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# A few Multiresolution Schemes for the Black-Scholes equation

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**Abstract**

In this note, we apply mutliresolution technics to a discretisation of the linear Black Schole equation, which is a convection diffusion equation. In particular, we describe a few numerical schemes in order to perform one step of decimation and reconstruction. This suggests a new discretisation method of this equation, which should easily extend to more than one dimension, and lead to very efficient numerical schemes.

## 1 Introduction

We are interested in the numerical simulation of the following equation :

$$\frac{\partial u}{\partial \tilde{t}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = r\left(u - S\frac{\partial u}{\partial S}\right) \quad S \in \mathbb{R}^+ \text{ and } \tilde{t} \in [0,T]$$

$$u(S,T) = \left[S - K\right]^+ \qquad S \in \mathbb{R}^+ \text{ and } \tilde{t} = T$$

where $S(\tilde{t})$ is a market price, $u(S,\tilde{t})$ is the option premium for a market price $S$ at time $\tilde{t}$, and $K$ is the price "at-the-money".

By means of the changes of variable $x = \log S$ and $t = T - \tilde{t}$, the last equation is transformed into

$$\frac{\partial u}{\partial t} - v\frac{\partial u}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + ru = 0 \quad x \in \mathbb{R} \text{ and } t \in [0,T]$$

$$u(x,0) = \left[e^x - K\right]^+ \qquad x \in \mathbb{R} \text{ and } t = 0$$

which is a convection diffusion equation submitted to an initial condition with $v = r - \frac{\sigma^2}{2}$.

Since $\sigma^2 > 0$, the singularity of the initial condition is immediately smoothed, and the solution belongs to $C^\infty(\mathbb{R}) \times \mathbb{R}^+$ . Thus one can reasonably expect that very few information is needed to represent the solution.

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The $x$ range must be reduced from $\mathbb{R}$ to a window $[a, b]$, hence the problem with boundary conditions :

$$\begin{cases} \dfrac{\partial u}{\partial t} - v\dfrac{\partial u}{\partial x} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} + ru = 0 & (x,t) \in [a,b] \times [0,T] \\ u(a,t) = 0 \quad \text{and} \quad e^{-b}\dfrac{\partial u}{\partial x}(b,t) = 1 & \text{for } t \in [0,T] \\ u(x,0) = \left[e^x - K\right]^+ & \text{for } t = 0 \end{cases} \tag{1}$$

Several numerical schemes are considered : explicit schemes, both in physical space (equation 3) and in coefficient space with complet multiresolution (equation 4), and implicit schemes (equation 6 in physical space and equation 7 in coefficient space with one step of multiresolution). Most of these different strategies for solving the linear Black-Sholes equation have been developped during the summer program of CEMRACS 2000 (cf. [4]).

# 2 Multiresolution analysis

## 2.1 Decimation and reconstruction

The material of this section is taken from [6, 7]. In [3], this technique was extended to the representation of function to unstructured meshes, and applied to fluid mechanics problems in [1, 2].

One considers a set of embedded meshes

$$\mathcal{U}_R \subset \cdots \subset \mathcal{U}_{L-1} \subset \mathcal{U}_L$$

where $\mathcal{U}_L$ is the finest mesh, *i.e.* the physical space, with $N_L$ points, and $\mathcal{U}_R$ the roughest mesh considered.

To simplify the text, we assume that $N_L = 2^L + 1$ and

$$\sharp \mathcal{U}_j = 2^j + 1 \qquad for\ all\ R \leq j \leq L$$

but we can handle more general cases (as explained in [7] or [3]), where any value of $N_L$ is admissible.

The loss of information from a mesh to another is completed so that the initial $2^L + 1$ datas can always be recovered : let $\mathcal{D}_j$ be these completions. They are defined as sets of $2^j$ elements[1].

