and (A.68) we obtain

$$\begin{aligned} I(h) &= 2 \int_{x'_h}^{x''_h} \sqrt{f(x) - h} dx \\ &= 2 \int_{-\sqrt{\alpha_0 - h}}^{\sqrt{\alpha_0 - h}} p'(u) \sqrt{(\alpha_0 - h) - y^2} dy \\ &= 2(\alpha_0 - h) \int_{-1}^1 p'(u \sqrt{\alpha_0 - h}) \sqrt{1 - u^2} du \\ &= -\frac{\pi}{\sqrt{-\alpha_1}} (h - \alpha_0) + \frac{3\alpha_2 \pi}{8\alpha_1^2 \sqrt{-\alpha_1}} (h - \alpha_0)^2 + O((h - \alpha_0)^3). \end{aligned}$$

Hence,

$$I(\alpha_0) = 0, \quad \frac{dI}{dh}(\alpha_0) = -\pi/\sqrt{-\alpha_1}, \quad \frac{d^2I}{dh^2}(\alpha_0) = 3\pi\alpha_2/4\alpha_1^2\sqrt{-\alpha_1}. \quad (\text{A.69})$$

It follows from (A.64) and the fact that $\alpha_0 > 0$ that K is a C^∞ -smooth function of h in an open neighborhood of $h = \alpha_0$. By combining this with (A.69) we obtain

$$\frac{dK}{dI}(0) = \frac{dK}{dh}(\alpha_0) / \frac{dI}{dh}(\alpha_0) = -\frac{\sqrt{-\alpha_1}}{\pi} \int_{-N}^N \frac{dy}{\sqrt{\alpha_0 - q(y)}} \quad (\text{A.70})$$

and

$$\begin{aligned} \frac{d^2K}{dI^2}(0) &= \left(\frac{d^2K}{dh^2}(\alpha_0) - \frac{dK}{dI}(0) \frac{d^2I}{dh^2}(\alpha_0) \right) / \left(\frac{dI}{dh}(\alpha_0) \right)^2 \\ &= \frac{\alpha_1}{4\pi^2} \left(2 \int_{-N}^N \frac{dy}{(\alpha_0 - q(y))^{3/2}} - \frac{3\alpha_2}{\alpha_1^2} \int_{-N}^N \frac{dy}{\sqrt{\alpha_0 - q(y)}} \right). \end{aligned} \quad (\text{A.71})$$

□

Proof of Corollary B.6. The Corollary follows directly from Theorem B.5 and (A.61). In fact, it follows from (A.61) that

$$f(x) = 4\epsilon^2\pi^2 \left(1 - 4\pi^2(x - 1/4)^2 + \frac{16\pi^4}{3}(x - 1/4)^4 + O((x - 1/4)^6) \right).$$

Hence,

$$\alpha_0 = c^2, \quad \alpha_1 = -4\pi^2 c^2, \quad \alpha_2 = \frac{16\pi^4}{3} c^2, \quad (\text{A.72})$$

where we set for simplicity $c := 2\pi\epsilon$. Then, in view of Theorem B.5 we obtain that

$$\frac{dK}{dI}(0) = -2 \int_{-N}^N \frac{dy}{\sqrt{1 + \sinh^2 2\pi y}}$$

and

$$\frac{d^2 K}{dI^2}(0) = -\frac{1}{2\epsilon\pi} \left(2 \int_{-N}^N \frac{dy}{(1 + \sinh^2 2\pi y)^{3/2}} - \int_{-N}^N \frac{dy}{\sqrt{1 + \sinh^2 2\pi y}} \right).$$

By passing to the variable $v = \sinh 2\pi y$ in the integrals above one obtains that

$$\frac{dK}{dI}(0) = -\frac{2}{\pi} \arctan(\sinh 2\pi N) < 0 \quad (\text{A.73})$$

and

$$\frac{d^2 K}{dI^2}(0) = -\frac{1}{2\epsilon\pi^2} \frac{\sinh 2\pi N}{\cosh^2 2\pi N} < 0 \quad (\text{A.74})$$

which completes the proof of the first two statements of the Corollary. It is clear that the spectrum of the Poincaré map of the elliptic billiard ball map at the elliptic fixed point is equal to $\{e^{\pm i\varphi}\}$ where

$$\varphi \equiv \pm 2\pi \frac{dK}{dI}(0) \pmod{\pi}.$$

The last statement of the Corollary then follows from the definition of the 4-elementary Poincaré maps and formula (A.73). \square

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