Hence a natural definition of global meshes $\mathcal{M}_j$ that contain $2^L + 1$ informations :

$$\mathcal{M}_j = \mathcal{U}_j \oplus \bigoplus_{k=j}^{L-1} \mathcal{D}_k$$

One step of *decimation*, *i.e.* the transit from a mesh $\mathcal{U}_j$ to a mesh $\mathcal{U}_{j-1} \oplus \mathcal{D}_{j-1}$, is given by

$$\begin{cases} u_p^{j-1} = u_{2p}^j & for\ p = 0 \ldots 2^{j-1} \\ d_p^{j-1} = u_{2p+1}^j - \dfrac{u_{2p}^j + u_{2p+2}^j}{2} & for\ p = 0 \ldots 2^{j-1} - 1 \quad \text{(called } \textit{details}\text{)} \end{cases}$$

[1]Indeed, $2^j$ datas are lost when we reduce the mesh from $\mathcal{U}_{j+1}$ to $\mathcal{U}_j$.

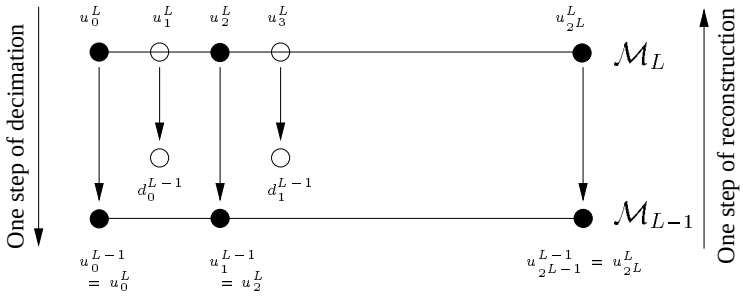


Fig. 1 : One step of decimation and reconstruction.

while *reconstruction* is given by

$$\begin{cases} u_{2p}^j = u_p^{j-1} & for\ p = 0 \dots 2^{j-1} \\ u_{2p+1}^j = \dfrac{u_p^{j-1} + u_{p+1}^{j-1}}{2} + d_p^{j-1} & for\ p = 0 \dots 2^{j-1} - 1 \end{cases}$$

Let $S_j$ be the decimation matrix sample at level $j$ :

$$S_j = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \dots & & & 0 & \dots & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & & 1 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & & 0 & \dots & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & & 0 & 1 & 0 & & \\ & & \ddots & \ddots & & & \ddots & \ddots & & \\ & & 0 & 1 & 0 & & & & \dots & 0 \\ & & & \frac{1}{2} & \frac{1}{2} & & & & 0 & 1 \\ & & & 0 & 1 & & & & & 0 \end{array}\right] \in \mathbb{M}_{(2^j+1)\times(2^j+1)}$$

and reconstruction sample

$$S_j^{-1} = \left[\begin{array}{ccccccccc} 1 & 0 & \dots & & & & & & \\ 0 & 0 & 1 & \dots & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 \\ \hline -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & & & & & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & & & & \ddots & & & & \\ & & & & & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right] \in \mathbb{M}_{(2^j+1)\times(2^j+1)}$$

Hence it is possible to define the $(2^L+1)\times(2^L+1)$ decimation operators by :

$$\left\{\begin{array}{l} M_{j-1} = \left[\begin{array}{c|c} S_j & 0 \\ \hline 0 & Id_{2^L-2^{L-j}} \end{array}\right] \times M_j \\ \\ M_{L-1} = S_L \quad and \quad M_L = Id_{2^L+1} \end{array}\right.$$

and the full reconstruction operators :

$$\left\{\begin{array}{l} M_{j-1}^{-1} = M_j^{-1} \times \left[\begin{array}{c|c} S_j^{-1} & 0 \\ \hline 0 & Id_{2^L-2^{L-j}} \end{array}\right] \\ \\ M_{L-1}^{-1} = S_L^{-1} \quad and \quad M_L^{-1} = Id_{2^L+1} \end{array}\right.$$

$M_{L-j}$ is the operator that makes $j$ steps of decimation, from mesh $\mathcal{M}_L$ to $\mathcal{M}_{L-j}$. Moreover, $M_R$ is the operator of full decimation, from the physical space $\mathcal{M}_L$ to coefficient space $\mathcal{M}_R$.

## 2.2 An explicit scheme

This kind of scheme has very strong CFL conditions. Nevertheless it is interesting to study because of its simplicity. The explicit scheme related to equation (1) can be written

$$\left\{\begin{array}{l} \dfrac{U_i^{n+1}-U_i^n}{\Delta t} - \dfrac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x} - \dfrac{\sigma^2}{2}\dfrac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} + rU_i = 0 \\ \\ \qquad \text{with} \quad F_{i+1/2}^n = \dfrac{1}{2}(vU_i^n + vU_{i+1}^n + |v|(U_{i+1}^n - U_i^n)) \\ \\ U_0^{n+1} = 0 \quad \text{and} \quad U_{2^L}^{n+1} = U_{2^L-1}^{n+1} + \Delta x\, e^{x_{max}} \\ \\ U_i^0 = \left[e^{x_{min}+i\Delta x} - K\right]^+ \end{array}\right. \tag{2}$$

Let be

$$\left\{\begin{array}{l} \alpha = \dfrac{\Delta t}{\Delta x}|v| \\ \\ \mu = \dfrac{\Delta t}{\Delta x^2}\dfrac{\sigma^2}{2} \\ \\ \beta = 1 - r\Delta t \end{array}\right.$$

If $v > 0$ (most of the time) the scheme (2) can be written

$$U^{n+1} = A \times U^n + V \tag{3}$$

with

$$A = \begin{bmatrix} 0 & 0 & & \dots & & 0 \\ \hline \mu & \beta-\alpha-2\mu & \mu+\alpha & & & \\ & \mu & \beta-\alpha-2\mu & \mu+\alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu & \beta-\alpha-2\mu & \mu+\alpha \\ \hline & & & \mu & \beta-\alpha-2\mu & \mu+\alpha \end{bmatrix} \quad and \quad V = \Delta x\, e^{x_{max}} \begin{bmatrix} 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{bmatrix}$$

This is the explicit scheme in physical space.

It is possible to write this scheme in the coefficient space, using the decimation operator. Let $\mathcal{M}_R$ be a minimal mesh[2]. Then $M_R$ is the full decimation operator :

$$\begin{array}{ccc} U^n & \xrightarrow{A\,U+V} & U^{n+1} \\ \uparrow M_R^{-1} & & \downarrow M_R \\ D^n & \xrightarrow{B\,D+W} & D^{n+1} \end{array}$$

The scheme written in the coefficient space is then

$$D^{n+1} = B \times D^n + W \tag{4}$$

with $B = M_R \times A \times M_R^{-1}$ and $W = M_R \times V$. The main drawbacks of this explicit scheme are, on the one hand, a lot of time steps, and on the other hand a lot of matrix multiplications.

## 2.3 Compression

Let $\mathcal{M}_R$ be the minimal admissible mesh and $\epsilon$ a tolerance.

At any time, it is possible to truncate the solution in coefficient space : given a set of details $\{d_k^j\} = \mathcal{D}_j$ at a level $j$, we define

$$\widetilde{d_k^j} = \begin{cases} d_k^j & if\ |d_k^j| > \epsilon_j \\ 0 & otherwise \end{cases}$$

with $\epsilon_j = \dfrac{\epsilon}{2^{j-R}}$.

Then the reconstructed solution after truncation $\widetilde{U^n} = M_R \times \widetilde{D^n}$ satisfies

$$\|\widetilde{U^n} - U^n\|_\infty \leq \epsilon$$

Since $U^n \in \mathbb{R}^{2^L+1}$, we need $2^L + 1$ informations to represent $U^n$ and $D^n$. After truncation, less information is needed within a tolerance $\epsilon$. This amount of information is quantified by the *compression*

[2]The mesh $\mathcal{M}_0$ contains only two points, so it can be useful to consider a cap in decimation.

*factor*

$$\Lambda = \frac{2^L}{2^R + \sum_{j=R}^{L-1} \sharp \left\{ d_k^j \; such \; as \; |d_k^j| > \epsilon_j \; , \; k = 0 \ldots 2^j - 1 \right\}}$$

# 3 Implicit Scheme

## 3.1 A possible implicitation

One of the possible implicitations of the scheme (2) is

$$\left\{ \begin{array}{l} \dfrac{U_i^{n+1} - U_i^n}{\Delta t} - \dfrac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x} - \dfrac{\sigma^2}{2} \dfrac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} + rU_i = 0 \\ \qquad \text{with} \quad F_{i+1/2}^n = \dfrac{1}{2}(vU_i^n + vU_{i+1}^n + |v|(U_{i+1}^n - U_i^n)) \\ U_0^{n+1} = 0 \quad \text{and} \quad U_{2^L}^{n+1} = U_{2^L-1}^{n+1} + \Delta x \, e^{x_{max}} \\ U_i^0 = \left[ e^{x_{min} + i\Delta x} - K \right]^+ \end{array} \right. \tag{5}$$

This scheme can be written

$$A \, U^{n+1} = B \, U^n + V \tag{6}$$

as

$$\left[ \begin{array}{cccccc} 1 & 0 & \ldots & & & \\ \hline -\mu & 1+2\mu & -\mu & & & \\ & -\mu & 1+2\mu & -\mu & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\mu & 1+2\mu & -\mu \\ \hline & & & 0 & -1 & 1 \end{array} \right] U^{n+1} = \left[ \begin{array}{cccccc} 0 & 0 & \ldots & & & \\ \hline 0 & \beta-\alpha & \alpha & & & \\ 0 & 0 & \beta-\alpha & \alpha & & \\ & & & \ddots & \ddots & \\ & & & & \beta-\alpha & \alpha \\ \hline & & & \ldots & 0 & 0 \end{array} \right] U^n + \left[ \begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ \hline \Delta x \, e^{x_{max}} \end{array} \right]$$

## 3.2 One step of multiresolution

Unlike the explicit scheme, we do not need to express the equation in the coefficient space. If we express the equation as a vector with firstly even nodes and then odds, the right member (flux and sources) can be written[3]

[3] First and last nodes are even, respectivly $0$ and $2^L$. This vector $P^n$ can be computed very fast.

$$
P^n = \left[\begin{array}{ccccccccc}
0 & & & \dots & & & & & 0 \\ \hline
0 & 0 & \beta-\alpha & \alpha & & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & & \beta-\alpha & \alpha & 0 & 0 \\
 & & & & & & \beta-\alpha & \alpha \\ \hline
0 & & & \dots & & & & & 0 \\ \hline
0 & \beta-\alpha & \alpha & 0 & \dots & & & & \\
0 & 0 & 0 & \beta-\alpha & \alpha & & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & 0 & 0 & \beta-\alpha & \alpha & 0
\end{array}\right] U^n + \left[\begin{array}{c}
0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline e^{x_{max}}\,\Delta x \\ \hline 0 \\ \vdots \\ 0
\end{array}\right]
\begin{array}{l}
\longleftarrow \text{first node, left boundary} \\
\\
\longleftarrow \text{even nodes} \\
\\
\longleftarrow \text{last node, right boundary} \\
\\
\longleftarrow \text{odd nodes} \\
\\
\end{array}
$$

The implicit scheme can be written as follows

$$(A \times S_L^{-1})\, D^{n+1} = B\, U^n + V$$

which means, in the base "even-then-odd" (*i.e.* after permutation)

$$C \times D^{n+1} = P^n \tag{7}$$

with

$$
C = \left[\begin{array}{cccccc|ccccc}
1 & 0 & 0 & & & 0 & 0 & 0 & \dots & \dots & 0 \\
-\mu/2 & 1+\mu & -\mu/2 & 0 & & 0 & -\mu & -\mu & 0 & \dots & 0 \\
0 & -\mu/2 & 1+\mu & -\mu/2 & 0 & \vdots & 0 & -\mu & -\mu & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & & -\mu/2 & 1+\mu & -\mu/2 & \vdots & & & -\mu & -\mu \\
0 & \dots & \dots & 0 & -1/2 & 1/2 & 0 & \dots & 0 & \dots & -1 \\ \hline
1/2 & 1/2 & 0 & & & 0 & 1+2\mu & 0 & 0 & & 0 \\
0 & 1/2 & 1/2 & 0 & & 0 & 0 & 1+2\mu & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & 0 & \ddots & 0 & \vdots \\
\vdots & & & 1/2 & 1/2 & \vdots & \vdots & & & & \vdots \\
0 & \dots & \dots & 0 & 1/2 & 1/2 & 0 & \dots & \dots & 0 & 1+2\mu
\end{array}\right]
$$

whose block sizes are

$$
\left[\begin{array}{c|c}
(2^{L-1}+1)\times(2^{L-1}+1) & (2^{L-1}+1)\times(2^{L-1}) \\ \hline
(2^{L-1})\times(2^{L-1}+1) & (2^{L-1})\times(2^{L-1})
\end{array}\right]
$$

### 3.3 Example

As an example, we compute the premium of the Call "C90 dec 98" on MATIF Notional Bonds, on october 17th 1990, 44 days before exercise. The updated market price was $K^\star = 97.98$.

This corresponds to the datas :

| Technical datas | | | | Market datas | | | | |
|---|---|---|---|---|---|---|---|---|
| $L$ | $N_L$ | $S_{min}$ | $S_{max}$ | $\sigma$ | $r$ | $K$ | $K^\star$ | $T$ |
| 10 | 1025 | 88 | 108 | 7.5% | 7.5% | 98.0 | 97.98 | 0.12 |

The exact Black formula gives a premium of 1.00, and the implicit multiresolution scheme gives 1.00 : The numerical interpolated value is

$$u(e^{-rT}K^\star, 0) = 1.004$$

with a tick of 0.01.

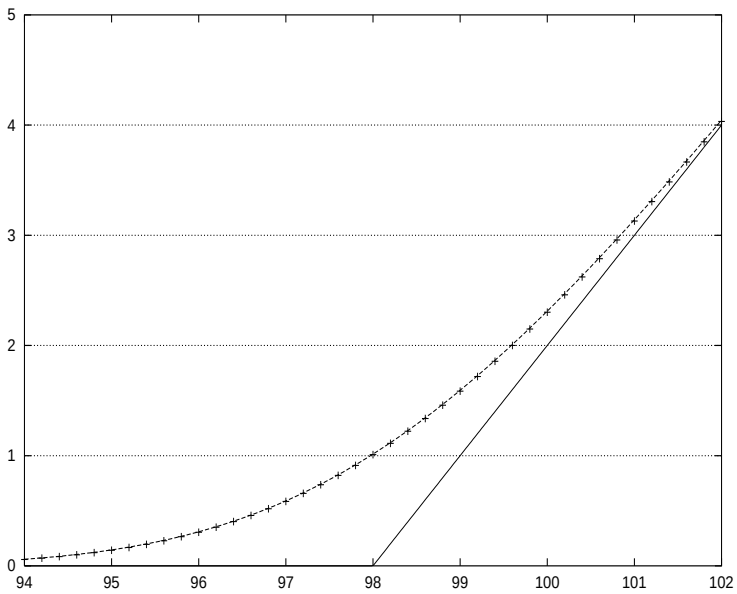


Fig. 2 : Premium of *C90 dec 98* on MATIF Notional Bonds, using one step of multiresolution with the implicit scheme. Points are the evaluations of the exact Black formula.

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